TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 306, Number 1, March 1988

ESTIMATES FOR $(\bar\partial - \mu\partial)^{-1}$ AND CALDERÓN'S THEOREM ON THE CAUCHY INTEGRAL

STEPHEN W. SEMMES

ABSTRACT. One can view the Cauchy integral operator as giving the solution to a certain $\bar\partial$ problem. If one has a quasiconformal mapping on the plane that takes the real line to the curve, then this $\bar\partial$ problem on the curve can be pulled back to a $\bar\partial-\mu\partial$ problem on the line. In the case of Lipschitz graphs (or chord-arc curves) with small constant, we show how a judicial choice of q.c. mapping and suitable estimates for $\bar\partial - \mu\partial$ gives a new approach to the boundedness of the Cauchy integral. This approach has the advantage that it is better suited to related problems concerning H^∞ than the usual singular integral methods. Also, these estimates for the Beltrami equation have application to quasiconformal and conformal mappings, taken up in a companion paper.

Let Γ be an oriented rectifiable Jordan curve in the plane that passes through ∞, and let Ω_+ and Ω_- denote its two complementary regions. Given a function f defined on Γ define its Cauchy integral $F(z) = C_\Gamma(f)(z)$ off Γ by

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)\,dw}{w-z}, \qquad z \notin \Gamma. \tag{0.1}$$

If F_+ and F_- are the restrictions of F to Ω_+ and Ω_0, and if f_+ and f_- denote their boundary values, then the classical Plemelj formula states that

$$f_\pm(z) = \pm\frac{1}{2}f(z) + \frac{1}{2\pi i}\,\mathrm{P.V.}\int_\Gamma \frac{f(w)\,dw}{w-z}, \qquad z \in \Gamma. \tag{0.2}$$

(Of course, one must worry about the existence of the limits.) The singular integral on the right side is also called the Cauchy integral of f.

The problem is to know when there are L^p estimates, i.e., $\|f_\pm\|_p \le C_p\|f\|_p$ for $1 < p < \infty$. If Γ is, say $C^{1+\varepsilon}$ and O.K. at ∞, then it is easy to deduce these estimates from the corresponding facts about the Hilbert transform. This does not work when Γ is not smooth.

If Γ is the graph of a Lipschitz function $A\colon R \to R$, the L^p boundedness was proved by Calderón [**Ca**] when $\|A'\|_\infty$ is small and by Coifman, McIntosh, and Meyer [**CMM**] in general. G. David [**Dv**] has shown that for each p, $1 < p < \infty$, the Cauchy integral is bounded on $L^p(\Gamma)$ if and only if Γ is regular, which means that there is a $K > 0$ so that for all $z_0 \in \mathbf{C}$ and all $R > 0$ the arclength measure of $\{z\colon |z-z_0| \le R\} \cap \Gamma$ is at most KR.

In this paper we shall give a new approach to the Cauchy integral. This approach is less powerful from the real-variable point of view, and we shall only be able to

Received by the editors May 1, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30E20; Secondary 42B20, 30C60.
Key words and phrases. $\bar\partial$, Cauchy integral, quasiconformal mapping, BMO, Carleson measures.

0002-9947/88 \$1.00 + \$.25 per page

obtain small constant results. However, it is in some respects more natural from the standpoint of complex analysis, and can be used for other problems.

The idea is to think of the Cauchy integral as the solution to a $\overline{\partial}$ problem at Γ, and then make a change of variables to reduce to a $\overline{\partial} - \mu\partial$ problem relative to the line, which is solved by summing a Neumann series.

Let us be more precise. Given $z \in \Gamma$ define the jump of F across Γ at z to be $f_+(z) - f_-(z)$. By (0.2) this is $f(z)$. Also, F is holomorphic off Γ, so that $\overline{\partial}F = 0$ on $\mathbf{C} \setminus \Gamma$. (Here $\overline{\partial} = \partial/\partial\overline{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, $\partial = \partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$.) These two properties can be reexpressed by saying that, in the distributional sense, $\overline{\partial}F = f\, dz_\Gamma$ on C, where dz_Γ denotes the usual measure on Γ. (If $z(s)$ is an arclength parameterization of Γ, then dz_Γ corresponds to $z'(s)|dz(s)|$. In asserting that $\overline{\partial}F = f\, dz_\Gamma$, we are using $\int_{\partial\Omega} g\, dz = \int_\Omega (\overline{\partial}g)\, d\overline{z} \wedge dz$, a form of Green's theorem.) If $\tilde{F}$ is another function with the same properties as F, then $F - \tilde{F}$ is entire, and a mild condition at ∞ will force it to be 0. Thus F is determined by $\overline{\partial}F = 0$ on $\mathbf{C} \setminus \Gamma$ and $\mathrm{jump}(F) = f$ on Γ.

Suppose $\rho\colon \mathbf{C} \to \mathbf{C}$ takes $\mathbf{R}$ to Γ. Let $G = F \circ \rho$ on $\mathbf{C} \setminus \mathbf{R}$ and $g = f \circ \rho$ on $\mathbf{R}$. Then $\overline{\partial}F = 0$ off Γ transforms into $(\overline{\partial} - \mu\partial)G = 0$ off $\mathbf{R}$, where $\mu = \rho_{\overline{z}}/\rho_z$ is the complex dilatation of ρ. (This is well known, and also follows from the calculations at the end of §7.) Also, the jump of G across $\mathbf{R}$ is given by g. Conversely, if we can find G with these properties, then we have F (assuming ρ^{-1} is reasonable).

It is convenient to change the problem a little more. Let $C(g)(z)$ denote the Cauchy integral on $\mathbf{R}$ of g, that is, defined by (0.1) with $\Gamma = R$. Thus $C(g)$ is holomorphic off $\mathbf{R}$ and has jump g across $\mathbf{R}$. Let $H = G - C(g)$ on $\mathbf{C} \setminus \mathbf{R}$. Then H has no jump across $\mathbf{R}$, and $(\overline{\partial} - \mu\partial)H = \mu C'(g)$ off R, where $C'(g)$ denotes the ordinary derivative of $C(g)$. Because H has no jump, one can think of this equation as holding on all of $\mathbf{C}$ when interpreted in the sense of distributions; i.e., there is no boundary piece.

Thus, to estimate the Cauchy integral on Γ, we need to understand two things: (a) what sort of mapping we can take ρ to be, and in particular what are the natural conditions on μ; and (b) what kind of estimates we can get for $(\overline{\partial} - \mu\partial)^{-1}$. It turns out that one can find bilipschitz mappings ρ so that μ will satisfy certain quadratic Carleson measure conditions, and when μ satisfies such conditions and is small one can solve $(\overline{\partial} - \mu\partial)H = \mu C'(g)$ with L^p or BMO estimates on the boundary values of H in terms of the corresponding norm on g. This will imply Calderón's theorem.

For convenience we shall restrict ourselves mostly to BMO estimates instead of L^p. In §11 we will show how to make the necessary changes to treat L^p. This does not really matter, because one can use the real-variable methods of Calderón and Zygmund to go from BMO to L^p estimates for the Cauchy integral. See [**Je**].

In §1 we review some definitions and basic facts, and §2 is devoted to bilipschitz mappings. In §3 we take care of some minor preliminaries, and estimates for $\overline{\partial}^{-1}$ are given in §4. We give estimates for $(\overline{\partial} - \mu\partial)^{-1}$ for a certain class of μ's in §5, and use that to prove the boundedness of the Cauchy integral on chord-arc curves with small constant (defined in §1).

In the remainder of this paper we develop these topics further and consider some variations. The class of μ's for which we can estimate $(\overline{\partial} - \mu\partial)^{-1}$ is enlarged in §6, and in §7 we look at the Calderón commutators in terms of perturbing $\overline{\partial}$. In §8 we show that an obvious way of trying to improve the estimates in §2 does not work, which would have allowed us to iterate the perturbation argument to estimate the Cauchy integral on all Lipschitz graphs. In §9 we show how the $\bar\partial - \mu\partial$ idea can be applied formally to a certain H^∞ problem, but that the estimates are missing in the interesting case. We discuss real-variable analogues of some of the estimates for $(\bar\partial - \mu\partial)^{-1}$ in §10. As mentioned before, we show how to replace BMO with L^p in §11.

There is a closely related problem of finding conditions on μ such that if $\rho\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism and $\rho_{\bar z} = \mu\rho_z$ then $\Gamma = \rho(\mathbf{R})$ is rectifiable and $\rho|_{\mathbf{R}}$ is locally absolutely continuous. This will be considered in a separate paper [**Se 2**].

Some general references for this paper are [**JK**, **G**, and **Je**]. The survey paper [**Se 1**] might also be useful. Some of the results here were announced in that paper.

I am grateful to R. R. Coifman and P. W. Jones for helpful suggestions and comments, and also to the National Science Foundation for partial support in the form of a postdoctoral fellowship.

1. Basic facts and definitions. It will always be true that $z = x + iy$, $w = u + iv$, and $\varsigma = \xi + i\eta$.

BMO is the space of locally integrable functions f on $\mathbf{R}$ such that

$$\|f\|_* = \sup_I \frac{1}{|I|}\int_I |f(x) - f_I|\,dx < \infty,$$

where I is any interval and $f_I = |I|^{-1}\int_I f(y)\,dy$. By the John-Nirenberg theorem, there are C, $\delta > 0$ such that if $f \in \mathrm{BMO}$, $\|f\|_* \le \delta$, then

$$\frac{1}{|I|}\int_I e^{|f(x)-f_I|}\,dx \le C.$$

A measure λ on $\mathbf{C}$ is called a Carleson measure (relative to $\mathbf{R}$) if for each $R > 0$ and $x \in \mathbf{R}$, $|\lambda|(\{w\colon |w - x| \le R\}) \le CR$, and the smallest such C is called the Carleson measure norm of λ. If we replace the condition $x \in \mathbf{R}$ by $x \in E$, where E is some fixed closed set in $\mathbf{C}$, then we say that λ is a Carleson measure with respect to E.

We are going to want to let singular integrals act on Carleson measures, and so we will need better L^p control locally. Suppose E is a closed set and $\delta(w) = \delta_E(w)$ is the distance of w to E. Let $a(w)$ be given. For $0 < \alpha < 1$ and $z \notin E$ let $B_z = B_{z,\alpha} = \{w\colon |w - z| < \alpha\delta(z)\}$ and define

$$\tilde a_r(z) = \tilde a_{r,\alpha}(z) = \left(\frac{1}{|B_z|}\int_{B_z} |a(w)|^r\,dw\right)^{1/r}, \tag{1.1}$$

$1 \le r \le \infty$. We say that a measure λ is an r-good Carleson measure (r-GCM) relative to E if λ is absolutely continuous with respect to Lebesgue measure, $\lambda = a(z)\,dx\,dy$, and if $\tilde a_r(z)\,dx\,dy$ is a Carleson measure. This condition is independent of α, $0 < \alpha < 1$, different α yielding equivalent norms. Notice that when $r = 1$, $\tilde a_r(z)\,dx\,dy$ is a Carleson measure iff $|a(z)|\,dx\,dy$ is. When $r > 1$ this puts extra control on $a(z)$ locally, but in the large nothing really changes. We denote the r-GCM norm of $a(z)$ by $\|a\|_{r\text{-GCM}}$.

This notion of r-GCM should be thought of only as a technical variation on Carleson measures. In particular, individual values of r are usually not so important, but only that $1 < r < \infty$. For some things, though, $r = 2$ will be particularly convenient, or the requirement $r > 2$ will be useful for getting nice estimates.

When $a(z)$ has some sort of subharmonicity, e.g., if $a(z) = \delta(z)|\nabla u(z)|^2$, u harmonic, then $\tilde{a}_{\infty,\alpha}(z) \leq C_\alpha \tilde{a}_{1,2\alpha}(z)$, and so $\lambda = a(z)\,dx\,dy$ is an ∞-GCM if λ is a Carleson measure.

If $f \in \mathrm{BMO}(\mathbf{R})$ and $C(f)$ denotes its Cauchy integral, then $|y|\,|C'(f)|^2\,dx\,dy$ is a Carleson measure relative to $\mathbf{R}$ (see [**G**]), and is hence an ∞-GCM. More generally, suppose $\psi(x)$ is a function on $\mathbf{R}$ such that $|\psi(x)| \leq C(1+|x|)^{-2}$ and $\int_{-\infty}^{\infty} \psi(x)\,dx = 0$. Define $\psi_y(x) = (1/|y|)\psi(x/|y|)$ for $y \in \mathbf{R}$. Then $|\psi_y * f(x)|^2|y|^{-1}\,dx\,dy$ is a Carleson measure, with norm dominated by $\|f\|_*^2$. (See [**CM1**, p. 148, or **Je**, Chapter 6].)

If we assume also that $|\psi^{(j)}(x)| \leq C(1+|x|)^{-2}$ for $j = 1, 2, \ldots, K$, then $|y^j\nabla^j(\psi_y * f(x))|^2|y|^{-1}\,dx\,dy$ is a Carleson measure for the same j's. In particular, if $K = 2$ and if

$$a(x,y) = \sum_{j=0}^{2} |y^j\nabla^j(\psi_y * f)(x)|^2|y|^{-2},$$

then $a(x,y)\,dx\,dy$ is a Carleson measure. This implies that $\tilde{a}_{1,\alpha}(x,y)\,dx\,dy$ is a Carleson measure, and a Sobolev embedding argument gives that if $b(x,y) = |\psi_y * f(x)|^2|y|^{-1}$, then

$$\tilde{b}_{\infty,\beta}(x,y) \leq C(\alpha,\beta)\tilde{a}_{1,\alpha}(x,y) \quad \text{if } \beta < \alpha.$$

Thus $|\psi_y * f(x)|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM if $|\psi^{(j)}(x)| \leq C(1+|x|)^{-2}$, $j = 0, 1, 2$, and $\int \psi(x)\,dx = 0$.

A variation of this is the following. Suppose $\theta(x)$ is a $C^\infty(\mathbf{R})$ function with compact support, with no condition on $\int \theta\,dx$, and $\theta_y(x) = (1/|y|)\theta(x/|y|)$. Then $|y\nabla(\theta_y * f(x))|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM with norm $\leq C\|f\|_*^2$. The reason is that there are $\psi^1, \psi^2 \in C^\infty(\mathbf{R})$ with compact support such that $\int \psi^i\,dx = 0$, $i = 1, 2$, $|y|(\partial\theta_y(x)/\partial x) = \psi_y^1$, and $y(\partial\theta_y(x)/\partial y) = \psi_y^2(x)$. Similarly,

$$|y^j\nabla^j\theta_y * f(x)|^2|y|^{-1}\,dx\,dy$$

is an ∞-GCM if $j \geq 1$.

Let Γ be an oriented rectifiable Jordan curve that passes through ∞, and let $z(t)$ be an arc length parameterization. We say that Γ is a chord-arc curve if $|s-t| \leq (1+K)|z(s)-z(t)|$ for all $s, t \in \mathbf{R}$, and the smallest such K is called the chord-arc constant of Γ. Examples include Lipschitz graphs and logarithmic spirals. Coifman and Meyer [**CM2**, **3**] showed that if $z'(t) = e^{ib(t)}$ where $b \in \mathrm{BMO}(\mathbf{R})$ has small enough norm, then Γ is a chord-arc curve, and has small constant. Conversely, they also showed that if Γ is a chord-arc curve with small constant, then there is a real-valued BMO function b with small norm such that $z'(t) = e^{ib(t)}$. In fact, they showed that $\|b\|_* \approx \sqrt{K}$.

The first part is simple. Suppose $b \in \mathrm{BMO}$ is real valued, $\|b\|_*$ is small, $s, t \in \mathbf{R}$, and $I = [s,t]$. Let $\beta_I = e^{ib_I}$, where as before b_I is the mean of b over I, so that

$|\beta_I| = 1$. Then

$$
\begin{aligned}
|z(s) - z(t) - \beta_I(s-t)| &= \left|\int_s^t (z'(u) - \beta_I)\, du\right| \\
&= \left|\int_s^t (e^{i(b(u)-b_I)} - 1)\, du\right| \\
&\leq \int_s^t |e^{i(b(u)-b_I)} - 1|\, du \\
&\leq \int_s^t |b(u) - b_I|\, du \leq |s-t|\, \|b\|_*,
\end{aligned}
\tag{1.2}
$$

so that $(1 + \|b\|_*)^{-1}|s-t| \leq |z(s) - z(t)|$. The converse is considerably harder.

If Γ is a chord-arc curve, then there is a mapping $\rho\colon \mathbf{C} \to \mathbf{C}$ such that $\rho(\mathbf{R}) = \Gamma$ and ρ is bilipschitz, i.e., $C^{-1}|z-w| \leq |\rho(z) - \rho(w)| \leq C|z-w|$. See [**T1**, **2**; **JK**]. This is often useful for dealing with the geometry of the complementary domains of Γ.

BMO(Γ) for a chord-arc curve Γ is defined exactly as for $\mathbf{R}$, except that arcs replace intervals.

2. Some well-behaved bilipschitz mappings. To carry out the program described in the introduction we must first find suitable mappings $\rho\colon \mathbf{C} \to \mathbf{C}$ such that ρ maps $\mathbf{R}$ to the given curve Γ. Here Γ will be the graph of a Lipschitz function with small norm, or more generally a chord-arc curve with small constant. These mappings will be bilipschitz, and their dilatations will satisfy certain quadratic Carleson measure conditions.

Good bilipschitz mappings ρ for Lipschitz graphs Γ were first constructed by Dahlberg, and his construction worked in $\mathbf{R}^n$ as well. This was simplified greatly by Kenig and Stein who found a simple formula, which will be presented in a moment. Unfortunately, this formula does not work for chord-arc curves, nor does it give the best sort of estimates. A construction of Tukia [**T2**] (modified slightly) does work for all chord-arc curves and gives the better estimates, but it is not very explicit, because it uses the Riemann mapping. When the chord-arc constant is small, one can find a formula again, which is essentially just the Beurling-Ahlfors construction, and which we discuss also in this section.

Let Γ be the graph of a Lipschitz function $A\colon \mathbf{R} \to \mathbf{R}$ with small norm. Let φ be a $C^\infty(\mathbf{R})$ function with compact support such that $\varphi(x) = \varphi(-x)$, $\int \varphi(x)\, dx = 1$, and set $\varphi_y(x) = (1/|y|)\varphi(x/|y|)$, $y \in \mathbf{R}$, $y \neq 0$. Define $\eta(x,y) = \varphi_y * A(x)$ if $y \neq 0$, $\eta(x,0) = A(x)$, and

$$\rho(x,y) = x + iy + i\eta(x,y). \tag{2.1}$$

This is the formula of Kenig and Stein.

PROPOSITION 2.2. *If $\|A'\|_\infty$ is small enough, then ρ is a bilipschitz map of $\mathbf{C}$ onto itself, $\rho(\mathbf{R}) = \Gamma$, $|\nabla^n \eta(x,y)| \leq C_n \|A'\|_\infty |y|^{-n+1}$ for $n \geq 1$, and for $n \geq 2$, $|y^{n-1}\nabla^n \rho(x,y)|^2 |y|^{-1}\, dx\, dy$ is an ∞-GCM with norm at most $C_n\|A'\|_\infty^2$.*

Such a mapping ρ was first found by Dahlberg, without the restriction that $\|A'\|_\infty$ be small. When $\|A'\|_\infty$ is not small one replaces iy by iLy in (2.1), where $L > 0$ is large enough.

Because $\partial\eta(x,y)/\partial x = \varphi_y * A'(x)$, $\|\partial\eta/\partial x\|_\infty \le C\|A'\|_\infty$, and also

$$|\nabla^{n-1}(\partial\eta(x,y)/\partial x)| \le C_n\|A'\|_\infty |y|^{-n+1}.$$

When $y \neq 0$, $y(\partial\varphi_y/\partial y) = \psi_y$, where ψ is C^∞, compactly supported, and $\int \psi\,dx = 0$. Thus if $\theta(x) = \int_{-\infty}^x \psi(u)\,du$, then θ is also compactly supported. Thus $y(\partial\varphi_y * A(x)/\partial y) = \psi_y * A(x) = |y|\theta_y * A'(x)$, and so $\|\partial\eta(x,y)/\partial y\|_\infty = \|\theta_y * A(x)\|_\infty \le C\|A'\|_\infty$. Also, $|\nabla^{n-1}(\partial\eta(x,y)/\partial y)| \le C_n\|A'\|_\infty y^{-n+1}$.

In particular, we get that $\eta(x,y)$ is a Lipschitz function on $\mathbf{C}$, with norm $\le C\|A'\|_\infty$. If $\|A'\|_\infty$ is small enough, then ρ is a small perturbation of the identity in the Lipschitz topology, and ρ is bilipschitz.

Let us check the Carleson measure estimates. Since $\partial\eta(x,y)/\partial x = \varphi_y * A'(x)$, $|y^j\nabla^j(\partial\eta(x,y)/\partial x)|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM with norm $\le C\|A'\|_*^2 \le C\|A'\|_\infty^2$ by the remarks of §1. Similarly, since $\partial\eta(x,y)/\partial y = (\operatorname{sgn} y)\theta_y * A'(x)$,

$$|y^j\nabla^j(\partial\eta(x,y)/\partial y)|^2|y|^{-1}\,dx\,dy$$

is an ∞-GCM, with the same norm estimate. This proves the proposition. An immediate consequence is the following.

COROLLARY 2.3. *If $\mu = \bar\partial\rho/\partial\rho$ is the complex dilatation of ρ, then for $j \ge 1$ $|y^j\nabla^j\mu|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM with norm $\le C(j)\|A'\|_\infty^2$. Also $\|\mu\|_{L^\infty(\mathbf{C})} \le C\|A'\|_\infty$.*

Now suppose that Γ is a chord-arc curve with small constant K. Let $z(t)$ be an arc length parameterization of Γ, so that $z'(t) = e^{ib(t)}$ for some real-valued $b \in \mathrm{BMO}(\mathbf{R})$ with $\|b\|_*$ small.

