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THE COHOMOLOGY REPRESENTATION OF AN ACTION OF C_p ON A SURFACE

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ABSTRACT. When a finite group G acts on a surface S, then $H^1(S;\mathbf{Z})$ posseses naturally the structure of a $\mathbf{Z}G$-module with invariant symplectic inner product. In the case of a cyclic group of odd prime order we describe explicitly this symplectic inner product space in terms of the fixed-point data of the action.

Introduction. Let S_g be a closed connected smooth oriented surface of genus g and let ϕ be a diffeomorphism of S_g which is periodic of period n. There are only a finite number of points on which $C_n \cong \langle\phi\rangle$ does not act freely; the action of C_n on the tangent spaces to these points will be called the fixed-point data of ϕ (see §1).

THEOREM A. *If $g \geq 2$, two periodic orientation-preserving diffeomorphisms which are isotopic* (*not necessarily through periodic maps*) *have the same order and fixed-point data. Two periodic maps with the same order and fixed-point data are conjugate in* $\mathrm{Diff}^+(S_g)$.

Thus the fixed-point data is well defined on the torsion elements of the mapping-class group and it distinguishes the conjugacy classes.

C_n acts on $H^1(S_g;\mathbf{Z})$ giving it the structure of a $\mathbf{Z}C_n$-module with C_n-invariant symplectic inner product, or symplectic $\mathbf{Z}C_p$-space for short. We shall now restrict ourselves to the case $n = p$, an odd prime. The fixed-point data is now described by a set of integers modulo p, $\{\beta_i\}$, one for each fixed point x_i, such that ϕ^{β_i} acts on the tangent space at x_i by rotation counterclockwise through $2\pi/p$.

Let $\mathbf{Z}[\lambda]$, $\lambda^p = 1$, be the ring of integers with a pth root of unity added. This can be regarded as a $\mathbf{Z}C_p$-module by letting a generator of C_p act as multiplication by λ. If u is a real unit of $\mathbf{Z}[\lambda]$ we can construct a C_p-invariant symplectic inner product on $\mathbf{Z}[\lambda]$, and obtain a symplectic $\mathbf{Z}C_p$-space, denoted $(u, \mathbf{Z}[\lambda])$, by

$$\langle x, y\rangle = \mathrm{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(\Delta^{-1}ux\bar{y}), \qquad x, y \in \mathbf{Z}[\lambda].$$

Δ is an imaginary generator of the different ideal of the extension $\mathbf{Q}(\lambda)/\mathbf{Q}$ as in (4.1).

Given any $\mathbf{Z}C_p$-lattice M we can construct the standard hyperbolic symplectic inner product on $M \oplus M^*$, which we denote $\overline{H}(M)$.

THEOREM B. *Suppose C_p acts on S_g with fixed-point data $B = \{\beta_1, \ldots, \beta_n\}$ and c is the number of disjoint pairs $\{\beta, -\beta\}$* (mod p) *in B. Then as a symplectic*

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$\mathbf{Z}C_p$-space, $H^1(S_g;\mathbf{Z})$ splits into an orthogonal direct sum as follows (where m is chosen to give the correct $\mathbf{Z}$-rank).

(i) $n=0$, $H^1(S_g;\mathbf{Z}) \cong \overline{H}(\mathbf{Z}) \oplus \overline{H}(\mathbf{Z}C_p)^m$;

(ii) $n \neq 0$, $n = 2c$, $H^1(S_g;\mathbf{Z}) \cong \overline{H}(\mathbf{Z}[\lambda])^{c-1} \oplus \overline{H}(\mathbf{Z}C_p)^m$;

(iii) $n \neq 0$, $n \neq 2c$, $H^1(S_g;\mathbf{Z}) \cong \bigoplus_{i=1}^{n-2c-2}(u_i,\mathbf{Z}[\lambda]) \oplus \overline{H}(\mathbf{Z}[\lambda])^c \oplus \overline{H}(\mathbf{Z}C_p)^m$.

The units u_i are given explicitly in terms of B by (6.1).

Other results in this direction have been obtained independently in [**7**].

1. Fixed-point data and the mapping-class group. The mapping-class group Γ_g is the set of components of $\mathrm{Diff}^+(S_g)$, i.e. the group of orientation-preserving diffeomorphisms up to isotopy. It acts naturally on $H^1(S_g;\mathbf{Z})$ and preserves the cup product so gives a homeomorphism

$$r\colon \Gamma_g \to \mathrm{Sp}(2g;\mathbf{Z}).$$

r is known to be onto and its kernel is torsion-free.

We are interested in the torsion elements of Γ_g. According to a theorem of Nielsen [**13**] any torsion element of Γ_g of order n can be realized by a diffeomorphism ϕ of S_g with $\phi^n = 1$ exactly, not just up to isotopy, and S_g can be given a complex structure which is invariant under ϕ.

The most important invariant of an orientation-preserving periodic diffeomorphism ϕ of S_g of period n is its fixed-point data, which we shall define formally as follows. The set of points of S_g at which $C_n \cong \langle\phi\rangle$ does not act freely is a finite set, $\mathrm{Sing}(\langle\phi\rangle)$. Let $\{x_i\}$ be a set of representatives of the orbits of $\mathrm{Sing}(\langle\phi\rangle)$ under $\langle\phi\rangle$. Let $\alpha_i = |\operatorname{stab}_{\langle\phi\rangle}(x_i)|$. ϕ^{n/α_i} generates $\operatorname{stab}_{\langle\phi\rangle}(x_i)$ so it acts faithfully by rotation on the tangent space at x_i. Let β_i be an integer such that $\phi^{\beta_i n/\alpha_i}$ acts by rotation through $2\pi/\alpha_i$ in the counterclockwise direction (clockwise defined in terms of the orientation of S_g), i.e. if S_g is given an invariant complex structure, $\phi^{\beta_i n/\alpha_i}$ acts as multiplication by $e^{2\pi/\alpha_i}$.

