SPACES OF GEODESIC TRIANGULATIONS OF THE SPHERE

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ABSTRACT. We study questions concerning the homotopy-type of the space $GT(K)$ of geodesic triangulations of the standard n -sphere which are (orientation-preserving) isomorphic to K . We find conditions which reduce this question to analogous questions concerning spaces of simplexwise linear embeddings of triangulated n -cells into n -space. These conditions are then applied to the 2-sphere. We show that, for each triangulation K of the 2-sphere, certain large subspaces of $GT(K)$ are deformable (in $GT(K)$) into a subspace homeomorphic to $SO(3)$. It is conjectured that (for $n = 2$) $GT(K)$ has the homotopy of $SO(3)$. In a later paper the authors hope to use these same conditions to study the homotopy type of spaces of geodesic triangulations of the n -sphere, $n > 2$.

0. Introduction. Spaces of geodesic triangulations of spheres were first studied by S. S. Cairns in the early 1940s. In [Cairns, 1941], he announced a determination of the homotopy-type of the space $GT(K)$ of all geodesic triangulations of S^2 which have an orientation preserving isomorphism onto a given triangulation, K . However, in [Cairns, 1944] he announced a defect in that determination and proved the weaker result that $GT(K)$ was path-connected. It is conjectured that $GT(K)$ has the homotopy type of $SO(3)$ for each geodesic triangulation K of S^2 . In this paper we extend Cairns results by showing (see Theorem 1, below) that a certain large subset of $GT(k)$ is deformable in $GT(K)$ to a subset homeomorphic to $SO(3)$. In doing so we define and study in §1 spaces $GT(K)$ of geodesic triangulations of S^n , $n \geq 2$, which are (orientation-preserving) isomorphic to a fixed triangulation K . We label $L \in GT(K)$ by the vertex map $f: K^0 \rightarrow L^0 \subset S^n$ induced by the isomorphism. In §1 we show that $GT(K) \cong O(n+1) \times GT(K; \vec{\sigma})$, where $GT(K; \vec{\sigma})$ is a certain subspace of $\{g \in GT(K) \mid g(a) = a\}$, where we consider $a \in K^0$ as the North Pole of S^n . In §2 we show that a subspace $G \subset GT(K; \vec{\sigma})$ deforms into $G \cap GT(K; S)$ [$GT(K; S) \equiv \{g \in GT(K; \vec{\sigma}) \mid g(K^0 \setminus a) \subset \text{Southern Hemisphere}\}$] whenever, for each $f \in G$, there is a (not necessarily continuously chosen) $g \in G \cap GT(K; S)$ such that $g(v)$ and $f(v)$, $v \in K^0$, are on the same longitudes. There is a gnomonic projection of the Southern Hemisphere onto $\mathbf{R}^n$ which allows us to identify $GT(K; S)$ with spaces of simplexwise linear embeddings of

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$K \setminus \text{int}(\text{st}(a))$ into $\mathbf{R}^n$. In §3 we restrict our attention to $n = 2$ and use the results of §2 to show that $G(\text{Long}) \equiv \{g \in \text{GT}(K; \vec{\sigma}) \mid \text{for some longitude } l, \text{ each 1-simplex in } g(K) \text{ intersecting } l \text{ either has a vertex on } l \text{ or is contained in } l\}$ deforms into $G(\text{Long}) \cap \text{GT}(K; S, c)$ where $\text{GT}(K; S, c) \equiv \{g \in \text{GT}(K; S) \mid g(\text{link}(a)) \text{ is strictly convex}\}$. A theorem of Ethan Bloch's [Bloch, 1985] concerning strictly convex embeddings of a triangulated 2-cell into $\mathbf{R}^2$ implies that (for $n = 2$) $\text{GT}(K; S, c)$ is contractible. As a consequence of Bloch's result and of 2.5 and 3.3 below, we have

THEOREM. (a) $G(\text{Long})$ is contractible to a point in $G(\text{Long}) \cup \text{GT}(K; S, c) \subset \text{GT}(K; \vec{\sigma})$.

(b) There is a deformation in $\text{GT}(K)$ of $\{i \circ g \in \text{GT}(K) \mid g \in G(\text{Long}), i \in \text{O}(3)\}$ to $\{i|K^0 \mid i \in \text{O}(3)\} \cong \text{O}(3)$.

We hope that the techniques of this paper will be useful in establishing the conjecture that $\text{GT}(K)$ has the homotopy type of $\text{SO}(3)$ for $n = 2$. In a later paper the authors will use the results of §§1 and 2 to study the homotopy-type of spaces of geodesic triangulations of S^n , $n > 2$. The only known results for $n > 2$ is a theorem of N. H. Kuiper [Kuiper, 1965] wherein he proves that $\text{GT}(K)$ has the homotopy type of $\text{O}(n+1)$ whenever K is the boundary of the $(n+1)$ -simplex.

Spaces of geodesic triangulations of spheres are related to smoothings of combinatorial manifolds in [Cairns, 1940], [Whitehead, 1961], and [Kuiper, 1965]. For further references and history concerning spaces of simplexwise linear embeddings see [BCH, 1984]. See [Ho, 1979], for results when $n > 2$.

