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ON THE SECOND FUNDAMENTAL THEOREM OF NEVANLINNA

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ABSTRACT. It is shown that a condition on the size of the exceptional set in the second fundamental theorem of Nevanlinna cannot be improved. The method is based on a construction of Hayman and also makes use of a quantitative version of a result of F. Nevanlinna about the growth of the characteristic function of a meromorophic function omitting a finite number of points

1. Introduction. The second fundamental theorem of Nevanlinna about the value distribution of meromorphic functions states that for a meromorphic function F in $|z| \leq r$ and $q > 2$ distinct values $a_1, a_2, \ldots, a_q$ of the complex extended plane we have the inequality

$$(q-2)T(r,F) < \sum_{\nu=1}^{q} N(r,a_\nu) - N_1(r) + S(r,F), \tag{1.1}$$

where $N_1(r)$ is positive and $S(r,F)$ is given by

$$S(r,F) = m\left(r, \frac{F'}{F}\right) + m\left\{r, \sum_{\nu=1}^{q} \frac{F'}{F-a_\nu}\right\} + q\log^+ \frac{3q}{\delta}$$
$$+ \log 2 + \log \frac{1}{|F'(0)|} \quad \text{if } |a_\mu - a_\nu| \geq \delta \text{ for } 1 \leq \mu < \nu \leq q,$$

with modifications if $F(0) = \infty$ or $F'(0) = 0$.

The quantity $S(r,F)$ will be, in general, negligible with respect to $T(r,F)$. More precisely

$$S(r,F) = O\{\log T(r,F)\} + O\{\log r\} \tag{1.2}$$

as $r \to \infty$ through all values if F has finite order, and outside an exceptional set of finite measure otherwise.

In particular (1.2) implies

$$S(r,F) = o(T(r,F)) \tag{1.3}$$

for a transcendental function F outside an exceptional set E_1 of finite measure.

From (1.1) and (1.3) we obtain

$$(q-2)T(r,F) < \left(\sum_{\nu=1}^{q} N(r,a_\nu)\right)(1+o(1)). \tag{1.4}$$

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But we could still have the relation (1.4) even if (1.3) fails. We can then consider the exceptional set E_2 outside which (1.4) is true. From what we have said above we deduce that we can choose

$$E_1, E_2 \text{ so that } E_2 \subset E_1. \tag{1.5}$$

In [**1**] some conditions on the size of E_1 were given; in particular, it was shown that we can take E_1 independent of λ so that

$$\int_{E_1} r^\lambda \, dr < \infty \quad \text{for every } \lambda \geq 0. \tag{1.6}$$

It is clear from (1.5) that we can also choose E_2 independent of λ so that

$$\int_{E_2} r^\lambda \, dr \leq \int_{E_1} r^\lambda \, dr < \infty \quad \text{for every } \lambda \geq 0. \tag{1.7}$$

In this paper we show that this condition is the best that one can obtain in this direction for the set E_2.

2. Statement of the main result. Precisely we shall prove

THEOREM 1. *For any function $\Phi(r)$ such that*

$$\Phi(r)/r^\lambda \to \infty \quad \textit{for every } \lambda \geq 0, \tag{2.1}$$

there exists an entire function such that the corresponding exceptional set E_2 satisfies

$$\int_{E_2} \Phi(r) \, dr = \infty. \tag{2.2}$$

The proof of this result is based on a construction of Hayman [**5**]. The function f shown is defined so that it approximates a sequence of polynomials with few zeros and i-values on a sequence of corresponding expanding circles. In this way $N(r,0), N(r,i)$ will not grow very quickly for the function f either.

We shall use a result due to F. Nevanlinna (see R. Nevanlinna [**8**]) to show that all the auxiliary functions in [**5**] have characteristic growing near $r = 1$ quicker than

$$C \log \frac{1}{1-r} \quad \text{for certain constant } C\text{'s}.$$

Using this fact we shall find precise estimates of the size of the set where

$$N(r,0) + N(r,i) + N(r,\infty) < \tfrac{1}{2} T(r,f), \tag{2.3}$$

and check that this set is big enough to satisfy (2.2).

3. Some preliminary results.

3.1. In the proof of Theorem 1 we shall use several auxiliary results. First of all we shall prove the above-mentioned result of F. Nevanlinna but we shall obtain precise quantitative values for the constants involved, whereas he only obtained a quantitative result.

LEMMA 3.1. *Suppose that $a_1, a_2, \ldots, a_{q-1}$ are distinct complex numbers such that for a certain δ, with $1 > \delta > 0$,*

$$|a_\mu - a_\nu| \geq \delta, \quad a_\mu \leq \frac{1}{\delta}, \qquad 1 \leq \mu < \nu \leq q-1, \tag{3.1}$$

and assume that F maps the unit disc conformally onto the universal covering surface over the Riemann sphere punctured at $a_1, \ldots, a_{q-1,\infty}$, satisfying

$$|F(0) - a_\mu| \geq \delta, \quad |F(0)| \leq \frac{1}{\delta}, \qquad \mu = 1, \ldots, q-1. \tag{3.2}$$

Then

$$T(r, F) \geq C_1(q) \log \frac{1}{1-r}, \qquad 1 - A_0 \left(\frac{1}{2}\delta^2\right)^{C_2(q)} \leq r < 1, \tag{3.3}$$

where

$$C_1(q) = \frac{1}{2(q-2)}, \quad A_0 = \frac{1}{3.000}, \quad \textit{and} \quad C_2(q) = 2q+3.$$

In the proof of Lemma 3.1 we shall use the magnitude $\alpha(w)$ defined on the w-plane punctured at $a_1, a_2, \ldots, a_{q-1,\infty}$ by

$$\alpha(w) = (1 - |z|^2)|F'(z)|, \qquad w = F(z). \tag{3.4}$$

This is a one-valued function of w. Next we consider the function

$$d(w) = \min_{1 \leq \mu \leq q-1} |w - a_\mu|,$$

and show that $\alpha(w)$ satisfies

(i) $\alpha(w) \geq d(w)$,

(ii) $\alpha(w) \leq 2d(w)(|\log d(w)| + \log \frac{1}{\delta} + \Gamma(\frac{1}{4})^4/2\pi^2)$.

(i) follows from the fact that the Riemann surface contains the disc $\{\xi | |\xi - w| < d(w)\}$, since by Schwarz's Lemma applied to the inverse function of

$$G(s) = F\left(\frac{s+z}{1+\bar{z}s}\right),$$

in this disc we have

$$|(G^{-1})'(w)| = \left|\frac{1}{G'(0)}\right| = \left|\frac{1}{F'(z)(1-|z|^2)}\right| \leq \frac{1}{d(w)},$$

i.e.

$$\alpha(w) = |F'(z)|(1 - |z|^2) \geq d(w).$$

(ii) follows from Landau's Theorem applied to

$$\Phi(s) = \frac{F((s+z)/(1+\bar{z}s)) - a_\mu}{a_\nu - a_\mu},$$

where a_μ is the nearest omitted value and a_ν is any other omitted value.

By Landau's Theorem as in [**6**],

$$|\Phi'(0)| \leq 2|\Phi(0)| \left\{ |\log|\Phi(0)|| + \frac{\Gamma(1/4)^4}{2\pi^2} \right\}. \tag{3.5}$$

On the other hand

$$|\Phi'(0)| = \frac{|F'(z)(1-|z|^2)|}{|a_\nu - a_\mu|} = \frac{\alpha(w)}{|a_\nu - a_\mu|} \tag{3.6}$$

and

$$|\Phi(0)| = \frac{|F(z) - a_\mu|}{|a_\nu - a_\mu|} = \frac{|d(w)|}{|a_\nu - a_\mu|}. \tag{3.7}$$

From (3.5), (3.6), and (3.7), we conclude

$$\begin{aligned}\alpha(w) &\le 2|d(w)| \left(|\log|d(w)|| + \log\frac{1}{|a_\nu - a_\mu|} + \frac{\Gamma(1/4)^4}{2\pi^2} \right) \\ &\le 2|d(w)| \left\{ |\log|d(w)|| + \log\frac{1}{\delta} + \frac{\Gamma(1/4)^4}{2\pi^2} \right\},\end{aligned}$$

i.e. (ii).

