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A CLASSIFICATION OF A CLASS OF 3-BRANCHFOLDS

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ABSTRACT. An n-orbifold is a topological space provided with a local modelling on (an open set in $\mathbf{R}^n$)/(a finite group action). Mainly, we deal with 3-branchfolds (i.e. 3-orbifolds with 1-dimensional singular locus). We define a map between two 3-branchfolds. With respect to this map, we prove some facts parallel to 3-manifold theorems. Using the facts, we classify a class of 3-branchfolds, analogous to Waldhausen's classification theorem of Haken manifolds.

The concept of orbifold is introduced by Satake [**4**] and renamed by W. Thurston (Thurston [**5**, Chapter 13]. It is a generalization of the concept of manifold. An n-orbifold is a connected and separable metric space which is locally homeomorphic to (an open set in $\mathbf{R}^n$)/(a finite group action) and each point of it is provided with an isotropy data.

Mainly, we deal with 3-branchfolds (i.e. 3-orbifolds with 1-dimensional singular locus) whose underlying spaces are orientable 3-manifolds. In this paper, we prove the branchfold version of the classification of Haken manifolds (due to F. Waldhausen [**5**]). For this purpose, we need to generalize important facts in the theory of 3-manifolds to those of 3-branchfolds via the functor (manifolds, continuous maps)$\rightarrow$(orbifolds, OR-maps).

After preparing a general theory of branchfolds, in §4, we formulate and prove the following:

BRANCHFOLD SPHERE THEOREM. *Let (S^2,a) be an elliptic sphere, (M,b) be a 3-branchfold which does not contain bad spheres, and $f\colon (S^2,a) \rightarrow (M,b)$ be a proper normal OR-map which is not extendable to an OR-map from the cone $C(S^2,a)$ of (S^2,a). Then there exist an elliptic sphere $(S^{2'},a')$ and a normal OR-embedding $g\colon (S^{2'},a') \rightarrow (M,b)$ which are not extendable to an OR-map from $C(S^{2'},a')$.*

The notation (M,b) is due to Kato [**3**].

Let ω be a class of 3-branchfolds whose element (M,b) satisfies the following conditions (1)–(5):

(1) The 1-dimensional singular locus is not empty.

(2) (M,b) is uniformizable.

(3) (M,b) is irreducible.

(4) For any component (F,b') of $\partial(M,b)$,

$$\operatorname{Ker}(i_*\colon \pi_1(F,b') \rightarrow \pi_1(M,b)) = 1,$$

where i is the inclusion.

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(5) $\partial(M - \overset{\circ}{U}(\Sigma_M))$ is incompressible in $M - \overset{\circ}{U}(\Sigma_M)$, where $U(\Sigma_M)$ is a regular neighborhood of Σ_M in M.

For the definition of the fundamental group of an orbifold (M, b), $\pi_1(M, b)$, see §0.

Our main result is as follows:

THE CLASSIFICATION THEOREM. *Let (M, b), (N, c) be elements of ω. If there exists an isomorphism $\phi\colon \pi_1(M_0) \to \pi_1(N_0)$ which is normal and the induced homomorphism $\overline{\phi}\colon \pi_1(M, b) \to \pi_1(N, c)$ is an isomorphism which is peripheral in $\pi_1(N - \Sigma_N)$, then (M, b) and (N, c) are OR-isomorphic.*

0. Preliminaries. Let X be a connected and separable n-dimensional metric space and b a function from X to natural numbers $\mathbf{N}$. We call the pair (X, b) an orbifold, if, for any point $x \in X$, there exist an open neighborhood X_x of x in X and a finite subgroup G_x of the orthogonal group $O(n)$, such that $X_x = G_x \setminus \mathbf{R}^n$ and, for any point $z \in X_x$, $b(z) = \#G_x(z)$, where $G_x(z)$ is the isotropy subgroup of z in G_x. We call X the underlying space of (X, b).

Let M be a connected and second countable n-dimensional (topological) manifold and G a group of homeomorphisms from M to M. If M and G satisfy the following properties,

(1) the action

$$\begin{aligned} G \times M &\to M \times M \\ (g, z) &\to (z, g(z)) \end{aligned}$$

is proper.

