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A BROUWER TRANSLATION THEOREM FOR FREE HOMEOMORPHISMS

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ABSTRACT. We prove a generalization of the Brouwer Translation Theorem which applies to a class of homeomorphisms (free homeomorphisms) which admit fixed points, but retain a dynamical property of fixed point free orientation preserving homeomorphsims. That is, if $h\colon M^2 \to M^2$ is a free homeomorphism where M^2 is a surface, then whenever D is a disc and $h(D) \cap D = \varnothing$, we have that $h^n(D) \cap D = \varnothing$ for all $n \neq 0$.

THEOREM. *Let h be a free homeomorphism of S^2, the two-sphere, with finite fixed point set F. Then each $p \in S^2 - F$ lies in the image of an embedding $\phi_p\colon (R^2, 0) \to (S^2 - F, p)$ such that:*

(i) *$h\phi_p = \phi_p\tau$, where $\tau(z) = z + 1$ is the canonical translation of the plane, and*

(ii) *the image of each vertical line under ϕ_p is closed in $S^2 - F$.*

The class of free homeomorphisms has been introduced and studied by M. Brown in connection with the dynamics of surface homeomorphisms.

Introduction. L. E. J. Brouwer, from 1909 to 1919, formulated and presented the Brouwer Plane Translation Theorem [**2**–**6**, **9**, **10**]. In modern terminology it can be stated thusly.

THEOREM. *Let h be a fixed point free orientation preserving self homeomorphism of R^2, the plane. Then each $p \in R^2$ lies in the image of an embedding $\phi_p\colon (R^2, 0) \to (R^2, p)$ such that:*

(i) *$h\phi_p = \phi_p\tau$, where $\tau(z) = z + 1$ is the canonical translation of the plane, and*

(ii) *the image of each vertical line under ϕ_p is closed in R^2.*

In his first proof of this theorem [**3**–**6**] he assumed that a simple curve bounded only two domains. Later [**7**] he discovered the Lakes of Wada continuum and realized the error in his proof. The second proof [**9**] utilized the Brouwer Translation Arc Lemma, which was proved using the No Retraction Theorem [**8**] and the concept of index of a fixed point. However, this proof contained serious gaps.

Since 1920 a succession of authors [**1**, **13**–**16**, **17**, **18**] have presented proofs of this theorem, each either correcting errors in the previous proof or embarking upon a simplification for the proof. For a more thorough discussion of this problem, the reader should consult [**11**] which presents the history of Brouwer's work and [**16**] for the history of the plane translation theorem.

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In this paper, we use the notion of *free homeomorphism*, introduced and developed by Brown [12]. A homeomorphism $h\colon M^2 \to M^2$, for an orientable surface M^2, is said to be *free*, if for each disc D, with the property that $h(D) \cap D = \emptyset$, it follows that $h^n(D) \cap D = \emptyset$ for all $n \neq 0$. As a consequence of the Brouwer Translation Arc Lemma, an orientation preserving fixed point free homeomorphism of the plane is a free homeomorphism. It should also be noted (cf. [12]) that since there exist no free homeomorphisms for M^n, n-manifolds, for $n > 2$, the concept of free homeomorphism is intimately related to the dynamics of surfaces. The interested reader should consult Brown [12] for a detailed study of free homeomorphisms.

We present a version of the Brouwer Translation Theorem for free homeomorphisms of S^2, the two-sphere, with finite fixed point set, which generalizes the standard Brouwer Plane Translation Theorem.

Let h be a fixed point free, orientation preserving homeomorphism of the plane. Then h has a unique extension to a homeomorphism $\tilde{h}$ of S^2 with precisely one fixed point K, where K is the point of compactification. Since the Euler characteristic of S^2 is two, the fixed point index of K is also two. By a result of Brown [12], $\tilde{h}$ and also h must be free homeomorphisms, and thus our theorem can be applied to obtain the classical Brouwer Plane Translation Theorem.

§1 of this paper contains the definitions and notation used in the sequel. In §2 we collect the work of Brown [12] on free homeomorphisms which will be used in the proof of the main theorem. In §3 we present the proof of this theorem.

I would like to express my greatest appreciation to my advisor Mort Brown for suggesting this problem, for his advice and his continued support throughout this research. I am also grateful to Jim Kister for clarifying the many subtleties in the problem and his ongoing interest in this work.

This paper contains work based upon the author's Ph.D. dissertation at the University of Michigan.

1. Preliminary definitions and notation. The unit interval will be denoted by I.

Let R^n denote euclidean n-space endowed with the standard norm $\|x\|$.

Let S^2 be the unit sphere $\{x \in R^3 \mid \|x\| = 1\}$ embedded in R^3. The distance $d(p, q)$ between two points p, q in S^2 is the length of the geodesic in S^2 from p to q using the metric inherited from R^3.

The notation $[p, q]$ (respectively (p, q)) will denote a homeomorph of the closed (resp. open) interval I with endpoints p and q in S^2.

A ball $B(p, r)$ in S^2 centered at $p \in S^2$ of radius r is the set $\{x \in S^2 \mid d(x, p) \leq r\}$. A disc D in S^2 is a homeomorph of $B(p, 1)$.

We will use the conventions $\operatorname{int}(X)$, $\operatorname{bd}(X)$, and $\overline{X}$ to denote the interior, boundary and closure of a set X in S^2 respectively.

A homeomorphism $h\colon S^2 \to S^2$ is said to be *free* [12] if h is an orientation preserving homeomorphism such that if D is any disc in S^2 and $D \cap h(D) = \emptyset$, then $D \cap h^n(D) = \emptyset$ for all $n \neq 0$.

As we will be considering a specific free homeomorphism, $h\colon S^2 \to S^2$, the notation X^n for $h^n(X)$ will be employed in the sequel, with the convention that $X^0 = X$. For the same reason let F denote the fixed point set of h.

The following two concepts are central to the proof of the main theorem.

DEFINITION. Let $h\colon S^2 \to S^2$ be an orientation preserving homeomorphism of the two-sphere. A disc $D \subset S^2$ is said to be *critical* if and only if $D \cap D^1 \neq \varnothing$ and $D \cap D^1 \subset \operatorname{bd}(D)$. The disc D is *at most critical* if D is critical or $D \cap D^1 = \varnothing$. A disc D is *not at most critical* if $\operatorname{int}(D) \cap \operatorname{int}(D^1) \neq \varnothing$, that is, D *overlaps* its image.