1. Spaces of geodesic triangulations of S^n . We consider S^n to be the unit sphere with the standard metric in $\mathbf{R}^{n+1}$. For any two nonantipodal points, $x \neq y \in S^n$, we denote the unique shortest geodesic (arc of a great circle) joining x to y by (xy) and call it a *geodesic segment*. Thus, if X is a subset of any open hemisphere then we say X is *convex* if $x, y \in X$ implies $(xy) \subset X$ and we define the *convex-hull* of X , $\text{ch}(X)$, as the smallest convex set containing X . A collection of not more than $n+1$ points $X \equiv \{b_0, b_1, \dots, b_n\} \subset S^n$ is said to be in *general position* if, for each $i \leq n+1$, on i distinct points in X are contained in an $(i-1)$ -hyperplane through the origin of $\mathbf{R}^{n+1}$. [An $(i-1)$ -hyperplane through the origin of $\mathbf{R}^{n+1}$ is said to intersect S^n in a *great $(i-2)$ -sphere*.] Using the fact that any collection of i points determines a unique great $(i-1)$ -sphere, it can be proved by induction that any general position collection of not more than $n+1$ points is contained in an open hemisphere. If $\{a_0, a_1, \dots, a_i\} \subset S^n$ are in general position then we define the convex-hull, $\text{ch}\{a_0, a_1, \dots, a_i\}$, to be a *geodesic i -simplex* and denote it by $(a_0 a_1 \cdots a_i)$. For any two Euclidean i -simplices, any 1-to-1 correspondence of the vertices extends uniquely to an (affine) linear map of the i -simplices. There seems to be no analogous result for geodesic simplices. However, for any two geodesic i -simplices Δ_1 and Δ_2 there is a *geodesic map* (a map which takes each geodesic segment to a geodesic segment) $f \equiv (p_2^{-1} \cdot p \cdot p_1): \Delta_1 \rightarrow \Delta_2$, where p_i is a *gnomic projection* (projection from the center of S^n) of Δ_i onto Π , some n -plane tangent to S^n in $\mathbf{R}^{n+1}$, and p is any projective map of $p_1(\Delta_1)$ onto $p_2(\Delta_2)$. Note that such a geodesic map is not uniquely defined. It is for this reason that we deal with "spaces of geodesic triangulations" instead of any sort of "space of simplexwise geodesic homeomorphisms."

Let K be a geodesic triangulation of S^n . (Note that each (geodesic) simplex of K must be contained in an open hemisphere of S^n .) If K^0 is the collection of vertices of K , then any orientation-preserving isomorphism of K onto another geodesic triangulation L is determined by a map $f: K^0 \rightarrow L^0 \subset S^n$. Thus we can consider the space of all such triangulations of S^n as

$$\text{GT}(K) \equiv \{f: K^0 \rightarrow S^n \mid f(K^0) \text{ induces an orientation-preserving isomorphism}\}.$$

Note that $\text{GT}(K)$ is a subspace of $(S^n)^\lambda$, where λ is the number of vertices of K .

Let $\sigma \equiv (aa_1 \cdots a_n)$ be a fixed ordered simplex in K . Let $S(aa_1 \cdots a_i)$ denote the great i -sphere containing $a, a_1, \dots, a_i$ and define the *ray* of $(aa_1 \cdots a_i)$, denoted $(\overrightarrow{aa_1 \cdots a_i})$, to be that hemisphere of $S(aa_1 \cdots a_i)$ which is bounded by the $(i-1)$ -sphere $S(aa_1 \cdots a_{i-1})$ and which contains a_i . Now define

$$\text{GT}(K; \vec{\sigma}) \equiv \{f \in \text{GT}(K) \mid f(a) = a \text{ and } f(aa_1 \cdots a_i) \subset (\overrightarrow{aa_1 \cdots a_i}), 1 \leq i \leq n\}.$$

Consider $\text{SO}(n+1)$, the space of all orientation-preserving isometries of S^n . Then, for each $f \in \text{GT}(K)$, there is a unique $g_f \in \text{SO}(n+1)$ such that $g_f(f(a)) = a$ and $g_f(f(aa_1 \cdots a_i)) \subset (\overrightarrow{aa_1 \cdots a_i}), 1 \leq i \leq n$. Clearly g_f varies continuously with respect to f . Since, for $g \in \text{SO}(n+1)$ and $f \in \text{GT}(K; \vec{\sigma})$, $g \circ f$ belongs to $\text{GT}(K)$, we have the following

PROPOSITION 1.1. *There is a homeomorphism*

$$\text{GT}(K) \xrightarrow{g_f \times g_f \circ f} \text{SO}(n+1) \times \text{GT}(K; \vec{\sigma}).$$

PROOF. The inverse homeomorphism is $(g, f) \rightarrow g^{-1} \circ f$. $\square$

Let S denote the open southern hemisphere with respect to a , considered as the North Pole. Now consider the subspaces

$$\text{GT}(K; S) \equiv \{f \in \text{GT}(K; \vec{\sigma}) \mid f(K^0 \setminus a) \subset S\}$$

and

$$\text{GT}(K; S, c) \equiv \{f \in \text{GT}(K; S) \mid f(\text{link}(a)) \text{ is strictly convex}\}.$$

PROPOSITION 1.2. *For $n = 2$, $\text{GT}(K; S, c)$ is contractible.*

PROOF. Let p be the gnomonic projection of S onto $\mathbf{R}^2$. Then p takes great circles to straight lines. Thus, for each $f \in \text{GT}(K; S)$, $p \circ f(K \setminus \text{st}(a))$ is a triangulated strictly convex triangulated 2-cell in $\mathbf{R}^2$. Thus $f \mapsto p \circ f$ defines a homeomorphism of $\text{GT}(K; S, c)$ onto the space E of strictly convex triangulated 2-cells in $\mathbf{R}^2$ which are orientation-preserving isomorphic to $K \setminus \text{st}(a)$, contain the origin, and fix the vertex $a_1 \in \text{st}(a)$ on a given ray. This space, E is easily seen to be homotopy equivalent to the space D defined in the proof of Theorem 1.1 in §4 [Bloch, 1985]. Ethan Bloch shows that D (and, therefore, E and $\text{GT}(K; S, c)$) are contractible. $\square$

In the next section we study conditions under which a subspace $G \subset \text{GT}(K; \vec{\sigma})$ will deform into $G \cap \text{GT}(K; S)$ or $G \cap \text{GT}(K; S, c)$.