(2) For any point $z \in M$, there exist a $G(z)$-invariant open neighborhood M_z of z in M and a finite subgroup G'_z of $O(n)$, such that $(M_z, G(z))$ is homeomorphic to $(\mathbf{R}^n, G'_z)$ as a pair. Then $G \setminus M$ is metrizable and we can define a function $b'\colon G \setminus M \to \mathbf{N}$ by $b'(x) = \#G(z)$, where $G \cdot z = x \in X$, $z \in M$. Then $(G \setminus M, b')$ is an orbifold. Conversely, if, for an orbifold (X, b), there exist M and G as above, such that $X = G \setminus M$ and $b = b'$, then we call (X, b) uniformizable and (M, G) is a uniformization of (X, b).

We call $\Sigma_X := \{x \in X | b(x) \geq 2\}$ the branch set of an orbifold (X, b). We define the stratification $\mathcal{S}_X$ of (X, b) to satisfy the following properties.

(1) $\mathcal{S}_X$ is a stratification of X.

(2) b is constant on each stratum of $\mathcal{S}_X$.

(3) If C and D are distinct strata of $\mathcal{S}_X$ such that $C \subset \overline{D}$, then $b(D)$ divides $b(C)$.

And we define $\mathcal{S}_X^{(k)}$ to be the set of all k-dimensional strata of $\mathcal{S}_X$. $X_0 = X - \Sigma_X$ is the only n-dimensional stratum. An n-orbifold (X, b) is said to be an n-branchfold, if $\dim \Sigma_X \leq n - 2$. Put $H = \pi_1(X_0)$ and $\{l_j | j \in J\} = \mathcal{S}_X^{(n-2)}$. We call μ_j a normal loop of l_j, if μ_j is a boundary loop of a disk in X which meets Σ_X transversally at exactly one point of l_j. We call l_j the center locus of μ_j. Put $\Omega(X, b) = \{\mu_j | \mu_j$ is a normal loop of $l_j \in \Sigma_X^{(n-2)}\}$. Putting $b_j = b(l_j)$ and $\mu^b = \{\mu_j^{b_j} | j \in J\}$, let $H\langle \mu^b \rangle$ be the normal closure of μ^b in H, obviously, $H\langle \mu^b \rangle$ independent of the choice of the base point of X. We define the regular neighborhood $U(x, (X, b))$ of x in (X, b) to satisfy the following properties (1) and (2):

(1) $U(x,(X,b))$ is a regular neighborhood of x in X.

(2) If $S \cap U(x,(X,b)) \neq \emptyset$, for an $S \in \mathcal{S}_X$, then $x \in \overline{S}$. We also denote $U(x,(X,b))$ to $B(x,(X,b))$ or (B_x, b'). And we define $\pi_1(X,b) := H/H\langle\mu^b\rangle$.

1. Maps between orbifolds.

DEFINITION 1.1. Let (X,b) and (Y,c) be orbifolds. We call a PL-map $f\colon X \to Y$ an OR-map (order respecting map) if $c(f(x))|b(x)$ for each $x \in X$. We shall denote the above f by $f\colon (X,b) \to (Y,c)$.

PROPOSITION 1.2. *Let (X,b) and (Y,c) be orbifolds. An OR-map $f\colon (X,b) \to (Y,c)$ induces a natural homomorphism $f_*\colon \pi_1(X,b) \to \pi_1(Y,c)$.*

PROOF. Let $X_0 = X - \Sigma_X$, $Y_0 = Y - \Sigma_Y$, $H = \pi_1(X_0)$ and $H' = \pi_1(Y_0)$. By the property of OR-map, we have that $f(X_0) \subset Y_0$. Hence, $(f|X_0)\colon X_0 \to Y_0$ induces a homomorphism $(f|X_0)_\#\colon H \to H'$. By the continuity of f, for any point x of $\mathcal{S}_X^{(n-2)}$ and any regular neighborhood $B'(f(x),(Y,c))$, there exists a regular neighborhood $B(x,(X,b))$ such that $f(B(x,(X,b))) \subset B'(f(x),(Y,c))$. We may assume that the normal loop of each $l \in \mathcal{S}_X^{(n-2)}$ is contained in $B(x,(X,b))$. Thus, $(f|X_0)_\#(H\langle\mu^b\rangle) \subset H'\langle\mu^c\rangle$. Hence, the proposition is proved.

