SHARP DISTORTION THEOREMS FOR QUASICONFORMAL MAPPINGS

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ABSTRACT. Continuing their earlier work on distortion theory, the authors prove some dimension-free distortion theorems for K -quasiconformal mappings in R^n . For example, one of the present results is the following sharp variant of the Schwarz lemma: If f is a K -quasiconformal self-mapping of the unit ball B^n , $n \geq 2$, with $f(0) = 0$, then $4^{1-K^2}|x|^K \leq |f(x)| \leq 4^{1-1/K^2}|x|^{1/K}$ for all x in B^n .

1. Introduction. Some fundamental distortion properties of K -quasiconformal mappings of the unit ball B^n in R^n are expressed by the two facts that these mappings (1) are Hölder continuous and (2) satisfy a generalized Schwarz lemma. Such results were first obtained in the plane case by L. V. Ahlfors [Ah1] and J. Hersch and A. Pfluger [HP]. Later the multidimensional case was studied by E. D. Callender [C], F. W. Gehring [G], Yu. G. Reshetnyak [R1, R2], B. V. Shabat [Sh], and O. Martio, S. Rickman, and J. Väisälä [MRV]. These n -dimensional results depend essentially on n and they contain constants that are unbounded as n tends to ∞ . The present authors [AVV] recently improved these results by showing that there also exists a dimension-free distortion theory of K -quasiconformal mappings in R^n .

In the present paper we continue our earlier work and examine two conjectures due to A. V. Sychev and to T. Iwaniec, respectively. After this we prove a dimension-free variant of a distortion theorem due to F. W. Gehring and B. G. Osgood [GO, Theorem 3].

Before stating the main results of this paper we introduce some necessary notation and recall those well-known results which will be used throughout. The Grötzsch and Teichmüller condensers in R^n are denoted by $R_{G,n}(s)$, $s > 1$, and $R_{T,n}(t)$, $t > 0$, respectively. Their (conformal) capacities are denoted as in [AVV] by

$$\gamma(s) = \gamma_n(s) = \text{cap } R_{G,n}(s), \quad s > 1,$$

$$\tau(t) = \tau_n(t) = \text{cap } R_{T,n}(t), \quad t > 0.$$

These functions satisfy the basic functional identity

$$(1.1) \quad \gamma(s) = 2^{n-1}\tau(s^2 - 1), \quad s > 1.$$

We sometimes omit the subscript n if there is no danger of confusion.

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Next, for $K > 0$ and $n \geq 2$, we define a homeomorphism $\varphi_K = \varphi_{K,n}: [0, 1] \rightarrow [0, 1]$ by $\varphi_K(0) = 0$, $\varphi_K(1) = 1$ and

$$(1.2) \quad \varphi_K(r) = \varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

when $0 < r < 1$. For $K \geq 1$ and $n \geq 2$, we also set

$$(1.3) \quad \varphi_K^*(r) = \varphi_{K,n}^*(r) = \sup\{|f(x)|: f \in QC_K(B^n), f(0) = 0, |x| = r\},$$

$r \in [0, 1]$, and $\varphi_{K,n}^*(1) = 1$, where $QC_K(B^n)$ is the set of all K -quasiconformal mappings f of B^n into B^n . By the quasiconformal Schwarz lemma [Sh; MRV, 3.1; W] we obtain

$$(1.4) \quad \varphi_{K,n}^*(r) \leq \varphi_{K,n}(r)$$

for all $K \geq 1$, $n \geq 2$, and $r \in [0, 1]$. By [LV, p. 64, Theorem 3.1],

$$(1.5) \quad \varphi_{K,2}^*(r) = \varphi_{K,2}(r)$$

for all $K \geq 1$ and $r \in [0, 1]$. Whether the n -dimensional analogue of (1.5) holds for $n \geq 3$ is an open problem. By [W; AVV, 4.10, 4.14], for all $n \geq 2$ and $K \geq 1$,

$$(1.6) \quad r^\alpha \leq \varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha \leq 2^{1-1/K} K r^\alpha, \quad \alpha = K^{1/(1-n)},$$

where $\lambda_n \in [4, 2e^{n-1})$ [An, (4)] is the Grötzsch ring constant depending only on n . The weaker bound $\varphi_{K,n}(r) \leq \lambda_n r^\alpha$ is given in [MRV, 2.8].

It was conjectured by A. V. Sychev [Sy, p. 89, Remark 2] that

$$(1.7) \quad \varphi_{K,n}^*(r) \leq 4^{1-\alpha} r^\alpha, \quad \alpha = K^{1/(1-n)},$$

for all $n \geq 3$ and $K \geq 1$. Since $\lambda_2 = 4$ [LV, p. 61], inequality (1.7) follows from (1.6) [W] when $n = 2$. But since $\lim_n \lambda_n = \infty$ [An], one cannot derive (1.7) from (1.6) and (1.4) for general n . Furthermore, it follows from [AVV], as we shall point out below in Remark 2.28, that (1.7) with $\varphi_{K,n}^*$ replaced by $\varphi_{K,n}$ is false. However, by using Lemma 2.3 below (an improved version of [Vu2, 3.5]) along with (1.6) and (1.4) we are able to prove our first theorem, which is very close to Sychev's conjecture.

1.8. THEOREM. *For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$ the following inequality holds:*

$$(1.9) \quad \varphi_{K,n}^*(r) \leq 4^{1-1/K^2} r^{1/K}.$$

This inequality is sharp when $K = 1$. Furthermore, the constant 4 cannot be replaced by a smaller one independent of K .

Our second theorem gives an affirmative answer to a conjecture communicated by T. Iwaniec to M. Vuorinen during the 1978 ICM in Helsinki. The theorem is applicable to a class of mappings that is larger than $QC_K(B^n)$, namely the class $QR_K(B^n)$ of all K -quasiregular mappings of B^n into B^n . In a less precise form our theorem is contained in [R1, R2; R3, p. 38; MRV, 3.1]. For the definition and some basic properties of quasiregular mappings the reader is referred to [MRV, I, R3].

1.10. THEOREM. For $n \geq 2$, $r \in (0, 1)$, and $K \in [1, \infty)$ there exists a number $a(r)$ with $\lim_{r \rightarrow 0} a(r) = 1$ such that if $f \in QR_K(B^n)$ then

$$|f(x) - f(y)| \leq a(r) \lambda_n^{1-\alpha} |x - y|^\alpha \leq a(r) 2^{1-1/K} K |x - y|^\alpha$$

for all $x, y \in \bar{B}^n(r)$, where $\alpha = K^{1/(1-n)}$.