We will concentrate upon the following sets as candidates for critical discs.

DEFINITION. A set $D \subset S^2$ is a *partial disc* whenever:

(i) $D = \overline{D_1 - D_2} \neq \varnothing$, where D_1, D_2 are balls in S^2,

(ii) $\operatorname{center}(D_1) \notin D_2$, and

(iii) $D \cap D_2$ is a nonempty, connected arc of $\operatorname{bd}(D_2)$.

Let the *radius* of D and the *center* of D be those of D_1.

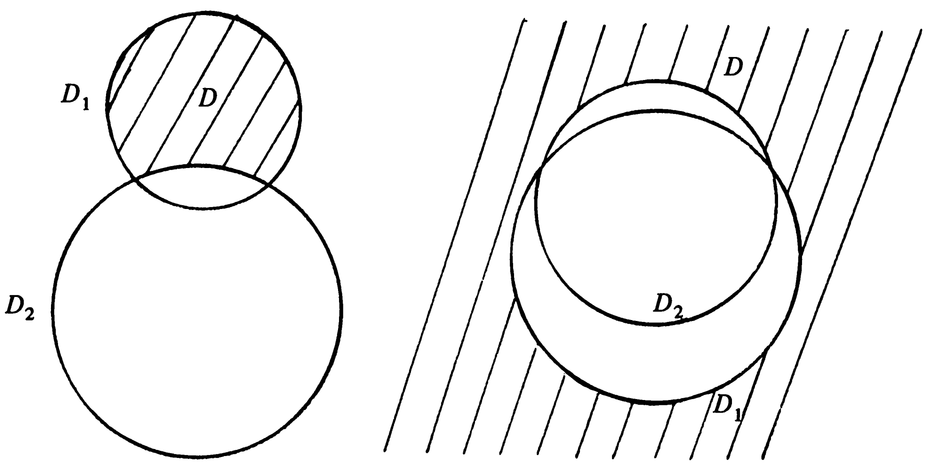

FIGURE 1

We will say that the partial disc D is situated *along* the ball D_2. Given an arc $[a, b]$ on $\operatorname{bd}(D_2)$, we can construct a partial disc D along D_2 which intersects D_2 only at $[a, b]$. This partial disc is then determined by the distance from $\operatorname{center}(D_2)$ to $\operatorname{center}(D)$. When this distance and the arc $[a, b]$ are specified, we denote the partial disc by $D(a, b)$.

For any homeomorphism $h\colon S^2 \to S^2$ and point $p \notin F$, there exists a small ball $B(p)$ centered at p such that $B(p) \cap B^1(p) = \varnothing$. We can construct a critical disc by enlarging the radius of $B(p)$ to get a ball $\tilde{B}(p)$ which intersects its image only in points along the boundary of $\tilde{B}(p)$.

We can extend this process to any small disc D containing p by appealing to the Schoenflies Theorem and radially extending the equivalent ball $\tilde{B}(p)$. This gives us a critical disc $\tilde{D}$, by conjugation, containing p.

In the sequel we will also be concerned with critical partial discs which do not require the inclusion of a specified point. To accomplish this we first choose a small partial disc D_0 along a given ball B such that $D_0 \cap D_0^1 = \varnothing$. We will then construct a continuous family of partial discs D_t, $t \in I$, along B such that:

(i) $\operatorname{radius}(D_t) > \operatorname{radius}(D_s)$, $t > s$,

(ii) $D_1 \cap D_1^1 \neq \varnothing$,

(iii) $D_s \cap D_t \neq \varnothing$ for all $s, t \in I$, and

(iv) if $x \in \operatorname{int}(D_t)$, $x \notin D_s$ and $t > s$, then there exists u with $t > u > s$ such that $x \in \operatorname{bd}(D_u)$.

These conditions ensure the existence of a critical partial disc D_t for some $t \in I$.

2. Free homeomorphisms and translation lines. We will list several properties of free homeomorphisms which will be used in the sequel. The reader is referred to Brown [**12**] for proofs of these and other properties.

DEFINITION. Let $h\colon S^2 \to S^2$ be a free homeomorphism. A half-open, half-closed arc $\alpha = [x, x^1)$ is a *translation arc* if $\alpha \cap h(\alpha) = \varnothing$. The invariant set $L_\alpha = \bigcup_{i=-\infty}^{\infty} h^i(\alpha)$ is called the *translation line* (generated by α).

PROPOSITION 1 [**12**, THEOREM 4.7]. *L_α is homeomorphic to the real line. (L_α is not necessarily a closed line however.)*

Thus we may naturally order L_α so that if $p \in L_\alpha$ then $p < p^1$.

PROPOSITION 2 [**12**, THEOREM 4.6]. (*Spanning property for translation arcs.*) *Let h be a free homeomorphism of S^2, L a translation line for h, and let $\alpha = [p, q)$ be a translation arc in L which generates L. Let C be a continuum satisfying* (1) *$C \cap \alpha = \varnothing$ and* (2) *C intersects both components of $(L - \alpha)$. Then $C \cap h(C) \neq \varnothing$.*

If C is a disc, $L \cap C$ is an arc $[p, q]$ on $\operatorname{bd}(C)$ *and $q > p^1$, then* $\operatorname{int}(C) \cap h(\operatorname{int}(C)) \neq \varnothing$. *Thus C overlaps its image and is therefore not critical (nor at most critical).*

PROPOSITION 3 [**12**, LEMMA 5.1]. *Let h be a homeomorphism of S^2 and let E be a disc such that $h(E) \subset \operatorname{int}(E)$. Then h is not free.*

PROPOSITION 4 [**12**, THEOREM 5.1]. *Let h be a free homeomorphism of S^2 and let D be a disk in S^2 such that $h(D) \subset D$. Then D has a fixed point on* $\operatorname{bd}(D)$.

The final proposition we need will be supplied with Brown's proof, as we will require a modification of it in our proof of the main theorem.