Now we define

$$\begin{aligned} &E_\mu = \{w \,|\, |w| - a_\mu| < \delta/2\}, \qquad \mu = 1, 2, \ldots, q-1, \\ &E_q = \{w \,|\, |w| > 2/\delta\}, \qquad E_{q+1} \text{ elsewhere.}\end{aligned} \tag{3.8}$$

Then if $F(z)$ is in E_μ, using (i) we obtain

$$\log((1-|z|^2)|F'(z)|) \ge \log|F(z) - a_\mu|, \tag{3.9}$$

and since $F(z) \in E_\mu$ we have $|F(z) - a_\mu| < \frac{1}{2}\delta < 1$, and so

$$\begin{aligned} m(r, a_\mu) = &\frac{1}{2\pi}\int_{E_\mu} \log\left|\frac{1}{F(re^{i\theta}) - a_\mu}\right| d\theta \\ &+ \frac{1}{2\pi}\int_{E'_\mu} \log^+\left|\frac{1}{F(re^{i\theta}) - a_\mu}\right| d\theta,\end{aligned} \tag{3.10}$$

where $E'_\mu = \mathbf{C}\backslash E_\mu$ is the complementary set of E_μ and where the first integral is taken over the set of values of θ such that $F(re^{i\theta}) \in E_\mu$ and similarly the second integral is taken over the set of values of θ such that $F(re^{i\theta}) \in E'_\mu$.

From (3.10) we deduce

$$\begin{aligned} &\frac{1}{2\pi}\int_{E_\mu} \log|F(re^{i\theta}) - a_\mu|\, d\theta \\ &\quad = -m(r, a_\mu) + \frac{1}{2\pi}\int_{E'_\mu} \log^+\left|\frac{1}{F(re^{i\theta}) - a_\mu}\right| d\theta \\ &\quad \ge -m(r, a_\mu), \qquad 1 \le \mu \le q-1.\end{aligned} \tag{3.11}$$

Next let us assume $w = F(z) \in E_q$; then we have

$$(1-|z|^2)|F'(z)| \ge \tfrac{1}{2}|F(z)|. \tag{3.12}$$

In fact, in this case $|F(z) - a_\mu| \ge |F(z)| - |a_\mu|$, and since $|a_\mu| \le 1/\delta \le \frac{1}{2}|F(z)|$, we obtain

$$|F(z) - a_\mu| \ge |F(z)| - \tfrac{1}{2}|F(z)| = \tfrac{1}{2}|F(z)|, \qquad \mu = 1, \ldots, q-1. \tag{3.13}$$

By (i), we deduce from (3.13)

$$(1-|z|^2)|F'(z)| = \alpha(w) \geq d(w) \geq \tfrac{1}{2}|F(z)|,$$

i.e. (3.12). From (3.12)

$$\frac{1}{2\pi}\int_{E_q} \log((1-r^2)|F'(re^{i\theta})|)\,d\theta \geq \frac{1}{2\pi}\int_{E_q} \log^+|F(re^{i\theta})|\,d\theta - \log 2, \tag{3.14}$$

where the integral over E_q has the same meaning as above.

On the other hand,

$$m(r,F) = \frac{1}{2\pi}\left(\int_{E_q} + \int_{E_q'}\right) \leq \frac{1}{2\pi}\int_{E_q} \log^+|F(re^{i\theta})|\,d\theta + \log\frac{2}{\delta}, \tag{3.15}$$

and so from (3.14) and (3.15)

$$\frac{1}{2\pi}\int_{E_q} \log((1-r^2)|F'(re^{i\theta})|)\,d\theta \geq m(r,F) - \log\frac{4}{\delta}. \tag{3.16}$$

In E_{q+1} we have by (i), $(1-|z|^2)|F'(z)| \geq d(w) \geq \frac{1}{2}\delta$, so that

$$\frac{1}{2\pi}\int_{E_{q+1}} \log((1-r^2)|F'(re^{i\theta})|)\,d\theta \geq \log\frac{\delta}{2}. \tag{3.17}$$

Now from (3.9), (3.11), (3.16), and (3.17) and using the fact that $\log|F'(z)|$ is harmonic we conclude

$$\begin{aligned}\log\{(1-r^2)|F'(0)|\} &= \frac{1}{2\pi}\sum_{\mu=1}^{q+1}\int_{E_\mu} \log\{(1-r^2)|F'(re^{i\theta})|\}\,d\theta \\ &\geq m(r,F) - \sum_{\mu=1}^{q-1} m(r,a_\mu) - \log\left(\frac{8}{\delta^2}\right).\end{aligned} \tag{3.18}$$

Since

$$m(r,a_\mu) \leq m(r,F) - \log|F(0)-a_\mu| + \log^+|a_\mu| + \log 2$$

(see Hayman [**3**, p. 5]), we deduce from (3.2) and (3.18)

$$\begin{aligned}-(q-2)m(r,F) \leq\ & \log(1-r^2) + \log|F'(0)| + \sum_1^{q-1}\log^+|a_\nu| \\ &+ (q-1)\log 2 + \log\left(\frac{8}{\delta^2}\right) - (q-2)\log\delta,\end{aligned}$$

i.e.

$$\begin{aligned}(q-2)T(r,F) \geq\ & \log\frac{1}{1-r^2} - \log|F'(0)| - \sum_1^{q-1}\log^+|a_\nu| \\ &- (q-1)\log\left(\frac{2}{\delta}\right) - \log\left(\frac{8}{\delta^2}\right).\end{aligned} \tag{3.19}$$

We also have, by hypotheses for $\mu = 1, 2, \ldots, q-1$,

$$\delta \leq |F(0)-a_\mu| \leq |F(0)| + |a_\mu| \leq \frac{2}{\delta},$$

so that

$$\delta \le d(F(0)) \le \frac{2}{\delta}.$$

Thus we obtain by (ii)

$$\begin{aligned} |F'(0)| &\le \frac{4}{\delta}\left(\log\frac{2}{\delta} + \log\frac{1}{\delta} + \frac{\Gamma(1/4)^4}{2\pi^2}\right) \\ &\le \frac{8}{\delta}\left(\log\frac{1}{\delta} + 3\right), \end{aligned}$$

i.e.

$$\begin{aligned} \log|F'(0)| &\le \log\frac{1}{\delta} + \log 8 + \log^+\log\frac{1}{\delta} + \log 3 + \log 2 \\ &< 2\,\log\frac{1}{\delta} + \log 48. \end{aligned} \tag{3.20}$$

We also have

$$\sum_{1}^{q-1} \log^+|a_\mu| \le (q-1)\log\frac{1}{\delta}. \tag{3.21}$$

From (3.19), (3.20), and (3.21), we conclude

$$\begin{aligned} (q-2)T(r,F) \ge\ & \log\frac{1}{1-r^2} - 2\,\log\frac{1}{\delta} - \log 48 - (q-1)\log\frac{1}{\delta} \\ & - (q-1)\log\frac{2}{\delta} - \log\left(\frac{8}{\delta^2}\right) \\ \ge\ & \log\frac{1}{1-r} - (2q+2)\log\frac{1}{\delta} - (q+3)\log 2 - \log 48, \end{aligned}$$

hence

$$(q-2)T(r,F) \ge \frac{1}{2}\log\frac{1}{1-r}$$

if

$$\frac{1}{2}\log\frac{1}{1-r} > \log\{48 \cdot 2^{(q+3)} \cdot \delta^{-(2q+2)}\},$$

i.e. if

$$r > 1 - \left\{\frac{1}{48} \cdot 2^{-(q+3)} \cdot \delta^{2q+2}\right\}^2.$$

Thus we get

$$T(r,F) \ge \frac{1}{2(q-2)}\log\frac{1}{1-r}$$

if

$$r > 1 - \frac{1}{3000}\left(\frac{1}{2}\delta^2\right)^{2q+3},$$

which is (3.3).

3.2.

LEMMA 3.2. *Let p be a positive integer and x a positive number. Then there exist a_1 such that $ex < |a_1| < 10ex$ and a function*

$$F(z) = a_1 z + a_{p+1} z^{p+1} + \cdots \tag{3.22}$$

regular in $|z| < 1$, univalent in $z < \sqrt{2} - 1$, and assuming the values $i, 0$ no more than once in $|z| < 1$.