We shall actually prove a sharp version of Theorem 1.10, which is perhaps new even for $n = 2$; the above simplified formulation sacrifices the sharpness.

For our third result we require the quasihyperbolic metric ℓ_D of a proper subdomain D of R^n . For $a, b \in D$ we set

$$(1.11) \quad \ell_D(a, b) = \inf_{\gamma} \int_{\gamma} d(x, \partial D)^{-1} ds$$

[GP, GO], where the infimum is taken over all rectifiable arcs γ joining a and b in D . We shall prove the following dimension-free version of a distortion theorem due to Gehring and Osgood [GO, Theorem 3].

1.12. THEOREM. Let $f: D \rightarrow D'$ be a K -quasiconformal mapping, where D and D' are proper subdomains of R^n . Then

$$\begin{aligned} \ell_{D'}(f(x), f(y)) &\leq c(K) \max\{\ell_D(x, y)^\alpha, \ell_D(x, y)\} \\ &\leq c(K) \max\{\ell_D(x, y)^{1/K}, \ell_D(x, y)\} \end{aligned}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$ and $c(K)$ is a constant depending only on K .

The proof of this theorem makes use of recent results on Teichmüller's modulus problem in R^n [Vu3]. We also obtain lower and upper bounds for the least constant $c(K)$ for which this theorem holds.

2. The extremal distortion function $\varphi_{K,n}^*(r)$. We shall adopt the relatively standard notation and terminology of [V]. Unit vectors in the directions of the rectangular coordinate axes in R^n are denoted by $e_1, \dots, e_n$. For $x \in R^n$ and $r > 0$ we let $B^n(x, r) = \{z \in R^n: |x - z| < r\}$, $S^{n-1}(x, r) = \partial B^n(x, r)$, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$.

The set of Möbius transformations in $\bar{R}^n = R^n \cup \{\infty\}$ is denoted by $GM(\bar{R}^n)$, and the subset of sense-preserving transformations by $M(\bar{R}^n)$. For nonempty $D \subset \bar{R}^n$ we let $GM(D) = \{f \in GM(\bar{R}^n): fD = D\}$ and $M(D) = \{f \in M(\bar{R}^n): fD = D\}$. We shall require the Poincaré metric $\rho(x, y)$ on B^n defined by

$$(2.1) \quad \tanh^2 \frac{\rho(x, y)}{2} = \frac{|x - y|^2}{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}$$

for $x, y \in B^n$ or, equivalently, by

$$(2.2) \quad \sinh^2 \frac{\rho(x, y)}{2} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}$$

for $x, y \in B^n$ (cf. [B, p. 40]). In particular,

$$\rho(0, x) = \log \frac{1 + |x|}{1 - |x|} = 2 \operatorname{arctanh} |x|$$

for $x \in B^n$. It is a basic fact that ρ is a $GM(B^n)$ -invariant function—that is, $\rho(x, y) = \rho(Tx, Ty)$ for all $T \in GM(B^n)$ and all $x, y \in B^n$.

The results in this section rely essentially on the following lemma, which is a sharpened version of [Vu2, 3.5]. By means of (1.1) one can show that this lemma is equivalent to the Schwarz lemma (1.4) if $n = 2$, while for $n \geq 3$ a different result is obtained (see (2.15) below).

2.3. LEMMA. *Let $K \geq 1$, $n \geq 2$, and $f \in QC_K(B^n)$. Then*

$$(2.4) \quad \sinh^2 \frac{\rho'}{2} \leq \tau^{-1} \left(\frac{1}{K} \tau \left(\sinh^2 \frac{\rho}{2} \right) \right)$$

and

$$(2.5) \quad \tanh \frac{\rho'}{4} \leq 2 \left(\tanh \frac{\rho}{4} \right)^{1/K},$$

where $\rho = \rho(x, y)$ and $\rho' = \rho(f(x), f(y))$ for all $x, y \in B^n$.

PROOF. The conformal invariant $\lambda_{B^n}(x, y)$ examined in [Vu2] has the following explicit expression (see [Vu2, 2.23]):

$$(2.6) \quad \lambda_{B^n}(x, y) = \frac{1}{2} \tau_n \left(\sinh^2 \frac{\rho(x, y)}{2} \right).$$

The proof of (2.4) now follows directly from (2.6) and [Vu2, Lemma 3.1(2), 2.20]. It follows from (2.6) and [Vu2, 2.14(2)] that

$$\lambda_{B^n}(x, y) \geq c_n \log(\coth \rho(x, y)/4),$$

while

$$\lambda_{fB^n}(f(x), f(y)) \leq \lambda_{B^n}(f(x), f(y)) \leq \frac{c_n}{2} \log \left(4 \coth^2 \frac{\rho(f(x), f(y))}{4} \right)$$

by (2.6) and [Vu2, (5.23)]. Here c_n is a constant depending only on n [V, 10.9]. These two inequalities together with [Vu2, 3.1(2)] yield inequality (2.5). $\square$

2.7. THEOREM. *If $f \in QC_K(B^n)$ and $f(0) = 0$, then*

$$|f(x)| / \left(1 + \sqrt{1 - |f(x)|^2} \right) \leq 2 \left(|x| / \left(1 + \sqrt{1 - |x|^2} \right) \right)^{1/K}$$

for all $x \in B^n$.