Let φ, ψ be C^∞ functions supported on $[-1,1]$, $\varphi \ge 0$, φ even, ψ odd, $\int \varphi(x)\,dx = 1$, and $\int \psi(x)x\,dx = 1$. Define $\rho\colon \mathbf{C} \to \mathbf{C}$ by

$$\begin{aligned} &\rho(x,y) = \varphi_y * z(x) + i(\operatorname{sgn} y)\psi_y * z(x), \qquad y \neq 0,\\ &\rho(x,0) = z(x), \end{aligned} \tag{2.4}$$

where $\varphi_y(x) = (1/|y|)\varphi(x/|y|)$, as always. This is essentially the Beurling-Ahlfors construction.

PROPOSITION 2.5. *Suppose Γ has small chord-arc constant, and ρ is as in (2.4). Then ρ is a bilipschitz map of $\mathbf{C}$ onto $\mathbf{C}$, $\rho(R) = \Gamma$, $\|\bar\partial\rho\|_{L^\infty(\mathbf{C})}$ and $\|\,|\partial\rho| - 1\|_{L^\infty(\mathbf{C})}$ are both small, $|\nabla^j\rho(x,y)| \le C(j)|y|^{-j+1}$, $j \ge 1$, and*

$$|\bar\partial\rho(x,y)|^2|y|^{-1}\,dx\,dy \quad \text{and} \quad |y^{j-1}\nabla^j\rho|^2|y|^{-1}\,dx\,dy,$$

$j \ge 2$, are ∞-GCM's with norm at most $C(j)\|b\|_*^2$. Also, if $\mu = \bar\partial\rho/\partial\rho$ is the complex dilatation, then $\|\mu\|_{L^\infty(\mathbf{C})}$ is small, and $|y^j\nabla^j\mu|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM for $j \ge 0$, with norm $\le C(j)\|b\|_*^2$.

Thus for the mapping (2.4) there are Carleson measure estimates for $\bar\partial\rho$ and μ (not just $\nabla\mu$). Thus ρ is not far from being conformal.

Because $\int\psi = 0$, $\psi_y * z(x) \to 0$ as $y \to 0$, and so $\rho(x,y)$ is continuous everywhere, even on $\mathbf{R}$. Since $z(x)$ is Lipschitz (with norm 1), ρ is Lipschitz and $|\nabla^j\rho(x,y)| \le C(j)y^{-j+1}$, just as in the proof of Proposition 2.2. Using $|e^{ix} - e^{iy}| \le |x-y|$, it is easy to show that $\|z'\|_* = \|e^{ib}\|_* \le 2\|b\|_*$, so that the ∞-GCM norm of $|y^{j-1}\nabla^j\rho(x,y)|^2|y|^{-1}\,dx\,dy$, $j \ge 2$, is at most $C(j)\|z'\|_*^2 \le C(j)\|b\|_*^2$.

Let us estimate $\overline{\partial}\rho$. For $y > 0$ there is a $\theta(x) \in C^\infty(\mathbf{R})$ with compact support such that $y\overline{\partial}\rho(x, y) = \theta_y * z(x)$. If $z(x) = x$ or $z(x) = 1$, then $\rho(z) = z$ or $\rho(z) = 1$, respectively, and in both cases $\overline{\partial}\rho = 0$. This implies that $\int \theta(x)\,dx = 0 = \int \theta(x)x\,dx$. Thus if $\nu(x) = \int_{-\infty}^x \theta(t)\,dt$, then ν has compact support and $\int \nu(x)\,dx = 0$. Hence $(1/y)\theta_y * z(x) = \nu_y * z'(x)$, and

$$|\overline{\partial}\rho(x, y)|^2|y|^{-1}\,dx\,dy = |\nu_y * z'(x)|^2|y|\,dx\,dy$$

is an ∞-GCM with norm $\le C\|z'\|_*^2 \le C\|b\|_*^2$. A similar argument works for $y < 0$, but $\theta(x)$ will be a little different, though having the same properties.

Also, $|\overline{\partial}\rho(x, y)| = |\nu_y * z'(x)| \le C\|z'\|_* \le C\|b\|_*$, and so $\overline{\partial}\rho$ is small. A similar argument gives that $y|\nabla^2\rho(x, y)| \le C\|b\|_*$, and hence is small, which we will need shortly.

Consider now $\partial\rho$. As before, there is a compact supported $\alpha \in C^\infty(\mathbf{R})$ such that $y\partial\rho(x, y) = \alpha_y * z(x)$ for $y > 0$. If $z(x) = 1$, $\rho(z) = 1$, $\partial\rho = 0$, which implies that $\int \alpha(x)\,dx = 0$. Hence $\beta(x) = \int_{-\alpha}^x (t)\,dt$ has compact support, and $\partial\rho(x, y) = (1/y)\alpha_y * z(x) = \beta_y * z'(y)$. If $z(x) = x$, then $z'(x) = 1$, $\rho(z) = z$, and $\partial\rho = 1$, so that $\int \beta(x)\,dx = 1$. Set $b(x, y) = (1/2y)\int_{x-y}^{x+y} b(t)\,dt$. Altogether, we get that

$$
\begin{aligned}
|\partial\rho(x, y) - e^{ib(x,y)}| &= |\beta_y * z'(x) - e^{ib(x,y)}| \\
&= \left|\int \beta_y(x, u)(z'(u) - e^{ib(x,y)})\,du\right| \\
&\le \int |\beta_y(x, u)|\,|e^{i(b(u)-b(x,y))} - 1|\,du \\
&\le \int |\beta_y(x - u)|\,|b(u) - b(x, y)|\,du \le C\|b\|_*.
\end{aligned}
\tag{2.6}
$$

A similar argument works for $y < 0$, but with a slightly different β. Therefore $\|\,|\partial\rho| - 1\|_\infty$ is small.

It remains to show that ρ is bilipschitz, i.e., that there is a $C > 0$ such that $C^{-1}|z - w| \le |\rho(z) - \rho(w)| \le C|z - w|$ for all $z, w \in \mathbf{C}$. Let $z_0 \in \mathbf{C} \setminus \mathbf{R}$ be given, and assume z satisfies $|z - z_0| \le \frac{1}{2}|y_0|$. By Taylor's theorem,

$$\rho(z) = \rho(z_0) + (z - z_0)\partial\rho(z_0) + (\overline{z} - \overline{z}_0)\overline{\partial}\rho(z_0) + \text{error}, \tag{2.7}$$

where $|\text{error}| \le C|z - z_0|\,\|b\|_*$, because $|\nabla^2\rho(x, y)| \le C|y|^{-1}\|b\|_*$. Since $|\overline{\partial}\rho(z_0)|$ and $|\,|\partial\rho(z_0)| - 1|$ are both dominated by $\|b\|_*$, we get

$$(1 - C\|b\|_*)|z - z_0| \le |\rho(z) - \rho(z_0)| \le (1 + C\|b\|_*)|z - z_0|$$

when $|z - z_0| \le \frac{1}{2}|y_0|$.

We want to do this for all z, z_0, and for this we need a lemma.

LEMMA 2.8. *Let $a \in \mathbf{R}$ and $h > 0$ be given, and set $I = [a - h, a + h]$, $b_I = |I|^{-1}\int_I b(x)\,dx$, $\beta_I = e^{ib_I}$, $\tilde{z}(x) = z(a) + \beta_I(x - a)$, and $\tilde{\rho}(w) = z(a) + \beta_I(z - a)$. Then for w such that $|w - a| \le \frac{1}{2}h$, $|\rho(w) - \tilde{\rho}(w)| \le Ch\|b\|_*$.*

In other words, around a and at the scale of h, $\rho(w)$ is approximately a translation and a rotation.

As in (1.1), for $t \in I$ one has that

$$
\begin{aligned}
(2.9)\qquad |z(t) - \tilde{z}(t)| = |z(t) - z(a) - \beta_I(t-a)| &\le \int_a^t |b(s) - b_I|\,ds \\
&\le \int_I |b(s) - b_I|\,ds \le |I|\,\|b\|_* = h\|b\|_*.
\end{aligned}
$$

Observe that $\tilde{\rho}$ is what you get from (2.4) if you substitute $\tilde{z}(x)$ for $z(x)$. This and (2.9) imply the lemma.

Let us finish proving that ρ is bilipschitz. Suppose $z, w \in C$ are given, and let us show that $C^{-1}|z-w| \le |\rho(z) - (w)|$ for some $C > 0$. If either $|z-w| \le \frac{1}{2}y$ or $|z-w| \le \frac{1}{2}v$, then we fall in the case we have already done, and so we may assume that $\frac{1}{2}y$ and $\frac{1}{2}v$ are $\le |z-w|$. Let $a = x$ and $h = 10|z-w|$, and let $\tilde{\rho}$ be as in the lemma. Then z and w both lie in $\{\varsigma: |\varsigma - a| \le \frac{1}{2}h\}$, so that

$$
\begin{aligned}
|z-w| = |\tilde{\rho}(z) - \tilde{\rho}(w)| &\le |\tilde{\rho}(z) - \rho(z)| + |\rho(z) - \rho(w)| + |\tilde{\rho}(w) - \rho(w)| \\
&\le Ch\|b\|_* + |\rho(z) - \rho(w)| \\
&\le 10C|z-w|\,\|b\|_* + |\rho(z) - \rho(w)|.
\end{aligned}
$$

If the chord-arc constant of Γ is small enough so that $10C\|b\|_* \le \frac{1}{2}$, then $\frac{1}{2}|z-w| \le |\rho(z) - \rho(w)|$, and ρ is bilipschitz.

3. Preliminary estimates for the Cauchy integral. Let Γ be a fixed chord-arc curve and let $\delta(w) = \delta_\Gamma(w) = \operatorname{dist}(w, \Gamma)$. Given a function f on Γ we define its Cauchy integral $F(z)$ on $\mathbf{C} \setminus \Gamma$ by (0.1). This makes sense if $f \in L^p(\Gamma)$, $1 < p < \infty$. If $f \in \mathrm{BMO}(\Gamma)$, the integral may diverge, but $F(z)$ can be defined modulo constants, because the integral defining $F(z) - F(w)$ does make sense for any $z, w \in \mathbf{C} \setminus \Gamma$.

Let us define precisely the jump of F across Γ using nontangential limits. For any given $K > 0$ and $z \in \Gamma$ define

$$
\begin{gathered}
A(z) = \{w \in \mathbf{C} \setminus \Gamma: |z-w| \le K\delta(w)\}, \\
B(z) = \{(z_1, z_2): z_1, z_2 \in A(z) \text{ and } K^{-1}|z - z_1| \le |z - z_2| \le K|z - z_1|\}, \\
j^*(z) = j^*(F)(z) = \sup\{|F(z_1) - F(z_2)|: (z_1, z_2) \in B(z)\},
\end{gathered}
$$

and

$$
(3.1)\qquad j(z) = j(F)(z) = \lim(F(z_1) - F(z_2)),
$$

where the limit is taken as both $z_1 \in \Omega_+$ and $z_2 \in \Omega_-$, $(z_1, z_2) \in B(z)$, tend to z.

LEMMA 3.2. *Suppose that Γ is a chord-arc curve and $f \in L^p(\Gamma)$, $1 < p < \infty$, or $f \in \mathrm{BMO}(\Gamma)$.*

(a) $\|j^*\|_p \le C_p\|f\|_p$ *if $p < \infty$, and if $f \in \mathrm{BMO}(\Gamma)$, then for each arc I there is a constant C_I such that for each $q < \infty$,*

$$
\left(|I|^{-1} \int_I |j^*(F - C_I)(z)|^q |dz|\right)^{1/q} \le C_q\|f\|_*;
$$

(b) $j(z) = f(z)$ *a.e. on* Γ;

(c) $\left(\int_\Gamma (\sup\{\delta(w)|F'(w)|: w \in A(z)\})^p |dz|\right)^{1/p} \le C_p\|f\|_p$ *if $p < \infty$, and* $\le C\|f\|_*$ *if $f \in$* BMO;

(d) $\delta(w)F'(w) \to 0$ *as* $w \to z$ *nontangentially for a.e.* $z \in \Gamma$;

(e) $|F(z)| \leq C_p \|f\|_p \delta(z)^{-1/p}$ *if* $p < \infty$, *and if* $p = \infty$,

$$|F(z) - F(w)| \leq C\|f\|_*(|\log(|z-w| + \delta(z) + \delta(w))| + |\log \delta(z)| + |\log \delta(w)|).$$

Note that (e) implies that F is a locally integrable tempered distribution on $\mathbf{C}$. As a distribution, $\overline{\partial} F = f\, dz_\Gamma$, where dz_Γ is the complex measure on Γ defined by $\langle g, dz_\Gamma \rangle = \int_\Gamma g\, dz$. Indeed, it is well known that if $\eta \in C^\infty(\mathbf{C})$ has compact support, then

$$\eta(z) = \frac{1}{\pi i} \iint_{\mathbf{C}} \frac{1}{w-z} \overline{\partial} \eta(w)\, du\, dv,$$

i.e., $(\pi i)^{-1} z^{-1}$ is the fundamental solution of $\overline{\partial}$. Chasing definitions and interchanging orders of integration gives $\overline{\partial} F = f\, dz_\Gamma$.

Lemma 3.2 is easy to prove. If $w \notin \Gamma$, then $\delta(w)(z-w)^{-2}\, dz$ is a nice mean zero bump on Γ, whose corresponding maximal function is controlled by the Hardy-Littlewood maximal function. From this it is easy to get (c) and (d) when $f \in L^p$, and the BMO case is also easy. The $p < \infty$ part of (e) follows from Holder's inequality. When $p = \infty$, one integrates F' along a path from z to w such that $\int_z^w |d\varsigma|/\delta(\varsigma)$ is controlled by the logarithm stuff. When $\Gamma = \mathbf{R}$ one can find such a path by taking hyperbolic geodesics. In general one reduces to $\Gamma = \mathbf{R}$ by a bilipschitz change of variables.

For (a) and (b) observe that

$$F(z_1) - F(z_2) = \frac{1}{2\pi i} \int_\Gamma \frac{z_2 - z_1}{(w-z_1)(w-z_2)} f(w)\, dw,$$

so that $j^*(f)$ is controlled by the Hardy-Littlewood maximal function (because of the definition of $A(z)$ and $B(z)$), which implies (a). By the residue theorem, if $z_1 \in \Omega_+$, $z_2 \in \Omega_-$, and $f(w) \equiv 1$, then $F(z_1) - F(z_2) = 1$, and (b) can be derived from this by standard approximation to the identity arguments.

4. Estimating $\overline{\partial}^{-1}$ of a good Carleson measure. As stated in the introduction, one of our goals is to get good estimates for $(\overline{\partial} - \mu\partial)^{-1}$, and as a first step we need to understand $\overline{\partial}^{-1}$. In this section we give estimates in terms of BMO and Carleson measures. We will discuss L^p analogues in §11.

Let Γ be a chord-arc curve, let Ω_+ and Ω_- denote its complementary regions, and let $\delta(w) = \delta_\Gamma(w) = \operatorname{dist}(w, \Gamma)$. Let $\theta\colon \mathbf{C} \to \mathbf{C}$ be a bilipschitz mapping such that $\theta(\mathbf{R}) = \Gamma$. It was pointed out at the end of §1 that such a θ always exists. For each $z \in \Gamma$ and $t \in \mathbf{R}$, define $r_t(z) = \theta(\theta^{-1}(z) + it)$. When $\Gamma = \mathbf{R}$, we take θ to be the identity and $r_t(x) = x + it$. Each $w \in \mathbf{C}$ can be represented as $\Gamma_t(z)$ for one $z \in \Gamma$ and one $t \in \mathbf{R}$. Using $\Gamma_t(z)$ we can define radial boundary values on Γ and a radial maximal function in the obvious way, for a given function on Ω_+ or Ω_-.

PROPOSITION 4.1. *Suppose* $a(z)\, dx\, dy$ *is an* r*-good Carleson measure on* Ω_+ *(relative to* Γ*),* $1 < r < \infty$. *Then there is a locally integrable function* $F(z)$ *on* Ω_+ *with the following properties*:

(a) *if* $\overline{\partial} F$ *and* ∂F *are defined in the distributional sense on* Ω_+, *then* $\overline{\partial} F = a$, ∂F *is locally integrable,* $|\partial F|\, dx\, dy$ *is an* r*-GCM on* Ω_+ *relative to* Γ, *and*

$$\|\partial F\|_{r\text{-GCM}} \leq C(r, \Gamma) \|a\|_{r\text{-GCM}};$$

(b) $\lim_{t\to 0^+} F(r_t(z)) = f(z)$ *exists for almost all* $z \in \Gamma$, *and* $f \in \mathrm{BMO}(\Gamma)$, $\|f\|_* \leq C(r,\Gamma)\|a\|_{r\text{-GCM}}$;

(c) F *satisfies a radial maximal function estimate, that for each arc* I *on* Γ *there is a constant* C_I *such that*

$$\frac{1}{|I|}\int_I \left(\sup_{0<t<|I|} |F(r_t(z)) - C_I|\right) |dz| \leq C(r,\Gamma)\|a\|_{r\text{-GCM}}. \tag{4.2}$$

If we also assume that $r > 2$, *then* F *is continuous on* Ω_+, *the boundary values in* (b) *exist nontangentially, and in* (c) *we can replace the integrand in* (4.2) *by its nontangential version,*

$$f_I^*(z) = \sup\{|F(w) - C_I| : w \in \Omega_+,\ 0 < |w - z| \leq 10\delta(w) \leq |I|\}.$$

The point of this lemma is that it allows us to compute $\overline{\partial}^{-1} a$ on all of $\mathbf{C}$ in terms of F and the Cauchy integral of a BMO(Γ) function. We will discuss this further after the proof.

The proof of the proposition is inspired by Jones [**Js1**], but it is much easier than that. Consider first the case where $\Gamma = \mathbf{R}$ and Ω_+ is the upper half-plane U. Define

$$K(z,\varsigma) = \frac{1}{z-\varsigma}\frac{\eta}{z-\overline{\varsigma}}, \tag{4.3}$$

so that $\overline{\partial}_z K(z,\varsigma) = \delta_\varsigma$, except for a multiplicative constant, which we ignore.

Let $\chi(\varsigma)$ denote the characteristic function of $\{\varsigma : |\varsigma - i| \geq \frac{1}{2}\}$, and define

$$F(z) = \iint_U [K(z,\varsigma) - K(i,\varsigma)\chi(\varsigma)] a(\varsigma)\, d\xi\, d\eta. \tag{4.4}$$

We subtract off $K(i,\varsigma)\chi(\varsigma)$ to make the integral converge at ∞. There is nothing special about i, and other point in Ω_+ would do, and using a different point will only change F by an additive constant.

The integral converges at ∞ because the kernel has enough decay and because $|a(\varsigma)|\, d\xi\, d\eta$ is a Carleson measure. $F(z)$ is locally integrable because $a(\varsigma)$ and $|z-\varsigma|^{-1}$ are. When $r > 2$, the integral converges absolutely, because $a(\varsigma) \in L^r_{\mathrm{loc}}(\Omega_+)$, since $a(\varsigma)$ is an r-GCM, and because $K(z,\cdot)$ lies in $L^q_{\mathrm{loc}}(\Omega_+)$ for all $q < 2$ and all $z \in \Omega_+$. One can also check that F is continuous on Ω_+ when $r > 2$. In any case, $\overline{\partial} F = a$ on Ω_+ in the sense of distributions.

Let us estimate ∂F, which is given by

$$\partial F(z) = -\iint_U \left[\frac{1}{(z-\varsigma)^2}\frac{\eta}{z-\overline{\varsigma}} - \frac{1}{z-\overline{\varsigma}}\frac{\eta}{(z-\overline{\varsigma})^2}\right] a(\varsigma)\, d\xi\, d\eta. \tag{4.5}$$

This integral converges at ∞, but for ς near z the first term should be interpreted as a principal value. The second term does not need to be taken as a principal value, since $|z-\varsigma|^{-1}$ is locally integrable, so that that term already gives rise to something locally integrable.

For $z \in U$ let $B_z = \{w : |w - z| \leq \frac{1}{10}y\}$ and let $\tilde{B}_z$ denote it double. Set

$$H(z) = \left(\frac{1}{|B_z|}\iint_{B_z} |\partial F(w)|^r\, du\, dv\right)^{1/r},$$

so that in the notation of (1.1) in §1, $H(z) = (\partial F)_r^{\sim}(z)$. Let $x_0 \in \mathbf{R}$ and $R > 0$ be given. We need to show that

$$\iint_{|z-x_0|\leq R} |H(z)|\, dx\, dy \leq CR\|a\|_{r\text{-GCM}}. \tag{4.6}$$

Let $a(z) = n(z) + f(z)$, the near and faraway parts, where $n(z)$ is supported on $\{z: |z - x_0| \leq 2R\}$, and $f(z)$ is supported on the complement. We shall assume that $a(z) = n(z)$, because the contribution from $f(z)$ is easier to deal with.

Let $\chi_z(\cdot)$ denote the characteristic function of $\tilde{B}_z$. We can split $a(\varsigma)$ into two pieces, $a_1(\varsigma) = a(\varsigma)\chi_z(\varsigma)$ and $a_2(\varsigma) = a(\varsigma)(1 - \chi_z(\varsigma))$, and this induces a splitting of $\partial F(w)$ into two pieces, which we denote as $G_1(q)$ and $G_2(w)$. These also depend on z, but for convenience we suppress this dependence in the notation.

