COMPLEMENTATION IN KREIN SPACES

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ABSTRACT. A generalization of the concept of orthogonal complement is introduced in complete and decomposable complex vector spaces with scalar product.

Complementation is a construction in the geometry of Hilbert space which was applied to the invariant subspace theory of contractive transformations in Hilbert space by James Rovnyak and the author [6]. The concept was later formalized by the author [3]. Continuous and contractive transformations in Krein spaces appear in the estimation theory of Riemann mapping functions [4]. It is therefore of interest to know whether a generalization of complementation theory applies in Krein spaces. Such a generalization is now obtained. The results are also of interest in the invariant subspace theory of continuous and contractive transformations in Krein spaces [5].

The vector spaces considered are taken over the complex numbers. A scalar product for a vector space $\mathcal{H}$ is a complex-valued function $\langle a, b \rangle$ of a and b in $\mathcal{H}$ which is linear, symmetric, and nondegenerate. Linearity means that the identity

$$\langle \alpha a + \beta b, c \rangle = \alpha \langle a, c \rangle + \beta \langle b, c \rangle$$

holds for all elements a, b, and c of $\mathcal{H}$ when α and β are complex numbers. Symmetry means that the identity

$$\langle b, a \rangle = \langle a, b \rangle^-$$

holds for all elements a and b of $\mathcal{H}$. Nondegeneracy means that an element a of $\mathcal{H}$ is zero if the scalar product $\langle a, b \rangle$ is zero for every element b of $\mathcal{H}$.

Every element b of $\mathcal{H}$ determines a linear functional b^- on $\mathcal{H}$ which is defined by $b^- a = \langle a, b \rangle$ for every element a of $\mathcal{H}$. The weak topology of $\mathcal{H}$ is the weakest topology with respect to which b^- is a continuous linear functional on $\mathcal{H}$ for every element b of $\mathcal{H}$. The weak topology of $\mathcal{H}$ is a locally convex topology having the property that every continuous linear functional on $\mathcal{H}$ is of the form b^- for an element b of $\mathcal{H}$. The element b is then unique.

The antispace of a vector space with scalar product is the same vector space considered with the negative of the given scalar product.

A fundamental example of a vector space with scalar product is a Hilbert space. A Krein space is a vector space with scalar product which is the orthogonal sum of a Hilbert space and the antispace of a Hilbert space.

If a vector space with scalar product has infinite dimension, then locally convex topologies exist, other than the weak topology, which have the same continuous linear functionals as the weak topology. The Mackey topology is the strongest such topology [2]. The Mackey topology of a Hilbert space is its norm topology. The topological properties of Krein spaces are the same as those of Hilbert spaces. A transformation of a Krein space into a Krein space is continuous for the Mackey topologies if, and only if, it is continuous for the weak topologies. Continuity is equivalent to the existence of an adjoint.

A transformation P of a Krein space $\mathcal{H}$ into itself is said to be nonnegative if $\langle Pc, c \rangle$ is nonnegative for every element c of $\mathcal{H}$. This condition implies that P is selfadjoint.

Such transformations appear in a construction of Hilbert spaces which are contained continuously in a Krein space.

THEOREM 1. *If P is a nonnegative transformation of a Krein space $\mathcal{H}$ into itself, then a unique Hilbert space $\mathcal{P}$ exists, which is contained continuously in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$.*

PROOF OF THEOREM 1. The theorem is already known in the case that $\mathcal{H}$ is a Hilbert space and that P is a nonnegative and contractive transformation. An elementary proof of the existence and uniqueness of $\mathcal{P}$ is then given by the Hilbert space theory of complementation [3]. The space $\mathcal{P}$ can also be constructed without any use of complementation theory if some knowledge of spectral theory for nonnegative transformations in Hilbert space is assumed. Write $P = Q^2$ for a nonnegative transformation Q. The desired space $\mathcal{P}$ is the range of Q considered with the unique scalar product such that Q acts as a partial isometry of $\mathcal{H}$ onto $\mathcal{P}$. It is easily seen that $\mathcal{P}$ is a Hilbert space, which is contained continuously in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. Uniqueness of $\mathcal{P}$ follows because the range of P is dense in $\mathcal{P}$ by the spectral theory of nonnegative transformations.

The theorem is also easily verified when $\mathcal{H}$ is a Hilbert space and P is not contractive. This result is immediately given by the spectral theory approach to the construction of $\mathcal{P}$. If complementation theory is used, an additional observation needs to be made. A positive number t exists such that the nonnegative transformation $Q = tP$ is contractive. It has been seen that a unique Hilbert space $\mathcal{Q}$ exists, which is contained continuously in $\mathcal{H}$, such that Q coincides with the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. The desired space $\mathcal{P}$ is the space $\mathcal{Q}$ with the scalar product

$$\langle a, b \rangle_{\mathcal{P}} = t \langle a, b \rangle_{\mathcal{Q}}.$$

If $\mathcal{H}$ is not a Hilbert space, then a Hilbert space $\mathcal{H}_+$ exists, which is contained continuously and isometrically in $\mathcal{H}$, such that the orthogonal complement of $\mathcal{H}_+$ in $\mathcal{H}$ is the antispace $\mathcal{H}_-$ of a Hilbert space. Let J be the transformation of $\mathcal{H}$ into itself which is the identity on $\mathcal{H}_+$ and which is minus the identity on $\mathcal{H}_-$. Then J is selfadjoint and unitary. The space $\mathcal{H}$ is a Hilbert space with the new scalar product $\langle Ja, b \rangle$. Since the transformation JP is nonnegative with respect to the new scalar product, a unique Hilbert space $\mathcal{P}_0$ exists, which is contained continuously in $\mathcal{H}$ with respect to the new scalar product, such that JP is the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ with respect to the new scalar product. Let $\mathcal{P}$ be the unique Hilbert space

such that J acts as an isometry of P_0 onto P. Then P is the unique Krein space, which is contained continuously in $\mathcal{H}$ with respect to the given scalar product, such that P is the adjoint of the inclusion of P in $\mathcal{H}$ with respect to the given scalar product.

This completes the proof of the theorem.