PROOF. The proof follows immediately from (2.5) and the observation that

$$\tanh \left(\frac{1}{4} \log \frac{1+s}{1-s} \right) = \frac{s}{1 + \sqrt{1-s^2}}$$

for $s \in [0, 1)$. $\square$

2.8. COROLLARY. For all $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$,

$$(2.9) \quad \varphi_{K,n}^*(r) \leq \min\{2, K\} 2^{1-1/K} r^{1/K}.$$

PROOF. Let $f \in QC_K(B^n)$ with $f(0) = 0$, and $|x| = r < 1$. Since there is nothing to prove if $|f(x)| \leq |x| = r$, we assume that $|f(x)| > |x|$. Then by Theorem 2.7

$$\begin{aligned} |f(x)| &\leq 2 \frac{1 + (1 - |f(x)|^2)^{1/2}}{(1 + (1 - |x|^2)^{1/2})^{1/K}} |x|^{1/K} \leq 2 \left(1 + (1 - |x|^2)^{1/2}\right)^{1-1/K} |x|^{1/K} \\ &\leq 2 \cdot 2^{1-1/K} |x|^{1/K}. \end{aligned}$$

From definition (1.3) it now follows that $\varphi_{K,n}^*(r) \leq 2^{2-1/K} r^{1/K}$. Finally, since $\varphi_{K,n}(r) \leq K \cdot 2^{1-1/K} r^{1/K}$ and $\varphi_{K,n}^*(r) \leq \varphi_{K,n}(r)$ by (1.6) and (1.4), we obtain (2.9). $\square$

2.10. COROLLARY. For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$, the extremal distortion function $\varphi_{K,n}^*$ satisfies

$$(2.11) \quad \varphi_{K,n}^*(r) \leq 4^{1-1/K^2} r^{1/K}$$

and

$$(2.12) \quad \varphi_{K,n}^*(r) \leq 8^{1-1/K} r^{1/K}.$$

Furthermore, the constant 4 in (2.11) cannot be replaced by a smaller absolute constant.

PROOF. One can derive both inequalities from (2.9) by elementary methods, considering separately the two cases $1 \leq K \leq 2$ and $K \geq 2$. We shall give details only for (2.11).

If $1 \leq K \leq 2$, then by (2.9) it suffices to prove that $4^{1-1/K^2} \geq 2^{1-1/K} K$ or, equivalently, that

$$(2.13) \quad 2^{1-2/K^2+1/K} \geq K.$$

Let $g(K) = (1 - 2K^{-2} + K^{-1})\log 2 - \log K$. Then $g(1) = g(2) = 0$ and $g''(K) < 0$ for $1 \leq K \leq 2$; (2.13) holds accordingly.

In case $K \geq 2$, (2.11) follows from (2.9) and the easily verified fact that $4^{1-1/K^2} \geq 2^{1-1/K} \cdot 2$ for such K .

To prove that 4 is the best constant in (2.11) we consider the case $n = 2$. Suppose that $\varphi_{K,2}(r) \leq d^{1-1/K^2} r^{1/K}$ for all $K \geq 1$ and all $r \in [0, 1]$. Then by [LV, p. 65]

$$4^{1-1/K} = \lim_{r \rightarrow 0} r^{-1/K} \varphi_{K,2}(r) \leq d^{1-1/K^2}.$$

Letting $K \rightarrow \infty$ yields $d \geq 4$ as desired. $\square$

The proof of Theorem 1.8 follows from (2.11) and the fact that $\varphi_{1,n}^*(r) = r$. It should be observed that for $n \geq 3$ and $K > 1$, the exponent $1/K$ in (2.11) is not equal to the best exponent $\alpha = K^{1/(1-n)}$ (see (1.4) and (1.6)). On the other hand, the upper bounds in Corollary 2.10 are bounded as K tends to ∞ , while the constant $2^{1-1/K} K$ in (1.6) fails to have this property.

2.14. REMARK. For a function $f \in QC_K(B^n)$ with $f(0) = 0$ one obtains from (2.11), (1.4), and (1.6) the following improved variant of the Schwarz lemma.

$$(2.15) \quad |f(x)| \leq \min\{2^{1-1/K}K|x|^\alpha, 4^{1-1/K^2}|x|^{1/K}\}$$

for all $x \in B^n$. By exploiting inequality (2.15) one can slightly improve the functions η and Θ constructed in [AVV, (5.22) and AVV, Theorem 6.2], respectively.

2.16. DEFINITION. For $n \geq 2$, $K \geq 1$, we set

$$(2.17) \quad \varphi_{1/K,n}^*(r) = \inf\{|f(x)| : f \in QC_K(B^n), f(0) = 0, |x| = r, fB^n = B^n\}$$

for $r \in [0, 1)$ and $\varphi_{1/K,n}^*(1) = 1$, extending definition (1.3).

2.18. REMARK. If $K \geq 1$ and $n \geq 2$ then, by (1.4) and the proof of [AVV, Theorem 4.10],

$$\varphi_{1/K,n}(r) \leq \varphi_{1/K,n}^*(r) \leq r^\beta \leq r^\alpha \leq \varphi_{K,n}^*(r) \leq \varphi_{K,n}(r),$$

where $\alpha = K^{1/(1-n)} = 1/\beta$. Moreover, by the proof of [AVV, Theorem 4.9],

$$\varphi_{1/K,n}^*(r) \leq \varphi_{\alpha,2}(r) \leq \varphi_{\beta,2}(r) \leq \varphi_{K,n}^*(r)$$

(cf. also (1.5)).

2.19. REMARK. It is known that $\varphi_{K,2}^* = \varphi_{K,2}$ for $K \geq 1$ (cf. (1.5)). One may also show that $\varphi_{1/K,2}^* = \varphi_{1/K,2}$ for $K \geq 1$ by considering the inverse of the Grötzsch extremal K -quasiconformal mapping of B^2 onto itself (cf. [LV, Theorem 3.1, p. 64]).

2.20. LEMMA. For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$,

$$\varphi_{1/K,n}^*(r) \geq 2^{1-K}K^{-K}r^\beta,$$

where $\beta = K^{1/(n-1)} = 1/\alpha$.

PROOF. By [AVV, Theorem 4.2 and (4.12)] we see that $\varphi_{1/K,n}^*(r) \geq \lambda_n^{1-\beta}r^\beta$. But from the proof of [AVV, Corollary 4.14] and the fact that $1 - \alpha = (\beta - 1)\alpha$ it follows that $\lambda_n^{\beta-1} \leq 2^{(1-1/K)/\alpha}K^{1/\alpha} \leq 2^{K(1-1/K)}K^K = 2^{K-1}K^K$. Thus $\lambda_n^{1-\beta} \geq 2^{1-K}K^{-K}$. $\square$

2.21. THEOREM. Let f be a K -quasiconformal mapping of B^n onto B^n with $f(0) = 0$. Then $|f(x)| \geq 2^{1-2K}|x|^K$ for all $x \in B^n$.