Furthermore for $x \geq 2$ the function F satisfies

$$T(r, F) \geq \frac{1}{192p} \log \frac{1}{1-r}, \qquad \mathbf{R}_p \leq r < 1, \tag{3.23}$$

where

$$\mathbf{R}_p = 1 - \frac{A_0}{p}(ex)^{-26p}$$

and A_0 is a positive constant.

Let E_1 be the set $\{0, \pm ex, i, \infty\}$ and let $\mathbf{R}$ be the universal covering surface over the complement of E_1. Let $\mathbf{R}_1, \mathbf{R}_2$ be the surfaces obtained by cutting $\mathbf{R}$ from ex to $+\infty$ along the real axis and $\mathbf{R}_3, \mathbf{R}_4$ those obtained by cutting $\mathbf{R}$ from $-ex$ to $-\infty$. Let $\mathbf{R}_5$ be the plane cut from ex to $+\infty$ and from $-ex$ to $-\infty$ along the real axis and finally let $\mathbf{R}_0$ be obtained by joining $\mathbf{R}_1, \mathbf{R}_2$ to $\mathbf{R}_5$ on the segment (ex, ∞) and $\mathbf{R}_3, \mathbf{R}_4$ to $\mathbf{R}_5$ along the segment $(-\infty, -ex)$.

The Riemann surface $\mathbf{R}_0$ obtained in this way contains none of the points $\pm ex, \infty$ in any sheet and contains the points 0 and i exactly once, those in the sheet $\mathbf{R}_5$. $\mathbf{R}_0$ is simply connected and since it does not contain points over $\pm ex, \infty$, $\mathbf{R}_0$ is hyperbolic.

Therefore there is a conformal map F_0 from the unit disc $\{z \,|\, |z| < 1\}$ onto $\mathbf{R}_0$,

$$F_0(z) = b_1 z + b_2 z^2 + \cdots, \qquad b_1 > 0.$$

By the construction of $\mathbf{R}_0$ and F_0 we deduce that F_0 assumes $i, 0$ precisely once and that it never assumes $\pm ex, \infty$. This implies that F_0 is subordinate to the function G which maps $\{z \,|\, |z| < 1\}$ onto the universal covering over the plane punctured at $\pm ex, \infty$ satisfying

$$G(0) = 0, \qquad G'(0) > 0.$$

This function G maps the sheet $\mathbf{R}_5$ onto a quadrilateral Q in the unit disc, bounded by four quarter circles, orthogonal to the circumference $\{z \,|\, |z| = 1\}$, joining the points $z = 1, i, -1, -i$ in the form $(1, i), (i, -1), (-1, -i)$, and $(-i, 1)$.

Q contains the disc $\{z \,|\, |z| < \sqrt{2} - 1\}$ and since $F_0(z)$ is subordinate to $G(z)$, $F_0(z)$ maps this disc onto a subset of the sheet $\mathbf{R}_5$; therefore $F_0(z)$ is univalent in $\{z \,|\, |z| < \sqrt{2} - 1\}$.

By Koebe's Theorem we have

$$ex > \frac{b_1(\sqrt{2} - 1)}{4},$$

and since the inverse function $z = \Phi(w)$ maps the disc $\{w \,|\, |w| < ex\}$ into the disc $\{z \,|\, |z| < 1\}$, we can apply Schwarz's Lemma and obtain

$$b_1^{-1} = \Phi'(0) < (ex)^{-1}.$$

Hence we have

$$ex < b_1 < \frac{4}{\sqrt{2}-1} ex < 10ex. \tag{3.24}$$

Thus we have constructed a function (3.22) in the case $p = 1$. When $p > 1$ we proceed as follows.

If p is odd so that $i^p \neq \pm 1$, we write $E_p = \{\pm(ex)^p, 0, i^p, \infty\}$ and define $F_p(z)$ as above with E_p instead of E_1 and

$$F(z) = \{F_p(z^p)\}^{1/p}. \tag{3.25}$$

A standard argument in the elementary theory of univalent functions shows that $F(z)$ has the required properties, i.e. it has an expansion (3.22) where $ex < |a_1| < 10ex$ is regular in $|z| < 1$, univalent in $|z| < \sqrt{2} - 1$ and assumes the values $i, 0$ no more than once in $|z| < 1$.

If p is even we consider the set $E'_p = \{\pm(ex)^p, 0, -i^{p+1}, \infty\}$ instead of E_p and the function $iF_p(-iz)$ instead of $F_p(z)$ and then define $F(z)$ by (3.25) again.

Finally we prove (3.23). We shall make use of Lemma 3.1.

We consider again the case p odd. The case p even follows with small modifications.

According to Lemma 3.1, any function F_5 mapping conformally the unit disc onto the universal covering surface over the Riemann sphere punctured at the points of $E_p = \{0, \pm(ex)^p, i^p, \infty\}$, such that

$$|F_5(0) - a| \geq \delta \quad \text{for every } a \in E_p, \qquad |F_5(0)| \leq \frac{1}{\delta}, \tag{3.26}$$

where $\delta \leq \min\{1, (ex)^p, (ex)^p - 1\}$ and $1/\delta \geq \{(ex)^p, 1\}$, i.e. such that (3.1) and (3.2) are satisfied, has a characteristic function $T(r, F_5)$ such that

$$T(r, F_5) \geq \frac{1}{6} \log \frac{1}{1-r}, \qquad 1 - A_0 \left(\frac{1}{2}\delta^2\right)^{13} \leq r < 1, \tag{3.27}$$

where A_0 is an absolute constant.

For $x \geq 2$, we can take $\delta = 1/(ex)^p$.

First we shall consider a translate F_{pa} of F_p where

$$F_{pa}(z) = F_p\left(\frac{z-a}{1-\bar{a}z}\right)$$

for some a such that $|a| < 1$.

We choose the branch of F_5 at $z = 0$, such that $F_5(0) = 1/(ex)^p$, so that for $x \geq 2$ (3.26) is satisfied and then consider a translate F_{pa} of F_p such that $F_{pa}(0) = 1/(ex)^p$. It is clear that we can define the function

$$\omega(z) = F_{pa}^{-1} \circ F_5 \tag{3.28}$$

in a neighborhood of $z = 0$, so that $\omega(0) = 0$, and by the construction of F_{pa} and F_5, ω can be continued without limit in the disc and since this is a simply connected set, we can make use of the monodromy theorem to obtain a function from the whole unit disc onto itself such that $\omega(0) = 0$.

From (3.28) we obtain

$$F_5(z) = F_{pa}(\omega(z)),$$

i.e. we conclude that F_5 is subordinate to F_{pa}. Hence F_{pa} also satisfies (3.27):

$$T(r, F_{pa}) \geq \frac{1}{6} \log \frac{1}{1-r}, \qquad 1 - A_0 \left(\frac{1}{2}(ex)^{-2p} \right)^{13} \leq r < 1. \tag{3.29}$$

Next, we shall prove a similar condition for F_p. To do this, we shall use the Ahlfors-Shimizu characteristic T_0 instead of the Nevanlinna characteristic T. T_0 and T differ only by a bounded term so that we obtain equivalent statements.

We recall that for a meromorphic function G, T_0 is defined by

$$T_0(r, G) = \int_0^r \frac{A(t)}{t} \, dt,$$

where $A(t)$ is the area, with due regard to multiplicity, of the image on the Riemann sphere of $\{z \mid |z| \leq t\}$ by G.

If G is regular then

$$T_0(r, G) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |G(re^{i\theta})|^2} \, d\theta - \log \sqrt{1 + G(0)^2}.$$

Therefore, using the inequality

$$\log^+ x \leq \log \sqrt{1 + x^2} \leq \log^+ x + \tfrac{1}{2} \log 2,$$

and the fact that for G regular

$$T(r, G) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |G(re^{i\theta})| \, d\theta,$$

we obtain

$$T(r, G) - \log \sqrt{1 + G(0)^2} \leq T_0(r, G) \leq T(r, G) - \log \sqrt{1 + G(0)^2} + \tfrac{1}{2} \log 2. \tag{3.30}$$

First we show that

$$\left| \frac{z+a}{1 + \bar{a}z} \right| \geq 1 - \frac{2}{1 - |a|}(1 - r), \qquad |z| = r. \tag{3.31}$$

In fact, we have

$$\left| \frac{z+a}{1 + \bar{a}z} \right| \geq \frac{r - a}{1 - |\bar{a}|r},$$

whence

$$1 - \left| \frac{z+a}{1 + \bar{a}z} \right| \leq 1 - \frac{r - |a|}{1 - |a|r} = \frac{(1-r)(1+|a|)}{1 - |a|r} \leq \frac{2}{1 - |a|}(1 - r),$$

which yields (3.31).