β_i is well defined modulo α_i, and β_i is prime to α_i, so there is no loss of information in considering just $\beta_i/\alpha_i \in \mathbf{Q}/\mathbf{Z}$ instead of α_i and β_i. By the "fixed-point data" of ϕ we shall mean the collection

$$\sigma(\phi) = (n, g|\beta_1/\alpha_1, \ldots, \beta_q/\alpha_q),$$

where n is the order of ϕ, g the genus of S_g, and the numbers $\beta_i/\alpha_i \in \mathbf{Q}/\mathbf{Z}$ are not ordered. If n is understood in the context we shall omit it.

According to another theorem of Nielsen [**12**], any two periodic elements of $\mathrm{Diff}^*(S_g)$ are conjugate in $\mathrm{Diff}^+(S_g)$ if and only if they have the same fixed-point data (which is the second part of Theorem A). This can be seen as follows. An action of C_n on S_g represents S_g as an n-fold regular cyclic branched covering of $S_{\bar{g}} = C_n\backslash S_g$. All such coverings with the same branching data can be shown to be equivalent (if we allow an automorphism of $S_{\bar{g}}$) by using the fact that $r\colon \Gamma_{\bar{g}} \to \mathrm{Sp}(2\bar{g};\mathbf{Z})$ is onto.

To prove the first part of Theorem A we consider two periodic diffeomorphisms ϕ, θ of S_g which are isotopic and have period n. $C_n \cong \langle\phi\rangle$ can act on $S^1 \cong \mathbf{R}/\mathbf{Z}$ by $\phi(x) = x + 1/n$, $x \in \mathbf{R}/\mathbf{Z}$. Let $\langle\phi\rangle$ act diagonally on $S_g \times S^1$ and let $F_\phi = \langle\phi\rangle\backslash S_g \times S^1$. F_ϕ is a Seifert fiber space with fibers the images of (x, S^1). Similarly we can construct F_θ.

We can also think of F_ϕ as $F_\phi = S_g \times [0,1]/\!\sim$ where $(s,0) \sim (\phi(s),1)$. If ϕ changes by an isotopy, the homeomorphism type of F_ϕ remains the same. Thus, as ϕ is isotopic to θ, $\pi_1(F_\phi) \cong \pi_1(F_\theta)$. Suppose $\sigma(\phi) = (n, g|\{\alpha_i/\beta_i\})$. Then the Seifert invariants of F_ϕ are $\{(\alpha_i, \beta_i)\}$. (Seifert invariants, considered as integers, are well defined to exactly the same extent as the alphas and betas in the fixed-point data.) Now if $g \geq 2$ the Seifert fiber structure of F is unique up to isotopy [**16**], so the Seifert invariants are the same both for the description F_ϕ and for F_θ. Thus $\sigma(\phi) = \sigma(\theta)$.

This can also be seen more directly by considering the usual presentation for $\pi_1(F_\phi)$:

$$\pi_1(F_\phi) \cong \mathrm{gp}\langle a-1, b_1, \dots, a_g, b_g, c_1, \dots, c_q, t|\ t \text{ is central},$$
$$c_i^{\alpha_i} = t^{\beta_i}, [a_i, b_1] \cdots [a_g, b_g] c_1, \dots, c_q = 1\rangle.$$

$\langle t\rangle$ is distinguished since it is the center, and the images in $\pi_1(F)/\langle t\rangle$ of the conjugates of $\langle c_i\rangle$ comprise exactly the torsion elements. So the pairs (α_i, β_i) can be read off from $\pi_1(F_\phi)$.

This completes the proof of Theorem A.

COROLLARY 1.1. *The fixed-point data σ is well defined on the torsion elements of Γ_g, $g \geq 2$, and takes different values on distinct conjugacy classes.*

REMARKS. (a) The set of possible values of σ is described in [**10**]. It is always the case that $\sum \beta_i/\alpha_i \in \mathbf{Z}$.

(b) Theorem A is probably folklore. It can also be proved by considering the action of the group of Teichmüller space (cf. [**6**]).

2. The image in $G_0(\mathbf{Z}G)$. Let G be a finite group acting on S_g preserving orientation. $M = H^1(S_g; \mathbf{Z})$ is a $\mathbf{Z}G$-lattice, i.e. a $\mathbf{Z}G$-module that is finitely generated and free over $\mathbf{Z}$. The cohomology group $H^1(S_g;\mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ by Poincaré duality. S_g can be given the structure of a finite G-CW-complex by lifting the cells from a CW-decomposition of $G\backslash S_g$. We shall insist that $\mathrm{Sing}(G)$ be contained in the 0-skeleton.

$G_0(\mathbf{Z}G)$ is the Grothendieck group of finitely generated $\mathbf{Z}G$-modules with relations coming from short exact sequences [**15**]. If C_* is the CW-chain complex of S_g then we can calculate its Euler characteristic $\chi(C_*) \in G_0(\mathbf{Z}G)$ either directly,

$$\chi(C_*) = [C_2] - [C_1] + [C_0],$$

or on homology,

$$\chi(C_*) = [H_2(S_g;\mathbf{Z})] - [H_1(S_g;\mathbf{Z})] + [H_0(S_g;\mathbf{Z})] = 2[\mathbf{Z}] - [M].$$

However C_2 and C_1 are free $\mathbf{Z}G$-lattices and $C_0 \cong H_0(\mathrm{Sing}(G);\mathbf{Z}) \oplus \mathbf{Z}G^m$. Thus

$$[M] = 2[\mathbf{Z}] - [H_0(\mathrm{Sing}(G);\mathbf{Z})] + m[\mathbf{Z}G]. \tag{2.1}$$

$H_0(\mathrm{Sing}(G);\mathbf{Z})$ is just a permutation module and is completely described by the alphas, so $[M]$ is known. Working over $\mathbf{Q}$, this gives $H_1(S_g;\mathbf{Q})$ as a $\mathbf{Q}G$-space and the alphas can be read off from $H_1(S_g;\mathbf{Q})$.