DEFINITION 1.3. OR-maps $f, g\colon (X,b) \to (Y,c)$ are called OR-homotopic if there exists an OR-map $F\colon (X,b) \times I \to (Y,c)$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, for any $x \in X$, where $(X,b) \times I := (X \times I, b \circ p_1)$, p_1 is a projection to the first factor.

REMARK 1.4. $F|((X - \Sigma_X) \times I)$ gives a homotopy between $f|(X - \Sigma_X)$ and $g|(X - \Sigma_X)$.

PROPOSITION 1.5. *Let $f, g\colon (X,b) \to (Y,c)$ be OR-maps. If f and g are OR-homotopic, then $f_* = g_*\colon \pi_1(X,b) \to \pi_1(Y,c)$.*

PROOF. Take any $[\sigma] \in \pi_1(X,b)$. We may assume σ is a map into $X - \Sigma_X$. Hence, $f \circ \sigma$ and $g \circ \sigma$ are maps into $Y - \Sigma_Y$. By the above remark, $[f \circ \sigma] = [g \circ \sigma]$ in $\pi_1(Y - \Sigma_Y)$. Hence, $[f \circ \sigma] = [g \circ \sigma]$ in $\pi_1(Y,c)$, that is $f_*([\sigma]) = g_*([\sigma])$ Q.E.D.

DEFINITION 1.6 An OR-map $f\colon (X,b) \to (Y,c)$ is called proper if $c(f(x)) = b(x)$, for any $x \in X$.

DEFINITION 1.7 A proper OR-map $f\colon (X,b) \to (Y,c)$ is called an OR-embed ding if $f\colon X \to Y$ is an embedding. An OR-embedding $f\colon (X,b) \to (Y,c)$ is called an OR-isomorphism if $f\colon X \to Y$ is a homeomorphism.

DEFINITION 1.8 When there is an OR-embedding $f\colon (X,b) \to (Y,c)$, we say that f or the image of f is a suborbifold of (Y,c).

2. Fuchsian complex. In this section, we deal with only 3-branchfolds of which underlying spaces are 3-manifolds. From now on, any 2-suborbifold (F,b') in a 3-branchfold (M,b) must have the following properties;

(1) F is properly embedded and 2-sided in M.

(2) The intersections of F and $\mathcal{S}_M^{(1)}$ are transversal.

(3) $F \cap \mathcal{S}_M^{(0)} = \emptyset$.

DEFINITION 2.1. Let $D^2(n)$ be a 2-orbifold (D^2, b), where Σ_{D^2} = a point $p \in \operatorname{Int} D^2$, $b(p) = n$, and D^2 is a 2-disk.

DEFINITION 2.2. A 2-suborbifold (F', b) of a 3-branchfold (M, b) is called incompressible, if for any 2-suborbifold $D^2(n)$ in (M, b) such that $D^2(n) \cap (F', b) = \partial D^2(n)$, there is a 2-suborbifold $D^2(m)$ in (F', b) such that $\partial D^2(n) = \partial D^2(m)$.

DEFINITION 2.3. $S^2(n_1, \dots, n_r) := (S^2, b)$, where $\Sigma_{S^2} = \{p_1, \dots, p_r\}$, $b(p_i) = n_i$, $(i = 1, 2, \dots, r)$, and S^2 is a 2-sphere.

DEFINITION 2.4. $S^2(\mathrm{bad}) := \{S^2(n), S^2(m,n)\}$, where $m, n \in \mathbf{Z}$, $m \neq n$.

$$S^2(\mathrm{elliptic}) := \{S^2(n,n), S^2(2,2,n), S^2(2,3,3), S^2(2,3,4), S^2(2,3,5)\}.$$

We call a 2-orbifold belonging to $S^2(\mathrm{bad})$ to be a bad sphere, a 2-orbifold belonging to $S^2(\mathrm{elliptic})$ to be an elliptic sphere.

DEFINITION 2.5. For an elliptic sphere (S^2, b), we define $C(S^2, b) := (CS^2, b')$, where $b'(z) = b(x)$, when $z \in Cx$-cone point, $x \in (S^2, b)$, or $b'(z) = \#\pi_1(S^2, b)$, when $z =$ cone point.