2. Deformation of subspaces of $\text{GT}(K; \vec{\sigma})$ into $\text{GT}(K; S)$. We will move f in $\text{GT}(K; \vec{\sigma})$ in such a way images of each vertex always stay on the same longitude of S^n with respect to a as the North Pole and $s \equiv a'$ as the South Pole. To keep track of these longitudes we use spherical coordinates (θ, φ) . For $x \in S^n \setminus \{a, s\}$,

let θx be the projection (along longitudes) of x onto E , the equatorial great $(n-1)$ -sphere of $\{a, s\}$ and let φx be the angle between x and E considered positive if x is closest to a and negative if x is closest to s . Then $(\theta x, \varphi x)$ are the spherical coordinates of x . For $f \in \text{GT}(K; \vec{\sigma})$ define

$$D_f \equiv \{g \in \text{GT}(K; \vec{\sigma}) \mid g^{-1}(s) = f^{-1}(s) \text{ and } \theta g(v) = \theta f(v), v \neq a, f^{-1}(s)\},$$

$$\begin{aligned} R_f \equiv \{g: K^0 \rightarrow S^n \mid g^{-1}(a) = a, g^{-1}(s) = f^{-1}(s), \theta g(v) = \theta f(v), \\ v \neq a, f^{-1}(s); \text{ and, for } (b_0 b_1 \cdots b_i) \in K, \text{ ch}\{g(b_0), g(b_1), \dots, g(b_i)\} \\ \text{is contained in a great } j\text{-sphere, } j < i, \text{ or } \text{ch}\{g(b_0), g(b_1), \dots, g(b_i)\} \\ \text{is an } i\text{-simplex with the same orientation as } (b_0 b_1 \cdots b_n)\}. \end{aligned}$$

In addition we shall need

$$D'_f \equiv D_f \cap \text{GT}(K; S), \quad D''_f \equiv D_f \cap \text{GT}(K; S, c),$$

and

$$\begin{aligned} R'_f &\equiv \{g \in R_f \mid g(K^0 \setminus a) \subset S\}, \\ R''_f &\equiv \{g \in R'_f \mid g(\text{link}(a)) \text{ is strictly convex}\}. \end{aligned}$$

Note that $\text{GT}(K; \vec{\sigma})$ is contained in the product $\prod\{S^n_v \mid v \in K^0, S^n_v = S^n\} \cong (S^n)^\lambda$. Each $f \in \text{GT}(K; \vec{\sigma})$ maps at most one vertex to s and thus, if $f^{-1}(s) = w$, then

$$D_f \subset P_w \equiv \{a_a\} \times \{s_w\} \times \prod\{S^n_v \setminus \{a, s\} \mid v \in K^0 \setminus \{a, w\}, S^n_v = S^n\},$$

where $a_a [s_w]$ denotes the point $a [s]$ on $S^n_a [S^n_w]$. If $f^{-1}(s) = \emptyset$, the empty set, then

$$D_f \subset P_\emptyset \equiv \{a_a\} \times \prod\{S^n_v \setminus \{a, s\} \mid v \in K^0 \setminus a, S^n_v = S^n\}.$$

We see that $P_\emptyset \cong [S^n \setminus \{a, s\}]^{\lambda-1}$ and $P_w \cong [S^n \setminus \{a, s\}]^{\lambda-2}$.

PROPOSITION 2.1. (a) $R_f [R'_f, R''_f]$ is the relative closure of $D_f [D'_f, D''_f]$ in $P_{f^{-1}(s)}$.

(b) There is a complete metric on each $P_{f^{-1}(s)}$ with respect to which $D_f, D'_f, D''_f, R_f, R'_f$, and R''_f are convex.

PROOF. Let $E \times \mathbf{R}$ have the usual metric and let

$$(\theta, t) \mapsto (\theta, \arctan(t)): E \times \mathbf{R} \rightarrow S^n - \{a, s\}$$

induce a metric on $S^n \setminus \{a, s\}$ and thus on $P_\emptyset \cong [S^n \setminus \{a, s\}]^{\lambda-1}$ and $P_w \cong [S^n \setminus \{a, s\}]^{\lambda-2}$. Let $g \in R_f \setminus D_f$ and notice that in this metric the geodesic from g to f is defined for $0 \leq t \leq 1$ by

$$\Lambda(t)(v) = (\theta f(v), \arctan[t \cdot \tan(\varphi g(v)) + (1-t) \cdot \tan(\varphi f(v))]).$$

Thus Λ moves each $g(v)$ to $f(v)$ along the longitude containing $f(v)$ and along this longitude Λ is linear with respect to $\tan \varphi$. Note that $\Lambda(t)|_{\text{st}(a)} [\Lambda(t)|_{\text{st}(f^{-1}(s))}]$ must be 1-to-1, since $\Lambda(t)(a) = a$ and $\Lambda(t)(f^{-1}(s)) = s$ and each longitude intersects both $\text{lk}(a)$ and $\text{lk}(f^{-1}(s))$ exactly once. If, for some t , $\Lambda(t)$ is not in $\text{GT}(K; \vec{\sigma})$ then there must be an n -simplex $\tau \in K \setminus [\text{st}(a) \cup \text{st}(f^{-1}(s))]$ such that $\Lambda(t)$ collapses