A Krein space P is said to be contained contractively in a Krein space $\mathcal{H}$ if P is a vector subspace of $\mathcal{H}$ and if the inequality

$$\langle a, a \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P$$

holds for every element a of P. If the inclusion of P in $\mathcal{H}$ is continuous, then a selfadjoint transformation P of $\mathcal{H}$ into itself exists which coincides with the adjoint of the inclusion of P in $\mathcal{H}$. If c belongs to $\mathcal{H}$, then

$$\langle Pc, c \rangle_{\mathcal{H}} = \langle Pc, Pc \rangle_P \geq \langle Pc, Pc \rangle_{\mathcal{H}} = \langle P^2 c, c \rangle_{\mathcal{H}}.$$

It follows that the inequality $P^2 \leq P$ is satisfied. A converse result holds.

THEOREM 2. *If P is a selfadjoint transformation of a Krein space $\mathcal{H}$ into itself which satisfies the inequality $P^2 \leq P$, then a unique Krein space P exists, which is contained continuously and contractively in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of P in $\mathcal{H}$.*

PROOF OF THEOREM 2. Note that $Q = 1 - P$ is a selfadjoint transformation such that $Q^2 \leq Q$. A construction will also be made of a unique Krein space $\mathcal{Q}$, which is contained continuously and contractively in $\mathcal{H}$, such that Q coincides with the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. Also it will be shown that $\mathcal{Q}$ is the complementary space to P in $\mathcal{H}$; that is, the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P + \langle b, b \rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in P and b in $\mathcal{Q}$, and every element c of $\mathcal{H}$ admits some decomposition for which equality holds.

Note also that if the space P is given, the space $\mathcal{Q}$ is the set of elements a of $\mathcal{H}$ such that

$$\langle a, a \rangle_{\mathcal{Q}} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle b, b \rangle_P]$$

is finite, where the least upper bound is taken over all elements b of P. Conversely, the space P is determined by a knowledge of the space $\mathcal{Q}$.

Neither space is known at the start of the proof, but some elements of the space P are known. These elements are in the range of P. The proposed scalar product in P of two such elements

$$\langle Pa, Pb \rangle_P = \langle Pa, b \rangle_{\mathcal{H}}$$

is well defined. It does not depend on the choice of representatives a and b. So a dense vector subspace of the desired space P is known, as well as scalar products of elements of the space. These should be sufficient to construct the desired space $\mathcal{Q}$. The space P is then obtained from a knowledge of $\mathcal{Q}$.

If a is an element of $\mathcal{H}$, define

$$\langle a, a \rangle_{\mathcal{Q}} = \sup[\langle a + Pb, a + Pb \rangle_{\mathcal{H}} - \langle Pb, b \rangle_{\mathcal{H}}],$$

where the least upper bound is taken over all elements b of $\mathcal{H}$. Note that the inequality

$$\langle a, a \rangle_{\mathcal{H}} \leq \langle a, a \rangle_{\mathcal{Q}}$$

is satisfied because it is permissible to choose $b = 0$ in the least upper bound. Let $\mathcal{Q}$ be the set of elements a of $\mathcal{H}$ for which $\langle a, a \rangle_{\mathcal{Q}}$ is finite. The identity

$$\langle wa, wa \rangle_{\mathcal{Q}} = |w|^2 \langle a, a \rangle_{\mathcal{Q}}$$

holds for every element a of $\mathcal{H}$ when w is a nonzero number. The identity also holds when $w = 0$ if a belongs to $\mathcal{Q}$.

Assume that a and b are elements of $\mathcal{H}$ and that t is a given number, $0 < t < 1$. If u and v are elements of $\mathcal{H}$, then the inequality

$$(1-t)\langle a + Pu, a + Pu \rangle_{\mathcal{H}} - (1-t)\langle Pu, u \rangle_{\mathcal{H}} + t\langle b + Pv, b + Pv \rangle_{\mathcal{H}} - t\langle Pv, v \rangle_{\mathcal{H}}$$
$$\leq \langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}} + t(1-t)\langle b - a, b - a \rangle_{\mathcal{Q}}$$

holds by the convexity identity and the definition of self-products in $\mathcal{Q}$. The inequality

$$(1-t)\langle a, a \rangle_{\mathcal{Q}} + t\langle b, b \rangle_{\mathcal{Q}} \leq \langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}}$$
$$+ t(1-t)\langle b - a, b - a \rangle_{\mathcal{Q}}$$

follows by the arbitrariness of u and v. Equality holds since the reverse inequality is obtained by a similar argument.

This verifies that $\mathcal{Q}$ is a vector space. It will be shown that a scalar product is defined on the space by

$$4\langle a, b \rangle_{\mathcal{Q}} = \langle a + b, a + b \rangle_{\mathcal{Q}} - \langle a - b, a - b \rangle_{\mathcal{Q}}$$
$$+ i\langle a + ib, a + ib \rangle_{\mathcal{Q}} - i\langle a - ib, a - ib \rangle_{\mathcal{Q}}.$$

The symmetry of a scalar product is immediate. A proof of linearity will be given.

It will first be shown that the identity

$$\langle wa, b \rangle_{\mathcal{Q}} = w\langle a, b \rangle_{\mathcal{Q}}$$

holds for all elements a and b of $\mathcal{Q}$ when w is a positive multiple of a power of i. The result is immediate from the definition of the product when w is a power of i. It is therefore sufficient to consider the case w positive. By the definition of the product, it is sufficient to verify the identity

$$\langle wa + b, wa + b \rangle_{\mathcal{Q}} - \langle wa - b, wa - b \rangle_{\mathcal{Q}} = w\langle a + b, a + b \rangle_{\mathcal{Q}} - w\langle a - b, a - b \rangle_{\mathcal{Q}}$$

and a similar identity with b replaced by ib. When $t = 1/(1+w)$, the identity reads

$$\langle (1-t)a + tb, (1-t)a + tb \rangle_{\mathcal{Q}} + t(1-t)\langle a - b, a - b \rangle_{\mathcal{Q}}$$
$$= \langle (1-t)a - tb, (1-t)a - tb \rangle_{\mathcal{Q}} + t(1-t)\langle a + b, a + b \rangle_{\mathcal{Q}}$$

which is a consequence of the convexity identity.