PROPOSITION 5 [**12**, LEMMA 4.1]. (*Translation arcs exist.*) *Let h be a free homeomorphism of S^2. If p and p^1 are in the same component of $S^2 - F$ then there exists a translation arc from p to p^1.*

PROOF. Let D be a small disk in $S^2 - F$ containing p in its interior. Enlarge the disk D by a continuous family of disks $D_t \subset S^2 - F$, $0 \leq t \leq 1$, such that $D_0 = D$, $D_t \subset \operatorname{int}(D_{t^1})$ whenever $t < t^1$, $D_t \cap h(D_t) = \varnothing$ for all $t < 1$, $\operatorname{int}(D_1) \cap h(\operatorname{int}(D_1)) = \varnothing$, and $\operatorname{bd}(D_1) \cap h(\operatorname{bd}(D_1)) \neq \varnothing$. Let $z \in \operatorname{bd}(D_1) \cap h(\operatorname{bd}(D_1))$, and let $h(y) = z$. So $y \in \operatorname{bd}(D_1)$.

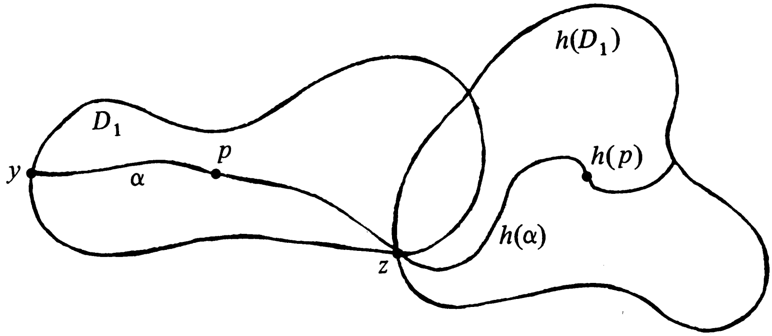

FIGURE 2

Let $\alpha = [y, z)$ be an arc in D_1 such that $p \in \alpha - y \subset \mathrm{int}(D_1)$. Then α is a translation arc containing p. So also is the union of the subarc $[p, z]$ of α with $[z, p^1)$ of $h(\alpha)$.

LEMMA 1. *If D is a critical disc and $D \cap F = \emptyset$ then $D^i \cap D^j = \emptyset$ for $|i - j| > 1$.*

PROOF. Without loss of generality assume that $D \cap D^n \neq \emptyset$ with $n > 1$ and that D^n does not intersect D^1 through D^{n-2}. Pick $q \in \mathrm{bd}(D) \cap \mathrm{bd}(D^n)$. Hence $q^{-n} \in \mathrm{bd}(D)$. Let $a = [q, q^{-n}]$ be an arc such that $\mathrm{int}(a) \subset \mathrm{int}(D)$. Hence

$$\begin{aligned} a \cap a^1 &= (q \cup q^{-n}) \cap (q^1 \cup q^{-n+1}) \\ &= (q \cap q^1) \cup (q \cap q^{-n+1}) \cup (q^{-n} \cap q^1) \cup (q^{-n} \cap q^{-n+1}). \end{aligned}$$

Since $D \cap F = \emptyset$, then $q \cap q^1 = \emptyset = q^{-n} \cap q^{-n+1}$. Since $D^n \cap D^1 = \emptyset$, then $q \cap q^{-n+1} = \emptyset$. Since h has no periodic points, $q^1 \neq q^{-n}$. Hence $a \cap a^1 = \emptyset$. But $q \in a \cap a^n$; hence the homeomorphism is not free, a contradiction.

LEMMA 2. *For each point $p \in S^2 - F$ and critical disc D with $p \in \mathrm{int}(D)$ and $D \cap F = \emptyset$, there exist three translation arcs α, u and v such that:*
(i) *$p \in \mathrm{int}(\alpha) \cap \mathrm{int}(D)$,*
(ii) *$u, v \subset \mathrm{bd}(D)$, and*
(iii) *$\mathrm{int}(u), \mathrm{int}(v)$, and L_α are disjoint.*

PROOF. Without loss of generality, by the Schoenflies Theorem let D be a ball. As in the proof of Proposition 5, let $\alpha = [y, y^1)$ be a translation arc through p with $y \in \mathrm{bd}(D)$. Let $\tilde{u}, \tilde{v}$ be the two arcs on $\mathrm{bd}(D)$ with endpoints y, y^1. Let $e, e^1 \in \tilde{u}$ be a point and its image such $\tilde{d}(e, e^1)$ is minimal, where $\tilde{d}(\ ,\ )$ is the distance along the arc $\tilde{u}$. Since $D \cap F = \emptyset$, $\tilde{d}(e, e^1) \neq 0$. We claim that $[e, e^1) \cap [e^1, e^2) = \emptyset$. If not then there exists an f such that $e < f^1 < e_1$ along $\tilde{u}^1$. The simple closed curve $\alpha \cup \tilde{u}$ is mapped to the simple closed curve $\alpha^1 \cup \tilde{u}^1$ with the same orientation, since h is orientation preserving. Hence the inverse image of f^1 must lie between e and e^1 along $\tilde{u}$. This implies that $\tilde{d}(f, f^1)$ is less than $\tilde{d}(e, e^1)$, a contradiction.

Let $u = [e, e^1] \cap \tilde{u}$ and similarly construct v. The arcs α, u and v are translation arcs, and, by Lemma 1, $\mathrm{int}(u), \mathrm{int}(v)$ and L_α are disjoint.

These arcs provide two translation lines L_u, L_v which are disjoint from L_α except possibly at the iterates of y and y^1, the endpoints of α.