The function $w = (z+a)/(1+\bar{a}z)$ maps the disc $\{z \mid |z| < t\}$, $t > |a|$, conformally onto a disc containing the origin, establishing also a one-to-one and continuous correspondence between the circumference $\{z \mid |z| = t\}$ and the boundary of that disc.

On the other hand by (3.31)

$$|w(z)| \geq 1 - \frac{2}{1 - |a|}(1 - t), \qquad |z| = t,$$

i.e. we can conclude that the disc contains the disc with center at the origin

$$\left\{w \,|\, |w| \le 1 - \frac{2}{1-|a|}(1-t)\right\}.$$

Now since $F_p(z) = F_{pa}((z+a)/(1+\bar{a}z))$, we deduce from the above considerations that for every solution in

$$\left\{w \,|\, |w| \le 1 - \frac{2}{1-|a|}(1-t)\right\}$$

of $F_{pa}(w) = \varsigma$, there is at least one solution of $F_p(z) = \varsigma$ in $\{z \,|\, |z| \le t\}$ and different solutions of the first equation give rise to different solutions of the last equation.

We now recall that (see Hayman [**3**, p. 13]),

$$A_{F_{pa}}(t) = \int_S n_{F_{pa}}(t,\varsigma)\, d\mu(\varsigma), \qquad A_{F_p}(t) = \int_S n_{F_p}(t,\varsigma)\, d\mu(\varsigma),$$

where S is the Riemann sphere, $d\mu$ is the normalized area element in S, and $n_{F_{pa}}(t)$ and $n_{F_p}(t)$ are the number of roots of $F_{pa}(w) = \varsigma$ and $F_p(z) = \varsigma$ in $|w| \le t, |z| \le t$, respectively.

Therefore we conclude from what we have said above

$$A_{F_p}(t) \ge A_{F_{pa}}\left(1 - \frac{2}{1-|a|}(1-t)\right), \qquad t > |a|. \tag{3.32}$$

On the other hand F_p is univalent in $\{z \,|\, |z| < \sqrt{2}-1\}$ and $F_p(0) = 0$. Thus by Koebe's Theorem the disc $\{w \,|\, |w| < b_1(\sqrt{2}-1)/4\}$, is covered by the image of $\{z \,|\, |z| < \sqrt{2}-1\}$ by F_p.

Therefore, since $F_p(-a) = F_{pa}(0) = (ex)^{-p}$, we can find a in $\{z \,|\, |z| < \sqrt{2}-1\}$ provided that

$$\frac{b_1(\sqrt{2}-1)}{4} > \frac{1}{(ex)^p}.$$

But, by (3.24), $|b_1| > (ex) > 10(ex)^{-p}$, since $x \ge 2$, so that the above condition is satisfied.

Now, by an inequality for univalent functions (see Hayman [**2**, p. 4]), we obtain

$$|F_p(-a)| \ge \frac{|b_1|\,|a|}{(1+|a|/(\sqrt{2}-1))^2} \ge \frac{|b_1|\,|a|}{4}$$

and since $|b_1| \ge (ex)^p$, we also have

$$|a| \le \frac{4|F_p(-a)|}{|b_1|} = \frac{4}{|b_1|(ex)^p} \le \frac{4}{(ex)^{2p}} \le \frac{1}{e^2} \le \frac{1}{7},$$

since $x \ge 2$.

Next we show that

$$1 - \frac{2}{1-|a|}(1-t) \ge t^4, \qquad \frac{2}{3} \le t \le 1. \tag{3.33}$$

In fact if $|a| < 1/7$, and $2/3 \le t \le 1$,

$$\begin{aligned} 1 - t^4 - \frac{2}{1-|a|}(1-t) &\ge 1 - t^4 - \frac{7}{3}(1-t) \\ &= (1-t)\left\{1 + t + t^2 + t^3 - \frac{7}{3}\right\} \\ &\ge (1-t)\left\{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} - \frac{7}{3}\right\} = \frac{2}{27}(1-t) > 0. \end{aligned}$$

From (3.30), (3.32), and (3.33) and since $F_p(0) = 0$ we have

$$\begin{aligned} T(r, F_p) + \frac{1}{2}\log 2 \ge T_0(r, F_p) &= \int_0^r A_{F_p}(t)\frac{dt}{t} \\ &\ge \int_{2/3}^r A_{F_{pa}}\left\{1 - \frac{2}{1-|a|}(1-t)\right\}\frac{dt}{t} \ge \int_{2/3}^r A_{F_{pa}}(t^4)\frac{dt}{t} \\ &= \int_{(2/3)^4}^{r^4} A_{F_{pa}}(s)\frac{ds}{s} \ge \frac{1}{4}\int_{1/5}^{r^4} A_{F_{pa}}(s)\frac{ds}{s} \\ &= \frac{1}{4}\left\{T_0(r^4, F_{pa}) - T_0\left(\frac{1}{5}, F_{pa}\right)\right\} \\ &> \frac{1}{4}\left\{T(r^4, F_{pa}) - T\left(\frac{1}{5}, F_{pa}\right) - \frac{1}{2}\log 2\right\}. \end{aligned} \tag{3.34}$$

Now we obtain a lower bound for $T(r^4, F_{pa})$ and an upper bound for $T(1/5, F_{pa})$. To do the first point, we make use of (3.29) and obtain

$$\begin{aligned} T(r^4, F_{pa}) &\ge \frac{1}{6}\log\frac{1}{1-r^4} \ge \frac{1}{6}\log\frac{1}{4(1-r)} \\ &= \frac{1}{6}\left(\log\frac{1}{1-r} - \log 4\right) \ge \frac{1}{12}\log\frac{1}{1-r} \end{aligned} \tag{3.35}$$

if

$$r^4 \ge 1 - A_0\left(\tfrac{1}{2}(ex)^{-2p}\right)^{13}, \tag{3.36}$$

and if

$$\log\frac{1}{1-r} \ge 2\log 4, \quad \text{i.e. } r \ge 1 - e^{-16},$$

both conditions can be included in (3.36) after decreasing A_0 if necessary.

Next we obtain an upper bound for $T(1/5, F_{pa})$.

F_p is univalent in $\{z \mid |z| < \sqrt{2} - 1\}$, therefore by an inequality for univalent functions (see Hayman [**2**, p. 4]), we get

$$|F_p(z)| < \frac{|b_1| r_0^2 |z|}{(r_0 - |z|)^2}, \qquad |z| < r_0 = \sqrt{2} - 1. \tag{3.37}$$

On the other hand for $|\xi| \le 1/5, |a| \le 1/7$, we have

$$\left|\frac{\xi - a}{1 - \bar{a}z}\right| \le \frac{|\xi| + |a|}{1 + |a|\,|\xi|} \le \frac{1/5 + 1/7}{1 + 1/35} = \frac{1}{3},$$

so that for $|\xi| \le 1/5$

$$F_{pa}(\xi) = F_p\left(\frac{\xi - a}{1 - \bar{a}\xi}\right) = F_p(z),$$

where $|z| \le 1/3$. Hence since $r_0 = \sqrt{2} - 1 > 2/5$, we conclude from (3.37)

$$|F_p(z)| \le \frac{|b_1|\,|z|}{(1 - |z|/r_0)^2} \le \frac{1/3|b_1|}{(1 - 5/6)^2} = 12|b_1| \le 120(ex)^p. \tag{3.38}$$

From (3.38), we obtain

$$T(1/5, F_{pa}) \le \log\{120(ex)^p\}. \tag{3.39}$$

Therefore if (3.36) holds, we have by (3.34), (3.35), and (3.39)

$$\begin{aligned} T(r, F_p) &\ge \frac{1}{4}\left\{\frac{1}{12}\log\frac{1}{1-r} - \log\{120(ex)^p\} - \frac{1}{2}\log 2\right\} - \frac{1}{2}\log 2 \\ &\ge \frac{1}{96}\log\frac{1}{1-r} \end{aligned} \tag{3.40}$$

if

$$\frac{1}{24}\log\frac{1}{1-r} \ge \log\{120(ex)^p\} + \frac{3}{2}\log 2,$$

and this will happen if

$$1 > r \ge 1 - (480(ex)^p)^{-24}. \tag{3.41}$$

Again we can put together (3.36) and (3.41) in

$$1 > r^4 \ge 1 - A_0(ex)^{-26p}, \tag{3.42}$$

with a new A_0.