The proof of linearity is completed by verifying the identity

$$\langle a + b, c \rangle_{\mathcal{Q}} = \langle a, c \rangle_{\mathcal{Q}} + \langle b, c \rangle_{\mathcal{Q}}$$

for all elements a, b, and c of $\mathcal{Q}$. By the definition of the product, the problem is to verify the identity

$$\langle a + b + c, a + b + c \rangle_{\mathcal{Q}} - \langle a + b - c, a + b - c \rangle_{\mathcal{Q}}$$
$$= \langle a + c, a + c \rangle_{\mathcal{Q}} - \langle a - c, a - c \rangle_{\mathcal{Q}} + \langle b + c, b + c \rangle_{\mathcal{Q}} - \langle b - c, b - c \rangle_{\mathcal{Q}}$$

and a similar identity with c replaced by ic. Because of the known identity

$$\langle a + b + 2c, a + b + 2c \rangle_\mathcal{Q} + \langle a - b, a - b \rangle_\mathcal{Q} = 2\langle a + c, a + c \rangle_\mathcal{Q} + 2\langle b + c, b + c \rangle_\mathcal{Q}$$

and a similar identity with c replaced by $-c$, it remains to verify the identity

$$2\langle a + b + c, a + b + c \rangle_\mathcal{Q} - 2\langle a + b - c, a + b - c \rangle_\mathcal{Q}$$
$$= \langle a + b + 2c, a + b + 2c \rangle_\mathcal{Q} - \langle a + b - 2c, a + b - 2c \rangle_\mathcal{Q}$$

which is true because

$$\langle a + b, 2c \rangle_\mathcal{Q} = 2\langle a + b, c \rangle_\mathcal{Q}.$$

Note that the inequality

$$\langle Pa + Qb, Pa + Qb \rangle_\mathcal{H} \le \langle Pa, a \rangle_\mathcal{H} + \langle Qb, b \rangle_\mathcal{H}$$

holds for all elements a and b of $\mathcal{H}$ because the inequality can be written

$$\langle P(1 - P)(a - b), a - b \rangle_\mathcal{H} \ge 0.$$

Equality holds when $a = b$. It follows that Qb belongs to $\mathcal{Q}$ for every element b of $\mathcal{H}$ and that the identity

$$\langle Qb, Qb \rangle_\mathcal{Q} = \langle Qb, b \rangle_\mathcal{H}$$

is satisfied.

It has not yet been shown that $\mathcal{Q}$ has a well-defined scalar product because nondegeneracy has not yet been verified. It is convenient to proceed first with the construction of the desired space $\mathcal{P}$.

If a is an element of $\mathcal{H}$, define

$$\langle a, a \rangle_\mathcal{P} = \sup[\langle a + b, a + b \rangle_\mathcal{H} - \langle b, b \rangle_\mathcal{Q}],$$

where the least upper bound is taken over all elements b of $\mathcal{Q}$. The set of elements a of $\mathcal{H}$ such that $\langle a, a \rangle_\mathcal{P}$ is finite is a vector space by the convexity identity

$$\langle (1 - t)a + tb, (1 - t)a + tb \rangle_\mathcal{P} + t(1 - t)\langle a - b, a - b \rangle_\mathcal{P}$$
$$= (1 - t)\langle a, a \rangle_\mathcal{P} + t\langle b, b \rangle_\mathcal{P}.$$

The definition

$$4\langle a, b \rangle_\mathcal{P} = \langle a + b, a + b \rangle_\mathcal{P} - \langle a - b, a - b \rangle_\mathcal{P}$$
$$+ i\langle a + ib, a + ib \rangle_\mathcal{P} - i\langle a - ib, a - ib \rangle_\mathcal{P}$$

has the linearity and symmetry properties of a scalar product on $\mathcal{P}$. If a belongs to $\mathcal{H}$, then Pa is an element of $\mathcal{P}$ which satisfies the identity

$$\langle Pa, Pa \rangle_\mathcal{P} = \langle Pa, a \rangle_\mathcal{H}.$$

If a belongs to $\mathcal{P}$ and if b belongs to $\mathcal{Q}$, then $c = a + b$ is an element of $\mathcal{H}$ which satisfies the inequality

$$\langle c, c \rangle_\mathcal{H} \le \langle a, a \rangle_\mathcal{P} + \langle b, b \rangle_\mathcal{Q}.$$

Every element c of $\mathcal{H}$ admits some such decomposition for which equality holds. It is obtained with $a = Pc$ and $b = (1 - P)c$.

The intersection $\mathcal{L}$ of $\mathcal{P}$ and $\mathcal{Q}$ is considered with the product

$$\langle u, u \rangle_\mathcal{L} = \langle u, u \rangle_\mathcal{P} + \langle u, u \rangle_\mathcal{Q}.$$

Assume that a is an element of $\mathcal{P}$ and that b is an element of $\mathcal{Q}$ such that $c = a+b$ and

$$\langle c, c \rangle_{\mathcal{H}} = \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}.$$

If u is an element of $\mathcal{L}$, then the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a + \lambda u, a + \lambda u \rangle_{\mathcal{P}} + \langle b - \lambda u, b - \lambda u \rangle_{\mathcal{Q}}$$

holds for every complex number λ. By the arbitrariness of λ, the inequality implies the identity

$$\langle a, u \rangle_{\mathcal{P}} = \langle b, u \rangle_{\mathcal{Q}}$$

and the inequality

$$0 \leq \langle u, u \rangle_{\mathcal{L}}.$$

But every element c of $\mathcal{P}$ is also an element of $\mathcal{H}$, and it admits a representation $c = c + 0$ with c in $\mathcal{P}$ and 0 in $\mathcal{Q}$. It follows that $c - Pc$ is an element of $\mathcal{L}$. If a is any element of $\mathcal{H}$, then the resulting identity

$$\langle Pa, c - Pc \rangle_{\mathcal{P}} = \langle (1 - P)a, c - Pc \rangle_{\mathcal{Q}}$$

implies the identity

$$\langle Pa, c \rangle_{\mathcal{P}} = \langle a, c \rangle_{\mathcal{H}}.$$

A similar argument shows that $\langle Qa, c \rangle_{\mathcal{Q}} = \langle a, c \rangle_{\mathcal{H}}$ for every element c of $\mathcal{Q}$ if a belongs to $\mathcal{H}$.