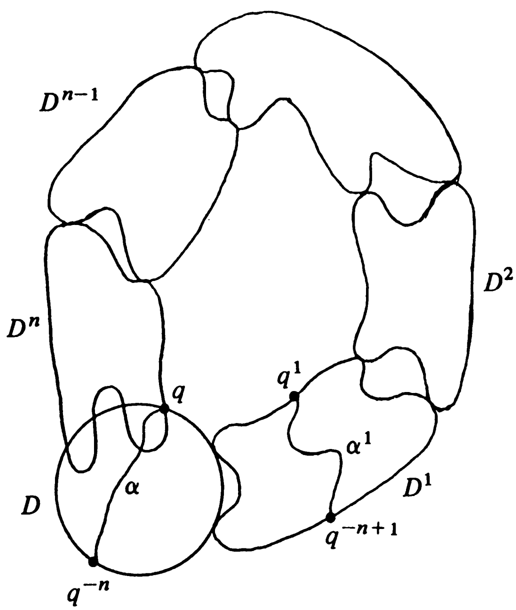

FIGURE 3

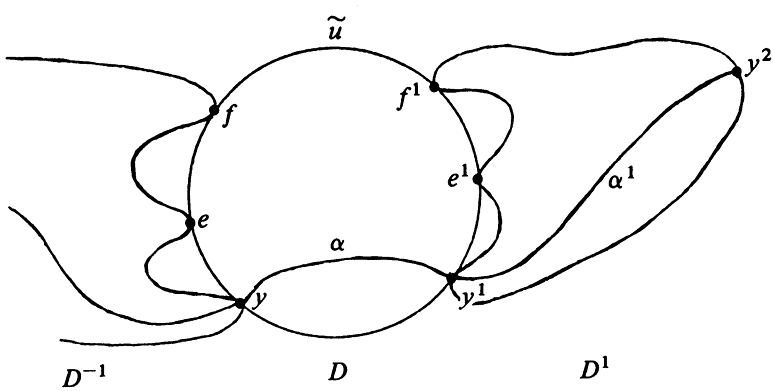

FIGURE 4

3. Main theorem. The following theorem is the version of the Brouwer Translation Theorem which will be proved.

THEOREM. *Let h be a free homeomorphism of S^2 with finite fixed point set F. Then each $p \in S^2 - F$ lies in the image of an embedding $\phi_p\colon (R^2, 0) \to (S^2 - F, p)$ such that*:

(i) *$h\phi_p = \phi_p\tau$, where $\tau(z) = z + 1$ is the canonical translation of the plane, and*
(ii) *the image of each vertical line under ϕ_p is closed in $S^2 - F$.*

As a consequence of this theorem, the image of the x-axis under ϕ_p is a translation line through p.

The proof will be divided into its local and global aspects. In the first part, using the results of §2, we construct L_u, L_v and L_α. We will then construct two

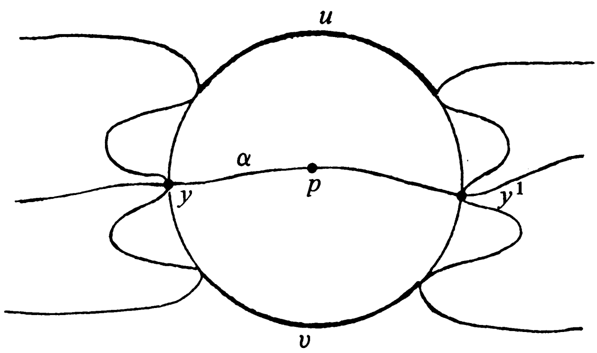

FIGURE 5

additional translation lines $L_{u(1)}$ and $L_{v(1)}$. The construction for $L_{u(1)}$ will be accomplished by showing that there exists a specific critical partial disc, part of whose boundary lies on L_u. We will think of $L_{u(1)}$ as above L_u, which is above L_α. A similar construction will be made for $L_{v(1)}$.

In the second part we will iterate this procedure, and, with carful control of the area of the associated partial discs, we will determine the existence of the embedding ϕ_p.

PROOF. By Proposition 4 there exists a translation line L_α through p. By Lemma 2 of §2 there exist two translation lines L_u and L_v "above" and "below" L_α.

By the Schoenflies Theorem we can conjugate h to obtain a homeomorphism $\tilde{h}$ such that D is mapped to a ball $D(0)$ centered at p. Without loss of generality denote $\tilde{h}$ by h.

The following proposition constructs either a simple arc from L_α to a fixed point, which does not intersect its image, or a critical partial disc which will yield a translation line "above" L_u.

PROPOSITION 6. *There exists either*
(i) *a critical partial disc* $D(1)$ *situated along* $D(0)$ *such that* $D(1) \cap D(0) \subset u$, *or*
(ii) *a simple arc* l *from* p *to a point* $q \in F$ *such that* $l \cap l^1 = q$ *and* $l \cap L_\alpha = p$.

PROOF. The arc $u = [C, C^1]$ for some point C on $\mathrm{bd}(D(0))$. For each $x \in (C, C^1)$ parametrize (C, x) by $t \in I$.

For each $t \in (C, x)$ we will construct a partial disc $D(t, x)$ such that $\mathrm{bd}\, D(t, x) \cap D(0) = [t, x] \subset \mathrm{bd}\, D(0)$. We need only show where the center of $D(t, x)$ is located. Let $r =$ radius of $D(0)$.

For each $x \in [C, C^1]$ and $t \in (C, x)$ define $r(t, x)$ to be the scalar valued function which is linear along each line segment of slope -1 in the (t, x)-plane, to have limiting value 0 on the line $x = t$, and limiting value $\pi - r$ on the lines $t = C$ and $x = C^1$ as indicated in Figure 6.

Let the center of the partial disc $D(t, x)$ be situated $r + r(t, x)$ from the center of $D(0)$ along the line bisecting the arc $[t, x] \subset \mathrm{bd}\, D(0)$ and passing through the center of $D(0)$.

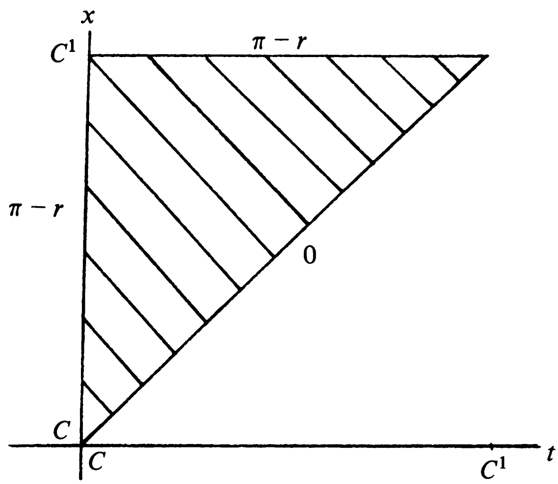

FIGURE 6

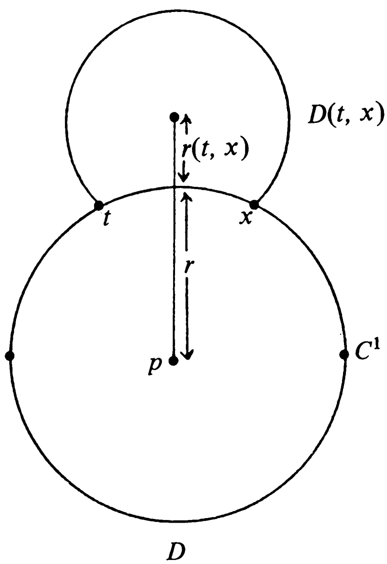

FIGURE 7

Since $t, x \in \text{bd}(D(0)) \cap \text{bd}(D(t,x))$ and $d(p, \text{center}(D(t,x))) > r$, we have constructed a partial disc $D(t,x)$. The family $D(t,x)$, $t \in (C,x)$, is a continuous family of partial discs (cf. §1) which has the following properties:

(i) as $t \to x$, $r(t,x) \to 0$,

(ii) as $t \to C$, $r(t,x) \to \pi - r$, and

(iii) if $y \in \text{int}(D(t,x))$ then there exists a $t_1 \in (t,x)$ on the arc u such that $y \in \text{int}(D(t_1,x))$.