The class of M in $G_0(\mathbf{Z}G)$ tells us more than just $\mathbf{Q} \otimes M$. For example in the case $G = C_p$, the cyclic group of order p, p prime, there is a decomposition

$$G_0(\mathbf{Z}C_p) \cong G_0(\mathbf{Q}C_p) \oplus \mathrm{Cl}(\mathbf{Z}[\lambda]),$$

where $\mathrm{Cl}(\mathbf{Z}[\lambda])$ is the ideal-class group of $\mathbf{Z}[\lambda]$, $\lambda^p = 1$, (cf. [**5**, §74]).

PROPOSITION 2.1. *The image of* $[M]$ *in* $\mathrm{Cl}(\mathbf{Z}[\lambda])$ *is* 0.

PROOF. This is true for every term on the right-hand side of (2.1) since $H_0(\mathrm{Sing}(G);\mathbf{Z})$ is a trivial $\mathbf{Z}G$-lattice.

3. The Atiyah-Singer signature. From now on we shall restrict to the case of a cyclic group C_p of prime order $p \neq 2$ and regard $H^1(S_g;\mathbf{Z})$ not only as a $\mathbf{Z}C_p$-lattice but as one equipped with a C_p-invariant symplectic inner product $\langle\ ,\ \rangle$ (a symplectic $\mathbf{Z}C_p$-space). Our main sources of results on these are [**2**, **4**]. Recall that if M is an inner-product space (not necessarily symplectic here) and N is a submodule then

$$N^{\perp} = \{m \in M \mid \langle n, m\rangle = 0 \text{ for all } n \in N\}.$$

M is metabolic if there exists a submodule N satisfying $N = N^{\perp}$. M is hyperbolic if there is a submodule N such that $M \cong N \oplus \mathrm{Hom}_{\mathbf{Z}}(N,\mathbf{Z})$, where the right-hand side is equipped with the inner product

$$\langle (n-1, n_1'), (n_2, n_2')\rangle = n_2'(n_1) + \varepsilon n_1'(n_2), \qquad n_1, n_2 \in N,\ n_1', n_2' \in \mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Z}).$$

$\varepsilon = \pm 1$ according to whether the product is symmetric or symplectic. We shall write $M = \overline{H}(N)$ in this case.

The Witt group $W_2(\mathbf{Z}, C_p)$ is formed from the semigroup of symplectic $\mathbf{Z}C_p$-lattices under orthogonal direct sum. One takes the Grothendieck group and then quotients out the submodule generated by the metabolic spaces. One defines similarly $W_2(\mathbf{Q}, C_p)$ and $W_2(\mathbf{R}, C_p)$; there are canonical injections between them,

$$W_2(\mathbf{Z}, C_p) \to W_2(\mathbf{Q}, C_p) \to W_2(\mathbf{R}, C_p).$$

An element of $W_2(\mathbf{R}, C_p)$ is determined by its Atiyah-Singer signature [**3**], which is defined as follows. Take a representative $(M, \langle\ ,\ \rangle)$ of a given class. Construct an invariant symmetric inner product $(\ ,\ )$ on M by averaging over C_p any symmetric inner product. Then for some automorphism A of M,

$$\langle x, y\rangle = (x, Ay), \qquad x, y \in M.$$

A^* is defined by $(A^*x, y) = (x, Ay)$. Let $J = A(AA^*)^{-1/2}$, where $(AA^*)^{1/2}$ is the positive square root of AA^*. Then $J^2 = -1$ and J commutes with the action of C_p on M so we can consider M to be a complex vector space, of half its real dimension, with J acting as i. This yields a complex representation of C_p which is well defined up to isomorphism and which we shall call $\mathrm{Hol}(M)$. It is not a Witt-class invariant, but if we pass to the representation ring $R(G)$ and define the Atiyah-Singer signature to be

$$\mathrm{ASsign}(G, M) = \mathrm{Hol}(M) - \overline{\mathrm{Hol}(M)} \in R(G),$$

then this is a Witt-class invariant. If $h \in C_p$ we can evaluate the character of $\mathrm{ASsign}(G, M)$ on h to get $\mathrm{ASsign}(h, M) \in \mathbf{C}$.

$\mathbf{C} \otimes M$ splits as $E_+ \oplus E_-$, corresponding to the two idempotents

$$e_1 = \tfrac{1}{2}(1 \otimes 1 - i \otimes J), \qquad e_2 = \tfrac{1}{2}(1 \otimes 1 + i \otimes J).$$

Since these are orthogonal, $1 \otimes J$ acts as i on E_+ and as $-i$ on E_-. As $\mathbf{C}C_p$-spaces, $\mathrm{Hol}(M) \cong E_+$ by $x \mapsto e_1(1 \otimes x)$ and similarly $\overline{\mathrm{Hol}(M)} \cong E_-$ (cf. [**1**]).

Now let S_g be an oriented Riemann surface on which C_p acts conformally (there is no loss of generality by the theorem of Nielsen mentioned in §1). We can identify $H^1(S_g;\mathbf{R})$ with the space of harmonic forms on S_g and can take

$$\langle \alpha,\beta\rangle = \int \alpha_\wedge \beta, \quad (\alpha,\beta) = \int \alpha_\wedge * \beta, \qquad \alpha,\beta \in H^1(S_g;R),$$

where $*$ is the conjugation operation on harmonic forms. Hence $* = -A = -J$. This means that E_+ and thus $\mathrm{Hol}(M)$ can be identified with the space of holomorphic forms on S_g since the latter are the harmonic forms ω which satisfy $*\omega = -i\omega$.