PROOF. First, if $|f(x)| \geq |x|$, there is nothing to prove. Next, suppose $|f(x)| < |x|$. Since $fB^n = B^n$, Theorem 2.7 applied to f^{-1} yields

$$|x|/\left(1 + \sqrt{1 - |x|^2}\right) \leq 2\left(|f(x)|/\left(1 + \sqrt{1 - |f(x)|^2}\right)\right)^{1/K}.$$

Solving this inequality for $|f(x)|$ and making some obvious estimates gives

$$\begin{aligned} |f(x)| &\geq \frac{1 + \sqrt{1 - |f(x)|^2}}{\left(1 + \sqrt{1 - |x|^2}\right)^K} \left(\frac{|x|}{2}\right)^K \\ &\geq \left(1 + \sqrt{1 - |x|^2}\right)^{1-K} \left(\frac{|x|}{2}\right)^K \geq 2^{1-2K}|x|^K, \end{aligned}$$

as desired. $\square$

2.22. COROLLARY. For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$,

$$\varphi_{1/K,n}^*(r) \geq \max\{2^{1-K}K^{-K}r^\beta, 2^{1-2K}r^K\}.$$

PROOF. This is clear by Lemma 2.20 and Theorem 2.21. $\square$

2.23. COROLLARY. For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$,

$$\varphi_{1/K,n}^*(r) \geq 4^{1-K^2}r^\beta.$$

PROOF. By Corollary 2.22 it suffices to show that

$$2^{1-K}K^{-K} \geq 4^{1-K^2} \quad \text{for } K \geq 1.$$

It is easy to see that this is equivalent to

$$h(K) \equiv K \log K + (1 + K - 2K^2) \log 2 \leq 0$$

for $K \geq 1$. But since $h(1) = 0$, $h'(1) = 1 - 3 \log 2 < 0$, and $h''(K) = 1/K - 4 \log 2 \leq 1 - 4 \log 2 < 0$ we have $h'(K) < 0$ for $K \geq 1$. Hence h is decreasing, so that $h(K) \leq 0$ for all $K \geq 1$. $\square$

2.24. THEOREM. For $n \geq 2$, $K \geq 1$, and $r \in [0, 1]$,

$$(2.25) \quad \varphi_{K,n}^{*2}(r) + \varphi_{1/K,n}^2(r') \leq 1$$

and

$$(2.26) \quad \varphi_{1/K,n}^{*2}(r) + \varphi_{K,n}^2(r') \geq 1,$$

where $r' = \sqrt{1 - r^2}$. In each case there is equality for all r when $n = 2$ or $K = 1$.

PROOF. We prove only (2.25), since the proof of (2.26) is similar. By Lemma 2.3, $\tau_n(\sinh^2(\rho'/2)) \geq \tau_n(\sinh^2(\rho/2))/K$ for $f \in QC_K(B^n)$, $f(0) = 0$, $\rho' = \rho(f(x), f(y))$, and $\rho = \rho(x, y)$. Using the identity $\sinh^2 A = (\tanh^2 A)/(1 - \tanh^2 A)$ and (1.1) we obtain

$$\gamma_n\left(\left(1 - \tanh^2(\rho'/2)\right)^{-1/2}\right) \geq (1/K)\gamma_n\left(\left(1 - \tanh^2(\rho/2)\right)^{-1/2}\right).$$

Now set $y = 0 = f(y)$ and $|x| = r$. Since $\rho(x, y) = 2 \operatorname{arctanh} r$ and $\rho(f(x), f(y)) = 2 \operatorname{arctanh} |f(x)|$, we obtain

$$(2.27) \quad \gamma_n\left(\left(1 - |f(x)|^2\right)^{-1/2}\right) \geq (1/K)\gamma_n\left(\left(1 - r^2\right)^{-1/2}\right).$$

Solving (2.27) for $|f(x)|$ gives

$$|f(x)|^2 \leq 1 - \varphi_{1/K,n}^2(\sqrt{1 - r^2}),$$

from which (2.25) follows when we take the supremum over all such f .

When $n = 2$, equality holds in (2.25) and (2.26) for each $K > 0$ and $r \in [0, 1]$. This fact follows immediately from (1.5) and 2.19. $\square$

2.28. REMARK. In view of (1.4) and Sychev's conjecture (1.7) it is of interest to observe that $\varphi_{K,n}(r)$ is not majorized by $4^{1-\alpha}r^\alpha$, $\alpha = K^{1/(1-n)}$, for $K > 1$ and large n . This is true because by [AVV, (4.5)]

$$r < \tanh(K \operatorname{arctanh} r) \leq \varphi_{K,n}(r)$$

for all $n \geq 2$, $K > 1$, and $r \in (0, 1)$, while $4^{1-\alpha}r^\alpha$ tends to r as n tends to ∞ . On the other hand, we do not know whether $\varphi_{K,n}(r)$ is majorized by $4^{1-1/K}r^{1/K}$ for $n \geq 3$ and $K > 1$.

3. A sharp bound for $|f(x) - f(y)|$. Several well-known Hölder-continuity theorems already existing in the literature have been cited in the introduction. Although some of these results yield the best Hölder exponent $\alpha = K^{1/(1-n)}$, it seems that very few give the correct limit behavior as $K \rightarrow 1$. We shall prove such a theorem, which is suggested by a problem of T. Iwaniec.

For $x \in B^n$ and $M \in (0, \infty)$ we let $D(x, M)$ denote the non-Euclidean ball $\{z \in B^n: \rho(x, z) < M\}$. The Euclidean diameter of a set E in R^n will be denoted by $d(E)$. It follows easily from the well-known formula for $D(z, M)$ [Ah2, p. 86, (4), (5); Vu2, (2.4)] that

$$d(D(z, M)) \leq d(D(0, M)) = 2 \tanh(M/2).$$

Next, given $x, y \in B^n$ let $z \in B^n$ be such that $\rho(x, z) = \rho(y, z) = \rho(x, y)/2$. Then

$$(3.1) \quad |x - y| \leq d(D(z, \rho(x, y)/2)) \leq 2 \tanh(\rho(x, y)/4)$$

for all $x, y \in B^n$.

Moreover, by [Ah2, (32), p. 27],

$$(3.2) \quad |x - y| = [x, y] \tanh \frac{\rho(x, y)}{2} \geq (1 - |x||y|) \tanh \frac{\rho(x, y)}{2}$$

for all $x, y \in B^n$, where $[x, y]^2 = |x|^2|y|^2 - 2x \cdot y + 1$.

In summary, we have

3.3. LEMMA. *If $T \in GM(B^n)$ and $x, y \in B^n$, then*

$$(1 - |Tx||Ty|) \tanh(\rho(x, y)/2) \leq |Tx - Ty| \leq 2 \tanh(\rho(x, y)/4).$$

The first bound is sharp if $Ty = 0$; the second is sharp if T is a rotation fixing the origin and $x = -y$.