Property (ii) implies that when $t \to C$, $\text{bd}(D(t,x)) \to \text{bd}(D(0))$.

Using the fact that $F \cap D(0) = \varnothing$ and Proposition 2, there exists a compact set G with $D(0) \subset \mathrm{int}(G)$, $G \cap F = \varnothing$, such that for t sufficiently close to x along u,

$$D(t,x) \cap D^1(t,x) = \varnothing, \qquad D(t,x) \cap (L_u - u) = \varnothing \quad \text{and} \quad D(t,x) \subset G.$$

Let t approach C along u until $D(t,x)$ either
(a) is critical, but does not meet $L_u - u$ nor a fixed point,
(b) meets a fixed point and is critical, but does not meet $L_u - u$, or
(c) meets $L_u - u$, and is at most critical.

If $t \to C$ then by (ii) above, $\mathrm{bd}(D(t,x)) \to \mathrm{bd}(D(0))$. But if $D(t,x)$ is at most critical, then $F \subset \mathrm{bd}(D(0))$ which is a contradiction. Hence by (ii) and (iii) the above are the only possible cases.

In case (a) the requirements of the proposition have been met. For case (b) take a simple arc l in $D(0) \cup D(x,t)$ from p to a fixed point q on $\mathrm{bd}(D(t,x))$ such that $\mathrm{int}(l) \subset \mathrm{int}(D(0) \cup D(t,x))$, and $l \cap L_\alpha = p$. Thus $l \cap l^1 = q$ and $l \cap L_\alpha = p$.

The final case constitutes the remainder of the proof of Proposition 6.

Case (c). Assume that for each $x \in (C, C^1)$ the partial disc so constructed intersects $L_u - u$ and is at most critical. Call this partial disc, the *partial disc associated to* x, $D(x)$.

We shall show that there exists an $x \in (C, C^1)$ such that $D(x)$ is not of case (c) by demonstrating that these partial discs induce a partition of the open interval into two nonempty, disjoint and relatively closed sets, thereby arriving at a contradiction.

LEMMA 3. $D(x) \cap (L_u - u) \subset (C^1, C^2) \cup (C^{-1}, C)$.

PROOF. The partial disc $D(x)$ cannot contain either C or C^1 by construction. If $D(x)$ meets $L_u - (C^{-1}, C^2)$ then there exists an arc on $\mathrm{bd}(D(x))$ which spans both x and x^1 or x and x^{-1}. Hence by the spanning property for translation arcs, $D(x)$ is not at most critical, that is, $\mathrm{int}(D(x)) \cap \mathrm{int}(D^1(x)) \neq \varnothing$.

DEFINITION. Call those $x \in (C, C^1)$ such that $D(x)$ meets (C^{-1}, C) (respectively (C^1, C^2)) *type-C* (resp. *type-C^1*).

LEMMA 4. *The two sets composed of $x \in (C, C^1)$ of type-C or type-C^1 are disjoint, nonempty and relatively closed in (C, C^1).*

PROOF. *Disjoint.* Let $D(x)$ be a partial disc such that x is of type-C and of type-C^1. Then there exists an arc on $\mathrm{bd}(D(x))$ which spans both C and C^1. By the spanning property of translation arcs, we have a contradiction.

Nonempty. Assume that $x \in (C, C^1)$ is of type-C^1. Let $M = D(x) \cap [C^1, C^2]$. Since M is compact, let $m(x)$ be the least element in M in the order $<$ of $[C^1, C^2]$. The point $m(x) \neq C^1$ or C^2. Consider the point $m^{-1}(x) \in (C, C^1)$ and the partial disc $D(m^{-1}(x))$. We claim that $m^{-1}(x)$ is of type-C. Assume to the contrary that $m^{-1}(x)$ is of type-C^1. If $x < m^{-1}(x)$ along (C, C^1) then the arc $[x, m(x)] \subset \mathrm{bd}(D(x))$ is an arc which spans both $m(x)$ and $m^{-1}(x)$ and hence, by the spanning property for translation arcs, $D(x)$ overlaps its image and is not at most critical.

Similarly we can show that $m^{-1}(x) < t$ where t is such that $D(x) = D(t,x)$. Let $m(m^{-1}(x))$ be, as defined above, the first point along $[C^1, C^2]$ such that $D(m^{-1}(x))$ intersects $[C^1, C^2]$. If $m(m^{-1}(x)) \geq m(x)$ along (C^1, C^2) then the arc

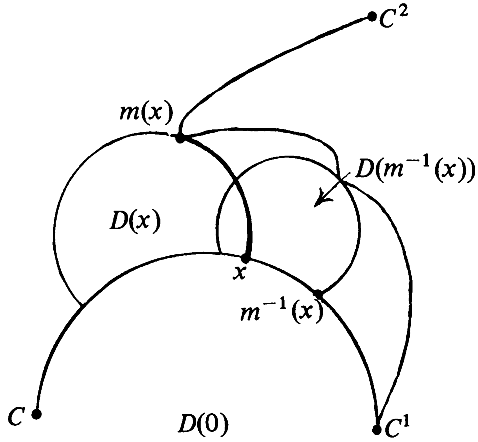

FIGURE 8

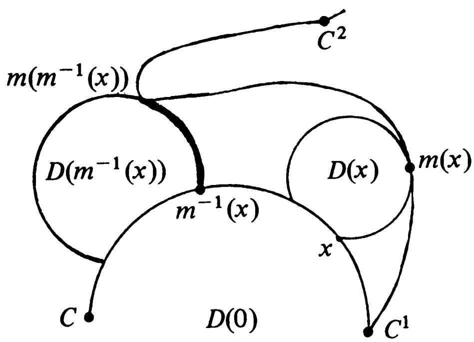

FIGURE 9

$[m^{-1}(x), m(m^{-1}(x))] \subset \mathrm{bd}(D(m^{-1}(x)))$ is an arc which spans both $m^{-1}(x)$ and $m(x)$. Thus by the spanning property for translation arcs, $D(m^{-1}(x))$ overlaps its image, a contradiction.