The importance of this is that the representation of C_p on the holomorphic forms is known in terms of the fixed-point data of the action by a formula of Eichler, which in the case of C_p is as follows (see [**9**]).

THEOREM 3.1. *Let ϕ be an automorphism of a Riemann surface S_g, $\sigma(\phi) = (p, g|\beta_1,\ldots,\beta_n)$. Let $\bar\beta_1$ be defined* (mod p) *by $\bar\beta_i\beta_i \equiv 1$* (mod p), *$i = 1,\ldots,n$. Then*

$$\chi_{\mathrm{Hol}(M)}(\phi) = 1 + \sum_{i=1}^{n} 1/(e^{2\pi\bar\beta_i/p} - 1).$$

We see that the Witt class of $H^1(S_g;\mathbf{Z})$ is completely determined by the fixed-point data.

4. The multisignature. Following Conner et al. [**2**, **4**] we consider the ring $\mathbf{Z}[\lambda]$, $\lambda^p = 1$. There is a Witt ring $W_2(\mathbf{Z},\mathbf{Z}[\lambda])$ formed from finitely generated projective $\mathbf{Z}[\lambda]$-modules equipped with a $\mathbf{Z}$-valued $\mathbf{Z}$-linear symplectic inner product which is invariant under multiplication by λ. Fix a generator $h \in C_p$: A $\mathbf{Z}[\lambda]$-module becomes a $\mathbf{Z}C_p$-module with h acting as multiplication by λ, and there is a canonical homomorphism

$$S\colon W_2(\mathbf{Z},\mathbf{Z}[\lambda]) \to W_2(\mathbf{Z}, C_p),$$

which is in fact an isomorphism.

The ring $\mathbf{Z}[\lambda]$ has an involution induced by $\lambda \mapsto \lambda^{-1}$ so it makes sense to define $\mathcal{H}_0(\mathbf{Z}[\lambda])$ to be the Witt group of finitely generated projective $\mathbf{Z}[\lambda]$-modules with $\mathbf{Z}[\lambda]$-valued Hermitian inner product (Hermitian-symmetric, not skew-symmetric). There is a homomorphism

$$T\colon \mathcal{H}_0(\mathbf{Z}[\lambda]) \to W_2(\mathbf{Z},\mathbf{Z}[\lambda])$$

which takes a representative $(M(\ ,\ ))$ to $(M,\langle\ ,\ \rangle)$ where

$$\langle\alpha,\beta\rangle = \mathrm{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(\Delta^{-1}(\alpha,\beta)), \qquad \alpha,\beta\in M.$$

Δ is an imaginary generator of the different ideal of the extension $\mathbf{Q}(\lambda)/\mathbf{Q}$. This is the inverse fractional ideal of

$$\{x \in \mathbf{Q}(\lambda) |\, \mathrm{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(xy) \in \mathbf{Z} \text{ for all } y \in \mathbf{Z}[\lambda]\}.$$

It is generated by $f'(\lambda)$ where f is the minimal polynomial of λ over $\mathbf{Q}$ and f' is its formal derivative. For our purposes we shall use instead

$$\Delta = \lambda^{-(p-3)/2} f'(\lambda) \tag{4.1}$$

since $\overline{\Delta} = -\Delta$.

PROPOSITION 4.1 [**2**]. *T gives a one-to-one correspondence between the two types of inner product even before Witt equivalence and so*

$$\mathcal{H}_0(\mathbf{Z}[\lambda]) \cong W_2(\mathbf{Z}, \mathbf{Z}[\lambda]).$$

REMARK. For $u \in \mathbf{Q}(\lambda + \bar{\lambda})$ and I an ideal of $\mathbf{Z}[\lambda]$ such that $uI\bar{I} = \mathbf{Z}[\lambda]$, let (u, I) represent the Hermitian space with underlying $\mathbf{Z}[\lambda]$-module I and product

$$(x, y) = ux\bar{y}, \qquad x, y \in I.$$

Then the (u, I) generate $\mathcal{H}_0(\mathbf{Z}[\lambda])$. One can also define $\mathcal{H}_0(\mathbf{Q}(\lambda))$ in similar fashion. It is generated by the $(u, \mathbf{Q}(\lambda))$.

Let us use the symbol P for a prime ideal in $\mathbf{Z}[\lambda + \bar{\lambda}]$. These are the finite primes of $\mathbf{Q}(\lambda + \bar{\lambda})$. There are also $(p-1)/2$ infinite primes corresponding to the embeddings of $\mathbf{Q}(\lambda + \bar{\lambda})$ in $\mathbf{R}$. They will be denoted by P_∞.

We are going to use the Hilbert symbol $(x, \sigma)_P = \pm 1$ for $x \in \mathbf{Q}(\lambda + \bar{\lambda})^*$ and P a prime (which may be infinite) (see [**4**, **14**]). For an infinite prime, $(x, \sigma)_P$ is $+1$ or -1 according to whether $x > 0$ or $x < 0$ respectively under P_∞.

The multisignature is a homomorphism

$$\text{multisign}\colon \mathcal{H}_0(\mathbf{Q}(\lambda)) \to \mathbf{Z}^{(p-1)/2},$$

where the entries of $\mathbf{Z}^{(p-1)/2}$ are indexed by the infinite primes. The P_∞-entry of multisign is the signature after applying P_∞. For convenience later we shall let the rth coordinate of $\mathbf{Z}^{(p-1)/2}$ correspond to the prime $P_r\colon \lambda + \bar{\lambda} \mapsto \varsigma^r + \bar{\varsigma}^r$. This depnds on the root of unity λ so we should write multisign(λ).