Let $H_1(z)$ and $H_2(z)$ denote the corresponding pieces of $H(z)$, so that for $i = 1, 2$

$$H_i(z) = \left(\frac{1}{|B_z|}\iint_{B_z} |G_i(w)|^r\, du\, dv\right)^{1/r}.$$

Let us show that

$$\left(|B_z|^{-1}\iint_{B_z} |G_1(w)|^r\, du\, dv\right)^{1/r} \leq C\left(|\tilde{B}_z|^{-1}\iint_{\tilde{B}_z} |a(\varsigma)|^r\, d\xi\, d\eta\right)^{1/r}. \tag{4.7}$$

This implies (by chasing definitions) that

$$\iint_{|z-x_0|\leq R} H_1(z)\, dx\, dy \leq CR\|a\|_{r\text{-GCM}}.$$

To prove (4.7) we look at the kernel. By definition,

$$G_1(w) = -\iint_U \left[\frac{1}{(w-\varsigma)^2}\frac{\eta}{w-\bar{\varsigma}} - \frac{1}{w-\bar{\varsigma}}\frac{\eta}{(w-\bar{\varsigma})^2}\right] a_1(\varsigma)\, d\varsigma\, d\eta.$$

Observe that

$$\begin{aligned}\frac{1}{(w-\varsigma)^2}\frac{\eta}{w-\bar{\varsigma}} &= \frac{1}{(w-\varsigma)^2}\left[\frac{\eta}{(\varsigma-\bar{\varsigma})} + \frac{\eta(w-\varsigma)}{(\varsigma-\bar{\varsigma})(w-\bar{\varsigma})}\right] \\ &= \frac{1}{(w-\varsigma)^2}\left[\frac{1}{2i} + \frac{1}{2i}\frac{w-\varsigma}{w-\bar{\varsigma}}\right].\end{aligned}$$

Thus

$$G_1(w) = -\iint_U \left[\frac{1}{2i}\frac{1}{(w-\varsigma)^2} + L(w,\varsigma)\right] a_1(\varsigma)\, d\xi\, d\eta,$$

where $|L(w,\varsigma)| \leq 100\eta^{-1}|w-\varsigma|^{-1}$ for $w_1\varsigma \in \tilde{B}_z$. Since $1/(w-\varsigma)^2$ is a Calderón-Zygmund kernel, it is bounded on L^p, $1 < p < \infty$ (but not L^1, and that is why we had to introduce r-GCM's). The kernel $L(w,\varsigma)$ induces a bounded operator on $L^p(\tilde{B}_z)$, $1 \leq p \leq \infty$, because $\iint_{|w|\leq\eta} \eta^{-1}|w|^{-1}\, du\, dv \leq C$. Altogether we get (4.7), since $a_1(\varsigma) = a(\varsigma)\chi_z(\varsigma)$.

Consider now $G_2(w)$. By definition,

$$\begin{aligned}|G_2(w)| &= \left|\iint_{\{\varsigma: |\varsigma-\chi_0|\leq 2R\}\setminus\tilde{B}_z} \left[\frac{1}{(w-\varsigma)^2}\frac{\eta}{w-\bar{\varsigma}} - \frac{1}{w-\varsigma}\frac{\eta}{(w-\bar{\varsigma})^2}\right] a_1(\varsigma)\, d\xi\, d\eta\right| \\ &\leq C\iint_{\{\varsigma: |\varsigma-x_0|\leq 2R\}} \frac{\eta}{|w-\bar{\varsigma}|^3}|a(\varsigma)|\, d\xi\, d\eta\end{aligned}$$

for $w \in B_z$. (We have used the fact that if $w \in B_z$ and $\varsigma \in U \setminus \tilde{B}_z$, then $|w-\varsigma|$ and $|w-\bar{\varsigma}|$ are comparable.) Call this last expression $T(w)$. Observe that $T(w) \leq CT(z)$ if $w \in B_z$ (by looking at the kernel), so that

$$H_2(z) = \left(|B_z|^{-1} \iint_{B_z} |T(w)|^r \, du\, dv\right)^{1/r} \leq CT(z).$$

By Fubini,

$$\begin{aligned}
\iint_{|z-x_0|\leq R} H_2(z)\, dx\, dy &\leq C \iint_{|z-x_0|\leq R} T(z)\, dx\, dy \\
&\leq C \iint_{|z-x_0|\leq R} \iint_{|\varsigma-x_0|\leq 2R} \frac{\eta}{|z-\bar{\varsigma}|^3} |a(\varsigma)|\, d\xi\, d\eta\, dx\, dy \\
&= C \iint_{|\varsigma-x_0|\leq 2R} \left(\iint_{|z-x_0|\leq R} \frac{\eta}{|z-\bar{\varsigma}|^3}\, dx\, dy\right) |a(\varsigma)|\, d\xi\, d\eta \\
&\leq C \iint_{|\varsigma-x_0|\leq 2R} |a(\varsigma)|\, d\xi\, d\eta \leq CR\|a\|_{r\text{-GCM}}.
\end{aligned}$$

Putting together the estimates for $H_1(z)$ and $H_2(z)$ gives (4.6), and hence part (a) of the proposition.

Parts (b) and (c) follow from (a) and Varopoulos **[V1]**, since $|\nabla F|\, dx\, dy$ is a Carleson measure. Let us prove the nontangential analogues when $r > 2$.

Let us show that F has nontangential boundary values. For $x \in \mathbf{R}$ define $f(x)$ formally by replacing z by x in (4.4). We must show that that integral converges for almost all x.

Let I be any interval and x_0 be its center, and set $\hat{I} = \{x \in U : x \in I,\ 0 < y < |I|\}$. Analogous to the $\chi(\varsigma)$ in (4.4), let $\tilde{\chi}(\varsigma)$ be the characteristic function of $\{\varsigma : |\varsigma - x_0 + i|I|\,| \geq \frac{1}{2}|I|\}$. With

$$C_I = \iint_U [K(x_0 + i|I|, \varsigma)\tilde{\chi}(\varsigma) - K(i,\varsigma)\chi(\varsigma)] a(\varsigma)\, d\xi\, d\eta,$$

the definition of $f(x)$ gives

$$|f(x) - C_I| \leq \iint_U |K(x,\varsigma) - K(x_0 + i|I|, \varsigma)\tilde{\chi}(\varsigma)|\, |a(\varsigma)|\, d\xi\, d\eta, \tag{4.8}$$

and from Fubini we get that $\int_I |f(x) - C_I|\, dx \leq C|I|\, \|a\|_{CM}$. In particular, the right side of (4.8) converges for a.e. x, and so the integral defining $f(x)$ converges a.e., and $f(x)$ is well defined.

Let us verify that F has f as its nontangential boundary values. Let I be any interval, and let us prove convergence on I. Let J denote the double of I, and let $\hat{J}$ be as above. We may suppose that $\operatorname{supp} a(\varsigma) \subseteq \hat{J}$, because the faraway part is easy to handle. For $z \in U$ let $\chi_1(\varsigma) = \chi_{1,z}(\varsigma)$ denote the characteristic function of $\{\varsigma : |\varsigma - z| < \frac{1}{10}y\}$, and let $\chi_2 = 1 - \chi$. Observe that

$$|K(z,\varsigma)\chi_2(\varsigma)| \leq C|K(u,\varsigma)|$$

if z lies in a cone with vertex U. Observe also that as $z \to u \in I$ nontangentially, $K(z,\varsigma)\chi_2(\varsigma)$ tends to $K(u,\varsigma)$. Because

$$\int_I \iint_{\hat{J}} |K(u,\varsigma)|\, |a(\varsigma)|\, d\xi\, d\eta\, d\eta \leq C \iint_{\hat{J}} |a(\varsigma)|\, d\xi\, d\eta \leq C\|a\|_{CM},$$

and in particular $\iint_{\hat{J}} |K(u,\varsigma)|\, |a(\varsigma)|\, d\xi\, d\eta < \infty$ for a.e. u, we conclude from the dominated convergence theorem that

$$F_2(z) = \iint_{\hat{J}} K(z,\varsigma)\chi_2(\varsigma)a(\varsigma)\,d\xi\,d\eta$$

converges to $f(u)$ a.e. and in $L'(|I|)$ as $z \to u$ nontangentially.

This leaves

$$F_1(z) = \iint_{\hat{J}} K(z,\varsigma)\chi_1(\varsigma)a(\varsigma)\,d\xi\,d\eta,$$

which we now show tends to 0. By Hölder's inequality, $|F_1(z)| \le Cy\tilde{a}_r(z)$, where $\tilde{a}_r(z)$ is defined by (1.1), with $\alpha = 1/10$. (It is here that we use the assumption $r > 2$.) One can check that if $u \in I$ and $|z-u| \le 10y$, then

$$ya_r(z) \le C \iint_{\substack{|u-s|\le 10t\\ .9y\le t\le 1.1y}} \tilde{a}_r(s,t)\,\frac{ds\,dt}{t},$$

so that

$$\int_I \left(\sup_{\substack{|z-u|\le cy\\ y\le\varepsilon}} |F_1(z)| \right) du \le \int_I C \left(\iint_{\substack{|s-u|\le 10t\\ t\le 1.1\varepsilon}} \tilde{a}_r(s,t)\,\frac{ds\,dt}{t} \right) du$$
$$\le C \int_{10J} \int_0^{1.1\varepsilon} \tilde{a}_r(s,t)\,ds\,dt.$$

This last integral goes to 0 as $\varepsilon \to 0$, because $\iint_{10J} \tilde{a}_r(s,t)\,ds\,dt \le C|I| < \infty$. Thus $F_1(z) \to 0$ a.e. as $z \to u$ nontangentially.

The proof of the nontangential version of (4.2) can be done using the calculations we just did.

This completes the proof of Proposition 4.1 in the case where $\Omega_+ = U$. Suppose now that Γ is a general chord-arc curve, and let $\theta\colon \mathbf{C} \to \mathbf{C}$ and $r_t(z)$ be as in the beginning of this section. If $\varsigma \in \mathbf{C}$, $\varsigma = r_t(z)$, $z \in \Gamma$, and $t \in R$, define $\varsigma^* = r_{-t}(z)$. Thus $\varsigma \mapsto \varsigma^*$ is a bilipschitz reflection across Γ. Define

$$K(w,\varsigma) = \frac{1}{w-\varsigma}\,\frac{\varsigma^*-\varsigma}{w-\varsigma^*}.$$

Clearly $\partial K(w,\varsigma)/\partial\overline{w} = C\delta_\varsigma(w)$. Define $F(w)$ as in (4.4), but replacing i by any fixed $w_0 \in \Omega_+$, and taking $\chi(\varsigma)$ to be the characteristic function of $\{\varsigma\colon |\varsigma - w_0| \ge \frac{1}{2}\delta_\Gamma(w_0)\}$. One can prove the proposition in general in the same way as when $\Omega_+ = U$. (For many of the estimates it is useful to use the bilipschitz map θ to reduce to calculations on U.) This concludes the proof of the proposition.

Proposition 4.1 allows us to compute $\overline{\partial}^{-1}$ of a good Carleson measure. Suppose Γ is a chord-arc curve (the line, for instance), and $a(z)\,dx\,dy$ is an r-GCM relative to Γ, $r > 1$. In particular $a(z)$ is a locally integrable tempered distribution on $\mathbf{C}$, and we want to find another tempered distribution G on $\mathbf{C}$ such that $\overline{\partial}G = a$ on $\mathbf{C}$.

Let a_+ and a_- denote the restrictions of a to Ω_+ and Ω_-, and let F_+ and F_- be functions on Ω_+ and Ω_- that satisfy the conclusions of Proposition 4.1 with respect to a_+ and a_-, respectively. Define F on $\mathbf{C} \setminus \Gamma$ by setting $F = F_+$ on Ω_+ and $F = F_-$ on Ω_-. From the proposition it follows that F is a tempered distribution on $\mathbf{C}$, and that $\overline{\partial} F = a$ on $\mathbf{C} \setminus \Gamma$, but not necessarily all of $\mathbf{C}$, because F may have a jump across Γ.

Let f_+ and f_- be the boundary values of F_+ and F_- on Γ, in the sense of (b) in the proposition. Let $b = f_+ - f_-$, the jump of F across Γ. Define the measure dz_Γ supported on Γ as before (see §2). Then $\overline{\partial} F = a + b\,dz_\Gamma$ on all of $\mathbf{C}$ in the sense of distributions. (Inessential multiplicative constants are ignored.) Here $b\,dz_\Gamma$ is interpreted as a measure, and it is also a tempered distribution. This equation comes from Green's theorem in the form $\int_\Gamma g\,dz = \int_{\Omega_+} (\overline{\partial} g)\,d\overline{z} \wedge dz$, applied to Ω_+ and Ω_-. For Ω_- there is a minus sign for the line integral because the orientation reverses.

Let $C_\Gamma(b)(z)$ denote the Cauchy integral of b,

$$C_\Gamma(b)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{b(w)}{w - z}\,dw, \qquad z \notin \Gamma.$$

As observed in §3, $C_\Gamma(b)$ is a tempered distribution, and $\overline{\partial}(C_\Gamma(b)) = b\,dz_\Gamma$ on $\mathbf{C}$, in the distributional sense. Thus if $G = F - C_\Gamma(b)$, then $\overline{\partial} G = a$ on $\mathbf{C}$ as tempered distributions. In particular, G has no jump across Γ, because F and $C_\Gamma(b)$ both have jump given by b.

If $\tilde{G}$ is another tempered distribution such that $\overline{\partial}\tilde{G} = a$ on $\mathbf{C}$, then $G - \tilde{G}$ is entire, and hence a polynomial. A mild growth condition at infinity will force it to be constant, which is natural, since we are working with BMO.

Thus we can estimate $\overline{\partial}^{-1}$ of an r-GCM. In the next two sections we use this to get estimates for $(\overline{\partial} - \mu\partial)^{-1} = \overline{\partial}^{-1}(I - \mu\partial\overline{\partial}^{-1})^{-1}$.

5. The Cauchy integral on chord-arc curves with small constant.

THEOREM 5.1. *There exists $\eta > 0$ such that if Γ is a chord-arc curve with constant $K \leq \eta$ then the Cauchy integral on Γ is bounded on* BMO(Γ). *More precisely, given $f \in$ BMO(Γ) there is a holomorphic function F on $\mathbf{C} \setminus \Gamma$ with the following properties*:

(a) *$F_+ = F|_{\Omega_+}$ and $F_- = F|_{\Omega_-}$ have nontangential boundary values a.e. on Γ, which we denote by f_+ and f_-, where $f = f_+ - f_-$ and $\|f_\pm\|_* \leq C\|f\|_*$*;

(b) *F_+ and F_- satisfy the nontangential version of* (4.2), *with the right side replaced by $C\|f\|_*$*;

(c) *$|F'(z)|^2 \delta_\Gamma(z)\,dx\,dy$ is a Carleson measure with norm $\leq C\|f\|_*^2$*;

(d) *F is given by the Cauchy integral* (0.1) *of f, modulo constants.*

The Cauchy integral on these curves was first estimated by Coifman and Meyer [**CM2, 3**]. We have stated the boundedness of the Cauchy integral in this somewhat indirect way because that is how it comes out in our proof, which is based on the following estimates for $(\overline{\partial} - \mu\partial)^{-1}$ of an r-GCM.

THEOREM 5.2. *For each r, $1 < r \leq 2$, there is a $\gamma > 0$ so that the following holds. Suppose $\mu \in L^\infty(\mathbf{C})$, $\|\mu\|_\infty \leq \gamma$, and $|\mu|^2|y|^{-1}\,dx\,dy$ is a Carleson measure with norm $\leq \gamma^2$. Then for each r-GCM $a(z)\,dx\,dy$ there is a function $G(z)$ on $\mathbf{C}$ such that $(\overline{\partial} - \mu\partial)G = a$ on $\mathbf{C} \setminus R$ and $G = H - C(b)$, where $C(b)$ denotes the Cauchy integral of b (on $\mathbf{R}$), and*

(i) $b \in$ BMO($\mathbf{R}$), $\|b\|_* \leq C\|a\|_{r\text{-GCM}}$;

(ii) *if* $H_+ = H|_U$ *and* $H_- = H|_L$, *where* U *and* L *denote the upper and lower half-planes, then* H_+ *and* H_- *satisfy* (b) *and* (c) *of Proposition* 4.1;

(iii) *if* h_+ *and* h_- *denote the boundary values of* H_+ *and* H_-, *then* $b = h_+ - h_-$, *so that* G *has no jump across* $\mathbf{R}$;

(iv) *if* ∇H *is defined on* $\mathbf{C} \setminus R$ *in the distributional sense, then* $|\nabla H|\,dx\,dy$ *is an* r-*GCM, and its norm is dominated by the norm of* a. *Furthermore, if* $2 < r < \infty$ *and* $|\mu|^2|y|^{-1}\,dx\,dy$ *is an* $r/2$-*GCM with small enough norm, then the same conclusion holds, and in* (ii) H_+ *and* H_- *will satisfy the nontangential version of* (b) *and* (c) *in Proposition* 4.1.

Loosely speaking, we think of G as satisfying $(\overline{\partial} - \mu\partial)G = a$ on all of $\mathbf{C}$. This is not precisely correct, because ∂G may not be locally integrable on $\mathbf{C}$, so that $\mu(\partial G)$ may not be defined as a distribution on all of $\mathbf{C}$. On $\mathbf{C} \setminus R$ it is O.K., because ∂G is locally integrable there. By the next lemma, though, $\mu C'(b)$ is locally integrable on $\mathbf{C}$, where $C'(b)$ denotes the derivative of $C(b)$ on $\mathbf{C} \setminus R$, which allows us to interpret $\mu(\partial G) = \mu(\partial H) - \mu C'(b)$ as a locally integrable distribution. Notice that this problem goes away if b is smooth or if μ vanishes near $\mathbf{R}$.

LEMMA 5.3. *If* $|\mu|^2|y|^{-1}\,dx\,dy$ *is a Carleson measure and* $b \in \mathrm{BMO}(\mathbf{R})$, *then* $|\mu C'(b)|\,dx\,dy$ *is a* 2-*GCM, with norm* $\leq C\|b\|_*\|\,|\mu|^2|y|^{-1}\,dx\,dy\|_{CM}^{1/2}$. *If* $|\mu|^2|y|^{-1}\,dx\,dy$ *is an* $r/2$-*GCM, then* $|\mu C'(b)|\,dx\,dy$ *is an* r-*GCM, with norm at most* $C\|b\|_*\|\,|\mu|^2|y|^{-1}\,dx\,dy\|_{r/2\text{-GCM}}^{1/2}$.

Since $b \in \mathrm{BMO}(\mathbf{R})$, $|C'(b)|^2|y|\,dx\,dy$ is a Carleson measure. It is also an ∞-GCM because of the harmonicity of $C'(b)$, by a remark in §1. Thus Hölder's inequality gives the estimate for $\mu C'(b)$.

Let us show how to derive Theorem 5.1 from Theorem 5.2 as outlined in the introduction. Let $\rho\colon \mathbf{C} \to \mathbf{C}$ be the bilipschitz mapping in Proposition 2.5, such that $\rho(\mathbf{R}) = \Gamma$. Let μ be the dilatation of ρ, $\rho_{\overline{z}} = \mu\rho_z$, so that $\|\mu\|_\infty$ is small and $|\mu|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM with small norm.

Let $f \in \mathrm{BMO}(\Gamma)$ be given, and let $f_0 = f \circ \rho$, so that $f_0 \in \mathrm{BMO}(\mathbf{R})$. By Lemma 5.3, if $a = \mu C'(f_0)$, then $|a(z)|\,dx\,dy$ is an ∞-GCM.

Let G be as in Theorem 5.2, with this μ and a, and with $r = 4$. (Any $r > 2$ will do; the point is to get nontangential estimates.) Define $F_0 = G + C(f_0)$, and let $F = F_0 \circ \rho^{-1}$. The fact that $(\overline{\partial} - \mu\partial)F_0 = 0$ off $\mathbf{R}$ transforms into $\overline{\partial}F = 0$ off Γ, so that F is holomorphic off Γ. Properties (a) and (b) in Theorem 5.1 follow from their analogues in Theorem 5.2, since they are preserved by bilipschitz mappings.

To prove (c) it is enough to show that $|\nabla F_0|^2|y|\,dx\,dy$ is a Carleson measure. Since $F_0 = H - C(b) + C(f_0)$, we need only worry about H, because b, $f_0 \in \mathrm{BMO}(\mathbf{R})$. We know (from Theorem 5.2) that $|\nabla H|\,dx\,dy$ is a 4-GCM, and hence a 2-GCM. This implies that $|\nabla H|^2|y|\,dx\,dy$ is a Carleson measure, for the same reason that $l^1 \subseteq l^2$. Let us check this.

Set $a(z) = |\nabla H(z)|$, and let $\tilde{a}_{2,\alpha}(z)$ be as in (1.1) with $\alpha = \frac{1}{10}$, say. Define $d(z) = |\nabla H(z)|^2|y|$, and let $\tilde{d}_{1,\alpha}(z)$ be as in (1.1) also. As remarked in §1, $d(z)\,dx\,dy$ is a Carleson measure if and only if $\tilde{d}_{1,\alpha}(z)\,dx\,dy$ is. By definitions, $\tilde{d}_{1,\alpha}(z) \leq Cy(\tilde{a}_{2,\alpha}(z))^2$, and $\tilde{a}_{2,\alpha}(z)$ is a Carleson measure. It follows that if $y\tilde{a}_{2,\alpha}(z)$ is bounded, then $\tilde{d}_{1,\alpha}(z)\,dx\,dy$ is a Carleson measure, as desired. Using the notation of §1 (around (1.1)),

$$y^2 \tilde{a}_{2,\alpha}(z) \le C \int_{B_{z,\alpha}} \tilde{a}_{2,\alpha}(z)\,dx\,dv$$
$$\le C \int_{\{w:|u-x|\le 2y\}} \tilde{a}_{2,\alpha}(w)\,du\,dv \le Cy,$$

since $\tilde{a}_{2,\alpha}(w)\,du\,dv$ is a Carleson measure. Thus $y\tilde{a}_{2,\alpha}(z)$ is bounded, which implies that $d(z)\,dx\,dy = |\nabla H(z)|^2|y|\,dx\,dy$ is a Carleson measure, and hence (c) holds.

Now consider (d). It is enough to show that the Cauchy integral of f_+ gives F_+ on Ω_+ and 0 on Ω_-, and similarly for f_-. This is proved by approximating Γ by closed curves inside Ω_+. For this it is important that the Cauchy integral of $f_+ \in \mathrm{BMO}$ is defined modulo constants, so that there is enough decay at ∞ for the approximation to work.

Thus Theorem 5.1 can be derived from Theorem 5.2. Notice that G is small compared to $\|f\|_*$, since μ is small, and hence $F \circ \rho^{-1} = F_0 = G + C(f_0)$ is a small perturbation of $C(f_0)$.

There is another result of Coifman and Meyer [**CM 2,3**] which is closely related and which can also be obtained from Theorem 5.2. Suppose that $b \in \mathrm{BMO}(\mathbf{R})$ is real valued and has small norm. By the John-Nirenberg lemma, $e^{b(t)}$ is locally integrable, and in fact $h(x) = \int_0^x e^{b(t)}\,dt$ is a homeomorphism of $\mathbf{R}$ onto itself. Define the operator V_h by $V_h(f) = f \circ h$. It is well known (see [**Js 2**]) that V_h is bounded on BMO. Let H denote the Hilbert transform, $Hf(x) = \mathrm{P.V.}\int_{-\infty}^{\infty} f(y)/(x-y)\,dy$, which is also bounded on BMO.