PROOF. Because T is an isometry in the Poincaré metric ρ the result follows from (3.1) and (3.2). $\square$

The main result of this section, perhaps new even for $n = 2$, is the following generalization of Lemma 3.3.

3.4. THEOREM. *Let f be a K -quasiregular mapping of B^n into B^n . Then*

$$(3.5) \quad |f(x) - f(y)| \leq b_K(\tanh \tfrac{1}{2}\rho(x, y))$$

for all $x, y \in B^n$, where $b_K(s) = 2\varphi_{K,n}(s)/(1 + \sqrt{1 - \varphi_{K,n}^2(s)})$. The result is sharp if f is a rotation fixing the origin and $x = -y$.

PROOF. If we let $t' = \frac{1}{2}\rho(f(x), f(y))$, it follows from (3.1) that

$$(3.6) \quad |f(x) - f(y)| \leq 2 \tanh \frac{t'}{2} = \frac{2 \tanh t'}{1 + \sqrt{1 - \tanh^2 t'}}.$$

Since $\tanh t' \leq \varphi_{K,n}(\tanh \frac{1}{2}\rho(x, y))$ by [MRV, 3.1] (cf. [Vu2, 3.3]) and since the right side of (3.6) is an increasing function of t' , we obtain (3.5). Since $\varphi_{1,n}(r) \equiv r$, the sharpness assertion follows from Lemma 3.3. $\square$

3.7. COROLLARY. *Under the hypotheses of Theorem 3.4,*

$$(3.8) \quad |f(x) - f(y)| \leq (2\lambda_n)^{1-\alpha} \rho(x, y)^\alpha, \quad \alpha = K^{1/(1-n)},$$

and

$$(3.9) \quad |f(x) - f(y)| \leq \varphi_{K,n}(a) + \varphi_{K,n}^2(a), \quad a = \tanh \frac{\rho(x, y)}{2},$$

for all $x, y \in B^n$.

PROOF. It follows from (3.1) and (1.6) (cf. [Vu2, 3.3(1)]) that

$$\begin{aligned} |f(x) - f(y)| &\leq 2 \tanh(\rho(f(x), f(y))/2) \\ &\leq 2\varphi_{K,n}(\tanh(\rho(x, y)/2)) \leq 2\lambda_n^{1-\alpha}(\rho(x, y)/2)^\alpha \end{aligned}$$

and (3.8) is proved. Formula (3.9) follows from (3.5) and the second part of the elementary inequality

$$(3.10) \quad \frac{4(1+x)}{3+2\sqrt{2}} \leq \frac{2}{1+\sqrt{1-x^2}} \leq 1+x$$

for $0 \leq x \leq 1$. $\square$

A drawback of the sharp inequality (3.5) is its relatively complicated form. In order to derive a more practical alternative form we recall some facts about elliptic integrals. As in [LV, p. 60] we let

$$\mu(r) = \frac{\pi \mathcal{K}(\sqrt{1-r^2})}{2\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2} dx,$$

for each $r \in (0, 1)$. We shall need the well-known inequality

$$(3.11) \quad \log(1/r) < \mu(r) < \log(4/r), \quad 0 < r < 1$$

(cf. [LV, p. 61]).

3.12. LEMMA. *For $n \geq 2$, $K \geq 1$, and $0 < r < 1$, the following inequalities hold:*

$$(3.13) \quad \frac{1+t^{2K}}{(1+t^K)^2} \leq \frac{1}{1+\sqrt{1-\varphi_{K,n}^2(r)}} \leq \frac{1}{2}(1+\varphi_{K,n}(r)),$$

where $t = ((1-r)/(1+r))^{1/2}$. Moreover,

$$(3.14) \quad \frac{1}{1+\sqrt{1-\varphi_{K,n}^2(r)}} \leq \frac{1+(t/2)^{2K}}{(1+(t/2)^K)^2}.$$

PROOF. By [AVV, (4.5)],

$$(3.15) \quad \frac{(1+r)^K - (1-r)^K}{(1+r)^K + (1-r)^K} \leq \varphi_{K,n}(r) \leq \tanh\left(\frac{K}{2}\mu\left(\frac{1-r}{1+r}\right)\right)$$

for $K \geq 1$. The lower bound in (3.13) follows from the lower bound in (3.15). The upper bound in (3.13) is a consequence of the second inequality in (3.10). Finally, from the upper bounds in (3.15) and (3.11) we see that

$$\varphi_{K,n}(r) \leq \tanh\left(K \log \frac{2}{t}\right) = \frac{1 - (t/2)^{2K}}{1 + (t/2)^{2K}},$$

and (3.14) follows by an easy computation. $\square$

3.16. REMARK. For very small values of r , say for $r \in (0, r_0)$, where $\lambda_n^{1-\alpha} r_0^\alpha = 1/2$ and $\alpha = K^{1/(1-n)}$, one can use inequality (1.6) instead of [AVV, (4.5)] to prove an estimate for $\varphi_{K,n}$ that is slightly different from those in Lemma 3.12.

3.17. PROOF OF THEOREM 1.10. Let $r \in (0, 1)$ and $x, y \in \bar{B}^n(r)$. By (3.5), (1.6), (2.1), and (3.13),

$$\begin{aligned} |f(x) - f(y)| &\leq b_K \left(\tanh \frac{\rho(x, y)}{2} \right) \\ &\leq \frac{2\lambda_n^{1-\alpha} (\tanh(\rho(x, y)/2))^\alpha}{1 + \sqrt{1 - \varphi_{K,n}^2(2r/(1+r^2))}} \\ &\leq \frac{\lambda_n^{1-\alpha} (1 + \varphi_{K,n}(2r/(1+r^2))) |x - y|^\alpha}{\left[|x - y|^2 + (1 - |x|^2)(1 - |y|^2) \right]^{\alpha/2}}. \end{aligned}$$

From these estimates and from [AVV, Theorem 4.4] it follows that we may choose

$$\begin{aligned} a(r) &= \left(1 + \varphi_{K,n} \left(\frac{2r}{1+r^2} \right) \right) (1-r^2)^{-\alpha} \\ &\leq \left(1 + \tanh \left(\frac{K}{2} \mu \left(\left(\frac{1-r}{1+r} \right)^2 \right) \right) \right) (1-r^2)^{-\alpha} \end{aligned}$$

in Theorem 1.10. Inequality (3.11) implies that $\lim_{r \rightarrow 0} a(r) = 1$. Finally, $\lambda_n^{1-\alpha} \leq 2^{1-1/K} K$ by [AVV, 4.14]. $\square$

3.18. REMARKS. (1) If we set $K = 1$ and $n = 2$ in (3.5) we find that the upper bound in Lemma 3.3 holds not just for $GM(B^n)$ but for all analytic functions $f: B^2 \rightarrow B^2$.