τ . In order for this to happen, there must be a longitude l which intersects the interior τ in a segment and such that $\Lambda(t)[l \cap \tau]$ is a point. Let $\alpha, \beta < \tau$ be the proper faces of τ which contain the endpoints of $l \cap \tau$. Both α and β intersect each longitude at most once and are therefore themselves not collapsed. Thus α and β satisfy the hypotheses of Lemma 2.3 below, and it now follows from Lemma 2.3 that $\tan(\varphi[\Lambda(t)(\alpha) \cap l])$ and $\tan(\varphi[\Lambda(t)(\beta) \cap l])$ vary linearly with respect to t and thus can be equal for at most one value of t and the orientation of the image changes sign at that point. Therefore $\Lambda(t)$ lies entirely in D_f except for the endpoint g . Now if $f \in \text{GT}(K; S)$ and $g \in R'_f [R''_f]$, then $\Lambda(t)$ will stay in $D'_f \subset \text{GT}(K; S)$ [$D''_f \subset \text{GT}(K; S, c)$]. Thus Proposition 2.1 is proved. $\square$

LEMMA 2.2. *If $(b_0 b_1 \cdots b_i)$ is an i -simplex in $S^n \setminus \{a, s\}$ then each longitude which intersects $(b_0 b_1 \cdots b_i)$ also intersects $\text{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\} \subset E$, i.e., if $x = (\theta x, \varphi x) \in (b_0 b_1 \cdots b_i)$ then $\theta x \in \text{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\}$.*

PROOF (BY INDUCTION ON i). For $i = 1$, $(b_0 b_1)$ is contained in the great 2-sphere $S(a, b_0, b_1)$ where the lemma clearly holds. Every point $x \in (b_0 b_1 \cdots b_i)$ can be expressed as a point on a segment $(b_i y)$ for some (unique) $y \in (b_0 b_1 \cdots b_{i-1})$. Thus, by induction, $\theta y \in \text{ch}\{\theta b_0, \theta b_1, \dots, \theta b_{i-1}\}$ and thus every longitude which intersects $(b_i y)$ also intersects $(\theta b_i \theta y) \subset \text{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\}$. $\square$

LEMMA 2.3. *Let $\tau \equiv (b_0 b_1 \cdots b_i)$ be an i -simplex in $S^n \setminus \{a, s\}$ with $\theta b_0, \theta b_1, \dots, \theta b_i$ distinct (and nonantipodal) in E . For each $\theta z \in \text{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\} = (\theta b_0 \theta b_1 \cdots \theta b_i)$, the simplex $(b_0 b_1 \cdots b_i)$ intersects the longitude line of θz in the point $z \equiv (\theta z, \varphi z)$, where, if $\theta b_0, \theta b_1, \dots, \theta b_i, \theta z$ are fixed then $\tan \varphi z$ is a linear combination of $\tan \varphi b_0, \tan \varphi b_1, \dots, \tan \varphi b_i$.*

PROOF (BY INDUCTION ON i). For $z \in (b_0 b_1 \cdots b_i)$, there is a $y \in (b_0 b_1 \cdots b_{i-1})$ such that $z \in (xy)$, where we set $b_i \equiv x$. Consider the 2-sphere $S(a, x, y) \subset \mathbf{R}^3$ with its center at the origin and $a = (0, 0, 1)$ and $(\theta z, 0) = (1, 0, 0)$. Then

$$x = (x_1, x_2, x_3) = (\cos(\theta x - \theta z) \cos \varphi x, \sin(\theta x - \theta z) \cos \varphi x, \sin \varphi x)$$

and

$$y = (y_1, y_2, y_3) = (\cos(\theta y - \theta z) \cos \varphi y, \sin(\theta y - \theta z) \cos \varphi y, \sin \varphi y).$$

The intersection of the line (in $\mathbf{R}^3$) $\overline{xy}$ with the plane $\theta = \theta z$ (or second cartesian coordinates equal 0) is

$$w = x \left[\frac{-y_2}{x_2 - y_2} \right] + y \left[\frac{x_2}{x_2 - y_2} \right]$$

and thus

$$\tan \varphi z = \tan \varphi w = \left[\frac{w_3}{w_1} \right] = \left[\frac{\sin(\theta y - \theta z) \tan \varphi x - \sin(\theta x - \theta y) \tan \varphi y}{\sin(\theta x - \theta y)} \right].$$

Thus $\tan \varphi z$ is a linear combination of $\tan \varphi b_i$ and $\tan \varphi y$. But, by induction, $\tan \varphi y$ is a linear combination of $\tan \varphi b_0, \tan \varphi b_1, \dots, \tan \varphi b_{i-1}$ and thus the conclusion of the lemma follows. $\square$

PROPOSITION 2.4. *Collections $\{D''_f \mid f \in \text{GT}(K; S, c)\}, \{D'_f \mid f \in \text{GT}(K; S)\}$, and $\{D_f \mid f \in \text{GT}(K; \vec{\sigma})\}$ are continuous decompositions.*

PROOF. We must show that $\{f_i\} \rightarrow f$ implies that $\{D_{f_i}\} \rightarrow D_f$, where the latter convergences are convergences of sets. Clearly, $\lim\{D_{f_i}\} \subset D_f$. To show

that $D_f \subset \lim\{D_{f_i}\}$, let $g \in D_f$ and define $g_i(v) \equiv (\theta f_i(v), \varphi g(v))$. Since $\{g_i\} \rightarrow g$, eventually g_i will be in $\text{GT}(K)$ and thus in D_{f_i} . The same argument will work for D'_f and D''_f . $\square$