Consider $V_h H V_h^{-1}$, which can be written as $\mathrm{P.V.}\int_{-\infty}^{\infty}[f(y)/(h(x)-h(y))]h'(y)\,dy$ by making the change of variables $y \to h(y)$. Coifman and Meyer proved that if $\|b\|_*$ is small enough, then $V_h H V_h^{-1} - H$ has small operator norm on BMO, $\le C\|b\|_*$ in fact. They used this to prove the boundedness of the Cauchy integral on chord-arc curves with small constant.

Let us show how we can estimate $V_h H V_h^{-1} - H$ using Theorem 5.2 and the same argument as for the Cauchy integral. As in (2.4) define $\rho(x,y) = \varphi_y * h(x) + i(\operatorname{sgn} y)\psi_y * h(x)$ when $y \ne 0$, $\rho(x,0) = h(x)$. This is a minor variation of the Beurling-Ahlfors extension, and if $\|b\|_*$ is small enough, $\rho(x,y)$ defines a quasiconformal (but not necessarily bilipschitz) map of $\mathbf{C}$ onto itself. If μ is the dilatation of ρ, one can show that $\|\mu\|_\infty$ is small and that $|\mu|^2|y|^{-1}\,dx\,dy$ is an ∞-GCM with small norm.

A function G defined on some domain is holomorphic iff $(\overline{\partial} - \mu\partial)(G \circ \rho) = 0$. The Hilbert transform is to holomorphic functions as $V_h H V_h^{-1}$ is to solutions of $(\overline{\partial} - \mu\partial)F = 0$. More precisely, if f is a given function on $\mathbf{R}$, then $f_\pm = \pm\frac{1}{2}f + Hf/2\pi i$ have holomorphic extensions $F_\pm$ to the upper and lower half-planes, by (0.2) applied to $\Gamma = \mathbf{R}$. By making the change of variable ρ, we see that if g is a given function on $\mathbf{R}$, then $g_\pm = \pm\frac{1}{2}g + (1/2\pi i)V_h H V_h^{-1} g$ have extensions $G_\pm$ to the upper and lower half-planes that satisfy $(\overline{\partial} - \mu\partial)G_\pm = 0$.

Suppose you are given $g \in \mathrm{BMO}(\mathbf{R})$. To estimate $V_h H V_h^{-1} g$, we must find G on $\mathbf{C} \setminus \mathbf{R}$ such that $(\overline{\partial} - \mu\partial)G = 0$ and G has jump g across R, and then estimate its boundary values, which will be given by $\pm\frac{1}{2}g + (1/2\pi i)V_h H V_h^{-1} g$ (from above and below). By Lemma 5.3, $|\mu C'(g)|\,dx\,dy$ is an ∞-GCM with small norm, since μ is small, and we can use Theorem 5.2 to find $E(z)$ on $\mathbf{C}\setminus\mathbf{R}$ such that $(\overline{\partial}-\mu\partial)E=\mu C'(g)$ on $\mathbf{C}\setminus\mathbf{R}$ and E has no jump across $\mathbf{R}$. Because $C'(g)$ is small, E is small, and in particular the boundary values of E have small BMO norm. Thus $G=E+C(g)$ is the function we are looking for, and its boundary values are a small perturbation of those of $C(g)$. This means that $V_hHV_h^{-1}-H$ has small operator norm on BMO. One can check that its norm is $\leq C\|b\|_*$ if $\|b\|_*$ is small enough.

It is worthwhile to compare our arguments with those of Coifman and Meyer. Theirs used singular integrals acting on the line, while ours deal more with operators acting on the whole plane. Also, for them the estimates for $V_hHV_h^{-1}-H$ were a stepping stone to get to the Cauchy integral, while for us they are both examples of the same thing.

Let us prove Theorem 5.2. We first proceed formally. We want to solve $(\overline{\partial}-\mu\partial)G=a$, and so we want to take $G=(\overline{\partial}-\mu\partial)^{-1}a=\overline{\partial}^{-1}(I-\mu T)^{-1}a$, where $T=\partial\overline{\partial}^{-1}$. From §4 we can get that μT is a bounded operator from r-GCM's to r-GCM's with small norm if μ satisfies the hypotheses of Theorem 5.2 (with small norm). Indeed, if $a(z)$ is a p-GCM, then from §4 we can write $\overline{\partial}^{-1}a=F-C(b)$, where $|\nabla F|$ is an r-GCM and $b\in \mathrm{BMO}(\mathbf{R})$. Thus $\mu Ta=\mu(\partial F)-\mu C'(g)$. The first term is an r-GCM because $|\nabla F|$ is and $\mu\in L^\infty(\mathbf{C})$, and the second term is handled using Lemma 5.3. Thus μTa is an r-GCM with small norm.

The problem with this argument is that it is not clear how to interpret $\mu C'(g)$, since $C'(g)$ is not locally integrable on $\mathbf{C}$. We had the same problem in stating Theorem 5.2. However, if we define $\mu C'(g)$ to be the pointwise product on $\mathbf{C}\setminus\mathbf{R}$, and ignore $\mathbf{R}$, then by Lemma 5.3 we get a locally integrable function on C. Theorem 5.2 was stated in such a way that if we define $\mu C'(g)$ in that way, then the preceding formal argument for getting G works. Let us verify that.

Suppose $a(z)$ is as in the statement of the theorem. To find G it is enough to find $G_j=H_j-C(b_j)$, $j=0,1,2,\ldots$, such that $\overline{\partial}G_0=a$, $\overline{\partial}G_{j+1}=\mu\partial G_j$ on $\mathbf{C}\setminus\mathbf{R}$, and such that H_j and b_j satisfy (i)–(iv) above, with norm estimates like $C^j\gamma^j\|a\|_{r\text{-GCM}}$, where γ is as in the statement of the theorem. For then we can take $G=\sum_{j=0}^{\infty}G_j$ if γ is small enough, and G will satisfy the conclusions of the theorem. (Formally, $G_j=G_j=\overline{\partial}^{-1}(\mu T)^j a$.)

By the remarks at the end of §4, we can find H_0 and b_0 satisfying (i)–(iv) and such that if $G_0=H_0-C(b_0)$, then $\overline{\partial}G_0=a$. On $\mathbf{C}\setminus\mathbf{R}$, $\partial G_0=\partial H_0-C'(b_0)$, and so $\mu(\partial G_0)=\mu(\partial H_0)-\mu C'(b_0)$ is an r-GCM, since ∂H_0 is and $\mu\in L^\infty(\mathbf{C})$ for the first term, and by Lemma 5.3 for the second. Using §4 again we can find G_1 so that $\overline{\partial}G_1=\mu(\partial G_0)$. Iterating this procedure gives the G_j.

Theorem 5.2 can be generalized to the case where $\mathbf{R}$ is replaced by any fixed chord-arc curve Γ, as long as you know that the Cauchy integral on Γ is bounded. The proof is the same as before. More precisely:

THEOREM 5.2$'$. *Suppose Γ is a chord-arc curve, and suppose the Cauchy integral on Γ is bounded on* $\mathrm{BMO}(\Gamma)$, *so that the conclusions of Theorem* 5.1 *hold for Γ. Then Theorem* 5.2 *holds with* $\mathbf{R}$ *replaced by Γ, e.g., Carleson measures are defined relative to Γ, $C(b)$ is replaced by $C_\Gamma(b)$, $|\mu|^2|y|^{-1}\,dx\,dy$ is replaced by $|\mu|^2\delta_\Gamma(z)^{-1}\,dx\,dy$, etc. The constant γ depends on the chord-arc constant of Γ and on the estimates for the Cauchy integral on Γ.*

Using this result one can show that the Cauchy integral on curves near Γ are also bounded on BMO. One would like to iterate to get estimates for the Cauchy integral on all chord-arc curves, or at least all Lipschitz graphs, starting with the real line. The problem with this is that the size of the allowed perturbation of Γ depends on the estimates for the Cauchy integral on Γ, and not merely the geometry. Thus if you try to iterate to get large perturbations, the estimates will blow up. We discuss this further in §8.

6. Estimates for $(\overline{\partial}-\mu\partial)^{-1}$ for more general μ. In Theorem 5.2 we obtained estimates for $(\overline{\partial}-\mu\partial)^{-1}$ when $|\mu|^2|y|^{-1}\,dx\,dy$ is a Carleson measure and is small. However, the μ that shows up for the mapping (2.1) does not satisfy this condition, but instead $|y\nabla\mu|^2|y|^{-1}\,dx\,dy$ is a Carleson measure.

We would like to have a version of Theorem 5.2 that also allows these types of μ's. The place where the proof breaks down is with Lemma 5.3: we needed the Carleson measure condition on μ to insure that $\mu C'(g)$ is a GCM if $g \in \mathrm{BMO}$. We needed $\mu C'(g)$ to be a GCM to compute $\overline{\partial}^{-1}$ of it using §4.

However, we can still compute $\overline{\partial}^{-1}(\mu C'(q))$ if $|yD\mu|^2|y|^{-1}\,dx\,dy$ is a Carleson measure, by integrating by parts. If $g \in \mathrm{BMO}(\mathbf{R})$, then $yC'(g)$ lies in $L^\infty(\mathbf{C})$ and has boundary values 0 a.e. on $\mathbf{R}$ by Lemma 3.2(c) and (d), in particular no jump across $\mathbf{R}$, and $\overline{\partial}(yC'(g)) = C'(g)$. Thus to compute $\overline{\partial}^{-1}(\mu C'(q))$, a good first guess is $y\mu C'(g)$. This function lies in $L^\infty(\mathbf{C})$ and has vanishing boundary values on $\mathbf{R}$, by Lemma 3.3(d), and $\overline{\partial}(y\mu C'(g)) = \mu C'(g) + y(\overline{\partial}\mu)C'(g)$. The last term is a GCM, so that $\overline{\partial}^{-1}$ of it can be computed by §4, and so $\overline{\partial}^{-1}(\mu C'(g)) = y\mu C'(g) - \overline{\partial}^{-1}(y(\overline{\partial}\mu)C'(g))$ can be computed.

(As in §5, $\mu C'(g)$ does not make sense as a distribution on all of $\mathbf{C}$, strictly speaking. Thus we interpret $F = \overline{\partial}^{-1}(\mu C'(g))$ to mean that $\overline{\partial}F = \mu C'(g)$ on $\mathbf{C}\setminus\mathbf{R}$ and F has no jump across $\mathbf{R}$. We use this interpretation throughout this section. The preceding computation does give such an F. Note that we can extend $\mu C'(g)$ from $\mathbf{C}\setminus\mathbf{R}$ to a distribution on $\mathbf{C}$ by identifying it with $\overline{\partial}F$.)

In this section we shall use this integration by parts to obtain an extension of Theorem 5.2 that allows this other class of μ's.

There are other conditions on μ that one can consider, for example, $\mu \in L^\infty(\mathbf{C})$, $y\nabla\mu \in L^\infty(\mathbf{C})$, and $|y^2\nabla^2\mu|^2|y|^{-1}\,dx\,dy$ a Carleson measure. For such a μ, $y(\overline{\partial}\mu)C'(g)$ will not be a GCM, and to compute its $\overline{\partial}^{-1}$ one must integrate by parts again:

$$\overline{\partial}^{-1}(y(\overline{\partial}\mu)C'(g)) = \tfrac{1}{2}y^2(\overline{\partial}\mu)C'(g) - \overline{\partial}^{-1}(\tfrac{1}{2}y^2(\overline{\partial}^2\mu)C'(g)).$$

The last term can be handled using §4. More generally, one could assume that $y^j\nabla^j\mu \in L^\infty(\mathbf{C})$ for $j = 0, 1, \ldots, K-1$ and $|y^K\nabla^K\mu|^2|y|^{-1}\,dx\,dy$ is a Carleson measure. To compute $\overline{\partial}^{-1}(\mu C'(g))$ for such a μ one must integrate by parts K times. We shall not consider these more general conditions on μ in detail because the algebra is messy and because it is not clear what the interesting examples are.

There is another extension which is more interesting, which is to replace $\mathbf{R}$ by a chord-arc curve Γ. In the above integration by parts we used the function y (e.g., in $\overline{\partial}(yC'(g)) = C'(g)$), which is good for $\mathbf{R}$ but not for Γ. There is a substitute for Γ, given by the following.

LEMMA 6.1. *Suppose Γ is a chord-arc curve. Then there exists a complex-valued function $\gamma = \gamma_\Gamma$ defined on C with the following properties*:

(a) $\delta_\Gamma(z)/C \le |\gamma(z)| \le C\delta_\Gamma(z)$;

(b) $|\nabla^j\gamma(z)| \le C_j\delta_\Gamma(z)^{j-1}$;

(c) *if* $j \ge 2$, *then* $|\delta_\Gamma(z)^{j-1}\nabla^j\gamma(z)|^2\delta_\Gamma(z)^{-1}\,dx\,dy$ *is an* ∞-*GCM*;

(d) $|\overline{\partial}\gamma(z) - 1|^2\delta_\Gamma(z)^{-1}\,dx\,dy$ *is an* ∞-*GCM*.

These estimates depend only on the chord-arc constant of Γ.

This function $\gamma(z)$ is a variation of the "adapted distance function" on Lipschitz domains in $\mathbf{R}^n$ of B. E. J. Dahlberg. There one requires $\gamma(z)$ be be real-valued and that (a), (b), and (c) hold, and also that $\partial\gamma/\partial y \ge C > 0$. Notice that if $\tilde{\gamma}(z)$ satisfies (a), (b), and (c) and also $|\overline{\partial}\tilde{\gamma}| \ge c > 0$, then $\gamma(z) = (\overline{\partial}\tilde{\gamma}(z))^{-1}\gamma(z)$ satisfies (a), (b), (c), and (d).

The proof of Lemma 6.1, which is based on conformal mappings estimates, will be given at the end of the section.

Let us show how one uses $\gamma(z)$ to compute $\overline{\partial}^{-1}$. For this, and for the rest of the section until we prove Lemma 6.1, we assume that the Cauchy integral on Γ is known to be bounded on $\mathrm{BMO}(\Gamma)$, so that the conclusions of Theorem 5.1 hold.

Suppose $g \in \mathrm{BMO}(\Gamma)$ and that we want to compute $\overline{\partial}^{-1}(C'_\Gamma(g))$. When $\Gamma = \mathbf{R}$ we multiplied by y. In the general case we try multiplying by γ: $\overline{\partial}(\gamma C'(g)) = (\overline{\partial}\gamma)C'_\Gamma(g) = C'_\Gamma(g) + (\overline{\partial}\gamma - 1)C'(g)$. By Lemma 6.1(c) and Lemma 5.3, $(\overline{\partial}\gamma - 1)C'_\Gamma(g)$ is a GCM, and so $\overline{\partial}^{-1}$ of it can be computed using §4. (Strictly speaking, we should extend Lemma 5.3 to the case where $\mathbf{R}$ is replaced by a general curve Γ, and where one requires that $|\mu|^2\delta_\Gamma(z)^{-1}\,dx\,dy$ is a Carleson measure relative to Γ. This extension is easy.) Thus we can compute $\overline{\partial}^{-1}(C'_\Gamma(g))$.

Suppose now that $\mu \in L^\infty(\mathbf{C})$ and $|\nabla\mu|^2\delta_\Gamma(z)\,dx\,dy$ is a Carleson measure (relative to Γ). To compute $\overline{\partial}^{-1}(\mu C'_\Gamma(g))$, we first observe that

$$\overline{\partial}(\gamma\mu C'_\Gamma(g)) = \mu C'_\Gamma(g) + (\overline{\partial}\gamma - 1)\mu C'_\Gamma(g) + \gamma(\overline{\partial}\mu)C'_\Gamma(g).$$

By Lemma 5.3 again, the last two terms are GCM's and $\overline{\partial}^{-1}$ of them can be obtained using §4. From this we get $\overline{\partial}^{-1}(\mu C'_\Gamma(g))$.

What we want to do now is to get an extension of Theorems 5.2 and 5.2′ that allows μ's such that $|\nabla\mu|^2\delta_\Gamma(z)\,dx\,dy$ is a Carleson measure with small norm. That is, we want estimates for $(\overline{\partial} - \mu\partial)^{-1}$. To do this we need not only to control $\overline{\partial}^{-1}(\mu C'_\Gamma(g))$, but also $\overline{\partial}^{-1}(\mu\partial\overline{\partial}^{-1})^j$, as in §5. This is achieved by more integration by parts. There is a complication, which is to properly choose the space of functions. In §5 we just used r-GCM's, but now other terms show up, like $\gamma\mu C'_\Gamma(g)$, which is not a GCM, and more generally, $(1/n!)\gamma^n C_\Gamma^{(n)}(g)$, because we have to repeat the integration by parts. We must first define our function space to incorporate these new terms, and afterwards the estimates for $(\overline{\partial} - \mu\partial)^{-1}$ will be easy.

Using $C^{-1}\delta_\Gamma(z) \le |\gamma(z)| \le C\delta_\Gamma(z)$ and the Cauchy integral formula one can show that there exists $D = D(\Gamma) > 0$ so that if $G(z)$ is holomorphic off Γ, then

$$\text{(6.2)}\qquad \begin{aligned} &\text{(i)}\quad \sup_{z\notin\Gamma} \frac{1}{j!}|\gamma(z)|^j|G^{(j)}(z)| \le D^j \sup_{z\notin\Gamma}|\gamma(z)|\,|G'(z)|,\\ &\text{(ii)}\quad \left\| \left|\frac{1}{j!}\gamma(z)^j G^{(j)}(z)\right|^2 \delta_\Gamma(z)^{-1}dx\,dy\right\|_{\infty\text{-GCM}}^{1/2}\\ &\qquad\qquad \le D^j\|\,|\gamma(z)G'(z)|^2\delta_\Gamma(z)^{-1}dx\,dy\|_{\infty\text{-GCM}}^{1/2}. \end{aligned}$$

Of course the GCM norm is taken relative to Γ.

For $2 \le r \le \infty$ define

$$M_{2,r} = M_{2,r}(\Gamma) = \{\mu \in L^\infty(\mathbf{C}): |\nabla\mu(z)|^2\delta_\Gamma(z)\,dx\,dy \text{ is an } r/2\text{-GCM relative to } \Gamma\},$$

and we give it the norm

$$\|\mu\|_{M_{2,r}} = \|\mu\|_{L^\infty} + \|\,|\nabla\mu|^2\delta_\Gamma(z)\,dx\,dy\|_{r/2\text{-GCM}}.$$

Here $\nabla\mu$ denotes the distributional gradient on $\mathbf{C}\setminus\Gamma$; we ignore anything that lives on Γ.

For $2 \le r \le \infty$ we define the space $U_r = U_r(\Gamma)$ to be the set of all locally integrable functions on $\mathbf{C}$ that can be represented as an l^1 sum of functions of the following two types:

(6.3) (i) $F - C_\Gamma(b)$, where $b \in \mathrm{BMO}(\Gamma)$, $\|b\|_* \le 1$,

$|\nabla F|\,dx\,dy$ is an r-GCM with norm ≤ 1, $F_+ = F|_{\Omega_+}$ and $F_- = F|_{\Omega_-}$ satisfy (b) and (c) of Proposition 4.1 with $\|f_\pm\|_* \le 1$ and with the left side of (4.2) at most 1, and where the jump of F across $\Gamma = f_+ - f_-$ is equal to b_j; also, when $r > 2$, we require that the nontangential versions of (b) and (c) in Proposition 4.1 hold, and that the left side of the nontangential version of (4.2) be at most 1;

(6.3) (ii) $m(z)D^{-j}(j!)^{-1}\gamma(z)^jC_\Gamma^{(j)}(b)$, where $j = 1, 2, 3, \ldots$,

$b \in \mathrm{BMO}(\Gamma)$, $\|b\|_* \le 1$, $m(z) \in M_{2,r}$, $\|m\|_{M_{2,r}} \le 1$.

By Lemma 3.2(c) and (d), and also (6.6), a function of type (6.3)(ii) is bounded by a constant depending only on Γ and has vanishing nontangential boundary values a.e. on Γ, from above or below. Since functions of type (6.3)(i) are locally integrable and have no jump across Γ, the same is true of all the elements of U_r. Also, the left side of (4.2) (or its nontangential version, when $r > 2$) will be bounded for all elements of $U_r(\Gamma)$, since that is true for $C_\Gamma(b)$ (because we are assuming that the Cauchy integral on Γ acting on BMO is bounded) and since functions of type (6.3)(ii) are bounded.

We define the $U_r(\Gamma)$ norm of a function f to be the infimum of $\sum|a_k|$ over all representations $f = \sum a_k\varphi_k$, where each φ_k is of the form (6.3)(i) or (ii). Note that many φ_k's may be of type (6.3)(ii) with the same j but different m's and g's. We consider constant functions to have norm 0.

Define $U_r'(\Gamma)$ to be the space of functions on $\mathbf{C}\setminus\Gamma$ which can be represented as an l^1 sum of functions of the following two types:

(6.4)(i) $a(z)$, where $|a(z)|\,dx\,dy$ is an r-GCM with norm ≤ 1;

(ii) $m(z)((j-1)!)^{-1}D^{-j+1}\gamma(z)^{j-1}C_\Gamma^{(j)}(b)$, where m, j, and b are as in (6.3)(ii).

The $U_r'(\Gamma)$ norm is defined analogous to the U_r norm. One should think of elements of $U_r'(\Gamma)$ as being derivatives of elements of $U_r(\Gamma)$, by the following.