5. The relation between the signatures. We can also consider ASsign to take values in $\mathbf{Z}^{(p-1)/2}$ by writing ASsign(h, M) in terms of $\varsigma^r - \bar{\varsigma}^r$, $r = 1, \ldots, (p-1)/2$, and taking as rth coordinatethe coefficient of $\varsigma^r - \bar{\varsigma}^r$.

Consider the diagram:

$$\begin{array}{ccc} \mathcal{H}_0(\mathbf{Z}[\lambda]) & \xrightarrow{S \circ T} & W_2(\mathbf{Z}C_p) \\ {\scriptstyle \text{multisign}}\big\downarrow & & \big\downarrow{\scriptstyle \text{ASsign}(h)} \\ \mathbf{Z}^{(p-1)/2} & \longrightarrow & \mathbf{Z}^{(p-1)/2} \end{array}$$

Both vertical maps are known to be injective and to have image

$$\{(m_1, \ldots, m_{(p-1)/2}) | m_1 \equiv \cdots \equiv m_{(p-1)/2} \pmod 2\}.$$

We should like to complete the bottom of the square so as to make the diagram commute. We shall actually do so for $\mathbf{Q}$ coefficients.

Let $(M, \langle\ ,\ \rangle)$ be a symplectic $\mathbf{Z}C_p$-space. $\langle\ ,\ \rangle$ can be extended to a skew Hermitian product on $\mathbf{C} \otimes M$ by

$$\langle a \otimes \alpha, b \otimes \beta \rangle = a\bar{b}\langle \alpha, \beta \rangle, \qquad a, b \in \mathbf{C},\ \alpha, \beta \in M.$$

Similarly for $(\ ,\ )$. Now suppose $x \in E_+ \cong \mathrm{Hol}(M)$:

$$i\langle x, x \rangle = \langle x, -ix \rangle = \langle x, -Jx \rangle = (x, x) > 0.$$

Similarly if $x \in E_-$ then $i\langle x, x \rangle < 0$, so we can test whether $x \in E_+$ or $x \in E_-$ by computing $i\langle x, x \rangle$.

Now consider a Hermitian space $(u, \mathbf{Q}(\lambda))$ and let us find its Atiyah-Singer signature after the map

$$\mathcal{H}_0(\mathbf{Q}(\lambda)) \to W_2(\mathbf{Q}, C_p) \to W_2(\mathbf{R}, C_p).$$

Let $\varsigma = e^{2\pi i/p}$. $h \in C_p$ acts on $\mathbf{R}\otimes_{\mathbf{Q}}\mathbf{Q}(\lambda)$ as $l\otimes\lambda$. In $\mathbf{C}\otimes_{\mathbf{Q}}\mathbf{Q}(\lambda)$ the ς^r-eigenvector of h is

$$e_r = 1\otimes 1 + \bar{\varsigma}^r \otimes \lambda + \cdots + \bar{\varsigma}^{(p-1)r} \otimes \lambda^{p-1}.$$

Since the ς^r-eigenspace has dimension one it must either be contained in E_+ or in E_-. We can decide which by considering

$$\begin{aligned}
i\langle e_r, e_r\rangle &= i\left\langle \sum_{s=0}^{p-1} \bar{\varsigma}^{sr}\otimes\lambda^s, \sum_{t=0}^{p-1}\bar{\varsigma}^{tr}\otimes\lambda^t\right\rangle = i\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\varsigma^{(t-s)r}\langle\lambda^s,\lambda^t\rangle \\
&= i\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\varsigma^{(t-s)r}\langle\lambda^{s-t},1\rangle = pi\sum_{v=0}^{p-1}\bar{\varsigma}^{vr}\langle\lambda^v,1\rangle \\
&= pi\sum_{v=0}^{p-1}\bar{\varsigma}^{vr}\operatorname{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(\Delta^{-1}u\lambda^v) \\
&= pi\sum_{v=0}^{p-1}\bar{\varsigma}^{vr}\operatorname{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(u\lambda^{v+(p-3)/2}/f'(\lambda)).
\end{aligned}$$

Now according to the formula for the dual basis with respect to $\operatorname{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}$, ([L]), the dual basis to $1, \lambda, \ldots, \lambda^{p-2}$ is

$$(1+\bar\lambda+\cdots+\bar\lambda^{p-2})/f'(\lambda), (1+\bar\lambda+\cdots+\bar\lambda^{p-3})/f'(\lambda), \ldots, (1+\bar\lambda)/f'(\lambda), 1/f'(\lambda).$$

So

$$\begin{aligned}
\operatorname{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(\lambda^r/f'(\lambda)) &= \operatorname{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}(\lambda^r(1/f'(\lambda))) \\
&= \begin{cases} 1 & \text{if } r = p-2, \\ -1 & \text{if } r = p-1, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Suppose that $u = \lambda^t + \bar\lambda^t$. Then

$$\begin{aligned}
i\langle e_r, e_r\rangle &= pi(\bar{\varsigma}^{r((p-1)/2+t)} + \bar{\varsigma}^{r((p-1)/2-t)} - \bar{\varsigma}^{((p+1)/2+t)} - \bar{\varsigma}^{r((p+1)/2-t)}) \\
&= pi(\varsigma^{r(p+1)/2} - \bar{\varsigma}^{r(p+1)/2})(\varsigma^{rt} + \bar{\varsigma}^{rt}) \\
&= -4p\sin(r(p+1)\pi/p)\cos(2rt\pi/p) \\
&= -4p\sin(r\pi + r\pi/p)\cos(2\pi rt/p) \\
&= (-1)^{r+1}4p\sin(r\pi/p)\cos(2\pi rt/p).
\end{aligned}$$

Let us define

$$\varepsilon(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0.\end{cases}$$

As $0 \le r \le (p-1)/2$, $\varepsilon\sin(r\pi/p) = 1$ so

$$\varepsilon(i\langle e_r, e_r\rangle) = (-1)^{r+1}\varepsilon\cos(2\pi rt/p) = (-1)^{r+1}(\lambda^t + \bar\lambda^t, \sigma)_{P_r},$$

where P_r is the infinite prime which takes $\lambda + \bar\lambda$ to $\varsigma^r + \bar{\varsigma}^r$.