PROPOSITION 2.5. *Let $G \subset \text{GT}(K; \vec{\sigma})$ be any subset satisfying, for each $f \in G$, $D_f \subset G$ and D'_f [or D''_f] is nonempty. Then there is a continuous function $h: G \rightarrow G' \equiv G \cap \text{GT}(K; S)$ [$G'' \equiv G \cap \text{GT}(K; S, c)$] which is homotopic to the identity and such that $h(D_f) \subset D'_f$ [D''_f].*

COROLLARY 2.6. *For $n = 2$, if G satisfies the hypotheses of Proposition 2.5, then G contracts to a point in $G \cup \text{GT}(K; S, c) \subset \text{GT}(K; \vec{\sigma})$ and*

$$\tilde{G} \equiv \{i \circ f \mid i \in \text{SO}(3) \text{ and } f \in G\} \subset \text{GT}(K)$$

deforms to $\text{SO}(3)$ in $\text{GT}(K)$.

PROOF OF PROPOSITION 2.5 We will now construct a continuous function $h: G \rightarrow G'$ [G''] such that $h(f) \subset D'_f$ [D''_f] for $f \in G$. If $\Lambda_f(t)$ is the geodesic defined in Proposition 2.1 joining f to $h(f)$, then $\Lambda_f(t)$ varies continuously with respect to f and thus defines the desired homotopy of h to the identity. Note that this homotopy moves vertices along longitudes into the Southern Hemisphere. We will give the proof for D'_f , the proof for D''_f is entirely similar.

The construction of $h(f)$ will depend upon whether or not the South Pole, $s \equiv a'$, belongs to $f(K^0)$. Let $G^0 \equiv \{g \in G \mid s \in g(K^0)\}$. Let $f \in G^0$; then, for all $f \in D_f$, we define $h(g) \equiv c(f)$, where $c(f) \in D'_f$ is a "center of gravity" given by Lemma 2.7 below. $c(f)$ varies continuously with respect to f in G^0 and with respect to f in $G \setminus G^0$. Now let $f \in G \setminus G^0$. Then, for some $(b_0 b_1 \cdots b_i) \in K$, $s \in \text{int}(f(b_0)f(b_1) \cdots f(b_i))$. For $0 \leq j \leq i$, define $fj: K^0 \rightarrow S^n$ by $fj(b_j) = s$ and $fj(v) = f(v)$, $v \neq b_j$. Now when $fj \in G$ then $fj \in G^0$ and, for any $g \in D'$, $gj \in G^0$ by Lemma 2.8. Furthermore, D_{fj} has dimension n less than the dimension of D_f , and D_{fj} sits in the boundary (in G) of D_f . Thus we must define $h|D_f$ in such a way that is is compatible with what has already been defined on D_{fj} , $0 \leq j \leq i$. To do this we construct (see Lemma 2.10 and Figure 2.2) arcs $\alpha_{fj}: [0, 1] \rightarrow (\text{closure in } G \text{ of } D'_f)$ such that α_{fj} varies continuously with respect to $f \in G \setminus G^0$, $\alpha_{fj}(0) = c(f)$, $\alpha_{fj}[0, 1] \subset D'_f$, and, if $fj \in G$, then $\alpha_{fj}(1) = c(fj)$. In the case that $fj \in G$ we cannot construct these arcs using the convex structure of Proposition 2.1 because in the structure $c(fj)$ is at infinity. We wish to define $h|D'_f$ so that

$$h(D_f) \subset A_f \equiv \bigcup \{\alpha_{fj}[0, 1] \mid 0 \leq j \leq i\}.$$

We do this as a function of the position of s in $\tau \equiv g(b_0 b_1 \cdots b_i)$, the simplex of $g(K)$ containing s in its interior, $g \in D_f$. For any face $\gamma \subset \tau$, let $c(\gamma)$ denote the center of gravity of γ . We coordinatize the position of s in τ with an i -tuple, $(t_1, t_2, \dots, t_i)$, $0 \leq t_j \leq 1$, in such a way that in the convex structure of S^n we have $s = (1 - t_i)c(\tau) + t_i x_{i-1}$, where x_{i-1} is in an $(i-1)$ -face $\gamma_{i-1} \subset \tau$ and $x_{i-1} = (1 - t_{i-1})c(\gamma_{i-1}) + t_{i-1} x_{i-2}$, where x_{i-2} is an $(i-2)$ -face γ_{i-2} of γ_{i-1} and so on until either we have defined $x_0 \equiv g(b_k)$, $0 \leq k \leq i$, or, for some $1 \leq j < i$, $x_j = c(\gamma_j)$ and thus $t_j = 0$ in which case we define $t_m = 0$, $1 \leq m < j$ (see Figure 2.1).