PROPOSITION 6.5. (a) *If $f \in U_r(\Gamma)$ and if ∂f and $\overline{\partial} f$ are defined on $\mathbf{C} \setminus \Gamma$ distributionally, then ∂f and $\overline{\partial} f$ lie in $U_r'(\Gamma)$, with norms dominated by the norm of f.*

(b) *If $g \in U_r'(\Gamma)$, then there is an $f \in U_r(\Gamma)$ such that $\overline{\partial} f = g$ on $\mathbf{C} \setminus \Gamma$, and with a norm estimate.*

The f in (b) is unique up to an additive constant. For suppose $f_1, f_2 \in U_r(\Gamma)$ satisfy $\overline{\partial} f_1 = \overline{\partial} f_2$ on $\mathbf{C} \setminus \Gamma$. Then $\overline{\partial}(f_1 - f_2) = 0$ on $\mathbf{C} \setminus \Gamma$, and hence on C, since $f_1 - f_2$ has no jump across Γ (because f_1 and f_2 do not). Thus $f_1 - f_2$ is entire, and hence a polynomial since it is a tempered distribution. Since $f_1 - f_2|_\Gamma$ lies in BMO, $f_1 - f_2$ must be a constant. In particular, $\overline{\partial}^{-1}$ is well defined as a map from U_r' into U_r.

Observe that although $g \in U_r'(\Gamma)$ is only defined on $\mathbf{C} \setminus \Gamma$, it can be extended to a tempered distribution on $\mathbf{C}$ by identifying it with $\overline{\partial} f$, f as in (b). Because f has no jump across Γ (by definition of $U_r(\Gamma)$), $\overline{\partial} f$ will not have a boundary term on Γ, and so this identification is reasonable.

Before proving Proposition 6.5, let us use it to estimate $(\overline{\partial} - \mu\partial)^{-1}$. The point is that if ∂, $\overline{\partial}$, and $\overline{\partial}^{-1}$ are defined by Proposition 6.5, then

$$(\overline{\partial} - \mu\partial)^{-1} = \overline{\partial}^{-1}(I - \mu\partial\overline{\partial}^{-1})^{-1}$$

can be dealt with as in §5.

For $2 \le r \le \infty$, define

$$N_{2,r} = N_{2,r}(\Gamma) = \{\mu \in L^\infty(\mathbf{C}) \colon |\mu|^2 \delta_\Gamma(z)^{-1}\, dx\, dy \text{ is an } r/2\text{-GCM relative to } \Gamma\},$$

and give it the norm

$$\|\mu\|_{N_{2,r}} = \|\mu\|_{L^\infty(\mathbf{C})} + \|\, |\mu|^2 \delta_\Gamma(z)^{-1}\, dx\, dy\|_{r/2\text{-GCM}}^{1/2}.$$

This is the sort of condition we considered in §5.

LEMMA 6.6. *If $g \in U_r'(\Gamma)$ and $\mu \in M_{2,r}(\Gamma)$ or $\mu \in N_{2,r}(\Gamma)$, then $\mu g \in U_r'(\Gamma)$, with*

$$\|\mu g\|_{U_r'} \le C\|\mu\|\,\|g\|_{U_r'},$$

where $\|\mu\|$ denotes the $M_{2,r}$ or the $N_{2,r}$ norm of μ.

This is easy to check. If $\mu \in M_{2,r}$, one can show that if g is of type (6.3)(ii), then so is μg. This uses the fact that if $m_1, m_2 \in M_{2,r}$, then $m_1 m_2 \in M_{2,r}$. If $\mu \in N_{2,r}$, one can show that if g is of type (6.3)(ii), then μg is an r-GCM, i.e., a term of type (6.3)(i), and with bounded norm. This uses (6.2)(ii), Lemma 5.3, and the fact that $|C_\Gamma'(b)|^2 \delta_\Gamma(z)\, dx\, dy$ is a Carleson measure, since $b \in \mathrm{BMO}$.

THEOREM 6.7. *Suppose $\mu \in L^\infty(\mathbf{C})$, $\mu = \mu_1 + \mu_2$, where $\mu_1 \in M_{2,r}$ and $\mu_2 \in N_{2,r}$, $2 \le r < \infty$, and that μ_1 and μ_2 have small enough norm (depending on Γ and the estimate for the Cauchy integral on Γ). Suppose $g \in U_r'(\Gamma)$. Then there is a $G \in U_r(\Gamma)$ such that $(\overline{\partial} - \mu\partial)G = g$ on $\mathbf{C} \setminus \Gamma$, and the $U_r(\Gamma)$ norm of G is dominated by the $U_r'(\Gamma)$ norm of g.*

This is proved in a similar way as Theorem 5.2 was proved. It is enough to find $G_j \in U_r$, $j = 0, 1, 2, \ldots$, with geometrically decreasing norms such that $\overline{\partial} G_0 = g$ and $\overline{\partial} G_{j+1} = \mu\partial G_j$ on $\mathbf{C} \setminus \Gamma$, for then $G = \sum_{j=0}^\infty G_j$ has the desired properties.

One can find G_0 using Proposition 6.5. If G_j has been obtained, then $\partial G_j \in U'_r$, $\mu\partial G_j \in U'_r$ by Lemma 6.6, and one can find $G_{j+1} \in U_r$ such that $\bar{\partial} G_{j+1} = \mu\partial G_j$ on $\mathbf{C} \setminus \Gamma$ by Proposition 6.5. The decrease of the norms of the G_j comes from the assumption that μ is small.

The basic example for Theorem 6.7 is when $g = \mu C'(b)$, $b \in \mathrm{BMO}(\Gamma)$, which is what shows up when estimating the Cauchy integral on a nearby curve, as in the introduction. In particular, one can use Theorem 6.7 and the mapping (2.1) to obtain estimates for the Cauchy integral on Lipschitz graphs with small constant, just like Theorem 5.1 and its proof. The mapping (2.1) depends on the Lipschitz graph in a very simple way, which makes it possible for us to compute in more detail what our $\bar{\partial}$ approach is doing to the Cauchy kernel. This is discussed in the next section.

Let us now prove Proposition 6.5. To show that ∂ and $\bar{\partial}$ take $U_p(\Gamma)$ into $U'_p(\Gamma)$ boundedly, we need to see what they do to terms of type (6.3)(i) and (ii). The derivative of a term of type (i) is trivially a term of type (6.4)(i). For a term of type (6.3)(ii) we have that

$$\begin{aligned}
&\partial\{m(z)D^{-j}(j!)^{-1}\gamma(z)^j C_\Gamma^{(j)}(b)\} \\
&\quad = (\partial m)(z)D^{-j}(j!)^{-1}\gamma(z)^j C_\Gamma^{(j)}(b) \\
&\qquad + m(z)D^{-j}((j-1)!)^{-1}(\partial\gamma)(z)\gamma(z)^{j-1}C_\Gamma^{(j)}(b) \\
&\qquad + m(z)D^{-j}(j!)^{-1}\gamma(z)^j C_\Gamma^{(j+1)}(b).
\end{aligned}$$

The last term is of type (6.4)(ii), by definition. So is the middle term, except for a constant factor, because $\partial\gamma \in M_{2,r}$ by Lemma 6.1. The first term on the right is an r-GCM of bounded norm, and hence of type (6.4)(i). This is because $|\partial m(z)|^2\delta_\Gamma(z)\,dx\,dy$ is an $r/2$-GCM and $|C'_\Gamma(b)|^2\delta_\Gamma(z)\,dx\,dy$ is an ∞-GCM with norm $\le C\|b\|_*^2 \le C$, and by using (6.2)(ii) and Schwarz's inequality, as in the proof of Lemma 5.3.

If we took $\bar{\partial}$ instead of ∂ the above calculation simplifies because the last term does not appear.

Now consider part (b) of Proposition 6.5. Given $g \in U'_r(\Gamma)$, we want to find $f \in U_r(\Gamma)$ such that $\bar{\partial} f = g$ on $\mathbf{C} \setminus \Gamma$. If g is of type (6.4)(i), we can find f of type (6.3)(i) by §4.

Suppose g is of type (6.4)(ii), i.e.,

$$g(z) = m(z)((j-1)!)^{-1}D^{-j+1}\gamma(z)^{j-1}C_\Gamma^{(j)}(b).$$

The idea is that when you multiply this by $\gamma(z)/j$, you get an approximation to $\bar{\partial}^{-1}$ of it, leaving terms that can be handled using §4. (Indeed, if $m \equiv 1$, $\gamma(z) = 2y$, and $\Gamma = R$, then that gives $\bar{\partial}^{-1} g$ exactly.) More precisely,

$$\begin{aligned}
&\bar{\partial}(m(z)(j!)^{-1}D^{-j+1}\gamma(z)^j C_\Gamma^{(j)}(b)) \\
&\quad = \bar{\partial} m(z)(j!)^{-1}D^{-j+1}\gamma(z)^j C_\Gamma^{(j)}(b) \\
&\qquad + m(z)((j-1)!)^{-1}D^{-j+1}\gamma(z)^{j-1}(\bar{\partial}\gamma(z) - 1)C_\Gamma^{(j)}(b) \\
&\qquad + m(z)((j-1)!)^{-1}D^{-j+1}\gamma(z)^{j-1}C_\Gamma^{(j)}(b).
\end{aligned}$$

The last term is what we want to take $\overline{\partial}^{-1}$ of. The left side is $\overline{\partial}$ of something of type (6.3)(ii), except for a factor of D. If the other two terms are r-GCM's, then one can find $\overline{\partial}^{-1}$ of them as terms of type (6.3)(i), by §4. As usual, one proves that they are r-GCM's using (6.2)(ii) and Schwarz's inequality. For the first term one uses also that $|\overline{\partial} m|^2 \delta_\Gamma(z)\,dx\,dy$ is an $r/2$-GCM, while for the second one uses that $|\overline{\partial}\gamma - 1|^2 \delta_\Gamma(z)^{-1}\,dx\,dy$ is an ∞-GCM, by Lemma 6.1.

That proves Proposition 6.5. Let us now come back and prove Lemma 6.1. We shall find such a function $\gamma(z)$ on Ω_+; one can find γ on Ω_- in the same way.

Let Φ be a conformal mapping of Ω_+ onto the upper half-plane such that $\Phi(\infty) = \infty$. Define $\gamma(z) = 2\overline{(\Phi'(z))}^{-1}\,\mathrm{Im}\,\Phi(z)$. Condition (a) follows from the fact that $\mathrm{Im}\,\Phi(z) \approx |\Phi'(z)|\delta_\Gamma(z)$. (See p. 22 of [**P1**] or (1.6) in [**JK**].) The distortion theorem for Schlicht functions applied to the restriction of $\Phi(w)$ to $\{w: |w - z| < \delta_\Gamma(z)\}$ implies that

$$(1/c)|\Phi'(w)| \leq |\Phi'(z)| \leq C|\Phi'(w)|$$

when $|w - z| \leq \frac{2}{3}\delta_\Gamma(z)$. This and Cauchy's integral formula yields $|\Phi^{(j)}(w)| \leq C(j)\delta_\Gamma(z)^{-j+1}|\Phi'(z)|$ if $j \geq 2$ and $|w - z| \leq \frac{1}{2}\delta_\Gamma(z)$. From these estimates one gets (b) easily.

To prove (c) and (d) it is enough to show that if $f(z) = \Phi''(z)\Phi'(z)^{-1}$, then for $l \geq 0$, $|\delta_\Gamma(z)^{l+1} f^{(l)}(z)|^2 \delta_\Gamma(z)^{-1}\,dx\,dy$ is an ∞-GCM relative to Γ. That this is enough follows from the estimates we just did and direct computation. (For example,

$$\begin{aligned}\overline{\partial}\gamma &= 2\overline{\partial}(\overline{\Phi'}^{-1})\,\mathrm{Im}\,\Phi + 2\overline{\Phi'}^{-1}\overline{\partial}(\mathrm{Im}\,\Phi)\\ &= -2\overline{\Phi'}^{-2}\overline{\Phi''}(\mathrm{Im}\,\Phi) + 1.)\end{aligned}$$

The Cauchy integral formula allows us to reduce to $l = 0$. Because $f(z)$ is holomorphic, it is enough (by a remark in §1) to show that $|f(z)|^2\delta_\Gamma(z)\,dx\,dy$ is a Carleson measure instead of an ∞-GCM.

Notice that $f = g'$, where $g = \log\Phi'$. If $\Psi = \Phi^{-1}$, then $\Phi'(z) = \Psi'(\Phi(z))^{-1}$, so that $g = -\log\Psi'(\Phi(z))$. Because Γ is a chord-arc curve, $\log\Psi'$ lies in BMOA of the upper half-plane, by a theorem of Lavrentiev and Pommerenke (see [**P2, JK**]). We are therefore reduced to the following.

LEMMA 6.8. *Suppose $h(z)$ lies in BMOA of the upper half-plane, and set $g(z) = h(\Phi(z))$ on Ω_+. Then $|g'(z)|^2\delta_\Gamma(z)\,dx\,dy$ is a Carleson measure (relative to the chord-arc curve Γ).*

This lemma is not hard to derive from the H^p theory on chord-arc domains in [**JK**]. We shall assume the reader is familiar with that paper.

To show that $|g'(z)|^2\delta_\Gamma(z)\,dx\,dy$ is a Carleson measure it is enough to show that for $k(z)$ lying in $H^2(\Omega_+)$,

$$\iint_{\Omega_+} |k(z)|^2|g'(z)|^2\delta_\Gamma(z)\,dx\,dy \leq C\|K\|_2^2. \tag{6.9}$$

Indeed, suppose z_0 and $R > 0$ are given as in the definition of Carleson measure. There is a $w_0 \in \Omega_-$ such that $\delta_\Gamma(w_0) \geq R/C$ and $|w_0 - z_0| \leq CR$. This can be shown using a bilipschitz mapping of $\mathbf{C}$ onto $\mathbf{C}$ that takes $\mathbf{R}$ to Γ. Define $k(z) = R^{3/2}(z - w_0)^{-2}$, which lies in $H^2(\Omega_+)$ with norm bounded independently of z_0 and

R. Applying (6.9) with this $k(z)$ shows that (6.9) implies that $|g'(z)|^2\delta_\Gamma(z)\,dx\,dy$ is a Carelson measure relative to Γ.

To prove (6.9) we reduce to the upper half-plane using the conformal mapping Ψ. The left side of (6.9) is equal to

$$\begin{aligned}\iint_{\mathrm{UHP}} &|k(\Psi(z))|^2|g'(\Psi(z))|^2\delta_\Gamma(\Psi(z))|\Psi'(z)|^2\,dx\,dy\\ &= \iint_{\mathrm{UHP}} |k(\Psi(z))|^2|(g\circ\Psi)'(z)|^2\delta_\Gamma(\Psi(z))\,dx\,dy.\end{aligned}$$

We noted before that $|\Phi'(z)|\delta_\Gamma(z)$ is comparable to $\operatorname{Im}\Phi(z)$. This implies that $\delta_\Gamma(\Psi(z)) \approx y|\Psi'(z)|$. Substituting this and $h = g\circ\Psi$ gives above

$$\leq C\iint_{\mathrm{UHP}} |k(\Phi(z))\Psi'(z)^{1/2}|^2|h'(z)|^2 y\,dx\,dy.$$

Because $h \in \mathrm{BMOA}$, $|h'(z)|^2y\,dx\,dy$ is a Carleson measure on the upper half-plane, this expression is dominated by the square of the $H^2(\mathrm{UHP})$ norm of $(k\circ\Psi)(\Psi')^{1/2}$, which is equivalent to the $H^2(\Omega_+)$ norm of k. This proves (6.9), and finishes the proof of Lemma 6.1.

7. A closer look at the Calderón commutators. It is worthwhile to compare our methods with those of Coifman, McIntosh, and Meyer [**CMM**]. To do this properly we need to understand what the $\bar\partial - \mu\partial$ approach does to the Calderón commutators.

Let $A:\mathbf{R}\to\mathbf{R}$ be Lipschitz, $\|A'\|_\infty$ small, and let Γ denote its graph. To avoid technicalities, let us make the a priori assumption that A' is compactly supported and Hölder continuous. In terms of the graph coordinates $x \mapsto x + iA(x)$ the principal value version of the Cauchy integral in (0.2) is given by

$$Rf(x) = \frac{i}{2\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{f(y)(1+iA'(y))}{(x-y)+i(A(x)-A(y))}\,dy.$$

The Calderón commutators are defined by

$$R_nf(x) = \mathrm{P.V.}\int_{-\infty}^{\infty}\frac{(A(x)-A(y))^n}{(x-y)^{n+1}}f(y)\,dy,$$

so that $R = \sum_{n=0}^\infty (i/2\pi)(-i)^nR_nM$, where M denotes the operator of multiplication by $1+iA'$.

Because a power series is a power series is a power series, we ought to be able to re-express the Calderón commutators in terms of the calculations in §§5 and 6. Well suited for this is the mapping given in (2.1), $\rho(x,y) = x+iy+i\eta(x,y)$, where $\eta(x,y) = \varphi_y * A(x)$ if $y\neq 0$ and $\eta(x,0) = A(x)$. In §§5 and 6 we worked with

$$(\bar\partial - \mu\partial)^{-1} = \bar\partial^{-1}(I-\mu\partial\bar\partial^{-1})^{-1} = \bar\partial^{-1}\sum_{j=0}^\infty(\mu\partial\bar\partial^{-1})^j,$$

where $\mu = (\bar\partial\rho)(\partial\rho)^{-1}$. We could expand μ into a series in A, substitute that into the series for $(\bar\partial-\mu\partial)^{-1}$, and then collect terms to get the Calderón commutators, but that would be a horrid mess. Instead we use a slight variation of the $\bar\partial - \mu\partial$ calculation to make the algebra tractable.

Suppose f is defined on Γ, say smooth and compactly supported for technical convenience, and let F be its Cauchy integral (0.1), so that $\overline{\partial}F = 0$ off Γ and the jump of F across Γ is f. Also, $F'(z)$ is locally integrable if f is, say, Hölder continuous. Let $G = F \circ \rho$, so that the jump of G across $\mathbf{R}$ is given by $g(x) = f(\rho(x)) = f(x + iA(x))$ and $(\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)G = \rho_z(\overline{\partial} - \mu\partial)G = 0$ off $\mathbf{R}$. Thus if we compute $(\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)G$ on all of $\mathbf{C}$ we get only a boundary term coming from the jump.

When you compute the distributional derivative $\partial G/\partial x$ on $\mathbf{C}$ you get the same thing as on $\mathbf{C}\backslash\mathbf{R}$; there is no contribution on $\mathbf{R}$ from the jump. When you compute $\partial G/\partial y$ on $\mathbf{C}$, though, you get not only the part from $\mathbf{C} \setminus \mathbf{R}$, but you also pick up a boundary term from the jump, namely $g(x)\,dx$, where dx denotes the measure on $\mathbf{C}$ that is supported on $\mathbf{R}$ and agrees with Lebesgue measure there. Because $\overline{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ and $\partial = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, the boundary terms for $\overline{\partial}G$ and ∂G are $\frac{1}{2}ig(x)\,dx$ and $-\frac{1}{2}ig(x)\,dx$, respectively. Thus, on $\mathbf{C}$, we get that

$$(\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)G = (\rho_z + \rho_{\overline{z}})(i/2)g\,dx.$$

Since $\rho_z + \rho_{\overline{z}} = \partial\rho/\partial x$, and since the right side lives on $\mathbf{R}$, we get

$$(\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)G = (i/2)(1 + iA'(x))g\,dx.$$

We can rewrite the left side as

$$((1 + i\eta_z)\overline{\partial} - i\eta_{\overline{z}}\partial)G = (I + i(\eta_z - \eta_{\overline{z}}T))\overline{\partial}G,$$

where $T = \partial\overline{\partial}^{-1}$, so that

$$\begin{aligned} G &= \overline{\partial}^{-1}(I + i(\eta_z - \eta_{\overline{z}}T))^{-1}\left(\frac{i}{2}Mg\,dx\right) \\ &= \left(\sum_{n=0}^{\infty}(-i)^n\overline{\partial}^{-1}(\eta_z - \eta_{\overline{z}}T)^n\right)\left(\frac{i}{2}Mg\,dx\right). \end{aligned} \tag{7.1}$$

The boundary values of F on Γ from above and below are given by (0.2). Thus the boundary values of G on $\mathbf{R}$ above and below are given by

$$\begin{aligned} \frac{1}{2}g \pm Rg &= \frac{1}{2}g \pm \sum_{n=0}^{\infty}\frac{i}{2\pi}(-1)^nR_nMg \\ &= \frac{1}{2}(1 + iA')^{-1}Mg \pm \sum_{n=0}^{\infty}\frac{i}{2\pi}(-i)^nR_nMg \\ &= \sum_{n=0}^{\infty}(-i)^n\left(\frac{1}{2}(A')^n \pm \frac{i}{2\pi}R_n\right)Mg. \end{aligned}$$

Since η is a linear function of A, we get by matching terms homogeneous of the same degree that the boundary values of

$$(-i)^n\overline{\partial}^{-1}(\eta_z - \eta_{\overline{z}} - T)^n\left(\frac{i}{2}Mg\,dx\right) \tag{7.2}$$

from above and below are given by

$$(-i)^n(\tfrac{1}{2}(A')^n \pm iR_n/2\pi)(Mg). \tag{7.3}$$

When $n = 0$, R_n is the Hilbert transform, $\overline{\partial}^{-1}(h\,dx)$ is given by the Cauchy integral of h, and this statement reduces to (0.2) for $\Gamma = R$.

This representation for the Calderón commutators should be compared with the $P_t - Q_t$ formulas in [**CMM**]. Here we are computing the nth Calderón commutator in terms of the nth power of an operator acting on functions on $\mathbf{C}$, while in [**CMM**] the nth Calderón commutator is represented as an integral of nth powers of operators acting on $\mathbf{R}$. Also, in [**CMM**] one works with spaces like $L^2(\mathbf{R})$, while here we must estimate the operators on the space of r-GCM's, or U_r and U_r' from §6.

It is natural that one can represent the nth Calderón commutator as an nth power, since one is looking for estimates like $\|R_n\| \le C^n$ to get Calderón's theorem.

We should explain how one estimates the right side of (7.1) using §6. This is not immediate because $Mg(x)\,dx$ does not lie in U_r', and so we must enlarge that space.

We should first get straight what estimates we are looking for. Unlike R, R_n is not bounded on BMO because it does not map constants to constants, and so we must instead consider R_n as an operator from L^∞ into BMO. We do not lose anything, because standard methods (see [**Je**]) allow one to show that R is bounded on BMO if it maps L^∞ into BMO boundedly. Thus we want to show that if $g \in L^\infty(\mathbf{R})$ then we can get BMO estimates for the terms in (7.1).