Now, from (3.25) and (3.40), we deduce

$$T(r, F) = \frac{1}{p}T(r^p, F_p) \ge \frac{1}{96p}\log\frac{1}{1 - r^p} \tag{3.43}$$

if

$$1 > r^{4p} \ge 1 - A_0(ex)^{-26p}, \tag{3.44}$$

whence

$$\begin{aligned} T(r, F) &\ge \frac{1}{96p}\log\frac{1}{p(1-r)} = \frac{1}{96p}\left(\log\frac{1}{1-r} - \log p\right) \\ &\ge \frac{1}{192p}\log\frac{1}{1-r} \end{aligned} \tag{3.45}$$

if, in addition to (3.44), we have

$$\log\frac{1}{1-r} \ge 2\log p, \quad \text{i.e. } r \ge 1 - \frac{1}{p^2}. \tag{3.46}$$

Using the fact that $4p(1-r) \ge 1 - r^{4p}$, we conclude that (3.44) and (3.46) will be satisfied if

$$1 > r \ge 1 - \frac{A_0}{p}(ex)^{-26p}, \tag{3.47}$$

with A_0 an absolute constant.

We have that (3.45) holds in the range (3.47), so that we have proved (3.23). This completes the proof of Lemma 3.2.

3.3.

LEMMA 3.3. *Suppose that $a_1, \dots, a_p$ are preassigned complex numbers, not all zero, and write $M = \sum_{\nu=1}^{p} |a_\nu|$ and $\mu > eM$.*

Then the function

$$\omega(z) = \frac{\sum_{\nu=1}^{p} a_\nu z^\nu + \mu z^{2p}}{\mu + \sum_{\nu=1}^{p} \bar{a}_\nu z^{2p-\nu}}, \tag{3.48}$$

has precisely $2p$ zeros and no poles in the disc

$$\{z \mid |z| \le \exp(-1/2p)\}. \tag{3.49}$$

Furthermore the function ω satisfies the following inequality:

$$1 - 16p^2(1-|z|) < |\omega(z)| < 1 - \frac{1}{8p}(1-|z|). \tag{3.50}$$

First we observe that $|\omega(z)| = 1$ for $|z| = 1$, ω has no poles in $\{z \mid |z| < 1\}$, and by Rouché's Theorem ω has $2p$ zeros in the unit disc. Next we show that these $2p$ zeros are in the smaller disc (3.49).

We write $r_p = \exp\{-1/2p\}$. Then for $r_p < |z| < 1$, we have

$$\left| \sum_{\nu=1}^{p} a_\nu z^\nu \right| \le M < \frac{\mu}{e} = \mu r_p^{2p} < |\mu z^{2p}|,$$

i.e. $\omega(z) \neq 0$ for $r_p < |z| < 1$, and therefore if z_ν, $1 \le \nu \le 2p$, are the zeros of ω in the unit disc, we have

$$|z_\nu| \le r_p, \qquad 1 \le \nu \le 2p.$$

We now consider the function

$$\Pi(z) = \prod_{\nu=1}^{2p} \frac{z - z_\nu}{1 - \bar{z}_\nu z}.$$

Then $\Phi(z) = \omega(z)/\Pi(z)$ is regular and not zero in $\{z \mid |z| \le 1\}$ and $|\Phi(z)| = 1$ on $|z| = 1$. Thus $|\Phi(z)| \equiv 1$ and

$$\omega(z) = e^{i\theta_0}\Pi(z) = e^{i\theta_0} \prod_{\nu=1}^{2p} \frac{z - z_\nu}{1 - \bar{z}_\nu z}, \tag{3.51}$$

where θ_0 is a constant such that $0 \le \theta_0 \le 2\pi$.

Using (3.49) we deduce that for $|z| = r > r_p$, we have

$$\left(\frac{r - r_p}{1 - r_p r} \right)^{2p} \le |\Pi(z)| = |\omega(z)| \le \left(\frac{r + r_p}{1 + r_p r} \right)^{2p}. \tag{3.52}$$

We write

$$x = \frac{r - r_p}{1 - r_p r}, \qquad y = \frac{r_p + r}{1 + r_p r},$$

and obtain

$$\begin{aligned} 1 - x^{2p} &= (1-x)(1 + x + \cdots + x^{2p-1}) < 2p(1-x) \\ &= 2p\frac{(1-r)(1+r_p)}{1 - r_p r} < \frac{4p}{1 - r_p}(1-r). \end{aligned} \tag{3.53}$$

On the other hand if $t = 1/2p$, so that $0 < t \leq 1/2$, we have $1 - e^{-t} = te^{-\tau}$, $0 < \tau < 1/2$, hence

$$1 - e^{-t} > te^{-1/2} > \frac{1}{2}t, \qquad r_p = e^{-t} < 1 - \frac{1}{2}t = 1 - \frac{1}{4p}.$$

Thus we obtain from (3.53)

$$1 - x^{2p} < \frac{4p}{1 - r_p}(1 - r) < 16p^2(1 - r). \tag{3.54}$$

Now, (3.52) and (3.54) yield the left-hand inequality of (3.50). To prove the right-hand inequality we obtain an upper bound for y^{2p}. We have

$$1 - y^{2p} \geq 1 - y = \frac{(1 - r_p)(1 - r)}{1 + r_p r} \geq \frac{1}{2}(1 - r_p)(1 - r) > \frac{1 - r}{8p}.$$

From this follows the right-hand inequality of (3.50).

3.4.

LEMMA 3.4. *Suppose given the complex numbers $a_1, a_2, \ldots, a_p$, not all zero, and write $M = \sum_{\nu=1}^{p} |a_\nu|$.*

Then there exists $F_p(z)$ regular in $|z| < 1$, assuming i and 0 no more than $2p$ times, and with a power series development

$$F_p(z) = a_1 z + a_2^2 + \cdots + a_p z^p + O(z^{p+1}), \tag{3.55}$$

near $z = 0$. Furthermore

$$|F_p(z)| < 40eM \quad \text{in } |z| < \frac{\sqrt{2} - 1}{2}, \tag{3.56}$$

and if $M \geq 2$, we have

$$T(r, F_p) \geq \frac{1}{8000p^3} \log \frac{1}{1 - r}, \qquad R'_p \leq r < 1, \tag{3.57}$$

where

$$R'_p = 1 - \frac{A_0}{p^3}(16eM)^{-1000p^4}.$$

Let $F(z)$ be the function whose existence is asserted in Lemma 3.2 with $x = M$. Let us write $a_1 = \mu$, then $eM < \mu < 10eM$, and consider the function

$$\omega(z) = \frac{\sum_{\nu=1}^{p} a_\nu z^\nu + \mu z^{2p}}{\mu + \sum_{\nu=1}^{p} \bar{a}_\nu z^{2p-\nu}},$$

as in Lemma 3.3.

Then we define

$$F_p(z) = F(\omega(z)). \tag{3.58}$$

One can check that

$$\omega(z) = \mu^{-1} \sum_{\nu=1}^{p} a_\nu z^\nu + O(z^{p+1}),$$

and so

$$F_p(z) = \mu\omega(z) + O(z^{p+1}),$$

whence we conclude (3.55).

The equation $\omega(z) = \varsigma$ has precisely $2p$ roots in $\{z \,|\, |z| < 1\}$ for any ς in $\{\varsigma \,|\, |\varsigma| < 1\}$, and since the equations $F(\omega) = 0$ and $F(\omega) = i$ have at most one root in the unit disc, we deduce that the equations $F_p(z) = 0$ and $F_p(z) = i$ have at most $2p$ roots in $\{z \,|\, |z| < 1\}$.

Since $F(z)$ is univalent in $\{z \,|\, |z| < r_0 = \sqrt{2} - 1\}$, we have by a classical inequality for univalent functions

$$|F(z)| < \frac{\mu r_0^2 |z|}{(r_0 - |z|)^2}, \qquad |z| < r_0.$$

By Schwarz's Lemma, $|\omega(z)| \leq |z|$ for $|z| < 1$ and so for $|z| = r_0/2$ we obtain

$$|F_p(z)| \leq \frac{\mu r_0^2 |\omega(z)|}{(r_0 - |\omega(z)|)^2} \leq 4\mu|\omega(z)| \leq 40eM|z| < 40eM,$$

which is (3.56).