Thus for $m^{-1}(x)$ to be of type-C^1, $m^{-1}(x) < t$ and $m(m^{-1}(x)) < m(x)$. In order to present a contradiction, we need to consider the manner in which $D(x)$ intersects (C^1, C^2) for any $x \in (C, C^1)$.

Let $s(x)$ be the simple closed curve from x to $m(x)$ (along $D(x)$) to C^1 (along u^1) to x (along u).

LEMMA 5. *The simple closed curve $s(x)$ does not separate C from C^2.*

PROOF. Assume that $s(x)$ separates C from C^2. Since h is orientation preserving, $\mathrm{int}(D^1(0))$ must intersect $D(x)$. Thus we must have that $\mathrm{bd}(D(x)) \cap (\mathrm{bd}(D^1(0)) - u^1) \neq \emptyset$. Let $\tilde{\alpha} \subset D(0)$ be an arc with endpoints C, C^1 passing through p such that $\mathrm{int}(\tilde{\alpha}) \subset \mathrm{int}(D(0))$. The arc $\tilde{\alpha}$ is a translation arc which generates a translation line $L_{\tilde{\alpha}}$. Let b be an arc from $y \in (t, x)$ to a point $z \in \mathrm{bd}(D(x)) \cap (\mathrm{bd}(D^1(0)) - u^1)$ such that $\mathrm{int}(b) \subset \mathrm{int}(D(x))$ and $y^1 \neq z$. Let a be an arc from p to y such that $\mathrm{int}(a) \subset \mathrm{int}(D(0))$ and $a \cap \tilde{\alpha} = p$. Let d be an arc from z to q with $q \geq p^1$ on $L_{\tilde{\alpha}}$ such that $\mathrm{int}(d) \subset \mathrm{int}(D^1(0))$ and $d \cap b = z$. Since

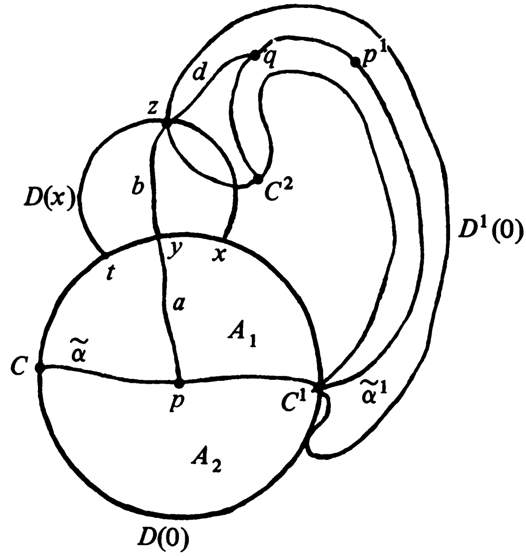

FIGURE 10

$\tilde{\alpha}$ separates $D(0)$ into two discs A_1 and A_2 with $\text{int}(a) \subset A_1$, we can choose d such that $\text{int}(d) \subset \text{int}(A_2^1)$ and $d \cap a^1 = \emptyset$.

The arc $a \cup b \cup d$ spans p and p^1 along $L_{\tilde{\alpha}}$, hence $(a \cup b \cup d) \cap (a^1 \cup b^1 \cup d^1) \neq \emptyset$. We will now show that each term in the expansion is empty by grouping these terms into those which have similar proofs and proving one of each type as an example.

Assume that $b \cap b^1 = \emptyset$.

(1) $a \cap a^1$, $d \cap d^1$. Since $\text{int}(a) \subset \text{int}(D(0))$ and $p \notin \text{bd}(D(0))$, if $a \cap a^1 \neq \emptyset$ then $y = y^1$, which is a contradiction.

(2) $a^1 \cap d$, $a^1 \cap b$, $b^1 \cap d$. Assume that $b^1 \cap d \neq \emptyset$. Then $b \cap d^{-1} \neq \emptyset$. However $d^{-1} \subset A_2$ and $b \subset D(x)$, yet $D(0) \cap D(x) = [t, x]$ and $[t, x] \cap A_2 = \emptyset$, which is a contradiction.

(3) $a \cap d^1$. Since $a \subset D(0)$ and $d^1 \subset D^2(0)$, $a \cap d^1 \subset D(0) \cap D^2(0) = \emptyset$, by Lemma 2.

(4) $a \cap b^1, b \cap d^1$. Consider $a \cap b^1$. Let g_1 be an irreducible arc in a from b to b^1 with endpoints y and w, where $w \in a \cap b^1$, and let g_2 be the arc in b^1 with endpoints y^1 and w. Let E^1 be a small disc containing g_2 such that $E \cap E^1 = \emptyset$. As in the proof of Proposition 5, enlarge E^1 until $E \cap E^1 \neq \emptyset$. Let β be a translation arc generated which contains g_2. The arc β can be constructed so that $\beta \cap g_1 = w$. Then g_1 spans y and y^1 along L_β. Hence $g_1 \cap g_1^1 \neq \emptyset$, which implies that $a \cap a^1 \neq \emptyset$, a contradiction.

Since these cases present a contradiction, $b \cap b^1 \neq \emptyset$ and thus $D(1)$ overlaps its image, a contradiction. Hence Lemma 5 is true.