DEFINITION 2.6. A 3-branchfold (M, b) is said to be irreducible if any elliptic 2-suborbifold (S^2, b') bounds $C(S^2, b')$ in the ambient orbifold (M, b).

DEFINITION 2.7. Define the associated Fuchsian complex $K_{(M,b)}$ of a 3-branchfold (M, b) as follows: Let $U(\Sigma_M)$ be the regular neighborhood of Σ_M, e_j^2 the 2-cell, $\mu_j \subset \partial U(\Sigma_M)$ the normal loop of $l_j \in \mathcal{S}^{(1)}$, $b_j = b(l_j)$ and ϕ_j the map from ∂e_j^2 to μ_j defined by $\phi_j(e^{i\theta}) = e^{ib_j\theta}$, $0 \leq \theta \leq 2\pi$. Let $(M - \operatorname{Int} U(\Sigma_M)) \cup (e_j, \phi_j)$ be the space constructed by attaching e^2's to $M - \operatorname{Int} U(\Sigma_M)$ with attaching maps ϕ_j's, respectively. We define $K_{(M,b)} = (M - \operatorname{Int} U(\Sigma_M)) \cup (e_j, \phi_j)$.

It is clear that (M, b) uniquely determines $K_{(M,b)}$ up to homeomorphism. Moreover, in the case where $\partial(M, b)$ contains no elliptic sphere, the converse holds.

PROPOSITION 2.8. *Let (M, b) and (N, c) be 3-branchfolds which have no elliptic spheres in their boundaries. If $K_{(M,b)}$ and $K_{(N,c)}$ are homeomorphic, then (M, b) and (N, c) are OR-isomorphic.*

PROOF. Let (M_0, b_0) be orbifold $(M, b) - \bigcup \operatorname{Int} U(x, (M, b))$, where $U(x, (M, b))$ is a regular neighborhood of x in $\operatorname{Int}(M, b)$, x is a vertex in the 0-strata. It is easy to see that (M_0, b_0) and (N_0, c_0) are homeomorphic, since $K_{(M,b)}$ and $K_{(N,c)}$ are homeomorphic. So (M, b) and (N, c) are OR-isomorphic, since both (M, b) and (N, c) do not contain elliptic spheres in their boundaries. Q.E.D.

PROPOSITION 2.9. $\pi_1(M, b) = \pi_1(K_{(M,b)})$.

PROOF. The kernel of the homomorphism $i_*\colon \pi_1(M - \operatorname{Int} \Sigma_M) \to \pi_1(K_{(M,b)})$ is $H\langle \mu^b \rangle$. On the other hand, $\pi_1(M, b) = H/H\langle \mu^b \rangle$ by the definition. Q.E.D.

3. Covering orbifold.

DEFINITION 3.1. Let $(\tilde{X}, \tilde{b})$ and (X, b) be orbifolds. An OR-map $p\colon (\tilde{X}, \tilde{b}) \to (X, b)$ is called an OR-covering if

(1) p is a proper OR-map, and

(2) $p\colon \tilde{X} \to X$ is a covering map.

We call $(\tilde{X}, \tilde{b})$ an OR-covering orbifold of (X, b). (In the usual sense, a covering orbifold is a branched covering of X with the branch sets Σ_X. But an OR-covering is a nonbranched covering.)

PROPOSITION 3.2. *Let $(\tilde{X},\tilde{b})$ and (X,b) be 3-branchfolds. If there exists an OR-covering $p\colon (\tilde{X},\tilde{b}) \to (X,b)$, then there exists a covering $q\colon K_{(\tilde{X},\tilde{b})} \to K_{(X,b)}$.*

PROOF. Put

$$K_{(\tilde{X},\tilde{b})} = (\tilde{X} - \operatorname{Int} U(\Sigma_{\tilde{X}})) \cup (e_j^2, \phi_j),$$
$$K_{(X,b)} = (X - \operatorname{Int} U(\Sigma_X)) \cup (e_i^2, \phi_i).$$