To prove (3.57), we shall make use of (3.23) and the definition of F_p in (3.58). We shall also use the Ahlfors-Shimizu characteristic T_0 instead of the Nevanlinna characteristic T as we did in the proof of Lemma 3.2.

First we show that

$$A_{F_p}(t) \geq A_F(1 - 16p^2(1 - t)), \qquad t \geq r_p = \exp\{-1/2p\}. \tag{3.59}$$

This follows from the definition of F_p in (3.58) and the fact that the number of roots in $\{z \,|\, |z| \leq t\}$ of any equation of the form

$$F_p(z) = \varsigma \tag{3.60}$$

is not less than the number of the roots in $\{\omega \,|\, |\omega| \leq 1 - 16p^2(1 - t)\}$ of the corresponding equation

$$F(\omega) = \varsigma. \tag{3.61}$$

To see this, suppose that ω_0 is a root of (3.61) in $\{\omega \,|\, |\omega| \leq 1 - 16p^2(1 - t)\}$. Then we prove that the equation

$$\omega(z) = \omega_0 \tag{3.62}$$

must have at least one solution in $\{z \,|\, |z| \leq t\}$, say z_0.

In fact, the image by ω of $\{z \,|\, |z| < t\}$ is a domain $D(t)$ contained in $\{\omega \,|\, |\omega < 1\}$ containing the origin, since $\omega(z)$ has $2p$ zeros in the smaller disc $\{z \,|\, |z| \leq \exp(-1/2p)\}$, and whose boundary is contained in the image of $\{z \,|\, |z| = t\}$. Then using the left-hand inequality of (3.50) we have

$$|\omega(z)| > 1 - 16p^2(1 - t), \qquad |z| = t, \; t > \exp\{-1/2p\}, \tag{3.63}$$

so that we can conclude that $D(t)$ contains the disc $\{\omega \,|\, |\omega| < 1 - 16p^2(1 - t)\}$, i.e. the equation (3.62) admits at least one solution z_0 in $\{z \,|\, |z| \leq t\}$.

Now

$$F_p(z_0) = F(\omega(z_0)) = F(\omega_0) = \varsigma,$$

i.e. z_0 is a solution of (3.60).

Therefore for every solution of (3.61) in $\{\omega \,|\, |\omega| \leq 1 - 16p^2(1 - t)\}$ there is at least one solution of (3.60) in $\{z \,|\, |z| \leq t\}$ and different solutions of (3.61) yield, in this way, different solutions of (3.60).

As in Lemma 3.2 we recall that (Hayman [**3**, p. 13])

$$A_{F_p}(t) = \int_S n_{F_p}(t,\varsigma)\,d\mu(\varsigma), \qquad A_F(t) = \int_S n_F(t,\varsigma)0\,d\mu(\varsigma),$$

where S is the Riemann sphere, $d\mu$ is the normalized area element in S, and $n_{F_p}(t)$, $n_F(t)$ are the number of roots of (3.60) and (3.61) respectively, in the disc of radius t.

From all these considerations we conclude (3.59).

To prove (3.57), we shall make use of the following inequality:

$$3n(1-t) > 1 - t^{3n} > n(1-t), \qquad \tfrac{1}{5} \le t^{3n} < 1,\ n \ge 1. \tag{3.64}$$

In fact, let us observe that the quotient

$$\frac{1-t^{3n}}{1-t} = 1 + t + t^2 + \cdots + t^{3n-1}$$

increases with t, so that it is enough to consider the case $t^{3n} = 1/5$ for the right-hand inequality in (3.64). The left-hand inequality is obvious.

We write $h = 1/n$, $a = \frac{1}{3}\log 5$, and note that

$$\frac{1-t}{1-t^{3n}} = \frac{5}{4}(1-e^{-ah}) < \frac{5ah}{4} = \frac{5\log 5}{12n} < \frac{1}{n},$$

which proves (3.64).

Now we make use of (3.59) and apply the right-hand inequality of (3.64) with $n = 16p^2$. Let $r' = 5^{-1/3n}$ and observe that $r' \ge r_p$. Then for $r > r'$ we obtain

$$\begin{aligned} T_0(r,F_p) - T_0(r',F_p) &= \int_{r'}^{r} A_{F_p}(t)\frac{dt}{t} \ge \int_{r'}^{r} A_F\{(1-n(1-t)\}\frac{dt}{t} \\ &\ge \int_{r'}^{r} A_F(t^{3n})\frac{dt}{t} = \frac{1}{3n}\int_{r'^{3n}}^{r^{3n}} A_F(s)\frac{ds}{s} \\ &= \frac{1}{3n}\left\{T_0(r^{3n},F) - T_0\left(\frac{1}{5},F\right)\right\}. \end{aligned} \tag{3.65}$$

Next we find an upper bound for $T_0(1/5,F)$ and a lower bound for $T_0(r^{3n},F)$.

To obtain an upper bound for $T_0(1/5,F)$, we observe that $F(z)$ is univalent in $|z| < r_0$, where $r_0 = \sqrt{2}-1 > 2/5$. Thus for $|z| = 1/5 < (1/2)r_0$, we have by an inequality for univalent functions (Hayman [**2**, p. 4]),

$$|F(z)| \le \frac{|F'(0)|\,|z|r_0^2}{(r_0-|z|)^2} \le 4|z|\,|F'(0)| \le \frac{4}{5}\mu < 8eM.$$

Hence

$$T_0\left(\tfrac{1}{5},F\right) \le T\left(\tfrac{1}{5},F\right) + \tfrac{1}{2}\log 2 \le \log(8eM) + \tfrac{1}{2}\log 2. \tag{3.66}$$

By (3.23) in Lemma 3.2

$$T(r^{3n},F) \ge \frac{1}{192p}\log\frac{1}{1-r^{3n}} \ge \frac{1}{192p}\left\{\log\frac{1}{1-r} - \log 3n\right\} \tag{3.67}$$

if

$$r^{3n} \ge 1 - \frac{A_0}{p}(eM)^{-26p}. \tag{3.68}$$

Thus by (3.65), (3.66), and (3.67) we have

(3.69)

$$T(r, F_p) \geq T_0(r, F_p) - \frac{1}{2}\log 2$$
$$\geq \frac{1}{3n}\left\{\frac{1}{192p}\left\{\log\frac{1}{1-r} - \log 3n\right\} - \log(8eM) - \frac{1}{2}\log 2\right\} - \frac{1}{2}\log 2$$

in the range (3.68).

From (3.69) we obtain

$$T(r, F_p) \geq \frac{1}{8000p^3}\log\frac{1}{1-r} \tag{3.70}$$

if (3.68) holds, and if in addition

$$r \geq 1 - A(16eM)^{-1000p^4}. \tag{3.71}$$

We make use again of the inequality

$$1 - r^{3n} \leq 3n(1-r) = 48p^2(1-r),$$

and conclude that (3.68) will be satisfied if

$$r \geq 1 - \frac{A_0}{p^3}(eM)^{-26p}. \tag{3.72}$$

(3.71) and (3.72) will be satisfied simultaneously if

$$r \geq 1 - \frac{A_0}{p^3}(16eM)^{-1000p^4}, \tag{3.73}$$

after decreasing A_0 if necessary.

But (3.70) and (3.73) yield (3.57) and the proof of Lemma 3.4 is complete.

4. Proof of Theorem 1. We prove Theorem 1 by showing that given an arbitrary function $\Phi(r)$ satisfying (2.1), there is an integral function f such that

$$N(r, 0, f) + N(r, i, f) + N(r, \infty, f) < \tfrac{1}{2}T(r, f), \tag{4.1}$$

for

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k - \gamma_k, \qquad k \geq k_0, \tag{4.2}$$

where

$$\lambda(r) = \frac{\log\Phi\left(\frac{1}{2}r\right)}{\log 2r}, \tag{4.3}$$

which tends to infinity as r tends to infinity by (2.1), $\{\rho_k\}$ is a sequence which grows to infinity very quickly, and $\{\gamma_k\}$ is a sequence decreasing to zero in such a way that

$$\gamma_k\rho_k^{\lambda(\rho_k)} \to 0 \quad \text{as } k \to \infty. \tag{4.4}$$

Thus increasing k_0 if necessary we ensure that all the intervals (4.2) belong to E_2.