If we repeat this for a general $u \in \mathbf{Q}(\lambda+\bar{\lambda})^*$ by writing $u = \sum_r a_r(\lambda^r + \bar{\lambda}^r)$, $a_r \in \mathbf{Q}$, then as $\mathrm{tr}_{\mathbf{Q}(\lambda)/\mathbf{Q}}$ is linear we see that

$$\varepsilon i\langle e_r, e_r\rangle = (-1)^{r+1}(u,\sigma)_{P_r}.$$

This number determines whether ς^r or $\bar{\varsigma}^r$ occurs in $\chi_{\mathrm{Hol}(M)}(h)$ and so is the coefficient of $\varsigma^r - \bar{\varsigma}^r$ as ASsign(h). Since the spaces $(u, \mathbf{Q}(\lambda))$ generate $\mathcal{H}_0(\mathbf{Q}(\lambda))$ we have shown

PROPOSITION 5.1. *As homomorphisms from $\mathcal{H}_0(\mathbf{Q}(\lambda))$ (or $\mathcal{H}_0(\mathbf{Z}[\lambda])$) to $\mathbf{Z}^{(p-1)/2}$,*

$$\mathrm{ASsign}(h) \circ S \circ T = W \circ \mathrm{multisign}(\lambda),$$

where $W(x_1, \ldots, x_{(p-1)/2}) = (x_1, -x_2, \ldots, (-1)^{r+1}x_r, \ldots, (-1)^{(p-1)/2}x_{(p-1)/2})$.

6. The case of genus $(p-1)/2$. Let us consider the simplest case of an action of C_p on a surface, with nonzero signature. This is an action on the surface of genus $(p-1)/2$ and necessarily has three fixed points (by an Euler-number argument). To these are associated three integers $0 < \beta_1, \beta_2, \beta_3 < p$, and $\beta_1 + \beta_2 + \beta_3 \equiv 0 \pmod p$. The cohomology representation must correspond to a Hermitian inner-product space (u, I), I an ideal of $\mathbf{Z}[\lambda]$, under the map T of §4 because $H^1(S_g;\mathbf{Q}) \cong \mathbf{Q}(\lambda)$ by §2 so the action of C_p on $H^1(S_g;\mathbf{Z})$ factors through $\mathbf{Z}[\lambda]$. However, according to Proposition 2.1, I must be principal so we can assume we have $(u, \mathbf{Z}[\lambda])$. We want to know u.

From Eichler's formula 3.1

$$\chi_{\mathrm{Hol}(M)} = 1 + \sum_{i=1}^{3} 1/(\varsigma^{\bar{\beta}_i} - 1),$$

where $\varsigma = e^{2\pi i/p}$. We shall need the formula (which is easy to check)

$$(\varsigma^r - 1)^{-1} = p^{-1}(\varsigma^r + 2\varsigma^{2r} + \cdots + (p-1)\varsigma^{(p-1)r}).$$

We continue to use the notation $\bar{x}$, when x is an integer, to mean some integer such that $x\bar{x} \equiv 1 \pmod p$. We write $D(x)$ for the integer satisfying $0 \le D(x) \le p-1$ and $D(x) \equiv x \pmod p$, so $D(x) = x - p\lfloor x/p \rfloor$. Then

$$(\varsigma^r - 1)^{-1} = p^{-1}\sum_{s=1}^{p-1} D(\bar{r}s)\varsigma^s,$$

$$\begin{aligned}(\varsigma^r - 1)^{-1} - (\bar{\varsigma}^r - 1)^{-1} &= p^{-1}\sum_{s-1}^{p-1}(D(\bar{r}s) - D(-\bar{r}s))\varsigma^s \\ &= p^{-1}\sum_{s=1}^{p-1}(2D(\bar{r}s) - p)\varsigma^s \\ &= p^{-1}\sum_{s=1}^{(p-1)/2}(2D(\bar{r}s) - p)(\varsigma^s - \bar{\varsigma}^s).\end{aligned}$$

Thus

$$\text{ASsign}(h) = p^{-1} \sum_{s=1}^{(p-1)/2} \sum_{i=1}^{3} (2D(\beta_i s) - p)(\varsigma^s - \bar{\varsigma}^s)$$
$$= \sum_{s=1}^{(p-1)/2} \left[2p^{-1} \left(\sum_{i=1}^{3} D(\beta_i s) \right) - 3 \right] (\varsigma^s - \bar{\varsigma}^s).$$

Since $\sum \beta_i \equiv 0 \pmod p$, $\sum \beta_i s \equiv 0 \pmod p$ and so $p^{-1} \sum D(\beta_i s) = 1$ or 2 as it must be a positive integer and $D(\beta_i s) < p$. So the coefficient of $\varsigma^s - \bar{\varsigma}^s$ in the above formula is -1 or $+1$ according to whether $p^{-1} \sum_{i=1}^{3} D(\beta_i s) = 1$ or 2 respectively.