$p\colon (\tilde{X},\tilde{b}) \to (X,b)$ induces a covering $p|(\tilde{X} - \operatorname{Int} U(\Sigma_{\tilde{X}}))\colon \tilde{X} - \operatorname{Int} U(\Sigma_{\tilde{X}}) \to X - \operatorname{Int} U(\Sigma_X)$, and lifts the normal loops of (X,b) to the normal loops of $(\tilde{X},\tilde{b})$. (Note $p\colon \tilde{X} \to X$ is a covering map.) Hence, we can define the desired q as follows; $q|(\tilde{X} - \operatorname{Int} U(\Sigma_{\tilde{X}})) := p|(\tilde{X} - \operatorname{Int} U(\Sigma_{\tilde{X}}))$, and $(q|e_j^2)\colon e_j^2 \to e_{i(j)}^2$ is a natural homeomorphism, where $p(l_j) = l_{i(j)}$. Q.E.D.

COROLLARY 3.3. *If $p\colon (\tilde{X},\tilde{b}) \to (X,b)$ is an OR-covering, then the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\tilde{X},\tilde{b}) & \xrightarrow{i_1} & \pi_1(K_{(\tilde{X},\tilde{b})}) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\ \pi_1(X,b) & \xrightarrow{i_2} & \pi_1(K_{(X,b)}) \end{array}$$

where i_1 and i_2 are the isomorphisms as in 2.9. Namely, p_* is an injection.

PROOF. It is obvious from 2.9 and the construction of q. Q.E.D.

COROLLARY 3.4. *Let $(\tilde{X},\tilde{b})$ and (X,b) be 3-branchfolds which have no elliptic spheres in their boundaries. If there exists an OR-covering $p\colon (\tilde{X},\tilde{b}) \to (X,b)$ and p_* is an isomorphism, then $(\tilde{X},\tilde{b})$ and (X,b) are OR-isomorphic.*

PROOF. From the above commutative diagram, $q_*\colon \pi_1(K_{(\tilde{X},\tilde{b})}) \to \pi_1(K_{(X,b)})$ is an isomorphism. Hence, $q\colon K_{(\tilde{X},\tilde{b})} \to K_{(X,b)}$ is a homeomorphism. Thus, by 2.8, $(\tilde{X},\tilde{b})$ and (X,b) are OR-isomorphic. Q.E.D.

REMARK 3.5. For an orbifold (Y,c) and a covering $p\colon X \to Y$, we can construct an orbifold structure, of X by $b = c \cdot p$. With this orbifold structure $p\colon (X,b) \to (Y,c)$ becomes an OR-covering.

4. Sphere theorem (an orbifold version). In this section, we assume that the underlying spaces of orbifolds are manifolds.

DEFINITION 4.1. Let (F,c) be a 2-branchfold and (M,b) be a 3-branchfold. We say an OR-map $f\colon (F,c) \to (M,b)$ is transversal, if $f(F)$ intersects transversally $S_M^{(1)}$ and $f(F) \cap S_M^{(0)} = \varnothing$.

For a map $f\colon X \to Y$ we define the singular set $S(f)$, of f to be the closure of $\{x \in X | \#(f^{-1}(f(x))) > 1\}$. We decompose $S(f)$ as a disjoint union, $S(f) = \bigcup_{i \geq 1} S_i(f)$, by $S_i(f) = \{x \in S(f) | \#(f^{-1}(f(x))) = i\}$. Putting $\Gamma_i(f) = f(S_i(f))$, we call the points of $\Gamma_1(f)$ branch points, $\Gamma_2(f)$ double points, $\Gamma_3(f)$ triple points, and so on. Let F be a 2-manifold, M be a 3-manifold and $f\colon M \to N$ be a general position map. (As to general position map, refer to Hempel [**1**].) We define the complexity of a general position map $f\colon (F,\partial F) \to (M,\partial M)$ to be the pair $c(f) = (t(f), d(f))$ where $t(f)$ is the number of triple points of f and $d(f)$ is the number of double curves of f, where a double curve is a connected component of the set of double points. We order complexities lexicographically; i.e. $c(f_1) < c(f)$ if either $t(f_1) < t(f)$ or $t(f_1) = t(f)$ and $d(f_1) < d(f)$.