FIGURE 2.1

Notice that the t_j are continuous functions of $g \in D_f$. Now we can define $h(g) \in A_f$, for $g \in D_f$, by $h(g) \equiv c(f)$ if any $t_j = 0$ and, otherwise;

$$h(g) \equiv \alpha_{fk}(t_k \times t_{i-1} \times \cdots \times t_1) \equiv \alpha_{fk} \text{ (product of the } t_j\text{'s)}.$$

The reader should be able to see that the h so defined is continuous. $\square$

LEMMA 2.7. *For each $g \in D_f \subset G$, we may choose a $c(g) \in D'_f$ [or D''_f] such that (a) if $g' \in D_f$, then $c(g') = c(g)$, and (b) $c(g)$ varies continuously on G^0 and on $G \setminus G^0$ (but not continuously on G).*

PROOF OF LEMMA 2.7. In order to define $c(g)$ we first define, for $f: K^0 \rightarrow S^n$, $E_f \equiv \{g: K^0 \rightarrow S^n \mid g^{-1}(a) = a, g^{-1}(s) = f^{-1}(s) \text{ and } \theta g(v) = \theta f(v), \text{ otherwise}\}.$

Note that $E_f \subset P_{f^{-1}(s)}$ which was defined before Proposition 2.1 and we consider it to inherit the metric defined on $P_{f^{-1}(s)}$. With this metric there is an isometry $\iota: E_f \rightarrow \mathbb{R}^{\lambda-1}$ [or $\mathbb{R}^{\lambda-2}$], where $\iota(g) \equiv \{\tan \varphi g(v) \mid v \in K \setminus g^{-1}\{a, s\}\}$. We compactify E_f by the embedding $\kappa_f: E_f \rightarrow S^{\lambda-1}$ [or $S^{\lambda-2}$], the gnomonic (central) projection with E_f considered as a Euclidean space tangent at its origin to the sphere $S^{\lambda-1}$ [or $S^{\lambda-2}$]. Since κ_f takes geodesics to geodesics, $\kappa_f(D_f)$ and $\kappa_f(D'_f)$ are convex. Define, for $g \in E_f$,

$$c(g) = \kappa_f^{-1}\{\text{center of gravity of } \kappa_f(D'_f)\}.$$

By looking at the definition of the center of gravity in terms of integrals, one may see that $c(g)$ varies continuously whenever the dimension of the convex sets D_f and D'_f does not change. For $E_f \subset G^0$, E_f (and therefore its open subsets D_f and D'_f) has dimension $\lambda - 2$. For $E_f \subset G \setminus G^0$, E_f (and D_f and D'_f) has dimension $\lambda - 1$. Thus, using Proposition 2.4, we see that $c(g)$ varies continuously on G^0 and on $G \setminus G^0$. $\square$

Let $\pi_j: E_f \rightarrow E_{fj}$ be defined by $\pi_j(g)(b_j) = s$ and $\pi_j(g)(v) = g(v)$, otherwise. We denote $\pi_j(g) \equiv gj: K^0 \rightarrow S^n$.

LEMMA 2.8. *If $f \in G \setminus G^0$ and $fj \in G^0$, then $\pi_j(D'_f) = D'_{fj}$ and $\pi_j(D''_f) = D''_{fj}$.*

PROOF OF 2.8 (AND 2.9). Since $fj \in G$, there is a $k \in G^0$ such that $fj \in E_k = E_{fj}$. Let (θ, φ) denote the spherical coordinates of S^n . For each $g \in D'_f$, define $gj_\varphi: K^0 \rightarrow S^n$ by $gj_\varphi(b_j) = (\theta g(b_j), \varphi)$ and $gj_\varphi(v) = g(v)$, otherwise. Let $\tau \equiv (b_j v_1 v_2 \cdots v_n) \in K$ be any n -simplex containing b_j . Then τ has the same orientation as $g(\tau)$ and as

$$k(\tau) = (sk(v_1)k(v_2) \cdots k(v_n))$$

which has the same orientation as

$$(sg(v_1)g(v_2) \cdots g(v_n)) = gj(\tau) = gj_{-(\pi/2)}(\tau),$$

since $\theta k(v_j) = \theta g(v_j)$. Thus s and $g(b_j)$ both lie on the same side of the great $(n-1)$ -sphere $S(g(v_1), g(v_2), \dots, g(v_n))$. Then, for $-(\pi/2) \leq \varphi \leq \varphi g(b_j)$, $gj_\varphi(b_j)$ lies also on this same side of the great $(n-1)$ -sphere and thus $gj_\varphi(\tau)$ has the same orientation as τ . Since this is true for all n -simplexes τ containing b_j , we conclude that $gj_\varphi \in D'_f$ for $-(\pi/2) < \varphi \leq \varphi g(b_j)$, and $\pi_j(g) = gj_{-(\pi/2)} \in D'_k \equiv D'_{fj}$. Thus we have also proved

LEMMA 2.9. *If $fj \in G^0$ and $g \in D'_f$, then $gj_\varphi \in D'_f [D''_f]$ for $-\pi/2 < \varphi \leq \varphi g(b_j)$.*

LEMMA 2.10. *If $f \in G \setminus G^0$, then there is an arc $\alpha_{fj}: [0, 1] \rightarrow D'_f \cup E_{fj} [D''_f \cup E_{fj}]$ such that α_{fj} varies continuously with respect to $f \in G \setminus G^0$, $\alpha_{fj}(0) = c(f)$, $\alpha_{fj}[0, 1) \subset D'_f [D''_f]$, and, if $fj \in G^0$, $\alpha_{fj}(1) = c(fj)$.*

PROOF OF 2.10. If $c(f)$ and $c(fj)$ were in the same E_f , then we could use the straight line joining them in the metric of Proposition 2.1. Instead, $c(fj)$ is at infinity in this metric, so we must be more subtle. Define

$$d(fj) \equiv \kappa_{fj}^{-1} \{\text{center of gravity of } \kappa_{fj}(\pi_j(D'_f))\}$$

as in the proof of Lemma 2.7. In the case that $fj \in G$ (i.e. $E_{fj} \cap G$ is nonempty), then $d(fj) = c(fj)$ because of Lemma 2.7. Let $k_\varphi: K^0 \rightarrow S^n$ be such that $k_\varphi(b_j) = \varphi$ and $k_\varphi(v) = d(fj)(v)$, $v \neq b_j$ (see Figure 2.2).