We need to take a closer look at η_z and $\eta_{\overline{z}}$. By Proposition 2.2, η_z and $\eta_{\overline{z}}$ both lie in M_2, with norm $\le C\|A'\|_\infty$. Also, η_z and $\eta_{\overline{z}}$ are smooth off $\mathbf{R}$, and their boundary values are given by $\frac{1}{2}A'(x)$, from both above and below. Indeed, $(\partial/\partial x)\eta(x, y) = (\partial/\partial x)(\varphi_y * A(x)) = \varphi_y * A'(x)$ tends to $A'(x)$ as $y \to 0$. Also

$$\text{(7.4)}\qquad \frac{\partial}{\partial y}\varphi_y * A(x) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y}\varphi_y(x-u)A(u)\,du \to 0 \text{ a.e.} \quad \text{as } y \to 0.$$

For a.e. x, $A(u) = A(x) + A'(x)(x-u) + o(|u-x|)$ as $u \to x$. Since $\int \varphi_y(u)\,du = 1$ and $\int \varphi_y(u)u\,du = 0$, the latter because we chose φ to be even, we get that

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\varphi_y(x-u)\,du = 0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial y}\varphi_y(x-u)(x-u)\,du.$$

Thus in the right side of (7.4) we can replace $A(u)$ by $o(|u-x|)$, from which (7.4) follows.

Thus η_z and $\eta_{\overline{z}}$ are well defined on $\mathbf{R}$. In particular, multiplication of $Mg(x)\,dx$ by η_z or $\eta_{\overline{z}}$ makes sense.

To use §6 to estimate (7.1), we need to enlarge U_r and U_r'. Define $\tilde{U}_r'$ just like we defined U_r' before, except we now allow also a third type of term, $h(x)\,dx$, where $h \in L^\infty(\mathbf{R})$, $\|h\|_\infty \le 1$, and (as before) dx denotes Lebesgue measure supported on $\mathbf{R}$. Similarly, we enlarge U_r by allowing it to contain also terms of the form $C(h)$, i.e., the Cauchy integral of h, where $h \in L^\infty(\mathbf{R})$ and $\|h\|_\infty \le 1$. This new larger space we call $\tilde{U}_r$. Thus the elements of $\tilde{U}_r$ are allowed to have jumps, unlike U_r, but the jumps must be bounded.

We also define $\tilde{M}_{2,r}$ to be the space of $\mu \in M_{2,r}$ such that μ is continuous on $\mathbf{C} \setminus \mathbf{R}$, has radial boundary values almost everywhere on $\mathbf{R}$, and these boundary values are the same from above and below (i.e., no jump). Thus η_z and $\eta_{\overline{z}}$ lie in $\tilde{M}_{2,r}$.

Let us indicate how one extends the result of §6 to $\tilde{U}_r$ and $\tilde{U}'_r$. Analogous to Proposition 6.5, ∂, $\overline{\partial}\colon \tilde{U}_r \to \tilde{U}'_r$ and $\overline{\partial}^{-1}\colon \tilde{U}'_r \to \tilde{U}_r$. Now, when we compute ∂ and $\overline{\partial}$, we must allow for boundary terms coming from the jumps. We need only check what happens to the new terms. If $C(b) \in \tilde{U}_r$, then $\overline{\partial}C(h) = \frac{1}{2}h\,dx$ and $\partial C(h) = C'(h) + \frac{1}{2}h\,dx$, both of which lie in $\tilde{U}'_r$. If $h\,dx \in \tilde{U}'_r$, then $\overline{\partial}^{-1}$ of it is $2C(h) \in \tilde{U}_r$. Analogous to Lemma 6.6, if $\mu \in \tilde{M}_{2,r}$ and $g \in \tilde{U}'_r$, then $\mu g \in \tilde{U}'_r$. One need only check the case where $g = h\,dx$, and that is O.K. because μ must be well defined on $\mathbf{R}$. Just as in Theorem 6.7 and its proof, one can estimate the terms of (7.1) as elements of $\tilde{U}_r$, and show that the series converges in $\tilde{U}_r$ if $\|A'\|_\infty$ is small enough.

(We have been sloppier than usual about distribution theory in the above computations. These things work out if we make a priori smoothness assumptions on f and A. If we do not, then we cannot define everything in terms of classical distribution theory, but instead we have to reinterpret ∂ and $\overline{\partial}$ slightly to make things work out, like we did in §§5 and 6.)

There is a more direct way of computing the relationship between the Calderón commutators and perturbations of $\overline{\partial}$, but which is less convenient for worrying about jumps and boundary values.

Suppose $\rho\colon \mathbf{C} \to \mathbf{C}$ is bilipschitz and that h is, say, Lipschitz and compactly supported. Then

$$\overline{\partial}(h \circ \rho) = ((\partial h) \circ \rho)\rho_{\overline{z}} + ((\overline{\partial} h) \circ \rho)\overline{\rho}_{\overline{z}},$$
$$\partial(h \circ \rho) = ((\partial h) \circ \rho)\rho_z + ((\overline{\partial} h) \circ \rho)\overline{\rho}_z.$$

Multiply the first by ρ_z and the second by $\rho_{\overline{z}}$, and subtract

$$(\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)(h \circ \rho) = ((\overline{\partial} h) \circ \rho)(|\rho_z|^2 - |\rho_{\overline{z}}|^2). \tag{7.5}$$

Define V_ρ, D_ρ, and M_ρ by $V_\rho(f) = f \circ \rho$, $D_\rho f = (\rho_z\overline{\partial} - \rho_{\overline{z}}\partial)f$, and $M_\rho f = (|\rho_z|^2 - |\rho_{\overline{z}}|^2)f$. Then (7.5) becomes

$$D_\rho V_\rho(h) = M_\rho V_\rho(\overline{\partial} h), \quad \text{or} \quad D_\rho = M_\rho V_\rho \overline{\partial} V_\rho^{-1}(h).$$

Thus

$$\begin{aligned} D_\rho^{-1}(h)(z) &= V_\rho \overline{\partial}^{-1} V_\rho^{-1} M_\rho^{-1}(h)(z) \\ &= V_\rho \left(\int_{\mathbf{C}} \frac{1}{\cdot - w} (V_\rho^{-1} M_\rho^{-1} h)(w)\, du\, dv \right)(z) \\ &= V_\rho \left(\int_{\mathbf{C}} \frac{1}{\cdot - \rho(w)} h(w)\, du\, dv \right)(z) \\ &= \int_{\mathbf{C}} \frac{1}{\rho(z) - \rho(w)} h(w)\, du\, dv, \end{aligned}$$

except for an unimportant multiplicative constant. In the third equality we used the change of variables formula and the fact that $|\rho_z|^2 - |\rho_{\overline{z}}|^2$ is the Jacobian of ρ.

Suppose $\rho(z) = z + i\eta(z)$, where η has small Lipschitz norm. Then

$$D_\rho^{-1}(h)(z) = \sum_{h=0}^{\infty} (-i)^n \int_{\mathbf{C}} \frac{(\eta(z) - \eta(w))^n}{(z - w)^{n+1}} h(w)\, du\, dv.$$

Also,

$$D_\rho^{-1} = (\overline{\partial} + i\eta_z\overline{\partial} - i\eta_{\overline{z}}\partial)^{-1} = \overline{\partial}^{-1}(I + i(\eta_z - \eta_{\overline{z}}T))^{-1}$$
$$= \sum_{n=0}^{\infty}(-i)^n\overline{\partial}^{-1}(\eta_z - \eta_{\overline{z}}T)^n,$$

where $T = \partial\overline{\partial}^{-1}$ as always. Thus

$$(-i)^n\overline{\partial}^{-1}(\eta_z - \eta_{\overline{z}}T)^n h(z) = (-i)^n \int_{\mathbf{C}} \frac{(\eta(z) - \eta(w))^n}{(z-w)^{n+1}} h(w)\, du\, dv,$$

except for a multiplicative constant for the right side.

8. Perturbations of Lipschitz graphs and why you cannot (in general) iterate to get all Lipschitz graphs. Let $A_0 \colon \mathbf{R} \to \mathbf{R}$ be a Lipschitz function and let A_1 be another, and let Γ_0 and Γ_1 denote their graphs. Suppose $\|A_0' - A_1'\|_\infty$ is small. As in §2, define mappings ρ_0 and ρ_1 of $\mathbf{C}$ onto itself by

$$\rho_j(x, y) = x + iLy + i\varphi_y * A_j(x), \qquad j = 0, 1,$$

where $L \geq C\|A_j'\|_\infty$ is large enough so that the ρ_j are bilipschitz. As before, $\rho_j(\mathbf{R}) = \Gamma_j$.

Define $\rho = \rho_1 \circ \rho_0^{-1}$. Then ρ is a bilipschitz map of $\mathbf{C}$ onto itself which is a small perturbation of the identity, since $\|A_0' - A_1'\|$ is small and $\rho(\Gamma_0) = \Gamma_1$. One can show that the dilatation $\mu = \rho_{\overline{z}}/\rho_z$ of ρ lies in $M_{2,\infty}(\Gamma_0)$, with small norm if $\|A_0' - A_1'\|$ is small, using Proposition 2.2 and Corollary 2.3.

Let C_0 and C_1 denote the Cauchy integral operators on Γ_0 and Γ_1. Suppose that we know that C_0 is bounded on BMO (so that the conclusion of Theorem 5.1 holds for Γ_0), and that we want to estimate C_1. As in the introduction and also §5, this reduces to the following: given $g \in \mathrm{BMO}(\Gamma_0)$, find G on $\mathbf{C} \setminus \Gamma_0$ with no jump on Γ_0 and BMO boundary values and satisfying $(\overline{\partial} - \mu\partial)G = \mu C_0'(g)$. This can be done using Theorem 6.7, since $C_0'(g) \in U_r(\Gamma_0)$ for every $r < \infty$. Thus the Cauchy integral on Γ_1 is bounded on $\mathrm{BMO}(\Gamma_0)$ if $\|A_0' - A_1'\|_\infty$ is small enough.

An interesting special case of this is when $A_1 = \lambda A_0$, λ close to 1. One can use these estimates to give another proof of the differential inequality of Calderón [**Ca**].

One would like to iterate this perturbation argument to go from the real line to all Lipschitz graphs. As pointed out at the end of §5, this does not work, because the estimates explode. It would be interesting if one could find a way of making this work. That would require better control on the relationship between μ and the geometry of Γ.

There is a natural way to try to fix this that works for some particularly nice Lipschitz domains, e.g. sawtooth domains. The point is that if we can find functions $\gamma(z)$ that satisfy somewhat better estimates than in Lemma 6.1 then one can get a better version of Theorem 6.7 that allows the iteration procedure to converge.

Let us be more precise. Let Γ be a Lipschitz graph (or a chord-arc curve), and suppose we know that there is a function $\gamma(z)$ on C that satisfies (a) and (b) of Lemma 6.1, but instead of (c) and (d) it satisfies

(8.1) (c′) $|\delta_\Gamma(z)^{j-1}\nabla^j\nu(z)|\, dx\, dy$ is an ∞-GCM if $j \geq 2$,

(d′) $|\overline{\partial}\gamma(z) - 1|\delta_\Gamma(z)^{-1}\, dx\, dy$ is an ∞-GCM.

These Carleson measure conditions are stronger than their quadratic analogues (c) and (d).

Using such a γ one can redo the set-up in §6. Define

$$M_{1,r} = M_{1,r}(\Gamma) = \{\mu \in L^\infty(C)\colon |\nabla\mu|\, dx\, dy \text{ is an } r\text{-GCM relative to } \Gamma\},$$

with the norm

$$\|\mu\|_{M_{1,r}} = \|\mu\|_\infty + \|\, |\nabla\mu|\, dx\, dy\|_{r\text{-GCM}}.$$

Here $\nabla\mu$ denotes the distributional gradient on $\mathbf{C}\setminus\Gamma$; we ignore any contribution on Γ. Define the spaces $V_r = V_r(\Gamma)$ and $V_r' = V_r'(\Gamma)$ exactly like U_r and U_r' in §6, except that $M_{2,r}$ is replaced by $M_{1,r}$ and one requires that γ satisfy (8.1).

With these definitions the analogues of Proposition 6.5, Lemma 6.6, and Theorem 6.7 will hold with estimates that do not depend on the boundedness of the Cauchy integral on Γ. Indeed, in §6 we needed the boundedness of the Cauchy integral on Γ to know that $|C_\Gamma'(g)|^2\delta_\Gamma(z)\, dx\, dy$ is an ∞-GCM if $g \in \mathrm{BMO}(\Gamma)$, which we needed to conclude that $|\nabla m|\,|C_\Gamma'(b)|\delta_\Gamma(z)\, dx\, dy$, $|\overline{\partial}\gamma - 1|\,|C'(g)|\, dx\, dy$, etc., are r-GCM's. If $m \in M_{1,r}$ and γ satisfies (8.1), then all we need is that $\delta_\Gamma(z)|C'(g)|$ is bounded, which we know trivially, by Lemma 3.2(c).

Suppose we know that we can find $\gamma(z)$ satisfying (8.1) for all Lipschitz graphs, with estimates depending only on the Lipschitz constant. Then the $M_{1,r} - V_r$ analogue of Theorem 6.7 would be true for all Lipschitz graphs, giving estimates for $(\overline{\partial} - \mu\partial)^{-1}$ if $\mu \in M_{1,r}$ has small norm, how small depending only on the Lipschitz constant.

Suppose Γ_0 is the graph of the Lipschitz function A_0, and Γ_1 is the graph of $A_1 = A_0$. Then the preceding paragraph would imply that if the Cauchy integral on Γ_0 is bounded, then it is also bounded on Γ_1 if $|\lambda - 1| \leq \varepsilon(\|A_0'\|)$. (Indeed, $\rho(x, y) = x + i\lambda y$ maps Γ_0 to Γ_1, and its dilatation μ is constant and small if $|\lambda - 1|$ is small. In particular, $\|\mu\|_{M_{1,\infty}}$ is small too.) One could then iterate to go from the boundedness of the Cauchy integral on graphs with small constant to all Lipschitz graphs.

On the surface it seems not unreasonable that one could find a $\gamma(z)$ satisfying (8.1) for all Lipschitz graphs. This would be true if Proposition 2.2 were true with $|\nabla^2\rho|\, dx\, dy$ a Carleson measure instead of $|\nabla^2\rho|^2|y|\, dx\, dy$. This could be done if we knew that for any Lipschitz function $f\colon \mathbf{R} \to \mathbf{R}$ there is a Lipschitz extension F to the upper half-plane such that $|\nabla^2 F|\, dx\, dy$ is a Carleson measure. Whether this could be done was an open question. It may not seem unreasonable, in view of the theorem of Varopoulos **[V1, 2]** that any $g \in \mathrm{BMO}(\mathbf{R})$ has an extension to the upper half-plane such that $|\nabla G|\, dx\, dy$ is a Carleson measure. However, the Lipschitz version is false; see Proposition 8.2 below.

It is not completely clear to me how to prove that $\gamma(z)$ satisfying (8.1) does not exist in general, but I do not believe that it could. One reason is the nonexistence of the good Lipschitz extension. The other is that if it did exist, the estimates that you would get for the Cauchy integral on Γ_1 in terms of the Cauchy integral of Γ_0 (Γ_0, Γ_1 as above) would be too good to be true.

Such a $\gamma(z)$ does exist for sawtooth domains, and in fact you can construct $\gamma(z)$ by hand, as was pointed to me by C. Kenig. Thus for sawtooth domains the above perturbation argument does work to give the boundedness of the Cauchy integral on all of them.

Let us return to the problem about Lipschitz extensions. Suppose $f \in L^\infty(\mathbf{R})$, and let $Pf(x,y)$ denote its Poisson extension to the upper half-plane. By Fefferman-Stein we know that $|\nabla Pf(x,y)|^2 y\,dx\,dy$ is a Carleson measure. Let us say that f is particularly nice if $|\nabla Pf(x,y)|\,dx\,dy$ is a Carleson measure. Let PNL^∞ denote the space L^∞ functions with norm $\|f\|_\infty + \|\,|\nabla Pf|\,dx\,dy\|_{CM}$. It is known that not every L^∞ function has this property; see [**G**, p. 237].

PROPOSITION 8.2. *Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is Lipschitz, $F(x,y)$ is a Lipschitz extension of f to the upper half-plane, and that $\nabla^2 f$, defined distributionally, is locally integrable and $|\nabla^2 F|\,dx\,dy$ is a Carleson measure. Then $f' \in PNL^\infty$. In particular, not all Lipschitz functions have such an extension.*

The converse is also true: if f is Lipschitz and $f' \in PNL^\infty$, then such an extension exists. You cannot simply take the Poisson integral, because that would not converge (too much growth at ∞), but if you use a compactly supported bump instead of the Poisson kernel it works. Using this, one can extend Proposition 2.2 and Corollary 2.3 to show that if Γ is the graph of A, $A' \in PNL^\infty$, then $y^{n-2}|\nabla^n \rho|\,dx\,dy$ are ∞-GCM's if $n \geq 2$. One can then use that to show that Lemma 6.1 remains valid for the graphs even if we replace (c) and (d) by (8.1). In particular, the perturbation argument described before applies to these graphs to show that their Cauchy integrals are bounded. Sawtooth domains are examples of these "particularly nice" Lipschitz domains.

The idea for the proof of Proposition 8.2 is the following. The characterizations of L^p or BMO in terms of the gradient of the Poisson integral are delicate, but the corresponding results for Lipschitz spaces or Besov spaces are less delicate and reduce to integrating the derivatives of the Poisson extension along the right paths. (See, e.g., [**S**, Chapter 5, §4, and p. 147] in particular.) Moreover, you do not even need the extensions to be harmonic or anything like that. This is how it is for Proposition 8.2, but it would not work that way for L^∞ instead of Lipschitz.

We shall need a preliminary fact.

LEMMA 8.3. *Suppose that $|a(x,y)|\,dx\,dy$ is a Carleson measure on the upper half-plane. Then so is $|P_y * a_y(x)|\,dx\,dy$, where $P_y(x)$ denotes the Poisson kernel and $a_y(x) = a(x,y)$.*

This is proved using standard near and far part arguments. Let $x_0 \in R$ and R be given, and let us indicate how to estimate the $|P_y * a_y(x)|\,dx\,dy$ measure of $\{z \in \mathrm{UHP}\colon |z - x_0| \leq R\}$. Let $a = a^1 + a^2$, where a^1 lives in $\{z\colon |z - x_0| \leq 2R\}$ and a^2 lives in the complement. One estimates the contribution from a^1 using $\|P_y * f\|_1 \leq \|f\|$, and the hypotheses that $|a(x,y)|\,dx\,dy$ is a Carleson measure; the contribution from the a^2 term is controlled using the decay of the Poisson kernel.

Let us prove Proposition 8.2. To show that $f' \in PNL^\infty$ we first show that

$$\left|\left(\frac{\partial^2}{\partial x^2}P_y\right) * f(x)\right|\,dx\,dy = \left|\frac{\partial}{\partial x}(P_y * f')(x)\right|\,dx\,dy$$

is a Carleson measure. Write

$$f(x) = \{F(x,0) + F(x,2y) - 2F(x,y)\} - F(x,2y) + 2F(x,y).$$

From the lemma it follows that

$$\left|\frac{\partial^2}{\partial x^2}P_y * F_y(x)\right|\,dx\,dy = \left|P_y * \left(\frac{2}{\partial x^2}F_y\right)(x)\right|\,dx\,dy$$

is a Carleson measure since $|\nabla^2 F|\,dx\,dy$ is, and similarly for the $F(x,2y)$ term.

Denote the expression in braces by $\Delta_y^2 F(x,y)$. If F is C^2 in y, then

$$\Delta_y^2 F(x,y) = \int_0^y \int_{-s}^{s} \frac{\partial^2}{\partial t^2} F(x, y+t)\,dt\,ds,$$

and by Fubini,

$$\begin{aligned} |\Delta_y^2 F(x,y)| &\le \int_{-y}^{y} (y - |t|) \left| \frac{\partial^2}{\partial t^2} F(x, y+t) \right| dt \\ &\le \int_0^{2y} t \left| \frac{\partial^2}{\partial t^2} F(x,t) \right| dt. \end{aligned}$$

Thus

$$\begin{aligned} \left| \frac{\partial^2}{\partial x^2} P_y * (\Delta_y^2 F_y)(x) \right| &= \left| \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2} P_y \right)(x-u) \Delta_y^2 F(u,y)\,du \right| \\ &\le C \int_{-\infty}^{\infty} \frac{y^3}{(y + |x-u|)^4} y^{-2} \int_0^{2y} t |\nabla^2 F(x,t)|\,dt\,du. \end{aligned} \tag{8.4}$$

It is not hard to show that if $|a(x,y)|\,dx\,dy$ is a Carleson measure and if $b(x,y) = y^{-2} \int_0^{2y} t|a(x,t)|\,dt$, then $b(x,y)\,dx\,dy$ is also a Carleson measure. By Lemma 8.3, $P_y * b_y(x)\,dx\,dy$ is also a Carleson measure. Since (8.4) tells us that $|\partial^2 P_y/\partial x^2 * (\Delta_y^2 F_y)(x)|$ is dominated by $P_y * b_y(x)$, we get that the former determines a Carleson measure.

Thus $|(\partial/\partial x)Pf'|\,dx\,dy$ is a Carleson measure, and we need to show that $|(\partial/\partial y)P(f')|/\,dx\,dy$ is. Since $(\partial/\partial x)P(f')$ is harmonic, $|(\partial/\partial x)P(f')|\,dx\,dy$ is an ∞-GCM, by a remark in §1. Thus, if $a(z) = |(\partial/\partial x)P(f')(x,y)|$, and $\tilde{a}(z) = \sup\{a(w) \colon |w - z| < \frac{1}{2}y\}$, then $\tilde{a}(z)\,dx\,dy$ is a Carleson measure. By the Poisson integral formula on the disk $\{w \colon |w-z| < \frac{1}{2}y\}$, $y|(\partial^2/\partial x^2)P(f')(x,y)| \le C\tilde{a}(x,y)$. Since $P(f')$ is harmonic, we get that

$$y \left| \frac{\partial^2}{\partial y^2} P(f')(x,y) \right| dx\,dy \le C\tilde{a}(z)\,dx\,dy$$

and so the left side is a Carleson measure. Since

$$\frac{\partial}{\partial y} P(f')(x,y) = \int_y^{\infty} \frac{\partial^2}{\partial t^2} P(f')(x,t)\,dt,$$

a simple argument allows us to conclude that $|(\partial/\partial y)P(f')(x,y)|\,dx\,dy$ is a Carleson measure. Thus $f' \in PNL^\infty$, and Proposition 8.2 is proved.