The nondegeneracy of a scalar product can now be verified in $\mathcal{P}$ and $\mathcal{Q}$. If a is an element of $\mathcal{P}$ which is orthogonal to every element of $\mathcal{P}$, then the identity

$$\langle a, Pb \rangle_{\mathcal{P}} = \langle a, b \rangle_{\mathcal{H}}$$

holds for every element b of $\mathcal{H}$. Since a is orthogonal in $\mathcal{H}$ to every element of $\mathcal{H}$, it is zero by the nondegeneracy of a scalar product in $\mathcal{H}$. The nondegeneracy of a scalar product is verified in $\mathcal{Q}$ by a similar argument.

If c is in $\mathcal{H}$, then $P(1 - P)c = (1 - P)Pc$ belongs to $\mathcal{P}$ because it is in the range of P and it belongs to $\mathcal{Q}$ because it is in the range of $Q = 1 - P$. So $P(1 - P)c$ belongs to $\mathcal{L}$. If a belongs to $\mathcal{L}$, then

$$\langle a, P(1 - P)c \rangle_{\mathcal{L}} = \langle a, P(1 - P)c \rangle_{\mathcal{P}} + \langle a, (1 - P)Pc \rangle_{\mathcal{Q}}$$
$$= \langle a, (1 - P)c \rangle_{\mathcal{H}} + \langle a, Pc \rangle_{\mathcal{H}} = \langle a, c \rangle_{\mathcal{H}}.$$

The properties of a scalar product in $\mathcal{P}$ and $\mathcal{Q}$ now imply the properties of a scalar product in $\mathcal{L}$. It has been shown that self-products are nonnegative in $\mathcal{L}$. It follows by the Schwarz inequality that

$$\langle c, c \rangle_{\mathcal{L}} > 0$$

whenever c is a nonzero element of $\mathcal{L}$.

Since $P(1 - P)$ is nonnegative, it has been shown in Theorem 1 that a unique Hilbert space $\mathcal{R}$ exists, which is contained continuously in $\mathcal{H}$, such that $P(1 - P)$ coincides with the adjoint of the inclusion of $\mathcal{R}$ in $\mathcal{H}$. A dense set of elements of $\mathcal{R}$ are of the form $P(1 - P)c$ with c in $\mathcal{H}$. Each such element of $\mathcal{R}$ belongs to $\mathcal{L}$. The identity

$$\langle P(1 - P)a, P(1 - P)b \rangle_{\mathcal{R}} = \langle P(1 - P)a, b \rangle_{\mathcal{H}} = \langle P(1 - P)a, P(1 - P)b \rangle_{\mathcal{L}}$$

holds for all elements a and b of $\mathcal{H}$.

Assume that c is an element of $\mathcal{R}$ which is the limit of a sequence of elements c_n of $\mathcal{R}$ in the range of $P(1 - P)$. Convergence is taken in the norm topology of $\mathcal{R}$, which coincides with the Mackey topology of $\mathcal{R}$. Since the inclusion of $\mathcal{R}$ in $\mathcal{H}$ is continuous, c_n converges to c in the Mackey topology of $\mathcal{H}$. It will be shown that c belongs to $\mathcal{L}$. If a is an element of $\mathcal{P}$ and if b is an element of $\mathcal{Q}$, then

$$\langle c + a, c + a \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}} = \lim[\langle c_n + a, c_n + a \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}}]$$

and

$$\langle c + b, c + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{\mathcal{Q}} = \lim[\langle c_n + b, c_n + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{\mathcal{Q}}].$$

It follows that

$$\langle c + a, c + a \rangle_{\mathcal{H}} - \langle a, a \rangle_{\mathcal{P}} + \langle c + b, c + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{\mathcal{Q}}$$
$$\leq \lim \langle c_n, c_n \rangle_{\mathcal{R}} = \langle c, c \rangle_{\mathcal{R}}.$$

The inequality

$$\langle c, c \rangle_{\mathcal{P}} + \langle c, c \rangle_{\mathcal{Q}} \leq \langle c, c \rangle_{\mathcal{R}}$$

is obtained by the arbitrariness of a and b. Thus the space $\mathcal{R}$ is contained contractively in the space $\mathcal{L}$.

Since the inclusion of $\mathcal{R}$ in $\mathcal{L}$ is isometric on a dense vector subspace of $\mathcal{R}$, and since self-products of nonzero elements of $\mathcal{L}$ are positive, it follows that $\mathcal{R}$ is contained isometrically in $\mathcal{L}$. Since $P(1 - P)$ coincides with the adjoint of the inclusion of $\mathcal{L}$ in $\mathcal{H}$, no nonzero element of $\mathcal{L}$ is orthogonal to $\mathcal{R}$. Since $\mathcal{R}$ is a Hilbert space, it contains every element of $\mathcal{L}$.

It can now be shown that $\mathcal{P}$ and $\mathcal{Q}$ are Krein spaces. Consider the Cartesian product $\mathcal{P} \times \mathcal{Q}$ with the Euclidean scalar product,

$$\langle (a, b), (c, d) \rangle_{\mathcal{P} \times \mathcal{Q}} = \langle a, c \rangle_{\mathcal{P}} + \langle b, d \rangle_{\mathcal{Q}}.$$

Then a transformation of $\mathcal{P} \times \mathcal{Q}$ into $\mathcal{H}$ is defined by taking (a, b) into $a + b$. The kernel of the transformation is the set of pairs $(c, -c)$ with c in $\mathcal{L}$. Since the transformation which takes c into $(c, -c)$ is an isometry of $\mathcal{L}$ into $\mathcal{P} \times \mathcal{Q}$, the kernel of the transformation which takes (a, b) into $a + b$ is a Hilbert space. Since the transformation is isometric on the orthogonal complement of its kernel and since $\mathcal{H}$ is a Krein space by hypothesis, the Cartesian product $\mathcal{P} \times \mathcal{Q}$ is a Krein space.

Nondegenerate closed subspaces of $\mathcal{P} \times \mathcal{Q}$ are formed by the elements which have first coordinate equal to zero and by the elements which have second coordinate equal to zero. Since these subspaces are orthogonal to each other and since they span the full space, it follows [1] that they are Krein spaces. This completes the proof that $\mathcal{P}$ and $\mathcal{Q}$ are Krein spaces.