Let η be the midpoint of the interval $Jf \equiv \{\varphi \mid k_\varphi \in D'_f\} \subset [-\pi/2, \theta]$. Let $\lambda: [0, 1] \rightarrow D'_f$ be the geodesic from $c(f)$ to k_η which is guaranteed by Proposition 2.1. Let $\lambda(t)_\varphi: K^0 \rightarrow S^n$ be defined by $\lambda(t)_\varphi(b_j) = (\theta \lambda(t)(b_j), \varphi)$ and $\lambda(t)_\varphi(v) = \lambda(t)(v)$, $v \neq b_j$. If $fj \notin G^0$, then $k_{\inf\{Jf\}} \subset R_f \subset E_f$ and thus (Proposition 2.1) the geodesic from $c(f)$ to $k_{\inf\{Jf\}}$ lies in D'_f except at its end. This geodesic can be seen to be $\lambda(t)_{\psi(t)}$, where

$$\psi(t) = \arctan[t \cdot \tan(\inf\{Jf\}) + (1-t) \cdot \tan \varphi c(f)].$$

If $fj \in G^0$, then define $\psi(t) \equiv -\pi/2$. It follows from 2.1 and 2.9 that, for $0 \leq t \leq 1$ and $\psi(t) < \varphi \leq \varphi \lambda(t)$, $\lambda(t)_\varphi$ belongs to D'_f . We can now define $\alpha_{fj}(t) \equiv \lambda(t)_{\mu(t)}$, where

$$\mu(t) \equiv t \cdot \varphi(\lambda(t)(b_j)) + (1-t) \cdot (\psi(t)).$$

It can be seen that α_{fj} varies continuously with respect to $f \in G \setminus G^0$ by noticing that, as $\inf\{Jf\} \rightarrow -\pi/2$, $\psi(t) \rightarrow -\pi/2$ for $0 < t \leq 1$. $\square$

FIGURE 2.2

3. Moving triangulations of S^2 into S along longitudes. In this section we restrict our attention to triangulations K of the 2-sphere S^2 and describe a subspace of $\text{GT}(K; \vec{\sigma})$ which satisfies the hypotheses of Proposition 2.5 and which is therefore contractible in $\text{GT}(K; \vec{\sigma})$. The reader can check that the triangulation of S^2 described in Figure 3.1 has no triangulation isomorphic to it which has all its vertices (except a) on the same longitudes and in S . However, for this particular example to work it is necessary that some edge be longer than $\pi/4$ and that the 2-simplices be “twisted” about the North Pole.

We now define a subspace of $\text{GT}(K; \vec{\sigma})$ which does not contain any twisted triangulations. We will then show that this subspace can be deformed into $\text{GT}(K; S, c)$. Let $G(\text{Long}) \equiv \{f \in \text{GT}(K; \vec{\sigma}) \mid \text{for some longitude } l \text{ (w.r.t } a), \text{ each 1-simplex in } f(K) \text{ intersecting } l \text{ either has a vertex on } l \text{ or is contained in } l\}$. Let $f \in G(\text{Long})$. Then, clearly, $D_f \subset G(\text{Long})$. Thus $G(\text{Long})$ satisfies the hypothesis of Proposition 2.5 if we can show that $D'_f \equiv D_f \cap \text{GT}(K; S, c)$ is nonempty for each $f \in G(\text{Long})$. Notice that Proposition 2.1 implies that D''_f is nonempty if there is a $g \in R''_f$. We shall now proceed to produce such a g .

DEFINITIONS AND CONVENTIONS. (a) Let $f \in G(\text{Long})$. f determines a triangulation of S^2 which we denote by $f(K)$. A subcomplex $T < f(K)$ is called *southern* if the underlying space of T is a ball which contains the South Pole $s \equiv a'$ and which is starlike with respect to s . (Note that $T^* \equiv f(K \setminus \text{st}(a))$ is clearly southern.)

(b) Furthermore, T is called *minimal* if it is southern and is either the star of s or contains no interior vertices.

(c) If T is a southern complex, then at least one of the vertices of ∂T lies on the distinguished longitude l . We will now define the *standard enumeration* of the vertices on ∂T . Let v_1 denote the element of $\partial T \cap l$ closest to s . The rest of the vertices on ∂T are enumerated starting at v_1 and continuing in the clockwise direction (with respect to s) along the 1-simplices of ∂T .

FIGURE 3.1

(d) Let T be a southern complex. A 2-simplex $(uv_i v_{i+1}) \in T$ with the edge $(v_i v_{i+1}) \in \partial T$ is called *normal* if $s \neq u$ and the longitude $(\overline{su})$ intersects ∂T in a point of $(v_i v_{i+1})$. A 2-simplex in T is called *spanning* if it has two edges on ∂T and is called *properly spanning* if it is spanning and does not contain s .

REMARK. Note that a 2-simplex may be both normal and properly spanning at the same time if two of its three consecutive vertices lie on the same longitude. However a nonproperly spanning 2-simplex cannot be normal.

PROPOSITION 3.1. *A nonminimal southern complex T always contains either a normal 2-simplex or a properly spanning 2-simplex.*

PROOF. Let $v_1, \dots, v_{j+1} = v_1$ be the standard enumeration of the vertices of ∂T . For each i , $1 \leq i \leq j$, let σ_i denote the 2-simplex in T with edge $(v_i v_{i+1})$, and let Δ_i denote the closed geodesic triangle $(sv_i v_{i+1})$. Note that Δ_i may be degenerate if v_i and v_{i+1} lie on the same longitude. Suppose that T has no properly spanning 2-simplices. We shall show in three cases that there is a normal 2-simplex in T .