(2) C. Carathéodory [Ca, Satz 1] proved that

$$|f(x) - f(y)|/|x - y| \leq 1$$

for distinct points $x, y \in \bar{B}^2(\sqrt{2} - 1)$ when $f: B^2 \rightarrow B^2$ is an analytic function with $f(0) = 0$. His result is closely related to Theorem 1.10, although no normalization at 0 is required in 1.10.

(3) For $f \in M(B^n)$ one may improve Theorem 1.10 in the following way. By conformal invariance of the hyperbolic metric, (2.2) yields the estimate

$$|f(x) - f(y)| \leq \frac{1}{1-r^2} |x - y|$$

for all $x, y \in \bar{B}^n(r)$, $r \in (0, 1)$.

4. Distortion of the quasihyperbolic metric. Let D, D' be domains in R^n , $L \geq 1$, and $f: D \rightarrow D'$ a homeomorphism such that

$$(4.1) \quad |x - y|/L \leq |f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in D$. Then we say that f is L -bi-Lipschitz. If (4.1) holds with $L = 1$ then f is called a *Euclidean isometry*.

4.2. LEMMA. *Let D, D' be proper subdomains of R^n and let $f: D \rightarrow D'$ be an L -bi-Lipschitz homeomorphism. Then*

$$(4.3) \quad L^{-2}k_D(a, b) \leq k_{D'}(f(a), f(b)) \leq L^2k_D(a, b)$$

for all $a, b \in D$, where k_D is the quasihyperbolic metric defined in (1.11). In particular, k_D is invariant under Euclidean isometries.

PROOF. It is enough to prove the second inequality, since the first follows from application of the second to f^{-1} .

Suppose γ is a rectifiable arc in D joining a to b . By Theorem 5.3 of [V, p. 12] we have

$$(4.4) \quad \int_{f \circ \gamma} d(y, \partial D')^{-1} |dy| \leq L \int_{\gamma} d(f(x), \partial D')^{-1} |dx|.$$

Choose $\zeta' \in \partial D'$ such that $d(f(x), \partial D') = |f(x) - \zeta'|$. Then there exists a sequence $y_m \in D'$ such that $\lim y_m = \zeta'$. Let $x_m = f^{-1}(y_m)$. By choosing a subsequence, we may assume that $x_m \rightarrow \zeta \in \partial D$. Then

$$(4.5) \quad |f(x) - y_m| = |f(x) - f(x_m)| \geq L^{-1}|x - x_m|.$$

Hence letting m tend to ∞ we get $d(f(x), \partial D')^{-1} \leq Ld(x, \partial D)^{-1}$ for all $x \in \gamma$. The second inequality in (4.3) follows if we combine (4.4) and (4.5), then take the infimum over all γ joining a and b in D . $\square$

We now define

$$j_D(x, y) = \log(1 + |x - y|/\min\{d(x), d(y)\}),$$

where $d(z) = d(z, \partial D)$ for $z \in D$ (cf. [GO; Vu2, (2.26)]).

4.6. COROLLARY. *Under the hypotheses of Lemma 4.2, we have*

$$L^{-2}j_D(x, y) \leq j_{D'}(f(x), f(y)) \leq L^2j_D(x, y)$$

for all $x, y \in D$.

PROOF. As in Lemma 4.2, it is enough to prove the second inequality. We may assume that $d(f(x), \partial D') \leq d(f(y), \partial D')$. Then

$$\begin{aligned} j_{D'}(f(x), f(y)) &= \log(1 + |f(x) - f(y)|/d(f(x), \partial D')) \\ &\leq \log(1 + L^2|x - y|/d(x, \partial D)) \\ &\leq L^2 \log(1 + |x - y|/d(x, \partial D)) \leq L^2j_D(x, y). \end{aligned}$$

Here we have used the proof of Lemma 4.2 and Bernoulli's inequality $t \log(1 + x) \geq \log(1 + tx)$ for $x \geq 0$ and $t \geq 1$ (cf. [Vu1, (2.33)]). $\square$

The next lemma generalizes [GO, Lemma 2].

4.7. LEMMA. For each $K \geq 1$ there exists a constant $a_1 \in ((\frac{1}{2}(\sqrt{3} - \sqrt{2}))^{8K}, 2(\sqrt{3} - \sqrt{2})^{8K})$ with the following property. If D, D' are proper subdomains of R^n , $n \geq 2$, and if $f: D \rightarrow D'$ is K -quasiconformal, then

$$|f(x) - f(y)|/d(f(x), \partial D') \leq 1/2$$

and

$$\frac{|f(x) - f(y)|}{d(f(x), \partial D')} \leq \frac{1}{2} \varphi_{K,n}^* \left(\frac{|x - y|}{a_1 d(x, \partial D)} \right) \leq \frac{K}{2^{1/K} a_1} \left(\frac{|x - y|}{d(x, \partial D)} \right)^\alpha$$

for $y \in B^n(x, a_1 d(x, \partial D))$.