The desired Krein space $\mathcal{P}$ has now been shown to exist. For the proof of uniqueness, consider Krein spaces $\mathcal{P}_+$ and $\mathcal{P}_-$, which are contained continuously and contractively in $\mathcal{H}$, such that the adjoints of the inclusions of $\mathcal{P}_+$ and $\mathcal{P}_-$ in $\mathcal{H}$ coincide with P. If c is any element of $\mathcal{H}$, then Pc belongs both to $\mathcal{P}_+$ and to $\mathcal{P}_-$. The identity

$$\langle Pa, Pb \rangle_{\mathcal{P}_+} = \langle Pa, b \rangle_{\mathcal{H}} = \langle Pa, Pb \rangle_{\mathcal{P}_-}$$

holds for all elements a and b of $\mathcal{H}$.

Let $\mathcal{Q}_+$ be the complementary space to $\mathcal{P}_+$ in $\mathcal{H}$ and let $\mathcal{Q}_-$ be the complementary space to $\mathcal{P}_-$ in $\mathcal{H}$. Then $\mathcal{Q}_+$ is the set of elements a of $\mathcal{H}$ such that

$$\langle a, a \rangle_{\mathcal{Q}_+} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{\mathcal{P}_+}] < \infty,$$

where the least upper bound is taken over all elements b of P_+, and Q_- is the set of elements a of $\mathcal{H}$ such that

$$\langle a, a \rangle_{Q_-} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{P_-}] < \infty,$$

where the least upper bound is taken over all elements b of P_-. Since P has a dense range in P, the least upper bounds are unchanged if they are taken over all elements b of the range of P. Since these elements have the same self-products in P_+ as in P_-, the spaces Q_+ and Q_- are isometrically equal.

But P_+ is the set of elements u of $\mathcal{H}$ such that

$$\langle a, a \rangle_{P_+} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{Q_+}] < \infty,$$

where the least upper bound is taken over all elements b of Q_+, and P_- is the set of elements a of $\mathcal{H}$ such that

$$\langle a, a \rangle_{P_-} = \sup[\langle a + b, a + b \rangle_{\mathcal{H}} - \langle b, b \rangle_{Q_-}] < \infty,$$

where the least upper bound is taken over all elements b of Q_-. Since Q_+ and Q_- are isometrically equal, P_+ and P_- are isometrically equal.

This completes the proof of the theorem.

A Krein space P is said to be contained boundedly in a Krein space $\mathcal{H}$ if it is a vector subspace of $\mathcal{H}$ and if a positive number t exists such that the inequality

$$\langle a, a \rangle_{\mathcal{H}} \leq t \langle a, a \rangle_P$$

holds for every element a of P. If the inclusion of P in $\mathcal{H}$ is continuous and if P is the adjoint of the inclusion of P in $\mathcal{H}$, then the inequality $P^2 \leq tP$ is satisfied. A converse result is an immediate consequence of Theorem 2.

THEOREM 3. *If P is a selfadjoint transformation of a Krein space $\mathcal{H}$ into itself which satisfies the inequality $P^2 \leq tP$ for a positive number t, then a unique Krein space P exists, which is contained continuously and boundedly in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of P in $\mathcal{H}$.*

PROOF OF THEOREM 3. Since the transformation $Q = t^{-1}P$ is selfadjoint and satisfies the inequality $Q^2 \leq Q$, a unique Krein space Q exists, which is contained continuously (and contractively) in $\mathcal{H}$, such that Q coincides with the adjoint of the inclusion of Q in $\mathcal{H}$. The desired space P is the space Q considered with the new scalar product

$$\langle a, b \rangle_P = t^{-1} \langle a, b \rangle_Q.$$

The desired properties of the space are easily verified. Uniqueness follows from uniqueness in Theorem 2.

This completes the proof of the theorem.

If P is an arbitrary selfadjoint transformation of a Krein space $\mathcal{H}$ into itself, then it is still true that a Krein space P exists, which is contained continuously in $\mathcal{H}$, such that P coincides with the adjoint of the inclusion of P in $\mathcal{H}$, but the space need not be unique.

For the construction of such a space P, it is sufficient by the proof of Theorem 1 to consider the case in which $\mathcal{H}$ is a Hilbert space. By the spectral theorem, $P = P_+ + P_-$ where P_+ and P_- are selfadjoint transformations with orthogonal ranges such that P_+ and $-P_-$ are nonnegative. The desired space P is the orthogonal sum of the spaces P_+ and P_- associated with P_+ and P_-.

Assume that a Krein space P is contained contractively in a Krein space $\mathcal{H}$. A complementary space Q to P in $\mathcal{H}$ is a Krein space, which is contained contractively in $\mathcal{H}$, such that the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P + \langle b, b \rangle_Q$$

holds whenever $c = a + b$ with a in P and b in Q, and such that every element c of $\mathcal{H}$ admits some decomposition for which equality holds.

If P is contained continuously in $\mathcal{H}$ and if P is the adjoint of the inclusion of P in $\mathcal{H}$, then Q is contained continuously in $\mathcal{H}$ and $1 - P$ is the adjoint of the inclusion of Q in $\mathcal{H}$. The minimal decomposition of an element c of $\mathcal{H}$ is unique. It is obtained with $a = Pc$ and $b = (1 - P)c$.

The overlapping space $\mathcal{L}$ of P and Q is the intersection of P and Q considered with the product

$$\langle a, b \rangle_{\mathcal{L}} = \langle a, b \rangle_P + \langle a, b \rangle_Q.$$

If P and Q are contained continuously in $\mathcal{H}$, the product is nondegenerate and the space $\mathcal{L}$ is a Hilbert space which is contained continuously in $\mathcal{H}$. The adjoint of the inclusion of $\mathcal{L}$ in $\mathcal{H}$ is $P(1 - P)$. It is easily seen that the inclusion of P in $\mathcal{H}$ is isometric if, and only if, $\mathcal{L}$ contains no nonzero element.

A condition is noted for the continuity of contractive inclusions of Krein spaces.

THEOREM 4. *Assume that Krein spaces P and Q are contained continuously and contractively in a Krein space $\mathcal{H}$. Let P be the adjoint of the inclusion of P in $\mathcal{H}$ and let Q be the adjoint of the inclusion of Q in $\mathcal{H}$. Then P is contained contractively in Q if, and only if, the inequality*

$$\begin{pmatrix} P - P^2 & P - PQ \\ P - QP & Q - Q^2 \end{pmatrix} \geq 0$$

is satisfied. The inclusion of P in Q is then continuous.