9. Problems concerning H^∞. Let Γ be a rectifiable Jordan curve in the plane that passes through ∞, and suppose E is a closed subset of Γ of positive length. Let $\Omega = \mathbf{C} \setminus E$, and let $H^\infty(\Omega)$ denote the space of bounded holomorphic functions on Ω. It is known (see [**M**]) that $H^\infty(\Omega)$ will always contain something besides constants. The proof is based on Calderón's theorem and a duality argument. One would of course like to understand $H^\infty(\Omega)$ better, in particular be able to construct $H^\infty(\Omega)$ functions explicitly, and to solve the corona problem for $H^\infty(\Omega)$.

If $\Gamma = \mathbf{R}$, a standard way to construct $H^\infty(\Omega)$ functions is to take a real-valued $L^\infty(E)$ function f, take its Cauchy integral $F(z)$, and then exponentiate,

$e^{F(z)} \in H^\infty(\Omega)$. Recently, Garnett and Jones [**GJ**] proved the corona theorem for $H^\infty(\Omega)$ for all E when $\Gamma = \mathbf{R}$, extending a result of Carleson [**Cr**] who proved it when E is uniformly thick. Jones [**Js3**] has obtained results for more general domains with thick boundary.

Understanding $H^\infty(\Omega)$ is infamitely connected to L^∞ estimates for the $\overline{\partial}$ problem on Ω. ("Infamitely" means infinitely, intimately, and infamously.) In view of this it is natural to try to use the methods of this paper to attack the problem when Γ is a chord-arc curve with small constant. In other words, we want to reduce from Γ to $\mathbf{R}$, just like we did for the Cauchy integral.

So let Γ be a chord-arc curve with small constant, let ρ be the bilipschitz mapping as in (2.4) and Proposition 2.5, and let μ be its dilatation. Let E be a closed subset of Γ of positive length, and let $E_0 = \rho^{-1}(E) \subseteq \mathbf{R}$, so that E_0 is closed and has positive length. Let $\Omega_0 = \mathbf{C} \setminus E_0$. If $F(z)$ is holomorphic on Ω, then $F_0 = F \circ \rho$ satisfies $(\overline{\partial} - \mu\partial)F_0 = 0$ on Ω_0. Let $H^\infty(\Omega_0, \mu)$ denote the set of all such functions on Ω_0 that are also bounded, so that $F_0 \in H^\infty(\Omega_0, \mu)$ iff $F_0 = F \circ \rho$, $F \in H^\infty(\Omega)$.

One would like to solve $\overline{\partial} F = a$ on Ω with $F \in L^\infty(\Omega)$ for given $a(z)$. This is equivalent to solving $(\overline{\partial} - \mu\partial)F_0 = a_0$ on Ω_0, with $f_0 = F \circ \rho$ and $a_0 = (a \circ \rho)\rho_{\overline{z}}^{-1}(|\rho_z|^2 - |\rho_{\overline{z}}|^2)$, by (7.5). If we understand $\overline{\partial}$ well on Ω_0, then we should be able to solve this by summing the series

$$F_0 = (I - \overline{\partial}^{-1}\mu\partial)^{-1}\overline{\partial}^{-1}a_0 = \sum_{j=0}^{\infty}(\overline{\partial}^{-1}\mu\partial)^j\overline{\partial}^{-1}a.$$

This is analogous to the Cauchy integral case in §5. There "understanding $\overline{\partial}$ on Ω_0" was replaced by "understanding the Cauchy integral on $\mathbf{R}$", which we do.

The problem with this program is that we do not really understand $\overline{\partial}$ on Ω_0 for a general set $E_0 \subseteq \mathbf{R}$. For that matter, it is not even clear what the correct notion of Carleson measure is. Garnett and Jones have estimates for $\overline{\partial}$ when the data a_0 lives on a certain part of Ω_0, the part where the boundary E_0 is thick, but not when a_0 lives on all of Ω_0, which is what we need here.

For uniformly thick E_0, $\overline{\partial}$ is sufficiently well understood on Ω_0 so that the above program can be carried through, and we will indicate how that works in this section. Because the corresponding $E \subseteq \Gamma$ will also be uniformly thick (since ρ is bilipschitz), the estimates for $\overline{\partial}$ on Ω are already contained in the work of Jones [**Js3**]. However, this method is different, and it may be useful in the case of general E.

Suppose E_0 satisfies Carleson's uniform thickness condition, i.e., there is a $\gamma > 0$ such that $|E_0 \cap (x - t, x + t)| \geq \gamma t$ for all $x \in E_0$ and $t > 0$. Let λ be a measure on Ω_0. We say that λ is a Carleson measure relative to E_0 if there is a $C > 0$ such that for all $R > 0$ and $u \in E_0$, $|\lambda|(\{z \in \Omega_0 : |z - u| \leq R\}) \leq CR$. This definition is not reasonable if E_0 is not uniformly thick. Define $\delta(z) = \delta_{E_0}(z)$. We can define r-GCM's relative to E_0 just as in §1. As with the Cauchy integral we will need $1 < r < \infty$ because there are singular integrals around, but here we will assume $r > 2$ also in order to get $L^\infty(\Omega_0)$, not just L^∞ estimates for the boundary values. (Taking $r = 3$ is fine.)

PROPOSITION 9.1. *Suppose $E_0 \subseteq \mathbf{R}$ is uniformly thick and $2 < r < \infty$. Assume that $\mu \in L^\infty(\mathbf{C})$ is such that $|\mu|^2|y|^{-1}\,dx\,dy$ is an $r/2$-GCM with small norm and that $\|\mu\|_\infty$ is also small, how small depending on the thickness constant γ and* *also r. Then for all $a(z)$ such that $|a(z)|\,dx\,dy$ is an r-GCM relative to E_0 there is an $F \in L^\infty(\Omega_0)$ such that ∇F is locally integrable, $(\bar\partial - \mu\partial)F = a$ on Ω_0, and $\|F\|_\infty \le C\|a\|_{r\text{-GCM}}$.*

COROLLARY 9.2. *Same hypotheses as above. If $G \in H^\infty(\Omega_0)$ then there is an $F \in H^\infty(\Omega_0, \mu)$ such that*

$$\|F - G\|_{L^\infty(\Omega_0)} \le C\|G\|_\infty(\|\mu\|_\infty + \|\,|\mu|^2 + |y|^{-1}\,dx\,dy\|^{1/2}_{r/2\text{-GCM}}).$$

It seems reasonable to conjecture that the corollary holds for arbitrary closed sets $E_0 \subseteq \mathbf{R}$.

As pointed out in the beginning of the section, you can construct $G \in H^\infty(\Omega_0)$ by taking exponentials of Cauchy integrals. By the corollary you can find $F \in H^\infty(\Omega_0, \mu)$ which is a small perturbation of G if μ is small.

As in §5 there are two basic classes of examples of μ's. The first comes from starting with a chord-arc curve Γ with small constant, taking ρ as in (2.4), so that $\rho(\mathbf{R}) = \Gamma$, and taking μ to be the dilatation of ρ. Then, as before, $F \in H^\infty(\Omega_{0,\mu})$ iff $F \circ \rho^{-1} \in H^\infty(\Omega)$, $\Omega = \mathbf{C} \setminus E$, $E = \rho^{-1}(E_0) \subseteq \Gamma$. Proposition 9.1 lets you solve $\bar\partial H = a$ on Ω with $H \in L^\infty(\Omega)$ when $|a(z)|\,dx\,dy$ is an r-GCM on Ω.

For the other example, suppose $b \in \mathrm{BMO}(\mathbf{R})$ is real-valued and has small norm. Define $h(x) = \int_0^x e^{b(t)}\,dt$, a homeomorphism of $\mathbf{R}$ to itself. As pointed out in §5, we can extend h to a quasiconformal map ρ of $\mathbf{C}$ to itself whose dilatation satisfies the hypotheses of Proposition 9.1. In this case $E = \rho^{-1}(E_0)$ still lies on $\mathbf{R}$. Notice that E is uniformly thick, because E_0 is and because $h'(x)$ is an A_∞ weight.

In both cases one is perturbing E_0 a little bit, and Corollary 9.2 implies that $H^\infty(\Omega)$ is then perturbed only a little. This should be compared with the work of Coifman and Meyer [**CM2**, **3**]. There one perturbed the real line into a chord-arc curve with small constant, or perturbed it with a homeomorphism h as in the preceding paragraph and concluded that the corresponding Hardy spaces H^2_+ and H^2_- (or their BMO analogues) moved only a little.

We should point out that one can extend the class of allowable μ's in the proposition and the corollary in a manner analogous to that in §6.

Corollary 9.2 can be proved by showing that $|\mu G'|\,dx\,dy$ is an r-GCM, using Proposition 9.1 to solve $(\bar\partial - \mu\partial)H = \mu G'$, and then setting $F = G - H$.

Proposition 9.1 is proved using the techniques of this paper, §5 in particular, and the $\bar\partial$ technology of [**Js1**] and [**GJ**]. There is another important ingredient. You need to know that if you solve $\bar\partial F = a$ on Ω_0 with $a(z)$ an r-GCM relative to E_0 and $F \in L^\infty(\Omega_0)$, then you can control ∂F also. The way to do this is to use §4 to obtain a different solution F on $\mathbf{C}\setminus R$, i.e., $\bar\partial\tilde F = a$ on $\mathbf{C}\setminus R$, with $\tilde F$ satisfying BMO estimates and also with estimates on $\partial\tilde F$. Then $F - \tilde F$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}$ and also in BMOA, and so one can get a quadratic Carleson measure estimate on $\partial(F - \tilde F)$ on $\mathbf{C} \setminus \mathbf{R}$. (This argument should be compared with the argument at the end of §4 for estimating $\partial\bar\partial^{-1}a$.)

We shall not give the details of the proof of Proposition 9.1. There is one other thing which is useful for the proof, though, and which is standard fare for $\bar\partial$ problems on these types of domains. If $I = [a, b]$ is any interval on $\mathbf{R}$, let $\hat I = \{z \in \mathbf{C}: |y| \le \min(|z - a|, |z - b|)\}$. Thus $\hat I$ is the union of two triangles in $\mathbf{C}$ with base I, one pointing up, the other down. Let $\{I_i\}$ denote the disjoint intervals whose interiors make up $\mathbf{R} \setminus E_0$. For studying a $\overline{\partial}$ problem on Ω_0, it is useful to consider separately $\Omega_2 = \bigcup_i \hat{I}_i$ and $\Omega_1 = \Omega_0 \setminus \Omega_2$. Most of the action occurs on Ω_1, while on Ω_2 one can deal with each $\hat{I}_i$ separately and directly. To patch Ω_1 and Ω_2 together, you solve a $\overline{\partial}$ problem. In particular, when doing the details of the argument in the preceding paragraph for estimating ∂F, one should really deal with Ω_1 and Ω_2 separately.

10. Real variable versions. Some of the things we have done have real-variable analogues. Take §4, for example, especially the remarks at the end. That told us the following. Suppose $|a(z)|\,dx\,dy$ is an r-GCM, $1 < r < \infty$, relative to the real line, say. Then $\overline{\partial}^{-1}(a) = F - C(b)$, where F satisfies the conclusions of Proposition 4.1 on the upper and lower half-planes, F has jump $b \in \mathrm{BMO}(\mathbf{R})$ across $\mathbf{R}$, and $C(b)$ denotes the Cauchy integral of b. Thus $\partial\overline{\partial}^{-1}(a) = \partial F - C'(b)$, where $C'(b)$ denotes the derivative of $C(b)$ off $\mathbf{R}$, which has a natural extension to a distribution on all of $\mathbf{C}$. Also, ∂F denotes the distributional derivative of F on $\mathbf{C} \setminus \mathbf{R}$. This is an r-GCM, and in particular it is locally integrable, and so it has a trivial extension to a distribution on $\mathbf{C}$. Because $F - C(b)$ has no jump across $\mathbf{R}$, $\partial\overline{\partial}^{-1}(a)$ has no boundary term on $\mathbf{R}$.

The operator $T = \partial\overline{\partial}^{-1}$ is sometimes called the two-dimensional Hilbert transform. It is given also by the singular integral

$$Tf(z) = \mathrm{P.V.} \int_{\mathbf{C}} \frac{f(\varsigma)}{(z-\varsigma)^2}\, d\varepsilon\, d\eta, \tag{10.1}$$

except for an inessential multiplicative constant.

Now suppose we are working on $\mathbf{R}^{n+1}$, and that T is a convolution singular integral with kernel $K(\cdot)$ defined on $\mathbf{R}^{n+1}$, which for simplicity we assume is homogeneous of degree $-(n+1)$ and is C^∞ away from the origin. Let us write points in $\mathbf{R}^{n+1}$ as (x,s) or (y,t), $x, y \in \mathbf{R}^n$, $s, t \in \mathbf{R}$. Define two functions ψ^+ and ψ^- on $\mathbf{R}^n$ by $\psi^\pm(x) = K(x, \pm 1)$, and for $s \in \mathbf{R}$, $s \neq 0$, define

$$\psi_s(x) = (1/|s|^n)\psi^\pm(x/|s|),$$

where we fake $+$ or $-$ depending on whether s is positive or negative. By homogeneity, $(1/|s|)\psi_s(x) = K(x,s)$.

We need a cancellation condition on $K(\cdot)$, that $\int_{\mathbf{R}^n} \psi^\pm(x)\,dx = 0$. This corresponds to $\int_{S^n_\pm} K(z')dz' = 0$, where S^n_+ and S^n_- denote the unit sphere in $\mathbf{R}^{n+1}$ intersected with $t > 0$ and $t < 0$, and where we use $(r, z') \in \mathbf{R}_+ \times S^n$ to denote polar coordinates for $\mathbf{R}^{n+1}$. Indeed, for each $\varepsilon > 0$,

$$\begin{aligned} \int_{S^n} K(z')\,dz' &= (-2\log\varepsilon)^{-1} \int_\varepsilon^{1/\varepsilon} \int_{S^n_+} K(z')\,dz'\,r^{-1}\,dr \\ &= (-2\log\varepsilon)^{-1} \int_\varepsilon^{1/\varepsilon} \int_{S^n_+} K(rz')\,r^n\,dz'\,dr \end{aligned}$$

since $K(\cdot)$ is homogeneous of degree $-(n+1)$. Converting this to rectangular coordinates, and using the condition $\int \psi^\pm(x)\,dx = 0$ and also easy kernel estimates, one can show that the double integral remains bounded, and hence the whole thing tends to 0, as $\varepsilon \to 0$. In particular, the condition $\int \psi^{\pm} = 0$ implies that $K(\cdot)$ is a Calderón-Zygmund operator.

A measure λ on $\mathbf{R}^{n+1}$ is called a Carleson measure if there is a $C > 0$ so that for each $x_0 \in \mathbf{R}^n$ and $R > 0$,

$$|\lambda|(\{(x,s) \in R^{n+1} : |(x,s) - (x_0, 0)| \leq R\}) \leq CR^n.$$

We define r-GCM's just as when $n = 1$, in §1.

PROPOSITION 10.2. *Suppose $|a(x,s)|\,dx\,ds$ is an r-GCM, $1 < r < \infty$, and that T and $K(\cdot)$ are as above. Then Ta can be written as $\tilde{a}(x,s) + (1/|s|)\psi_s * b(x)$, where $|\tilde{a}(x,s)|\,dx\,ds$ is an r-GCM and $b \in \mathrm{BMO}(\mathbf{R}^n)$.*

In the case where T is given by (10.1), $(1/|s|)\psi_s * b(x)$ corresponds to $C'(b)$. In general, just as with the Cauchy integral,

$$\left|\frac{1}{|s|}\psi_s * b(x)\right|^2 |s|\,dx\,ds = |\psi_s * b(x)|^2 \frac{dx\,ds}{|s|}$$

is an ∞-GCM. (See p. 85 of [**Je**], for instance.)

Notice that although $(1/|s|)\psi_s * b(x)$ is locally integrable on $\mathbf{R}^{n+1} \setminus \mathbf{R}^n$, it is not generally on $\mathbf{R}^{n+1}$. However, it does have a natural extension to a distribution on $\mathbf{R}^{n+1}$. Also, suitable a priori assumptions on a force b to be nice enough so that $(1/|s|)\psi_s * b(x)$ is locally integrable on $\mathbf{R}^{n+1}$.

Let us prove the proposition. We shall content ourselves with the case where $a(x,s)$ lies in $L^2(\mathbf{R}^{n+1})$ to avoid technicalities, but the estimates will not depend on this assumption.

By definition,

$$Ta(x,s) = \mathrm{P.V.} \iint_{\mathbf{R}^{n+1}} K(x-y, s-t) a(y,t)\,dy\,dt.$$

This is not a Carleson measure because $K(\cdot)$ does not decay quickly enough at ∞. The $(1/|s|)\psi_s * b(x)$ term comes in as a correction to provide that decay.

Let $\varphi(x)$ be a C^∞ function on $\mathbf{R}^n$ with compact support such that $\int \varphi(x)\,dx = 1$, and define $\varphi_s(x) = (1/|s|^n)\varphi(x/|s|)$ for $s \in \mathbf{R}$, $s \neq 0$. Define

$$b(x) = \int_{\mathbf{R}} \int_{\mathbf{R}^n} \varphi_t(x-y) a(y,t)\,dy\,dt. \tag{10.3}$$

This is the balayage of $a(y,t)$, and it is well known (and easily verified) that $b \in \mathrm{BMO}(\mathbf{R}^n)$ when $|a(y,t)|\,dy\,dt$ is a Carleson measure. If $a(y,t)$ is not in $L^2(\mathbf{R}^{n+1})$, the integral needs to be defined modulo constants to make sense. See [**G**, pp. 229–230].

Let us write $Ta(x,s)$ as

$$\begin{aligned} &\mathrm{P.V.} \iint_{\mathbf{R}^{n+1}} [K(x-y, s-t) - (K(\cdot, s) * \varphi_t)(x-y)] a(y,t)\,dy\,dt \\ &\quad + \iint_{\mathbf{R}^{n+1}} (K(\varphi, s) * \varphi_t)(x-y) a(y,t)\,dy\,dt. \end{aligned} \tag{10.4}$$

Here $K(\cdot, s) * \varphi_t$ stands for the convolution in the $\mathbf{R}^n$ variables. The second term is equal to

$$\int_{\mathbf{R}^n} K(x-u, s) \int_{\mathbf{R}} (\varphi_t * a(\cdot, t))(u)\,dt\,du = \frac{1}{|s|}\psi_s * b(x).$$

Call the first term in (10.4) $a'(x,s)$. The proof that it is an r-GCM is very similar to the proof of part (a) of Proposition 4.1. We want to prove the analogue of (4.6) with ∂F replaced by a' in the definition of $H(z)$. As before, a' can be split into two pieces, G_1 and G_2. The G_1 part is the localized part, and it is controlled by proving the analogue of (4.7). This is proved using the boundedness of T on L^r and an easy estimate for $(K(\cdot,s)*\varphi_t)(x-y)$, similar to the proof of (4.7).

For the G_2 part, just as in §4, one uses Fubini and an estimate on the decay of the kernel. In our case we need to look at the kernel

$$L((x,s),(y,t)) = K(x-y,s-t) - (K(\cdot,s)*\varphi_t)(x-y)$$

when $|(x,s)-(y,t)| \geq |s|/2$. When $|(x,s)-(y,t)| \geq |s|/2$ and $|t| \leq \frac{1}{10}(|x-y|+|s|)$ we have that

$$|L((x,s),(y,t))| \leq C|t|(|x-y|+|s|+|t|)^{-n-2}. \tag{10.5}$$

This follows from

$$|\nabla_x K(x,s)| + \left|\frac{\partial}{\partial s}K(x,s)\right| \leq C(|x|+|s|)^{-n-2},$$

and it is helpful to write

$$L((x,s),(y,t)) = [K(x-y,s-t)-K(x-y,s)] + [K(x-y,s)-(K(\cdot,s)*\varphi_t)(x-y)].$$

When $|(x,s)-(y,t)| \geq |s|/2$ and $|t| \geq \frac{1}{10}(|x-y|+|s|)$, (10.5) still holds. This is verified by applying $|K(x,s)| \leq C(|x|+|s|)^{-n-1}$ to the first term in $L((x,s),(y,t))$, and by writing the second as $(1/|s|)\psi_s*\varphi_t(x-y)$, estimating it using $\int \psi(x)\,dx = 0$ and the smoothness of φ.

Using (10.5) one can estimate the G_2 part as in §4. That is how Proposition 10.2 is proved.

There are some n-dimensional problems in which Proposition 10.2 shows up that are analogous to what we did for $\bar\partial$ and the Cauchy integral in $\mathbf{C}$. We shall give two examples, connected to perturbations of the Dirac operator and the Laplacian.

Recall that the Clifford algebra $C_n(\mathbf{R})$ is the algebra generated by an identity $e_0 = 1$ and n other elements $e_1,\dots,e_n$ satisfying the relations $e_i^2 = -1$, $i = 1,2,\dots,n$, and $e_ie_j = -e_je_i$ if $1 \leq i,\ j \leq n$, $i \neq j$. Define the Dirac operator $\mathcal{D}$ by $\mathcal{D} = \sum_{i=0}^n e_{ii}$, where $\partial_i = \partial/\partial x_i$. (Unlike our earlier (x,s) notation, we now let $x = (x_0,x_1,\dots,x_n)$ denote an element of $\mathbf{R}^{n+1}$.) A $C_n(\mathbf{R})$-valued function f defined on a domain in $\mathbf{R}^{n+1}$ is Clifford analytic if $\mathcal{D}f = 0$. A basic reference for Clifford analysis is [**BDS**].