But since $0 < p^{-1} D(\beta_3 s) < 1$,

$$p^{-1} \sum_{i=1}^{3} D(\beta_i s) = \begin{cases} 1 & \text{if } p^{-1}(D(\beta_1 s) + D(\beta_2 s)) < 1, \\ 2 & \text{if } p^{-1}(D(\beta_1 s) + D(\beta_2 s)) > 1. \end{cases}$$

So the coefficient of $\varsigma^s - \bar{\varsigma}^s$ is

$$- \varepsilon \sin\left((D(\beta_1 s) + D(\beta_2 s))\pi/p\right)$$
$$= -\varepsilon \sin((\beta_1 s + \beta_2 s)/p - \lfloor \beta_1 s/p \rfloor - \lfloor \beta_2 s/p \rfloor \pi)$$
$$= (-1)^{1 + \lfloor \beta_1 s/p \rfloor + \lfloor \beta_2 s/p \rfloor} \varepsilon \sin((\beta_1 + \beta_2) s\pi/p).$$

The decision to eliminate β_3 was quite arbitrary. We could have eliminated any one of the betas and obtained the same result. Moreover, since we are only interested in the ± 1, we can multiply together the three formuas and still have the correct number. Hence the coefficient of $\varsigma^s - \bar{\varsigma}^s$ is

$$-\varepsilon \sin((\beta_1 + \beta_2)s\pi/p) \sin((\beta_1 + \beta_3)s\pi/p) \sin((\beta_2 + \beta_3)s\pi/p).$$

So the P_s term of the multisignature is

$$(-1)^s \varepsilon \sin((\beta_1 + \beta_2)s\pi/p) \sin((\beta_1 + \beta_3)s\pi/p) \sin((\beta_2 + \beta_3 s\pi/p).$$

We now want to find a unit $v \in \mathbf{Z}[\lambda + \bar{\lambda}]$ for which the Hilbert symbols take these values. First note that

$$(-1)^s = \varepsilon \sin((p+1)s\pi/p), \quad (\text{for } 0 < s \leq (p-1)/2)$$
$$= \varepsilon (\sin((p+1)s\pi/p))^{-3}$$

(which will be more convenient).

Let

$$\xi = -\varsigma^{(p+1)/2} = e^{i\pi/p}, \qquad \mu = -\lambda^{(p+1)/2} \in \mathbf{Z}[\lambda],$$

$$\text{multisign}(\lambda)_{P_s} = \varepsilon(-1)^s \prod_{1 \leq j < k \leq 3} \sin((\beta_j + \beta_k)s\pi/p)$$
$$= \varepsilon \prod_{i \leq j < k \leq 3} \frac{\sin((\beta_j + \beta_k)s\pi/p)}{\sin((p+1)s\pi/p)}$$
$$= \varepsilon \prod_{i \leq j < k \leq 3} \frac{\xi^{(\beta_j + \beta_k)s} - \bar{\xi}^{(\beta_j + \beta_k)s}}{\xi^{(p+1)s} - \bar{\xi}^{(p+1)s}}.$$

So let us take

$$v = \prod_{1 \leq j < k \leq 3} \frac{\mu^{(\beta_j + \beta_k)} - \bar{\mu}^{(\beta_j + \beta_k)}}{\mu^{(p+1)} - \bar{\mu}^{(p+1)}}, \tag{6.1}$$

which is easily seen to be a unit of $\mathbf{Z}[\lambda]$. Then

$$\operatorname{multisign}(\lambda, (v, \mathbf{Q}(\lambda))) = \operatorname{multisign}(\lambda, H^1(S_g)).$$

Therefore a suitable candidate for $(u, \mathbf{Z}[\lambda])$ is $(v, \mathbf{Z}[\lambda])$ since they have the same multisignature. In fact they are isomorphic: This can be seen as follows. u and v both have Hilbert symbol 1 at every finite prime since they are units; they have equal Hilbert symbols at every infinite prime by the multisignature. Thus by the Hasse Cyclic-Norm Theorem **[14]**, $v = ua\bar{a}$ for some $a \in \mathbf{Q}(\lambda)$. Consider the prime factorization of the ideal $a\mathbf{Z}[\lambda]$. Since $a\bar{a}$ is a unit, $a\mathbf{Z}[\lambda]$ can only contain non-self-conjugate prime ideals in the form $\mathcal{P}^{\alpha}\bar{\mathcal{P}}^{-\alpha}$. By the Approximation Theorem **[11]** there is a $b \in \mathbf{Q}(\lambda)$ such that $\operatorname{ord}_{\mathcal{P}}(b) = -\alpha$ and $\operatorname{ord}_{\bar{\mathcal{P}}}(b) = 0$ for each of these pairs. Then $\operatorname{ord}_{\mathcal{P}}(bb^{-1}a) = 0$ at every single finite prime of $\mathbf{Z}[\lambda]$ so $c = b\bar{b}^{-1}a$ is a unit of $\mathbf{Z}[\lambda]$, and $v = uc\bar{c}$.

The isomorphism $(v, \mathbf{Z}[\lambda]) \to (u, \mathbf{Z}[\lambda])$ is just multiplication by c. It is a module isomorphism since c is a unit, and as for the product,

$$u(cx)(\overline{cy}) = uc\bar{c}x\bar{y} = ux\bar{y}, \qquad x, y \in \mathbf{Z}[\lambda].$$

This proves

PROPSITION 6.1. *When C_p acts on a surface of genus $(p-1)/2$ and $h \in C_p$ has fixed-point data $\sigma(h) = (p, (p-1)/2|\beta_1/p, \beta_2/p, \beta_3/p)$, the corresponding Hermitian inner-product space is isomorphic to $(v, \mathbf{Z}[\lambda])$, where v is given by* (6.1).

REMARK. A necessary and sufficient condition for an action to exist with the fixed-point data $\sigma(h) = (p, (p-1)/2|\beta_1/p, \beta_2/p, \beta_3/p)$ is that $\beta_1 + \beta_2 + \beta_3 \equiv 0 \pmod p$ **[10]**.