PROOF OF THEOREM 4. Assume that P is contained contractively in Q. Let S be the complementary space to Q in $\mathcal{H}$. Then the inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P + \langle b, b \rangle_S$$

holds whenever $c = a + b$ with a in P and b in S.

If u is an element of $\mathcal{H}$, then Pu is an element of P such that

$$\langle Pu, Pu \rangle_P = \langle Pu, u \rangle_{\mathcal{H}}.$$

If v is an element of $\mathcal{H}$, then $(1 - Q)v$ is an element of S such that

$$\langle (1 - Q)v, (1 - Q)v \rangle_S = \langle (1 - Q)v, v \rangle_{\mathcal{H}}.$$

In this notation the previous inequality reads

$$\langle Pu + (1 - Q)v, Pu + (1 - Q)v \rangle_{\mathcal{H}} \leq \langle Pu, u \rangle_{\mathcal{H}} + \langle (1 - Q)v, v \rangle_{\mathcal{H}}.$$

The desired matrix inequality follows.

If on the other hand the matrix inequality is assumed, then the last inequality holds for all elements u and v of $\mathcal{H}$. The inequality

$$\langle c, c \rangle_{\mathcal{H}} \leq \langle a, a \rangle_P + \langle b, b \rangle_S$$

follows when $a = Pu$ is in the range of P, $b = (1 - Q)v$ is in the range of $1 - Q$, and $c = a + b$. But every element a of P is the limit of a sequence of elements a_n of the range of P. Convergence is in the Mackey topologies of P and $\mathcal{H}$, and every element b of S is the limit of a sequence of elements b_n in the range of $1 - Q$. Convergence is in the Mackey topologies of S and $\mathcal{H}$. Then $c = a + b$ is the limit of the sequence of elements $c_n = a_n + b_n$. Convergence is in the Mackey topology of $\mathcal{H}$. Since

$$\langle a, a \rangle_P = \lim \langle a_n, a_n \rangle_P, \quad \langle b, b \rangle_S = \lim \langle b_n, b_n \rangle_S, \quad \langle c, c \rangle_{\mathcal{H}} = \lim \langle c_n, c_n \rangle_{\mathcal{H}},$$

the inequality

$$\langle c_n, c_n \rangle_{\mathcal{H}} \le \langle a_n, a_n \rangle_P + \langle b_n, b_n \rangle_S$$

implies the inequality

$$\langle c, c \rangle_{\mathcal{H}} \le \langle a, a \rangle_P + \langle b, b \rangle_S.$$

The inequality implies that P is contained contractively in Q.

It will be shown that the inclusion of P in Q is continuous. If c is an element of $\mathcal{H}$, then Qc is an element of Q in the domain of the adjoint of the inclusion of P in Q. The action of the adjoint on Qc is Pc. It has been shown that the adjoint of the inclusion of P in Q has a dense domain in Q. It follows that the inclusion of P in Q has a closed graph. Continuity of the inclusion follows by the closed graph theorem.

This completes the proof of the theorem.

A factorization of continuous and contractive transformations is given as an application of complementation theory.

THEOREM 5. *The kernel of a continuous and contractive transformation T of a Krein space A into a Krein space B is a Hilbert space N which is contained continuously and isometrically in A, and T acts as an isometric transformation of the orthogonal complement of N in A onto a Krein space M which is contained continuously and contractively in B.*

PROOF OF THEOREM 5. Since T is continuous, an adjoint transformation T^* of B into A exists. Since T is contractive, T^*T is a selfadjoint transformation of A into itself which satisfies the inequality $T^*T \le 1$. It follows that TT^* is a selfadjoint transformation of B into itself which satisfies the inequality

$$(TT^*)^2 \le TT^*.$$

By Theorem 2, a unique Krein space M exists, which is contained continuously and contractively in B, such that TT^* coincides with the adjoint of the inclusion of M in B.

Consider the Cartesian product $A \times B$ as a Krein space with the Cartesian product of the scalar products of A and B. The elements of the space are to be realized as column vectors with upper entry in A and lower entry in B. There is a corresponding representation of transformations of $A \times B$ into itself as two-by-two matrices of transformations. The inequality $T^*T \le 1$ implies that the selfadjoint matrix

$$\begin{pmatrix} 1 & T^* \\ T & TT^* \end{pmatrix}$$

satisfies the inequality

$$\begin{pmatrix} 1 & T^* \\ T & TT^* \end{pmatrix}^2 \le 2 \begin{pmatrix} 1 & T^* \\ T & TT^* \end{pmatrix}.$$

By Theorem 3, a unique Krein space $\mathcal{G}$ exists, which is contained continuously and boundedly in $\mathcal{A} \times \mathcal{B}$, such that the selfadjoint matrix coincides with the adjoint of the inclusion of $\mathcal{G}$ in $\mathcal{A} \times \mathcal{B}$. An isometric transformation of $\mathcal{A}$ onto $\mathcal{G}$ is defined by taking an element a of $\mathcal{A}$ into the pair $\begin{pmatrix} a \\ Ta \end{pmatrix}$.

If b is any element of $\mathcal{B}$, then

$$\begin{pmatrix} T^*b \\ TT^*b \end{pmatrix}$$

is an element of $\mathcal{G}$ and TT^*b is an element of $\mathcal{M}$ such that the identity

$$\left\langle \begin{pmatrix} a \\ Ta \end{pmatrix}, \begin{pmatrix} T^*b \\ TT^*b \end{pmatrix} \right\rangle_{\mathcal{G}} = \langle Ta, TT^*b \rangle_{\mathcal{M}}$$

holds for every element a of $\mathcal{A}$ such that Ta belongs to $\mathcal{M}$.

It will now be shown that every element a of $\mathcal{A}$ has this property and that the inequality

$$\langle Ta, Ta \rangle_{\mathcal{M}} \le \langle a, a \rangle_{\mathcal{A}}$$

is satisfied. It is easily seen that every element a of $\mathcal{A}$ in the range of T^* has the desired property. Also ever element a of $\mathcal{A}$ in the kernel of T has the desired property because T is assumed to be contractive. Since the kernel of T and the range of T^* are orthogonal subspaces of $\mathcal{A}$, every element of $\mathcal{A}$ which belongs to the span of the kernel of T and the range of T^* has the desired property. This verifies that a dense set of element of $\mathcal{A}$ has the desired property.