Let $A(x), B(x)$ (resp. $A(m^{-1}(x))$, $B(m^{-1}(x))$) be the two components of $S^2 - s(x)$ (resp. $S^2 - s(m^{-1}(x))$) with $C, C^2 \in B(x)$ (resp. $B(m^{-1}(x))$). If $D(m^{-1}(x))$ does not intersect $(x, m(x))$ along $D(x)$ then

$$A(x) \subset A(m^{-1}(x)), \quad B(x) \supset B(m^{-1}(x)) \quad \text{and} \quad s(x) \cap s(m^{-1}(x)) = \emptyset.$$

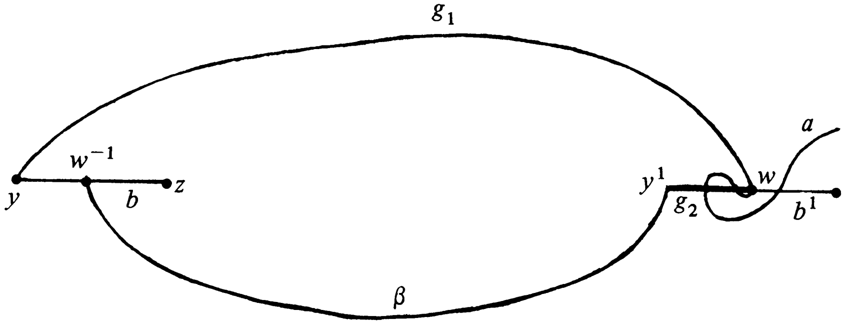

FIGURE 11

However, since $C^2 \in B(m^{-1}(x))$ and $m(m^{-1}(x)) < m(x)$ on (C^1, C^2) we must have that $(m(x), C^2)$ along (C^1, C^2) intersects $\mathrm{int}(D(m^{-1}(x)))$. Thus $\mathrm{int}(D(m^{-1}(x)))$ meets $D(1)$, a contradiction. Hence $D(m^{-1}(x))$ meets $(x, m(x))$ along $D(x)$. But $D(x)$, $D(m^{-1}(x))$ are constructed from balls, thus

$$m(x) \in \mathrm{int}(D(m^{-1}(x))).$$

Therefore $(C^1, C^2) \cap \mathrm{int}(D(m^{-1}(x))) \neq \varnothing$ which again implies that $\mathrm{int}(D(m^{-1}(x)))$ meets $D(1)$, a contradiction.

Hence given a point x of type-C^1, we can construct a point $m^{-1}(x)$ of type-C. The reverse statement can be proven similarly.

Closed. Let $x_1, x_2, \ldots$ be a sequence of points in (C, C^1) of type-C^1 such that $x \in (C, C^1)$ is a limit point. Associated to each x_i are points t_i, m_i where t_i is such that $D(x_i) = D(t_i, x_i)$ and m_i is $m(x_i)$ as noted above. Pick subsequences of x_i, t_i, m_i which converge to x, t, m respectively and relabel this subsequence x_i, t_i, m_i. We claim that this subsequence determines a partial disc $D(x)$ of type-C^1. Let $D(t, x)$ be the partial disc at t, x. If $\mathrm{int}(D(t, x)) \cap (L_u - (C, C^1)) \neq \varnothing$ then there exist infinitely many i such that $\mathrm{int}(D(t_i, x_i)) \cap (L_u - (C, C^1)) \neq \varnothing$, a contradiction. Similarly $D(t, x)$ is at most critical. Thus by the spanning property, $m \neq C^2$. By the continuity of $r(t, x)$, $m \in L_u - (C, C^1)$ and $m \neq C^1$ else $x = C^1$. Since $D(t, x)$ is at most critical, we have that $\mathrm{int}(D(t, x)) \cap (L_u - (C, C^1)) = \varnothing$, but $m \in (C^1, C^2)$, hence x is of type-C^1. A similar proof holds for the set of points of type-C.

Since the open arc cannot be represented as the disjoint union of two relatively closed, nonempty sets, there exists an $x \in (C, C^1)$ satisfying Proposition 6.

This completes the proof of Proposition 6 and the local part of the main theorem. We will now iterate the above process and construct the embedding ϕ_p.

Construction of ϕ_p. Let ϕ_p map the x-axis to the translation line L_α such that $(0,0) \mapsto p$ and $(-\frac{1}{2}, 0), (\frac{1}{2}, 0) \mapsto y, y^1$, where $\alpha = [y, y^1]$ along L_α, and $h\phi_p = \phi_p \tau$ on the x-axis. By Proposition 6 there exists a simple arc l from p to a fixed point q or there exists a critical partial disc $D(1)$ with the above-mentioned properties. If the simple arc l exists, let ϕ_p map the positive y-axis $\cup$ origin to $l - q$ so that $(0,0) \mapsto p$. The embedding ϕ_p can then be extended to the entire upper half-plane.

In the other case, $D(1)$ exists and either there exists a fixed point or a translation arc $u(1)$ on $\overline{\mathrm{bd}(D(1)) - \mathrm{bd}(D(0))}$. If the fixed point occurs, proceed as above to obtain ϕ_p. Otherwise, $u(1)$ generates a translation line $L_{u(1)}$ "above" L_u. Repeat Proposition 6 to obtain $D(2), D(3), \ldots$. If we arrive at a $D(N)$ with a fixed point

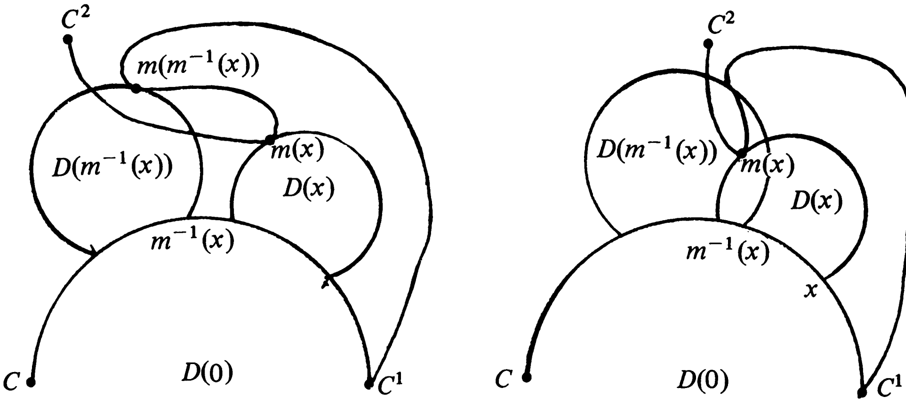

FIGURE 12

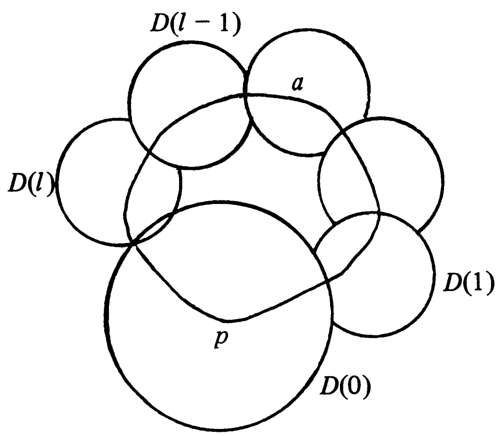

FIGURE 13

on $\mathrm{bd}(D(N)) - \mathrm{bd}(D(N-1))$, then, as above, we obtain ϕ_p defined on the upper half-plane.