PROOF. The second assertion follows easily from the first one and from definitions (1.3), (1.4), and (1.6). Hence it suffices to prove the first assertion. Since only the maximal inscribed ball $B^n(x, d(x, \partial D))$ will matter in the proof, we may assume that $D = B^n$ and $x = 0$. Let

$$r_{D'}(u, v) = \frac{|u - v|}{\min\{d(u, \partial D'), d(v, \partial D')\}}$$

for $u, v \in D'$. Then by [Vu3, Theorem 4.5],

$$(4.8) \quad \lambda_{D'}(f(x), f(y)) \leq 4\tau(r_{D'}(f(x), f(y))),$$

where $\lambda_{D'}$ represents a conformal invariant studied in [Vu2]. By (4.8), [Vu2, Lemma 3.1(2)], (2.6), and (2.2), we obtain

$$r_{D'}(f(0), f(y)) \leq \tau^{-1} \left(\frac{1}{8K} \tau \left(\frac{|y|^2}{1 - |y|^2} \right) \right).$$

Next, if we require that

$$\tau^{-1} \left(\frac{1}{8K} \tau \left(\frac{|y|^2}{1 - |y|^2} \right) \right) \leq \frac{1}{2}$$

and solve for $|y|$, we obtain the upper bound

$$|y|^2 \leq \tau^{-1}(8K\tau(\frac{1}{2}))/\tau^{-1}(8K\tau(\frac{1}{2})).$$

Thus we may choose a_1 as $[\tau^{-1}(8K\tau(\frac{1}{2}))/\tau^{-1}(8K\tau(\frac{1}{2}))]^{1/2}$ in the theorem or, in light of (1.1) and (1.2),

$$(4.9) \quad a_1 = \sqrt{1 - \varphi_{8K,n}^2(\sqrt{2/3})}.$$

Finally, from (3.13) and (3.14), with $r = \sqrt{2/3}$ and with K replaced by $8K$, we obtain

$$\frac{2(t/2)^{8K}}{1 + (t/2)^{16K}} \leq a_1 = \sqrt{1 - \varphi_{8K,n}^2(r)} \leq \frac{2t^{8K}}{1 + t^{16K}},$$

with $t = \sqrt{3} - \sqrt{2}$. Thus

$$(\frac{1}{2}(\sqrt{3} - \sqrt{2}))^{8K} < a_1 < 2(\sqrt{3} - \sqrt{2})^{8K}. \quad \square$$

4.10. LEMMA. *If D is a proper subdomain of R^n then*

$$(4.11) \quad \ell_D(x, y) \geq j_D(x, y)$$

for all $x, y \in D$ and

$$(4.12) \quad \ell_D(x, y) \leq \log(1 + |x - y|/(d(x) - |x - y|))$$

for $|x - y| < d(x)$, where $d(x) = d(x, \partial D)$.

PROOF. Inequality (4.11) is [GP, (2.2)], and (4.12) is [Vu1, Lemma 2.11]. $\square$

4.13. PROOF OF THEOREM 1.12. In [GO, Theorem 3] this result was proved with the constant $c = 4(4\lambda_n^2)^{1/\alpha}$, which tends to ∞ as n tends to ∞ [An]. We need to show that we can replace this with a universal constant c that does not depend on n . It suffices to prove the first inequality in Theorem 1.12 since

$$\max\{\ell_D(x, y), \ell_D(x, y)^\alpha\} = \ell_D(x, y)^\alpha \leq \ell_D(x, y)^{1/K}$$

for $\ell_D(x, y) \leq 1$, and

$$\max\{\ell_D(x, y), \ell_D(x, y)^\alpha\} = \ell_D(x, y) = \max\{\ell_D(x, y), \ell_D(x, y)^{1/K}\}$$

for $\ell_D(x, y) \geq 1$.

Suppose first that

$$(4.14) \quad |x - y| \leq a_1 d(x, \partial D),$$

where a_1 is as in Lemma 4.7. Then by Lemma 4.7,

$$(4.15) \quad |f(x) - f(y)|/d(f(x), \partial D') \leq 1/2$$

and hence by (4.12)

$$(4.16) \quad \begin{aligned} \ell_{D'}(f(x), f(y)) &\leq \log\left(1 + \frac{|f(x) - f(y)|}{d(f(x), \partial D') - |f(x) - f(y)|}\right) \\ &\leq \frac{2|f(x) - f(y)|}{d(f(x), \partial D')} \leq 1 \end{aligned}$$

since $\log(1 + t) \leq t$ for $t \geq 0$.

On the other hand, since $|x - y|/d(x, \partial D) \leq a_1$ and since $(1/t)\log(1 + t)$ is decreasing for $t > 0$ we see by (4.11) that

$$(4.17) \quad \ell_D(x, y) \geq \log\left(1 + \frac{|x - y|}{d(x, \partial D)}\right) \geq \frac{\log(1 + a_1)}{a_1} \frac{|x - y|}{d(x, \partial D)}.$$

Then by Lemma 4.7 we obtain

$$(4.18) \quad |f(x) - f(y)| \leq \frac{1}{2} d(f(x), \partial D') \varphi_{K,n}^*(|x - y|/(a_1 d(x, \partial D))).$$

Thus by (4.16), (4.18), (1.4), (1.6), and (4.17) we have

$$(4.19) \quad \begin{aligned} \ell_{D'}(f(x), f(y)) &\leq \varphi_{K,n}^*\left(\frac{|x - y|}{a_1 d(x, \partial D)}\right) \leq 2^{1-1/K} K \left(\frac{|x - y|}{a_1 d(x, \partial D)}\right)^\alpha \\ &\leq 2^{1-1/K} K \left(\frac{\ell_D(x, y)}{\log(1 + a_1)}\right)^\alpha. \end{aligned}$$

Next suppose that

$$(4.20) \quad |x - y| > a_1 d(x, \partial D)$$

and choose $z_1, \dots, z_{m+1}$ on a quasihyperbolic geodesic in D joining x and y [GO] so that $z_1 = x$, $z_{m+1} = y$, and

$$|z_l - z_{l+1}|/d(z_l, \partial D) = a_1, \quad |z_m - z_{m+1}|/d(z_m, \partial D) \leq a_1$$

for $l = 1, \dots, m-1$. Then by (4.11)

$$(4.21) \quad \ell_D(x, y) \geq \sum_{l=1}^{m-1} \ell_D(z_l, z_{l+1}) \geq (m-1)\log(1 + a_1).$$

Hence

$$(4.22) \quad m \leq 1 + \ell_D(x, y)/\log(1 + a_1).$$

Then by (4.12), the definition of the number a_1 , and (4.22),

$$(4.23) \quad \begin{aligned} \ell_{D'}(f(x), f(y)) &\leq \sum_{l=1}^m \ell_{D'}(f(z_l), f(z_{l+1})) \\ &\leq \sum_{l=1}^m \log \left(1 + \frac{|f(z_l) - f(z_{l+1})|}{d(f(z_l), \partial D') - |f(z_l) - f(z_{l+1})|} \right) \\ &\leq \left(1 + \frac{\ell_D(x, y)}{\log(1 + a_1)} \right) \log 2. \end{aligned}$$

Since $\ell_D(x, y) \geq j_D(x, y) \geq \log(1 + a_1)$, (4.23) now yields

$$(4.24) \quad \ell_{D'}(f(x), f(y)) \leq 2(\log 2)\ell_D(x, y)/\log(1 + a_1),$$

provided (4.20) holds.