If a is an element of $\mathcal{A}$ which is a limit in the Mackey topology of $\mathcal{A}$ of a sequence of elements a_n of $\mathcal{A}$ which have the desired property, then Ta is the limit of Ta_n in the Mackey topology of $\mathcal{B}$. If b is any element of $\mathcal{B}$, the identity

$$\langle Ta + (1 - TT^*)b, Ta + (1 - TT^*)b \rangle_{\mathcal{B}}$$
$$= \lim \langle Ta_n + (1 - TT^*)b, Ta_n + (1 - TT^*)b \rangle_{\mathcal{B}}$$

implies that

$$\langle Ta + (1 - TT^*)b, Ta + (1 - TT^*)b \rangle_{\mathcal{B}}$$
$$\le \langle b, (1 - TT^*)b \rangle_{\mathcal{B}} + \lim \sup \langle Ta_n, Ta_n \rangle_{\mathcal{M}}.$$

Since the inequality

$$\langle Ta_n, Ta_n \rangle_{\mathcal{M}} \le \langle a_n, a_n \rangle_{\mathcal{A}}$$

holds for every index n and since

$$\langle a, a \rangle_{\mathcal{A}} = \lim \langle a_n, a_n \rangle_{\mathcal{A}},$$

it follows that the inequality

$$\langle Ta + (1 - TT^*)b, Ta + (1 - TT^*)b \rangle_{\mathcal{B}}$$
$$\le \langle a, a \rangle_{\mathcal{A}} + \langle b, (1 - TT^*)b \rangle_{\mathcal{B}}$$

is satisfied. By the arbitrariness of b, Ta belongs to $\mathcal{M}$ and the inequality $\langle Ta, Ta \rangle_{\mathcal{M}} \le \langle a, a \rangle_{\mathcal{A}}$ is satisfied.

It has now been shown that a contractive transformation of $\mathcal{G}$ into $\mathcal{M}$ is defined by taking $\left(\begin{smallmatrix} a \\ Ta \end{smallmatrix}\right)$ into Ta. Since the adjoint transformation takes TT^*b into

$$\begin{pmatrix} T^*b \\ TT^*b \end{pmatrix}$$

for every element b of $\mathcal{B}$, it has a dense domain in $\mathcal{B}$. The transformation of $\mathcal{G}$ into $\mathcal{M}$ is continuous by the closed graph theorem. Since the adjoint transformation is isometric on a dense subset of $\mathcal{M}$, it is an isometric transformation of $\mathcal{M}$ into $\mathcal{G}$. The range of this adjoint transformation is a Krein subspace of $\mathcal{G}$ whose orthogonal complement is the set of elements of $\mathcal{G}$ of the form $\left(\begin{smallmatrix} a \\ Ta \end{smallmatrix}\right)$ for an element a of $\mathcal{A}$ in the kernel of T.

This completes the proof that the kernel of T is a Krein space. The set of such pairs is a Krein subspace of $\mathcal{G}$. The remaining assertions of the theorem are easily verified.

A continuous transformation of a Krein space $\mathcal{A}$ into a Krein space $\mathcal{B}$ is said to be a partial isometry if its kernel is a Krein space which is contained continuously and isometrically in $\mathcal{A}$ and if the transformation is an isometry on the orthogonal complement of its kernel. Theorem 5 states that a continuous and contractive transformation of a Krein space into a Krein space is the composition of a contractive partial isometry and a continuous and contractive inclusion.

A transformation T of a Krein space $\mathcal{A}$ into a Krein space $\mathcal{B}$ is said to be bounded if a positive number t exists such that the inequality

$$\langle Ta, Ta \rangle_{\mathcal{B}} \leq t \langle a, a \rangle_{\mathcal{A}}$$

holds for every element a of $\mathcal{A}$.

A factorization theorem for transformations which are continuous and bounded is a corollary of Theorem 5.

THEOREM 6. *The kernel of a continuous and bounded transformation T of a Krein space $\mathcal{A}$ into a Krein space $\mathcal{B}$ is a Hilbert space $\mathcal{N}$ which is contained continuously and isometrically in $\mathcal{A}$, and T acts as an isometric transformation of the orthogonal complement of $\mathcal{N}$ in $\mathcal{A}$ onto a Krein space $\mathcal{M}$ which is contained continuously and boundedly in $\mathcal{B}$.*

PROOF OF THEOREM 6. By hypothesis a positive number t exists such that the continuous transformation $T_0 = t^{-1}T$ of $\mathcal{A}$ into $\mathcal{B}$ is contractive. Since the kernel of T is equal to the kernel of T_0, it is a Hilbert space by Theorem 5. By Theorem 5, T_0 acts as an isometry of the orthogonal complement of $\mathcal{N}$ in $\mathcal{A}$ onto a Krein space $\mathcal{M}_0$ which is contained continuously and contractively in $\mathcal{B}$. Let $\mathcal{M}$ be the unique Krein space, which is contained continuously and boundedly in $\mathcal{B}$, such that multiplication by t acts as an isometry of $\mathcal{M}_0$ onto $\mathcal{M}$. Then T acts as an isometry of the orthogonal complement of $\mathcal{N}$ in $\mathcal{A}$ onto $\mathcal{M}$.

This completes the proof of the theorem.

An analogous factorization does not hold for every continuous transformation of a Krein space into a Krein space. The kernel of the transformation need not be a Krein space.

A useful property of complementation is its preservation under contractive partially isometric transformations.

THEOREM 7. *Let π be a contractive partially isometric transformation of a Krein space $\mathcal{H}$ onto a Krein space $\mathcal{H}'$. Assume that P is a Krein space which is contained continuously and contractively in $\mathcal{H}$ and that π acts as a contractive partial isometry of P onto a Krein space P' which is contained continuously and contractively in $\mathcal{H}'$. Then π acts as a contractive partial isometry of the complementary space Q to P in $\mathcal{H}$ onto the complementary space Q' to P' in $\mathcal{H}'$.*

PROOF OF THEOREM 7. Let T be the adjoint of π as a transformation of $\mathcal{H}$ into $\mathcal{H}'$. Since π is a partial isometry of $\mathcal{H}$ onto $\mathcal{H}'$, T is an isometry of $\mathcal{H}'$ into $\mathcal{H}$ which satisfies the identity $\pi T = 1$.