Case (a). *There are no spanning 2-simplices of any kind in T and $\text{lk}(s) \cap \partial T$ is empty.* This implies that $\sigma_i \neq \Delta_i$ for all $1 \leq i \leq j$. Let $\sigma_i = (u_i v_i v_{i+1})$. We will label the σ_i as follows: For each i , if $u_i \in \Delta_i$, then σ_i is normal and is left unlabelled. Otherwise, $u_i \in \Delta_k \setminus \Delta_i$, $k \neq i$. If $k > i$, the σ_i is labelled (+), and, if $k < i$, the σ_i is labelled (-). Since $u_i \neq s$, this labelling is defined. The assumption of the nonexistence of a spanning 2-simplex implies that a 2-simplex is either normal or is labelled exactly once. Suppose that neither σ_1 nor σ_j is normal; then σ_1 is necessarily labelled (+) and σ_j is necessarily labelled (-). If k is the smallest integer such that σ_k is negatively labelled, then σ_{k-1} is the desired normal

2-simplex, since it cannot be labelled (+) without running into σ_k . This last fact is due to the restrictions imposed by the distinguished longitude l .

Case (b). There are no spanning 2-simplices and $\text{lk}(s) \cap \partial T$ is not empty. Since T is nonminimal, it follows that for some $1 \leq i \leq j$, $s \notin \sigma_i$. Let i_1, i_2 be such that $1 \leq i_1 \leq i \leq i_2 \leq j$, where i_1 (respectively, i_2) is the least (greatest) integer such that $s \notin \sigma_{i_1}$ ($s \notin \sigma_{i_2}$). If $i_1 = i = i_2$, then σ_i is normal. Otherwise apply the labelling procedure used in Case (a) to the 2-simplices σ_k , $i_1 \leq k \leq i_2$, noting that σ_{i_1} is either labelled (+) or is normal and σ_{i_2} is either labelled (−) or is normal. The rest of the argument in Case (a) now follows to establish the existence of a normal 2-simplex σ_k , $i_1 \leq k \leq i_2$.

Case (c). T contains a spanning 2-simplex, $(v_i v_{i+1} v_{i+2})$, where $s \in (v_i v_{i+1} v_{i+2})$. If $i = 1$, then by hypothesis there exists a vertex in the region bounded by $v_1, v_3, v_4, \dots, v_{j+1} = v_1$. Apply the labelling procedure used in Case (a) to the 2-simplices $\sigma_3, \sigma_4, \dots, \sigma_j$, noting that σ_3 is either normal or is labelled (+) and σ_j is either normal or is labelled (−). The other possibilities, where $i = j$ or $1 \neq i \neq j$, can be treated similarly, each time arriving at the existence of a normal 2-simplex. $\square$

PROPOSITION 3.2. *Let T be a southern complex. Then the subcomplex obtained from T by removing either a normal 2-simplex or a properly spanning 2-simplex is also southern.*

PROOF. One can see that this is true by drawing (or imagining) a representative picture for each of the two cases. $\square$

THEOREM 3.3. *For each $f \in G(\text{Long})$ there exists a $g \in D_f''$.*

PROOF. By Proposition 2.1, it is enough to construct a $g \in R_f''$. Let $T^* = f(K \setminus \text{st}(a))$, as above. We shall construct a function $h: T^* \rightarrow S$ with the following properties:

(i) $h(v)$ and v are on the same longitude for each vertex $v \in T^*$.

(ii) h takes $(v_0 v_1 v_2) \in T^*$ to $\text{ch}\{h(v_0), h(v_1), h(v_2)\}$ which is either a 2-simplex with the same orientation as $(v_0 v_1 v_2)$ or a geodesic segment of length less than π .

(iii) $h(\partial T^*) = h(f(\text{lk}(a)))$ is strictly convex.

Clearly, $h \circ f$ will be the required $g \in R_f''$. We construct h in the following manner. First, for $v \in \partial T^*$, define $h(v) = (\theta v, -\pi/4)$. We define h on the interior of T^* , one 2-simplex at a time. Now suppose that h has been defined on $T^* \setminus \text{int}(T)$, where T is some southern subcomplex of T^* such that conditions (i), (ii), and (iii) are satisfied, and where in addition:

(iv) $h(\partial T)$ bounds a convex region containing s in its interior.

Let $v_1, \dots, v_{j+1} = v_1$ be the standard enumeration of ∂T . Now two cases arise:

Case 1. T is minimal. In this case h is extended to the rest of T in the only possible way, thus completing the proof.

Case 2. T is not minimal. By Proposition 3.1, T either contains a properly spanning 2-simplex or a normal simplex. If T contains a properly spanning 2-simplex $(v_r v_{r+1} v_{r+2})$ then define $h(v_r v_{r+1} v_{r+2}) = (h(v_r) h(v_{r+1}) h(v_{r+2}))$. The fact that orientation is preserved follows from condition (iv). If T contains a normal simplex $(uv_i v_{i+1})$ then define $h(u) = (\overrightarrow{su}) \cup (v_i v_{i+1})$ and note that $\text{ch}\{h(u), h(v_i), h(v_{i+1})\} = (v_i v_{i+1})$.

In either case properties (i)–(iv) are preserved and the remaining subcomplex is southern by Proposition 3.2. Thus, eventually, T will be minimal and thus h will be defined on all of T^* . $\square$

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