We show that we can assume $\lambda(r)$ increasing. In fact, since $\lambda(r) \to \infty$ as $r \to \infty$ there exists r_N such that for $r \geq r_N, \lambda(r) \geq N$, then we define $\lambda_1(r)$ by

$$\lambda_1(r) = N, \qquad r_N \leq r < r_{N+1},$$

and $\Phi_1(r)$ by the relation

$$\lambda_1(r) = \frac{\log \Phi_1\left(\frac{1}{2}r\right)}{\log 2r}. \tag{4.5}$$

Then it is clear that $\lambda(r) \geq \lambda_1(r)$ and therefore by (4.3) and (4.5) $\Phi(r) \geq \Phi_1(r)$. Hence it is enough to prove that there exists f satisfying (4.1) in a set satisfying (2.2) with $\Phi_1(r)$ instead of $\Phi(r)$.

We shall also assume $\rho_k \geq \rho_0 = 2$ and $\lambda(\rho_0) \geq 0$, i.e. $\varphi(1) \geq 1$. Then

$$\rho_k^{-\lambda(\rho_k)} \leq 1, \qquad \text{so } \rho_k - \rho_k^{-\lambda(\rho_k)} \geq \tfrac{1}{2}\rho_k. \tag{4.6}$$

Then using (4.3), (4.4), and (4.6) we obtain

$$\begin{aligned}\int_{E_2} \Phi(r)\, dr &= \int_{E_2} (4r)^{\lambda(2r)}\, dr \geq \sum_{k_0} \rho_k^{\lambda(\rho_k)} \cdot \{\rho_k - \gamma_k - (\rho_k - \rho_k^{-\lambda(\rho_k)})\} \\ &= \sum_{k_0} \rho_k^{\lambda(\rho_k)} \cdot \rho_k^{-\lambda(\rho_k)}(1 - \gamma_k \rho_k^{\lambda(\rho_k)}) > (1 + o(1)) \sum_{k_0} 1 = \infty.\end{aligned}$$

Next we proceed to construct the function f and the sequence $\{\rho_k\}$. Once $\{\rho_k\}$ has been defined, we will define $\gamma_k = \rho_k^{-(\lambda(\rho_k)+1)}$, and since $\rho_k \to \infty$, (4.4) is satisfied.

We define f by

$$f(z) = \sum_1^\infty b_n z^n, \tag{4.7}$$

and proceed to construct the coefficients b_n successively.

We set $b_1 = 1$. Now let $\{p_k\}$ be a strictly increasing sequence of positive integers such that $p_1 = 1$. We assume that we have already defined b_n for $n \leq p_k$ and proceed to construct b_n for $p_k < n \leq p_{k+1}$.

Simultaneously we shall inductively define the increasing sequence of positive numbers $\{\rho_k\}$ with $\rho_0 = 2$ and tending rapidly to infinity and the increasing sequence $\{p_k\}$.

Let us assume that ρ_k and p_{k+1} have already been chosen and let $F_k(z)$ be the function defined in Lemma 3.4 with $p = p_k$ and

$$a_n = b_n \rho_k^n \quad \text{for } 1 \leq n \leq p_k. \tag{4.8}$$

Then if a_n are the coefficients of $F_k(z)$ for all n, we define b_n by (4.8) for $p_k < n \leq p_{k+1}$.

Now we assume that ρ_{k-1} and p_k have already been chosen. Then we take ρ_k so large that the following conditions are satisfied:

$$\rho_k > 40 e B_k \left(\frac{2\rho_{k-1}}{A_0}\right)^{p_k+1}, \tag{4.9}$$

where

$$A_0 = \frac{\sqrt{2} - 1}{2}, \qquad B_k = \sum_{\nu=1}^{p_k} |b_\nu|,$$

$$\rho_k > \frac{16 p_k^2}{2 - \sqrt{2}}, \tag{4.10}$$

(4.11)
$$\rho_k^{\lambda(\rho_k)+1} \geq \frac{p_k^3}{A_0}(50C_k\rho_k^{P_k})^{1000}p_k^4,$$

where $C_k = \sum_{\nu=1}^{p_k}|b_\nu|$, which can be done since $\lambda(\rho)$ tends to infinity with ρ, and finally

(4.12)
$$\frac{1}{8000p_k^3}(\lambda(\rho_k)+1)\log\rho_k - \log 2 > 8p_k\log\frac{\rho_k}{\delta},$$

where δ is a number such that $0 < \delta < \frac{1}{2}$, for instance we can take $\delta = \frac{1}{3}$, and again (4.12) can be satisfied if ρ_k is sufficiently large since $\lambda(\rho)$ tends to infinity as ρ tends to infinity.

Now we show that if ρ_k is chosen so that (4.9)–(4.12) are satisfied we can define p_{k+1} so that the inductive definition of the sequences $\{\rho_k\}, \{p_k\}$ is finished and therefore also the definition of $f(z)$ in (4.7), and we check that $f(z)$ constructed in this way satisfies (4.1).

First of all we show that if ρ_k satisfies (4.9) then

(4.13)
$$|b_n| < (2\rho_{k-1})^{-n}, \qquad p_k < n \leq p_{k+1}.$$

In fact by (3.56) in Lemma 3.4 and Cauchy's inequality we have

(4.14)
$$|a_n| < 40eM_k\left(\frac{\sqrt{2}-1}{2}\right)^{-n}, \qquad p_k < n \leq p_{k+1},$$

where

$$M_k = \sum_{\nu=1}^{p_k}|b_\nu|\rho_k^\nu < \rho_k^{p_k}\cdot\sum_{\nu=1}^{p_k}|b_\nu|,$$

which is bigger than $\rho_1 \geq 2$.

Writing

$$A_0 = \frac{\sqrt{2}-1}{2}, \qquad B_k = \sum_{\nu=1}^{p_k}|b_\nu|,$$

we deduce from(4.8) and (4.14) for $p_k < n \leq p_{k+1}$

$$|b_n| \leq 40e\rho_k^{p_k-n}A_0^{-n}B_k.$$

Whence we conclude (4.13) if

$$\rho_k^{n-p_k} < 40e\left(\frac{2\rho_{k-1}}{A_0}\right)^n B_k,$$

i.e.

$$\rho_k > (40eB_k)^{1/(n-p_k)}\cdot\left(\frac{2\rho_k - 1}{A_0}\right)^{n/(n-p_k)},$$

and this will be satisfied for all $n > p_k$ if (4.9) holds since $B_k \geq |b_1| = 1$.

From (4.13), we deduce that $f(z)$ given by (4.7) is an integral function.

Next we show that all the roots of the equations

(4.15)
$$F_k\left(\frac{z}{\rho_k}\right) = 0 \quad\text{and}\quad F_k\left(\frac{z}{\rho_k}\right) = i$$

are in the disc $\{z \mid |z| < \rho_k - \rho_k^{-\lambda(\rho_k)}\}$.

In fact, let us recall that $F_k(z) = F\{\omega(z)\}$, where

$$\omega(z) = \frac{\sum_{\nu=1}^{p_k} a_\nu z^\nu + \mu z^{2p_k}}{\mu + \sum_{\nu=1}^{p_k} \bar{a}_\nu z^{2p_k-\nu}},$$

where $eM_k < \mu < 10eM_k$ and $M_k = \sum_{\nu=1}^{p_k} |a_\nu| = \sum_{\nu=1}^{p_k} |b_\nu| \rho_k^\nu$.

The equations $F(\omega) = 0$ and $F(\omega) = i$ have at most one root in $\{\omega \mid |\omega| < 1\}$. Let ω_0 and ω_1, respectively, be these roots in case there are any.

The roots of (4.15) are the same as those of

$$\omega\left(\frac{z}{\rho_k}\right) = \omega_0 \quad \text{and} \quad \omega\left(\frac{z}{\rho_k}\right) = \omega_1. \tag{4.16}$$

Therefore we can deal with (4.16) instead of (4.15).