There is a natural analogue of the Cauchy kernel, by

$$E(x) = c_n|x|^{-n-1}\left(x_0e_0 - \sum_{i=1}^n x_ie_i\right).$$

This kernel gives the fundamental solution of $\mathcal{D}$, i.e., it is the kernel of $\mathcal{D}^{-1}$. There is also a Cauchy integral formula. As when $n=1$, if $f(u)$ is a $C_n(\mathbf{R})$-valued function on $\mathbf{R}^n$, and if

$$F(x) = E(f)(x) = \int_{\mathbf{R}^n} E(u-x)f(u)\,du, \qquad x \in \mathbf{R}^{n+1}\setminus\mathbf{R}^n,$$

is its Cauchy integral, then $\not{D}F = 0$ off $\mathbf{R}^n$ and the jump of F across $\mathbf{R}^n$ is given by f.

Now suppose that J is a differential operator given by $J = \sum_{i=0}^n \mu_i \partial_i$, where each μ_i is Clifford valued, $\mu_i \in L^\infty(\mathbf{R}^{n+1})$, and $|\mu_i(x)|^2 |x_0|^{-1}\, dx$ is an $r/2$-GCM, $1 < r < \infty$, with small norms. Consider the problem of solving $(\not{D} - J)F = 0$ on $\mathbf{R}^{n+1} \backslash \mathbf{R}^n$ and F has jump f across $\mathbf{R}^n$, $f \in \mathrm{BMO}(\mathbf{R}^n)$ given, with BMO estimates on the boundary values of F (and a maximal function estimate, and estimates on the gradient, etc.).

This can be done like the proof of Theorem 5.2. Let us sketch the details, allowing sloppiness we did not allow earlier. If we set $G = F - E(f)$, then we are reduced to solving $(\not{D} - J)G = JE(f)$, on $\mathbf{R}^{n+1} \setminus \mathbf{R}^n$ where G has no jump across $\mathbf{R}^n$. One can show that $|\nabla E(f)(x)|^2 |x_0|\, dx$ is an ∞-GCM, and that $|JE(f)|\, dx$ is an r-GCM, analogous to Lemma 5.3. We want to define G by

$$G = (\not{D} - J)^{-1}(JE(f)) = \not{D}^{-1}(I - J\not{D}^{-1})^{-1}(JE(f)).$$

For each i, the operator $T_i = \partial_i \not{D}^{-1}$ satisfies the conclusions of Proposition 10.2. Indeed, each T_i can be written as a linear combination of the identity and an operator that satisfies the hypotheses of the proposition. This is because each T_i can be expressed as a combination of second Riesz transforms, by looking at the Fourier transform side, and the second Riesz transforms can be written as such a linear combination, as is shown later in this section. The conditions on the μ_i's imply that $J\not{D}^{-1}$ takes r-GCM's to r-GCM's, and with small norm. We did the analogous computation for $\mu T = \mu \partial \overline{\partial}^{-1}$ in §5. Thus $(I - J\not{D})^{-1}$ takes r-GCM's to r-GCM's. Hence G is given by $\not{D}^{-1}$ of an r-GCM, and one can show that this forces it to have BMO boundary values. Indeed, one computes $\not{D}^{-1}$ using the Cauchy integral $E(x)$. When computing the boundary values on $\mathbf{R}^n$ of $\not{D}^{-1}$ of an r-GCM, the e_0 component of $E(x)$ gives rise to a balayage of the Carleson measure, and hence lives in BMO, while the e_i components of $E(x)$ give rise to Riesz transforms of that balayage, and so they also are in BMO. (Of course, these things must be interpreted modulo constants to make the integrals converge at ∞.)

The problem with all this is that the perturbation $\not{D} - J$ of $\not{D}$ does not seem to arise as naturally in higher dimensions as it does in $\mathbf{R}^2$, e.g., induced by bilipschitz changes of variables. If instead of requiring that $|\mu_i|^2 |x_0|^{-1}\, dx$ be an $r/2$-GCM, we could get away with requiring that $|\nabla \mu_i|^2 |x_0|\, dx$ be an $r/2$-GCM, with μ_i still Clifford valued, then the perturbation $\not{D} - J$ would arise more naturally. For example, such perturbations come about when considering the n-dimensional version of (2.1). By §6, we can allow such perturbations when $n = 1$. Unfortunately, there we used integration by parts tricks that seem to be peculiar to $n = 1$. (They do not work for the Clifford algebra because of noncommutativity.)

It would be nice if one could make something like this work more naturally in $\mathbf{R}^{n+1}$, $n > 1$. Notice that in the above argument we did not use all the information that Proposition 10.2 gave us. We used the fact that $\psi_t * b(x)$ satisfies a quadratic Carleson measure condition, but we did not use the cancellation it has. When $n = 1$, those integrations by parts did take advantage of the cancellation. One would like to find some substitute in $\mathbf{R}^n$.

Now let us consider perturbations of the Laplacian. The estimates we get are not new; see [**FJK** and **Dh**] for much more general results. The point here is more the connection with what we have been doing. Let ∇^* denote the divergence operator, and let $A(x)$ be a bounded $(n+1)\times(n+1)$ matrix-valued function on $\mathbf{R}^{n+1}$ such that $\|A(x,s)\|_\infty$ is small and $|A(x,s)|^2|s|^{-1}\,dx\,ds$ is an $r/2$-GCM with small norm. Consider the operator $\nabla^*(I+A)\nabla = \Delta + \nabla^* A\nabla$. We want to look at this similar to the way we looked at $\overline{\partial}-\mu\partial$ and $\mathcal{D}-J$.

Suppose $F(x,s)$ is harmonic on $\mathbf{R}^{n+1}\setminus\mathbf{R}^n$ and has BMO boundary values on $\mathbf{R}^n$ from above and below. We do not require that these two sets of boundary values coincide. We do require that F be the Poisson integral of its boundary values on $\mathbf{R}^{n+1}_+$ and $\mathbf{R}^{n+1}_-$. Let us show that we can find G on $\mathbf{R}^{n+1}$ such that G has no jump across $\mathbf{R}^n$, $\nabla^*(I+A)\nabla G = -(\nabla^* A\nabla)F$ on $\mathbf{R}^{n+1}\setminus\mathbf{R}^n$, and G has BMO boundary values on $\mathbf{R}^n$ that are small in norm compared to F's.

This problem should be compared to the problems we considered earlier, like $(\overline{\partial}-\mu\partial)H = \mu C'(g)$ and $(\mathcal{D}-J)H = JE(f)$. Notice that $G+F$ satisfies $\nabla^*(I+A)\nabla(G+F)=0$ on $\mathbf{R}^{n+1}\setminus\mathbf{R}^n$ and the boundary values of $G+F$ differ from those of F only slightly. This implies, for example, BMO estimates for the Dirichlet problem for $\nabla^*(I+A)\nabla$ on $\mathbf{R}^{n+1}_+$ or $\mathbf{R}^{n+1}_-$, by iteration.

It follows from Fefferman-Stein that $|\nabla F|^2|s|\,dx\,ds$ is an ∞-GCM. Thus (like in Lemma 5.3), $|A\nabla F|\,dx\,ds$ is an r-GCM. Hence we may as well consider the more general problem of $\nabla^*(I+A)\nabla G = \nabla^*\alpha$ on $\mathbf{R}^{n+1}\setminus\mathbf{R}^n$ where α is vector valued, $|\alpha(x,s)|\,dx\,ds$ is an r-GCM, $\nabla^*\alpha$ is defined distributionally, and where G has no jump and BMO$(\mathbf{R}^n)$ boundary values.

Let $\Lambda = \Delta^{1/2}$, $R=\nabla\Lambda^{-1}$, the vector of Riesz transforms, and $R^* = \nabla^*\Lambda^{-1}$. Rewrite $(\Delta+\nabla^* A\nabla)G = \nabla^*\alpha$ as $\Lambda(I+R^*AR)\Lambda G = \Lambda R^*\alpha$, so that

$$\begin{aligned} G &= \Lambda^{-1}(I+R^*AR)^{-1}R^*\alpha \\ &= \sum_{j=0}^{\infty}\Lambda^{-1}(-1)^j(R^*AR)^jR^*\alpha = \sum_{j=0}^{\infty}\Lambda^{-1}(-1)^jR^*(ARR^*)^j\alpha. \end{aligned}$$

Note that all of this makes sense if we assume a priori that $\alpha\in L^2(\mathbf{R}^{n+1})$, since $\|A\|_\infty$ is small. We want to get BMO estimates that do not depend on that assumption.

Let us show that the conclusion of Proposition 10.2 holds for RR^*. We need to show that if $P(x)$ is a polynomial on $\mathbf{R}^{n+1}$ that is homogeneous of degree 2, and if $(Tf)^\wedge(\xi) = (P(\xi)/|\xi|^2)\hat{f}(\xi)$, then T satisfies the conclusions of the proposition. If $P(x)=|x|^2$, then $T=I$, and this is trivial. Thus we may assume that $P(x)$ is harmonic. By Theorem 5, p. 73 of **[8]** we know that

$$TF(x) = \text{P.V.}\int_{\mathbf{R}^{n+1}} K(x-y)f(y)\,dy,$$

where $K(x) = c_n(P(x)/|x|^{n+3})$.

We must show that this kernel satisfies the cancellation condition at the beginning of this section. As before, we rename the (x,s) coordinates as $(x_0,x') = (x_0,x_1,\dots,x_n)$, where $x_0=s$. If $P(x)=x_ix_j$, $i\neq j$, the cancellation condition is clear. Thus we can assume that $P(x)=\sum_{i=0}^n\beta_ix_i^2$ where $\sum_{i=0}^n\beta_i=0$. If $\beta_0=0$, then it is easy to check the cancellation condition using the symmetry among the other x_i's. This leaves $P(x)=nx_0^2-|x'|^2$. In this case

$$K(x_0,x') = \frac{n|x_0|^2-|x'|^2}{(|x_0|^2+|x'|^2)^{(n+3)/2}},$$

and the cancellation condition becomes

$$\int_{\mathbf{R}^n} \frac{n - |x'|^2}{(1+|x'|^2)^{(n+3)/2}}\, dx' = 0.$$

In polar coordinates this reduces to

$$\int_0^\infty \frac{n - r^2}{(1+r^2)^{(n+3)/2}} r^{n-1}\, dr = \int_0^\infty \frac{d}{dr}\left(\frac{r^n}{(1+r^2)^{(n+1)/2}}\right) dr = 0.$$

Thus the conclusion of Proposition 10.2 holds for R^*R, so that AR^*R takes r-GCM's to r-GCM's with small norm if A is small. Hence $\sum_{j=0}^{\infty}(AR^*R)^j\alpha$ defines an r-GCM that we will call γ. Define

$$G = \lambda^{-1}R^*\gamma = \lambda^{-2}\nabla^*\gamma = \Delta^{-1}\nabla^*\gamma = \Delta^{-1}\left(\sum_{i=0}^{n}\frac{\partial}{\partial x_i}\gamma_i\right),$$

where γ_i denotes the ith component of the vector-valued function γ. We know that Δ^{-1} is given by convolution with $c_n|x|^{-(n+1)+2}$, except when $n = 2$, which we leave to the reader, so that

$$G(x) = c_n \int_{\mathbf{R}^{n+1}} \sum_{j=0}^{n} \frac{\partial}{\partial x_j}\frac{1}{|x-y|^{n-1}}\gamma_j(y)\, dy.$$

We want to show that on $\mathbf{R}^n$, i.e., when $x_0 = 0$, this defines a BMO function. For $j = 0$, $(\partial/\partial x_0)(1/|x-y|^{n+1})$ gives precisely the Poisson kernel when $x_0 = 0$. Thus the $j = 0$ piece is just the balayage of the Carleson measure $\gamma_0(y)$, and so lives in $\mathrm{BMO}(\mathbf{R}^n)$. For the other j's,

$$\frac{\partial}{\partial x_j}\frac{1}{|x-y|^{n-1}} = c_n\frac{(x_j - y_j)}{|x-y|^{n+1}}$$

is a conjugate Poisson kernel, and so you get the Riesz transform of the balayage of a Carleson measure, which is still in BMO. Thus we get the desired BMO estimates for the boundary values of G.

Unfortunately, as in the case of perturbations of $\not{D}$, these perturbations of Δ do not arise naturally from changes of variables when $n > 1$. The reason is the following. The condition on $A(x,s)$ forces it to vanish when $s = 0$. If $\rho\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is quasiconformal, the condition that the restriction of its maximal dilatation K to $\mathbf{R}^n$ be identically 1 is much stronger when $n > 1$ than when $n = 1$. Indeed, when $n = 1$, Proposition 2.5 gives lots of such mappings.

11. Analogues for L^p, $1 < p < \infty$, instead of BMO. Using the tent spaces of Coifman, Meyer, and Stein [**CMS**] we can obtain L^p analogues of our earlier estimates. Let us recall a few of the definitions and properties. We restrict ourselves to the upper half-plane U since the extension to chord-arc domains is easily made, using a bilipschitz mapping.

For $x \in \mathbf{R}$ let $N(x)$ be the cone $\{(y,t) \in U\colon |x-y| \le t\}$. Given q, $0 < q \le \infty$, and a measurable function $f(y,t)$ defined on U, set

$$A_q(f) = \left(\iint_{N(x)} |f(y,t)|^q t^{-2}\, dy\, dt\right)^{1/q}$$

if $q < \infty$ and $A_\infty(f) = \sup\{|f(y,t)|: (y,t) \in U\}$. For $0 < p < \infty$ define the tent space $T_q^p = \{f: A_q(f) \in L^p(\mathbf{R})\}$.

Another important functional is C_q, $0 < q < \infty$, defined by

$$C_q(f)(x) = \sup_{I \ni x} \left\{ \frac{1}{|I|} \iint_{\hat{I}} |f(y,t)|^q \frac{dy\,dt}{t} \right\}^{1/q},$$

where I is any interval containing x and $I = \{(y,t): y \in I,\ 0 < t < |I|\}$. When $0 < q < p < \infty$, $\|C_q(f)\|_p$ and $\|A_q(f)\|_p$ are equivalent norms (Theorem 3 of [**CMS**]). With this characterization of T_q^p the natural endpoint space T_q^∞ is $\{f: |f(y,t)|^q t^{-1}\,dy\,dt$ is a Carleson measure$\}$, and that is how T_q^p is defined. With these definitions, there is a natural duality between T_q^p and $T_{q'}^{p'}$, $1 \le p, p', q, q' \le \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

For L^p the space $\{f: tf \in T_1^p\}$ plays the same role that Carleson measures do for BMO. For example, it follows immediately from duality that the balayage of an element of T_1^p lies in $L^p(\mathbf{R})$, $1 < p < \infty$. Another example is the result of Varopoulos, that if F is defined on U and $|\nabla F|\,dx\,dy$ is a Carleson measure, then F has radial boundary values in BMO($\mathbf{R}$). By the same proof, if $t|\nabla F| \in T_1^p$ then the radial boundary values f satisfy $f^\#(x) \le CC_1(t|\nabla F|)(x)$, where $f^\#$ denotes the Fefferman-Stein sharp function of f. Thus $\|f\|_p \le C_p\|f^\#\|_p \le C_p\|t|\nabla F|\,\|_{T_1^p}$ if f is not too bad at ∞.

Similarly, $\{tg \in T_2^\infty\} = \{g: |g|^2 t\,dy\,dt$ is a Carleson measure$\}$ is to BMO as $\{g: tg \in T_2^p\}$ is to L^p: both characterize the gradient of the Poisson integral of a function in BMO or L^p.

As with Carleson measures, we define the good tent spaces $r - GT_q^p = \{f: \tilde{f} \in T_q^p\}$, where

$$\tilde{f}(z) = \tilde{f}_\alpha(z) = \left(|B_z|^{-1} \int_{B_z} |f(w)|^r\,du\,dv\right)^{1/r},$$

$B_z = \{w: |w - z| \le \alpha\delta(z)\}$, $0 < \alpha < 1$. As in §1, this condition does not depend on α, and when $r = q$, $q - GT_q^p = T_q^p$.

There is a tent-space L^p analogue of Proposition 4.1. For simplicity we continue to restrict ourselves to the upper half-plane U. Suppose $ya(z)$ lies in $r - GT_1^p$, $1 < p < \infty$ and $1 < r < \infty$. Define

$$F(z) = \iint K(z,\varsigma)a(\varsigma)\,d\xi, d\eta, \tag{11.1}$$

where $K(z,\varsigma)$ is given by (4.3). Unlike (4.4), we do not need to subtract off a constant; the integral converges at ∞ because p is finite.

The function $f(x)$ obtained by formally replacing z by $x \in \mathbf{R}$ in (11.1) lies in L^p, as can be shown using duality. (In fact, f is the balayage of a.) One can show that $y\partial F(z)$ lies in $r - GT_1^p$ by estimating $C_1(y|\partial F(z)|)$, similar to the BMO case. A Varopoulos type argument shows that F has radial boundary values in L^p, and, analogous to (4.2),

$$(Mf)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I \sup_{0<t<|I|} |F(x+it)|\,dx \tag{11.2}$$

lies in L^p. (Unlike (4.2), we can get rid of C_I, because p is finite.) When $r > 2$, F is continuous, has nontangential maximal function in L^p, and the boundary values exist nontangentially.

§5 should undergo similar changes. For Lemma 5.3, we need that if $g \in L^p$ and μ is as before (i.e., $\mu \in r/2 - GT_2^\infty$), then $t\mu C'(g) \in r - GT_i^p$. For this we need that $f \in T_2^p$ and $a \in T_2^\infty$ imply $af \in T_1^p$ if $1 < p < \infty$. This is a variation of remark (b) following Theorem 3 in §6 in [**CMS**], and it is proved using the C_1 functional and the duality between T_2^1 and T_2^∞ (part (a) of Theorem 1, §4 in [**CMS**]).

§6 can also be suitably modified. We do not have to change the definition of $M_{2,r}$, but U_r and U_r' need to be replaced by $U_{p,r}$ and $U_{p,r}'$ with L^p taking the place of BMO, and tent spaces replacing Carleson measures. Similarly, for §10, in Proposition 10.2, r-GCM should be replaced by $\{f : tf \in r - GT_1^p\}$ and BMO by L^p. We omit the details.

REFERENCES

[**BA**] A. Beurling and L. Ahlfors, *The boundary correspondence under quasiconformal mappings*, Acta. Math. **96** (1956), 125–142.

[**BDS**] F. Brackx, R. Delanghe, and F. Sommer, *Clifford analysis*, Pitman Press, 1982.

[**Ca**] A. P. Calder6n, *Cauchy integrals on Lipschitz graphs and related operators*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1324–1327.

[**Cr**] L. Carleson, *On H^∞ in multiply connected domains*, Conference on Harmonic Analysis in honor of Antoni Zygmund, Vol. 2, Wadsworth, Belmont, Calif., 1983, pp. 349–372.

[**CM1**] R. Coifman and Y. Meyer, *Au-dela des opérateurs pseudo-différentiels*, Astérisque **57** (1978).

[**CM2**] ——, *Le théorème de Calderón par les méthodes de variable reele*, C. R. Acad. Sci. Paris Sér. A **289** (1979), 425–428.

[**CM3**] ——, *Une généralisation du théorème de Calderón sur l'integral de Cauchy*, Fourier Analysis (Proc. Conf., El Escorial, Spain, 1979) (M. de Guzman and I. Peral, eds.), 1980.

[**CMM**] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Ann. of Math. (2) **116** (1982), 361–387.

[**CMS**] R. Coifman, Y. Meyer, and E. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62** (1985), 304–335.

[**Dh**] B. E. J. Dahlberg, *On the absolute continuity of elliptic measures*, preprint.

[**Dv**] G. David, *Opérateurs intégraux singuliers sur certaines courbes du plan complex*, Ann. Sci. Ecole Norm. Sup. **17** (1984), 157–189.

[**FJK**] E. Fabes, D. Jerison, and C. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic harmonic measure*, Ann. of Math. (2) **119** (1984), 121–141.

[**G**] J. B. Garnett, *Bounded analytic functions*, Academic Press, 1981.

[**GJ**] J. B. Garnett and P. W. Jones, *The Corona theorem for Denjoy domains*, Acta Math. **155** (1985), 29–40.

[**Je**] J. L. Journé, *Calderón-Zygmund operators, pseudo-differential operators, and the Cauchy integral of Calderón*, Lecture Notes in Math., vol. 994, Springer-Verlag, Berlin and New York, 1983.

[**Js1**] P. W. Jones, *L^∞ estimates for the $\bar\partial$ problem in a half-plane*, Acta Math. **150** (1983), 137–152.

[**Js2**] ——, *Homeomorphisms of the line that preserve* BMO, Ark. Mat. **21** (1983), 229–231.

[**Js3**] ——, *On L^∞ solutions to $\bar\partial$ in domains with thick boundary* (to appear).

[**JK**] D. Jerison and C. Kenig, *Hardy spaces, A_∞, and singular integrals on chord-arc domains*, Math. Scand. **50** (1982), 221–247.

[**M**] D. Marshall, *Removable sets for bounded analytic functions*, Linear and Complex Analysis Problem Book, Lecture Notes in Math., no. 1043, Springer-Verlag, Berlin and New York, 1984, pp. 485–490.

[**P1**] Chr. Pommerenke, *Univalent functions*, Vanhoeck and Ruprecht, Gottingen, 1975.

[**P2**] ——, *Boundary behaviour of conformal mappings*, Aspects of Contemporary Complex Analysis (D. Brannen and J. Clunie, eds.), Academic Press, 1980.

[S] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[Se1] S. Semmes, *The Cauchy integral, chord-arc curves, and quasiconformal mappings*, Proc. Bieberbach Conf. (Purdue University, 1985) (A. Baernstein, P. Duren, A. Marden, and D. Drasin, eds.), Math. Surveys, no. 21, Amer. Math. Soc., Providence, R.I., 1986.

[Se2] ——, *Quasiconformal mappings and chord-arc curves*, Trans. Amer. Math. Soc. **302** (1988), 233–263.

[V1] N. Varopoulos, BMO *functions and the* $\overline{\partial}$ *equation*, Pacific J. Math. **71** (1977), 221–273.

[V2] ——, *A remark on* BMO *and bounded harmonic functions*, Pacific J. Math. **714** (1977), 257–259.

[T1] P. Tukia, *The planar Shonflies theorem for Lipschitz maps*, Ann. Acad. Math. Sci. Fenn. Ser. A I Math. **5** (1980), 49–72.

[T2] ——, *Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane*, Math. Sci. Fenn. Ser. AI Math. **6** (1981), 89–94.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77251