7. Higher genus. In order to deal with surfaces of higher genus we need an equivariant decomposition of the surface. As the action is determined up to conjugacy by the fixed-point data it suffices to build up, from the smaller pieces, some action on a surface with the correct fixed-point data. Since we are dealing only with C_p all the alphas are equal to p and we shall write $(g|\beta_1, \dots, \beta_n)$ for the surface, determined up to equivariant isomorphism, on which $\sigma(h) = (p, g|\beta_1/p, \dots, \beta_n/p)$. β_i will be called the index of the corresponding fixed point. As mentioned in §1, $\sum \beta_i \equiv 0 \pmod p$. So as to have enough building blocks we allow actions on tori and spheres.

The operations for constructing new actions are:

(a) connected sum at a fixed point. Suppose we have two actions on two surfaces S and S' such that S has a fixed point of index β_1 and S' has a fixed point of index $-\beta_1$. Then we can cut out two discs around these fixed points and join the surfaces along the boundary.

(b) Adding p handles to a connected surface so that they are permuted by C_p.

(c) Joining one surface to another by p cylinders that are permuted by C_p.

PROPOSITION 7.1. *Every action of C_p on a surface can be obtained from the following using operations* (a), (b), (c).

(i) *An action on a surface of genus $(p-1)/2$ (with three fixed points).*

(ii) *A free action on a torus.*

(iii) *An action on a sphere (with two fixed points).*

REMARK. (b) is unnecessary; we could perform (c) with an action on a torus instead.

PROOF. Look at the indices $\beta_1, \dots, \beta_n$. Collect all the pairs $(\beta, -\beta)$ and put them aside. Suppose we are left with $\beta_1, \dots, \beta_r$. If $r = 0$ carry on to the next stage. Otherwise $r \geq 3$ (or we would have $(\beta_1, -\beta_1)$). Let $\gamma_1 = -\beta_1 - \beta_2 : \gamma_1 \neq 0$ since this would imply $\beta_1 = -\beta_2$. Thus we can form $((p-1)/2|\beta_1, \beta_2, \gamma_1)$. Now let $\gamma_2 = \gamma_1 - \beta_3$; if $\gamma_2 \neq 0$ we can form $((p-1)/2| - \gamma_1, \beta_3, \gamma_2)$. The connected sum at the fixed points corresponding to $\pm\gamma_1$ is $(p-1|\beta_1, \beta_2, \beta_3, \gamma_2)$.

Continue with $\gamma_3 = \gamma_2 - \beta_4$ etc. Until $\gamma_{r'-1} = 0$ in which case we have constructed $((r' - 2(p-1)/2|\beta_1, \dots, \beta_{r'})$. Now start again with $\beta_{r'+1}$ to construct an action on another surface in the same way. The resulting surfaces can be joined using operation (c) to give $((r-2)(p-1)/2|\beta_1, \dots, \beta_r)$.

Now for the pairs $(\beta, -\beta)$. Each of these can be realized on a sphere, $(0|\beta, -\beta)$. These and the result of the first stage, if any, can be joined by operation (c). The result is $((n-2)(p-1)/2|\beta_1, \dots, \beta_n)$, unless $n = 0$ in which case we use the free action on a torus, $(1|\varnothing)$.

The indices are now correct. A look at the Euler number shows that the genus is the least possible, for

$$\chi(S_g) = p\chi(S_{\bar{g}}) - n(p-1)$$

where $S_{\bar{g}} = C_p \backslash S_g$. Hence $g = (n-2)(p-1)/2 + p\bar{g}$. So $g \geq (n-2)(p-1)/2$, and if $n = 0$, $g \geq 1$. If necessary the genus can be increased by using operation (b). This completes the proof of the theorem.

We need to know what this decomposition does to cohomology. Clearly (a) gives an orthogonal direct sum. Operation (b) gives an orthogonal direct sum with $\overline{H}(\mathbf{Z}C_p)$ since it involves a connected sum with p tori. Each torus contributes $\overline{H}(\mathbf{Z})$ and they are permuted.

To deal with (c) first consider the case of joining a sphere $(0|\gamma, -\gamma)$ to another sphere $(0|\sigma, -\sigma)$ using (c), obtaining $U_{\gamma,\sigma}$ say. Let c_i be a cycle in $H_1(U)$ represented by a loop around the ith handle. $\sum c_i = 0$ and the c_i are orthogonal under the intersection pairing; they generate a submodule C of $H_1(U)$ and $C \cong \mathbf{Z}[\lambda]$. Let d be a cycle which goes up one cylinder and down another. Let $d_i = h^i d$: Again the d_i are orthogonal and generate a submodule D of $H_1(U)$ with $D \cong \mathbf{Z}[\lambda]$. An argument with a Mayer-Vietoris sequence shows that $H_1(U) \cong C \oplus D$, so under duality $H^1(U) \cong \overline{H}(\mathbf{Z}[\lambda])$.

Suppose we wish to join S to T using (c). We may assume that S and T have fixed points p_s, p_t of indices γ, δ respectively, for if T, say, did not have a fixed point we could perform (b) on S instead. Form $u_{\gamma,\delta}$ as above and join it to S at p_s using (a). Join the result to T at p_t to get V,

$$H^1(V) \cong H^1(S) \oplus H^1(T) \oplus \overline{H}(\mathbf{Z}[\lambda]).$$

But V is isomorphic to the space obtained by joining S to T using (c) (because it has the same fixed-point data).

This information together with Proposition 7.1 implies Theorem B of the Introduction.

REMARK. Ewing [**8**] shows that the index in $W_2(\mathbf{Z}, C_p)$ of the submodule generated by the spaces which occur as $H^1(S_g; \mathbf{Z})$ for an action of C_p on S_g is equal to $h^*(p)$, the ideal class number of $\mathbf{Z}[\lambda + \bar{\lambda}]$.

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