The equation $\omega(z) = \varsigma$ has precisely $2p_k$ roots for $|\varsigma| < 1$, in particular the equations $\omega(z) = \omega_0$ and $\omega(z) = \omega_1$ have in total $4p_k$ roots. Let us write $s = \max\{|\omega_0|, |\omega_1|\}$; then we show that $s < \sqrt{2} - 1$, so that it is less than a number between zero and one independent of k. In fact $\omega_0 = 0$ and since F is univalent in $|z| < \sqrt{2} - 1$, Koebe's $\frac{1}{4}$-theorem implies $|\omega_1| < \sqrt{2} - 1$ provided that $\mu > 4$ and since

$$\mu > eM_k > M_k = \sum_{\nu=1}^{p_k} |a_\nu| = \sum_{\nu=1}^{p_k} |b_\nu| \rho_k^\nu$$

it will be enough to have $\rho_1 > 4$, which is implied by the assumptions $\rho_0 = 2$ and (4.9). Therefore by the left-hand inequality of (3.50), all the roots of the two equations above must satisfy

$$1 - 16p_k^2\left(1 - \left|\frac{z}{\rho_k}\right|\right) < s,$$

i.e.

$$\left|\frac{z}{\rho_k}\right| < 1 - \frac{1-s}{16p_k^2}. \tag{4.17}$$

We now make use of (4.10) and obtain

$$\rho_k > \frac{16p_k^2}{2-\sqrt{2}} = \frac{16p_k^2}{1-(\sqrt{2}-1)} > \frac{16p_k^2}{1-s}. \tag{4.18}$$

From (4.18), we get

$$1 - \frac{1-s}{16p_k^2} < 1 - \frac{1}{\rho_k} < \frac{\rho_k - \rho_k^{-\lambda(\rho_k)}}{\rho_k}. \tag{4.19}$$

Therefore all the roots of (4.16) are in $\{z \mid |z| < \rho_k - \rho_k^{-\lambda(\rho_k)}\}$, and the same happens for the roots of (4.15), which was what we were trying to prove.

Since the set $\{z \mid \rho_k - \rho_k^{-\lambda(\rho_k)} \le |z| \le \rho_k - \gamma_k\}$ is compact and the functions $F_k(z/\rho_k)$ and $F_k(z/\rho_k) - i$ have no zeros in it, there is $\varepsilon_k > 0$ such that

$$\left|F_k\left(\frac{z}{\rho_k}\right)\right| > \varepsilon_k \quad \text{and} \quad \left|F_k\left(\frac{z}{\rho_k}\right) - i\right| > \varepsilon_k. \tag{4.20}$$

Finally, we will see that p_{k+1}, which has not been determined yet, can be take so that (4.1) holds.

We write

$$F_k\left(\frac{z}{\rho_k}\right) = \sum_{n=1}^{\infty} B_n z^n,$$

and note that the series is absolutely uniformly convergent in $\{z|\rho_k - \rho_k^{-\lambda(\rho_k)} \leq |z| \leq \rho_k - \gamma_k\}$, so that we may choose p_{k+1} so large that

$$\sum_{n=p_{k+1}}^{\infty} |B_n|\,|z|^n < \tfrac{1}{2}\varepsilon_k \tag{4.21}$$

in this set.

We also have by (4.13)

$$\sum_{n=p_{k+1}+1}^{\infty} |b_n|\,|z|^n \leq \sum_{n=p_{k+1}+1}^{\infty} 2^{-n} = 2^{-p_{k+1}},$$

for the same values of z, i.e. taking p_{k+1} large enough we can get that

$$\sum_{n=p_{k+1}+1}^{\infty} |b_n|\,|z|^n < \tfrac{1}{2}\varepsilon_k. \tag{4.22}$$

And then from (4.21) and (4.22)

$$\left|f(z) - F_k\left(\frac{z}{\rho_k}\right)\right| = \left|\sum_{p_{k+1}+1} (b_n - B_n)z^n\right| < \varepsilon_k \tag{4.23}$$

in $\{z|\rho_k - \rho_k^{-\lambda(\rho_k)} \leq |z| \leq \rho_k - \gamma_k\}$.

With the choice of p_{k+1} the inductive construction of the sequences $\{\rho_k\}$ and $\{p_k\}$ is complete the therefore also of the function f. Furthermore we conclude from (4.20), (4.23), and Rouché's Theorem that the equations

$$f(z) = 0 \quad \text{and} \quad F_k\left(\frac{z}{\rho_k}\right) = 0$$

have equally many roots in $\{z|\,|z| \leq r\}$ for those values of r in

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k - \gamma_k$$

and the same is true for the equations $f(z) = i$ and $F_k(z/\rho_k) = i$.

By Lemma 3.4, the number of the roots of all these equations is at most $2p_k$.

By (4.13) we have $|b_n| \leq 1$ for all n. Thus for $0 < |z| = \rho < 1/2$ we have

$$|f(z)| \leq \sum_{n=1}^{\infty} \rho^n \leq \frac{\rho}{1-\rho} \leq 2\rho < 1.$$

We also have for such ρ

$$|f(z)| \geq \rho - \sum_{n=2}^{\infty} \rho^{\nu} = \rho - \frac{\rho^2}{1-\rho} = \frac{\rho - 2\rho^2}{1-\rho} > 0.$$

Therefore there is δ, $0 < \delta < \frac{1}{2}$, such that the equations $f(z) = 0$ and $f(z) = i$ have no roots different from $z = 0$ in $\{z|\,|z| < \delta\}$.

We deduce from this that for $\rho_k - \rho_k^{-\lambda(\rho_k)} \le r \le \rho_k - \gamma_k$, we have

$$n(t,i) = 0, \quad t < \delta, \qquad \text{and} \qquad n(t,i) \le 2p_k, \quad t \le r,$$

so that

$$N(r,i) = \int_\delta^r \frac{n(t,i)}{t}\,dt \le 2p_k \log\frac{r}{\delta}, \tag{4.24}$$

and since $f(z)$ has a simple zero at the origin and no other zeros in $\{z|0 < |z| < \delta\}$, we obtain similarly

$$N(r,0) = \int_\delta^r \frac{n(t,0)}{t}\,dt + \log\delta \le 2p_k \log\frac{r}{\delta}. \tag{4.25}$$

By Lemma 3.4 we have

$$T\left(r, F_k\left(\frac{z}{\rho_k}\right)\right) \ge \frac{1}{8000p_k^3}\log\frac{1}{1 - r/\rho_k} \tag{4.26}$$

in $\rho_k - \rho_k^{-\lambda(\rho_k)} \le r \le \rho_k - \gamma_k$ if

$$\frac{\rho_k - \rho_k^{-\lambda(\rho_k)}}{\rho_k} = 1 - \rho_k^{-(\lambda(\rho_k)+1)} \ge 1 - \frac{A_0}{p_k^3}(16eM_k)^{-1000p_k^4}, \tag{4.27}$$

where M_k is given by $M_k = \sum_{\nu=1}^{p_k} |a_\nu|$, and by (4.8)

$$M_k = \sum_{\nu=1}^{p_k} |a_\nu| \le \rho_k^{p_k} \sum_{\nu=1}^{p_k} |b_\nu| = C_k \rho_k^{p_k}$$

where C_k depends on the b_ν, $\nu \le p_k$.

Hence (4.27) is satisfied provided that

$$\rho_k^{\lambda(\rho_k)+1} \ge \frac{p_k^3}{A_0}(50C_k\rho_k^{p_k})^{1000p_k^3}$$

which is (4.11).

Therefore (4.26) holds for those r in

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \le r \le \rho_k - \gamma_k.$$

Finally we have by (4.12)

$$\begin{aligned} T(r, f(z)) &\ge T\left(r, F_k\left(\frac{z}{\rho_k}\right)\right) - \log 2 \ge \frac{1}{8000p_k^3}(\lambda(\rho_k)+1)\log\rho_k - \log 2 \\ &> 8p_k \log\frac{\rho_k}{\delta} \end{aligned} \tag{4.28}$$

for r such that

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \le r \le \rho_k - \gamma_k.$$

Thus from (4.24), (4.25), and (4.28) we conclude

$$N(r,0) + N(r,i) + N(r,\infty) \le \tfrac{1}{2}T(r,f)$$

in $\rho_k - \rho_k^{-\lambda(\rho_k)} \le r \le \rho_k - \gamma_k$.

This proves (4.1) and the proof of Theorem 1 is complete.

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