Finally by (4.19) and (4.24) we obtain, for all $x, y \in D$,

$$(4.25) \quad \begin{aligned} \ell_{D'}(f(x), f(y)) &\leq \max\{b_1 \ell_D(x, y)^\alpha, b_2 \ell_D(x, y)\} \\ &\leq \max\{b_1, b_2\} \max\{\ell_D(x, y)^\alpha, \ell_D(x, y)\}, \end{aligned}$$

where $b_1 = 2^{1-1/K}K(\log(1 + a_1))^{-\alpha}$ and $b_2 = (2\log 2)/\log(1 + a_1)$. From Lemma 4.7 it follows easily that

$$\max\{b_1, b_2\} \leq 2K/\log(1 + a_1).$$

Therefore, by (4.25), we may choose $c(K) = 2K/\log(1 + a_1)$ in Theorem 1.12. $\square$

Next, we define

$$(4.26) \quad \bar{c}(K) = \inf\{c(K) : \text{Theorem 1.12 holds with } c(K)\}.$$

We show that $\bar{c}(K) \rightarrow \infty$ as $K \rightarrow \infty$ and provide quantitative lower and upper bounds. To this end the following two lemmas are needed.

4.27. LEMMA. *Let $G = \mathbb{R}^n \setminus \{0\}$, let $x, y \in G$, and let φ be the angle determined by the segments $[0, x]$ and $[0, y]$, $\varphi \in [0, \pi]$. Then*

$$\ell_G(x, y) = \sqrt{\varphi^2 + \log^2(|x|/|y|)}.$$

PROOF. This formula is developed in [MO, §2]. $\square$

Now let $\lambda(K) = (\mu^{-1}(\pi/2K)/\mu^{-1}(\pi K/2))^2$ denote the well-known distortion coefficient of Lehto, Virtanen, and Väisälä [LVV] (cf. [LV, pp. 81, 82, 106–108]). In particular, $\lambda(1) = 1$ and $\lambda(K)$ tends to ∞ as K tends to ∞ .

4.28. LEMMA. *There exists a K -quasiconformal mapping $f: R^n \rightarrow R^n$ such that $f(0) = 0$, $f(\infty) = \infty$, $f(e_1) = e_1$, and $f(-e_1) = -\lambda(K^{1/(n-1)})e_1$.*

PROOF. This follows by rotation of the two-dimensional extremal quasiconformal mapping of [LVV] (cf. [AVV, Theorem 1.14]). $\square$

4.29. THEOREM. *The constant $\bar{c}(K)$ in (4.26) satisfies*

$$(4.30) \quad [1 + \pi^{-2} \log^2 \lambda(K)]^{1/2} \leq \bar{c}(K) \leq 2K [1 + (2(\sqrt{3} + \sqrt{2}))^{8K}].$$

PROOF. For the lower bound fix f as in Lemma 4.28. Then with $D = R^n \setminus \{0\}$, we have by Lemma 4.27 the relations

$$\ell_D(e_1, -e_1) = \pi$$

and

$$\begin{aligned} \ell_D(f(e_1), f(-e_1)) &= \ell_D(e_1, -\lambda(K^{1/(n-1)})e_1) \\ &= [\pi^2 + \log^2 \lambda(K^{1/(n-1)})]^{1/2}. \end{aligned}$$

Thus by the definition of $\bar{c}(K)$ we have

$$[\pi^2 + \log^2 \lambda(K^{1/(n-1)})]^{1/2} \leq \pi \bar{c}(K).$$

Choosing $n = 2$ yields the desired lower estimate.

Finally, by the choice of $c(K)$ in the proof of Theorem 1.12, the estimate $\log(1 + a_1) > a_1/(1 + a_1)$, and Lemma 4.7, we have

$$\bar{c}(K) \leq \frac{2K}{\log(1 + a_1)} < 2K(1 + 1/a_1) < 2K [1 + (2(\sqrt{3} + \sqrt{2}))^{8K}],$$

and the upper estimate follows. $\square$

We next obtain dimension-free versions of distortion results due to Gehring and Osgood [GO, Lemmas 2 and 3].

4.31. LEMMA. *If f is a K -quasiconformal mapping of $\bar{R}^n$ with $f(\infty) = \infty$, then*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} + 1 \leq B \left(\left(\frac{|x - y|}{|x - z|} \right)^K + 1 \right),$$

where $B = (1/a_1) \max\{2, \frac{1}{2}(\min\{2, K\})^K\}$.

PROOF. Let $D = R^n \setminus \{y\}$, $D' = R^n \setminus \{f(y)\}$. Then $|f(x) - f(y)| = d(f(x), \partial D')$, $|x - y| = d(x, \partial D)$. We may assume that

$$|f(x) - f(z)|/|f(x) - f(y)| \leq 2/B,$$

since otherwise the result holds trivially. Then $|f(x) - f(z)| \leq a_1 d(f(x), \partial D')$. So by (2.9) and Lemma 4.7 applied to f^{-1} we get

$$\begin{aligned} \frac{|x - z|}{|x - y|} &\leq \frac{1}{2} \varphi_{K,n}^* \left(\frac{|f(x) - f(z)|}{a_1 |f(x) - f(y)|} \right) \\ &\leq 2^{-1/K} \min\{2, K\} \left(\frac{|f(x) - f(z)|}{a_1 |f(x) - f(y)|} \right)^{1/K}, \end{aligned}$$

or

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq \frac{(\min\{2, K\})^K}{2a_1} \left(\frac{|x - y|}{|x - z|} \right)^K \leq B \left(\frac{|x - y|}{|x - z|} \right)^K,$$

and the result follows. $\square$

4.32. COROLLARY. *Theorem 4 and Corollary 3 in [GO] hold with constants that do not depend on n .*

PROOF. In [GO, Theorem 4] we can choose $c = 2/\alpha = 2\beta \leq 2K$ (as indicated in [GO]) and $d = \log B \leq \log(\max\{2, 2(\min\{2, K\})^K\}/a_1)$. In [GO, Corollary 3] we may choose $c_1 = c(K)$, $c_2 = 2\beta \leq 2K$, $d_2 = \log B$. Then the proof follows as in [GO]. $\square$

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