Let P be the adjoint of π as a transformation of P into P'. Since π acts as a partial isometry of P onto P', P is an isometry of P' into P which satisfies the identity $\pi P = 1$.

If c is a given element of $\mathcal{H}'$, consider the minimal decomposition $c = a + b$ of c as an element of $\mathcal{H}'$ with a as the element of P' and b as the element of Q'. Let $Tc = u + v$ be the minimal decomposition of Tc as an element of $\mathcal{H}$ with u as the element of P and with v as the element of Q.

If s is any element of P, then

$$\langle u, s \rangle_P = \langle Tc, s \rangle_{\mathcal{H}}$$

by the theory of minimal decompositions. By the definition of T,

$$\langle Tc, s \rangle_{\mathcal{H}} = \langle c, \pi s \rangle_{\mathcal{H}'}.$$

Since πs belongs to P',

$$\langle c, \pi s \rangle_{\mathcal{H}'} = \langle a, \pi s \rangle_{P'}$$

by the theory of minimal decompositions. Since the identity $\langle u, s \rangle_P = \langle a, \pi s \rangle_{P'}$ then holds for every element s of P, $u = Pa$. It follows that $\pi u = a$ and $\pi v = b$. Since the identities $\langle c, c \rangle_{\mathcal{H}'} = \langle Tc, Tc \rangle_{\mathcal{H}}$ and $\langle a, a \rangle_{P'} = \langle u, u \rangle_P$ are satisfied, the identity $\langle b, b \rangle_{Q'} = \langle v, v \rangle_Q$ is satisfied.

Assume that s is an element of Q which is orthogonal in Q to elements which are obtained from the range of T under the adjoint of the inclusion of Q in $\mathcal{H}$. Since s is then orthogonal in $\mathcal{H}$ to the range of T, it is in the kernel of π. Since π is contractive as a transformation of $\mathcal{H}$ into $\mathcal{H}'$, the self-product $\langle s, s \rangle_{\mathcal{H}}$ is nonnegative. Since the inclusion of Q in $\mathcal{H}$ is contractive, the self-product $\langle s, s \rangle_Q$ is nonnegative.

A dense vector subspace of elements s of Q have the property that πs belongs to Q' and that the inequality

$$\langle \pi s, \pi s \rangle_{Q'} \leq \langle s, s \rangle_Q$$

is satisfied. These elements are those which are sums of the two kinds of elements previously considered. One kind are the elements which are obtained from the range of T under the adjoint of the inclusion of Q in $\mathcal{H}$, and the other kind are the elements which are orthogonal to the elements obtained from the range of T under the adjoint of the inclusion of Q in $\mathcal{H}$.

An arbitrary element s of Q is the limit in the Mackey topology of Q of a sequence of elements s_n of Q such that πs_n belongs to Q' and such that the inequality

$$\langle \pi s_n, \pi s_n \rangle_{Q'} \leq \langle s_n, s_n \rangle_Q$$

is satisfied. Since the inclusion of $\mathcal{Q}$ in $\mathcal{H}$ is continuous, s is the limit of s_n in the Mackey topology of $\mathcal{H}$. Since π is continuous as a transformation of $\mathcal{H}$ in to $\mathcal{H}'$, πs is the limit of πs_n in the Mackey topology of $\mathcal{H}'$. Since the identity

$$\langle \pi s + a, \pi s + a \rangle_{\mathcal{H}'} = \lim \langle \pi s_n + a, \pi s_n + a \rangle_{\mathcal{H}'}$$

holds for every element a of $\mathcal{P}'$, where

$$\langle \pi s_n + a, \pi s_n + a \rangle_{\mathcal{H}'} \leq \langle a, a \rangle_{\mathcal{P}'} + \langle \pi s_n, \pi s_n \rangle_{\mathcal{Q}'},$$

and since

$$\langle s, s \rangle_{\mathcal{Q}} = \lim \langle s_n, s_n \rangle_{\mathcal{Q}},$$
$$\langle \pi s + a, \pi s + a \rangle_{\mathcal{H}'} \leq \langle a, a \rangle_{\mathcal{P}'} + \langle s, s \rangle_{\mathcal{Q}}.$$

Since a is an arbitrary element of $\mathcal{P}'$, it follows that πs belongs to $\mathcal{Q}'$ and that the inequality $\langle \pi s, \pi s \rangle_{\mathcal{Q}'} \leq \langle s, s \rangle_{\mathcal{Q}}$ is satisfied.

Assume that c is any element of $\mathcal{H}$, that v is the element of $\mathcal{Q}$ obtained from Tc under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$, and that b is the element of $\mathcal{Q}'$ obtained from c under the adjoint of the inclusion of $\mathcal{Q}'$ in $\mathcal{H}'$. If s belongs to $\mathcal{Q}$, then

$$\langle s, v \rangle_{\mathcal{Q}} = \langle s, Tc \rangle_{\mathcal{H}} = \langle \pi s, c \rangle_{\mathcal{H}'} = \langle \pi s, b \rangle_{\mathcal{Q}'}.$$

This verifies that the adjoint of π as a transformation of $\mathcal{Q}$ into $\mathcal{Q}'$ takes b into v. Since the transformation of $\mathcal{Q}$ into $\mathcal{Q}'$ given by the action of π has a densely defined adjoint, it is continuous by the closed graph theorem. Since the adjoint has been shown to be isometric on a dense subset of $\mathcal{Q}'$, it is an isometry. The range of the isometry is a Krein space which is contained continuously and isometrically in $\mathcal{Q}$, and which has $Q\pi$ as the adjoint of its inclusion in $\mathcal{Q}$. Since the kernel of the restriction of π to $\mathcal{Q}$ is the orthogonal complement of the range of Q, it is a Krein space which is contained continuously and isometrically in $\mathcal{Q}$. The restriction of π to the range of Q is an isometry of the range of Q onto $\mathcal{Q}'$.

This completes the proof of the theorem.

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