LEMMA 6. *$D^i(j) \cap D^k(l) \neq \varnothing$ if and only if $k = i$ and $|l - j| \leq 1$, or $l = j$ and $|k - i| \leq 1$. Hence the interior of the discs are disjoint.*

PROOF. Without loss of generality $i = 0$, $j = 0$. Assume that $D(0) \cap D^k(l) \neq \varnothing$ and k, $l \geq 0$.

If $l = 0$ then by Lemma 1, $k = -1, 0$ or 1.

Thus $l = j$ and $|k - i| \leq 1$.

Assume that $k = 0$ and $l > 1$. Let a be a simple closed curve from $p \in D(0)$ through $D(0), D(1), \ldots, D(l)$ to $D(0)$ such that a lies in $\mathrm{int}(\bigcup_{i=0}^{l} D(i))$ except possibly where $D(l)$ meets $D(0)$. The simple closed curve a separates S^2 into two discs E_1, E_2 such that $h(E_1) \subset E_1$. By Proposition 4, E_1 must have a fixed point on its boundary. However the only possible fixed point must be on the intersection of $D(l)$ and $D(0)$. By construction $D(0)$ does not intersect the fixed point set.

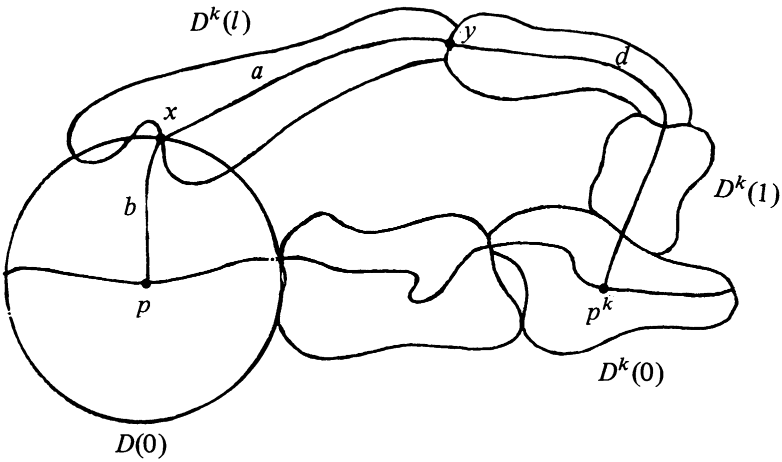

FIGURE 14

Assume $k \cdot l > 0$. Let $x \in \mathrm{bd}(D^k(l)) \cap \mathrm{bd}(D(0))$ and $y \in \mathrm{bd}(D^k(l)) \cap \mathrm{bd}(D^k(l-1))$ be the endpoints of an arc a such that $\mathrm{int}(a) \subset \mathrm{int}(D^k(l))$. Let b be an arc with endpoints x, p such that $\mathrm{int}(b) \subset \mathrm{int}(D(0))$. Let d be an arc with endpoints y, p^k such that $\mathrm{int}(d) \subset \mathrm{int}(\bigcup_{i=0}^{l-1} D^k(i))$. Then the arc $a \cup b \cup d$ spans p, p^k along L_α. But a, b can be chosen such that $d \cap b^1 = \varnothing$. Hence $a \cap a^1 \neq \varnothing$, and $D^k(l)$ is not at most critical, a contradiction.

LEMMA 7. *Either there exists a fixed point y such that each neighborhood of y contains infinitely many $D(i)$, or some $D(N)$ has a fixed point on its boundary.*

PROOF. Assume that none of the $D(i)$ has a fixed point on its boundary. Let K be any compact set not meeting F. Since F is finite, there exists an $\varepsilon > 0$ such that any partial disc D of radius less than ε, not having a fixed point on its boundary and intersecting K, has the property that $D \cap D^1 = \varnothing$. If the radius of the partial disc D is r, then $\mathrm{area}(D) > \pi r^2/2$. Hence, by Lemma 6, area $\bigcup_{i=1}^n D(i) > n\pi\varepsilon^2/2$. Since $\mathrm{area}(K) < \mathrm{area}(S^2)$, n must be finite. Thus all but a finite number of the $D(i)$ do not intersect K.

LEMMA 8. *The sequence $D(i)$ converges to a single fixed point. That is, for any neighborhood N of q, a fixed point, all but a finite number of the $D(i)$ are contained in N.*

PROOF. Assume the $D(i)$ have only two points $q_1, q_2 \in F$ as limit points. Consider two balls $B(q_1, \varepsilon), B(q_2, \varepsilon)$ centered at q_1 and q_2 of radii $\varepsilon < d(q_1, q_2)/3$. Since q_1, q_2 are limit points of the $D(i)$, there exist infinitely many i, j such that $D(i) \cap \mathrm{int}(B(q_1, \varepsilon)) \neq \varnothing$ and similarly $D(j) \cap \mathrm{int}(B(q_2, \varepsilon)) \neq \varnothing$. Consider subsequences $D(i_1), D(i_2), \ldots, D(j_1), D(j_2), \ldots$ such that $i_1 \leq j_1 < i_2 \leq \cdots$. Since the collection $D(i)$ is connected and $D(i)$ intersects only $D(i+1), D(i-1)$, there exist integers n_k, $k = 1, 2, \ldots$, such that $i_k \leq n_k \leq j_k$ and such that the following holds:

$$D(n_k) \cap (S^2 - (\mathrm{int}\, B(q_1, \varepsilon) \cup \mathrm{int}\, B(q_2, \varepsilon)) \neq \varnothing.$$

However, the set $S_2 - (\operatorname{int} B(q_1, \varepsilon) \cup \operatorname{int} B(q_2, \varepsilon))$ is compact. As in the proof of Lemma 7 we arrive at a contradiction. This proof requires only a slight modification for the case of more than two fixed points.

We now find a simple arc l from p to a fixed point q lying in $\overline{\bigcup D(i)}$ such that $l \cap l^1 = q$. Let ϕ_p map the positive y-axis to $l - q$. Then ϕ_q can be extended to the upper half-plane.

By substituting v for u in the above proposition and lemmas, ϕ_p can be extended to R^2, having the requisite properties.

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