PROPOSITION 4.2. *Let (S^2, a) be a bad sphere and (M, b) be a 3-branchfold. If $f\colon (S^2, a) \to (M, b)$ be a proper and transversal OR-map, then there exist a bad sphere $(S^{2'}, a')$ and a transversal OR-embedding $g\colon (S^{2'}, a') \to (M, b)$.*

PROOF. Since M is a manifold, we may assume that $f\colon S^2 \to M$ is general position as a map. Moreover, by relation of the dimensions of S^2, M, and Σ_M, f is transversal.

If $c(f) = 0$, then f is clearly an OR-embedding. As the next step of the induction, assuming that $\Gamma_1(f) = \varnothing$ and that the conclusion holds for all maps f' (orbifolds (M', b'), etc.) such that $c(f') < c(f)$ and $\Gamma_1 f' = \varnothing$. We show that for f the conclusion holds.

1. If f has a simple closed double curve, then the conclusion holds. In fact, among such simple closed double curves of f, take the inner most one in S^2. The two components of the preimage of the curve, J_1 and J_2, bound disks D_1 and D_2, giving rise to suborbifolds (D_1, a_1) and (D_2, a_2), respectively.

Case 1. $a_1 = a_2 = 1$.

We define an OR-map $f_1\colon S^2 \to (M, b)$ by using $f((S^2, a) - \operatorname{Int} D_2) \cup f(D_1)$. Then f_1 is a proper OR-map from a bad sphere to (M, b). Since $c(f_1) < c(f)$, the conclusion holds from the inductive hypothesis.

Case 2. $a_1 = 1$ and $(D_2, a_2) = D^2(n)$.

Then $(f(D_1) \cup f(D_2), a_1 \cup a_2)$ is a bad sphere in (M, b).

Case 3. $a_1 = 1$ and $(D_2, a_2) = D^2(n, m)$.

Define an OR-map f_1 as in Case 1. Then f_1 is a proper OR-map from a bad sphere to (M, b). Since $c(f_1) < c(f)$, the conclusion holds from the inductive hypothesis.

Case 4. $(D_1, a_1) = D^2(m)$ and $(D_2, a_2) = D^2(n)$.

Define an OR-map f_1 as in Case 1. Then f_1 is a proper OR-map from a bad sphere to (M, b). Since $c(f_1) < c(f)$, the conclusion holds from the inductive hypothesis.

2. *Tower construction.* Forget the orbifold structures of (S^2, a) and (M, b) and construct a tower of height n in a similar way to the tower construction of the sphere theorem of 3-manifolds,

$$\begin{tikzcd}
 & V_n \arrow[r, phantom, "\underset{i_n}{\subset}"] & M_n \arrow[dl, "p_n"] \\
 & V_{n-1} \arrow[r, phantom, "\underset{i_{n-1}}{\subset}"] & M_{n-1} \arrow[dl, "p_{n-1}"] \\
 & \vdots & \vdots \\
 & V_1 \arrow[r, phantom, "\underset{i_1}{\subset}"] & M_1 \arrow[dl, "p_1"] \\
S^2 \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uuur, "f_{n-1}"] \arrow[uuuur, "f_n"] & V_0 \arrow[r, phantom, "\underset{i_0}{\subset}"] & M_0
\end{tikzcd}$$

where $f_0 = f$, $M_0 = M$, V_j is a regular neighborhood of $f_j(S^2)$, p_j is a covering map, i_j is an inclusion map, f_j is a lift of f_{j-1}, $\pi_1(V_n)$ is a finite group and $\pi_1(V_j)$ is an infinite group when $j < n$. V_0 has an orbifold structure as the restriction of the orbifold structure of (M, b) and M_1 has an orbifold structure such that $p_1 \colon M_1 \to V_0$ is an OR-map, which is introduced in §3. We denote those orbifolds (V_0, b'_0) and (M_1, b_1), respectively. Since $b_1(f_1(x)) = b_0(p_1 f_1(x)) = b_0(f(x))|a(x)$ for any point $x \in S^2$, $f_1 \colon (S^2, a) \to (M_1, b_1)$ become an OR-map. Similarly, V_j and M_j have orbifold structures (V_j, b'_j) and (M_j, b_j), respectively, and $p_j \colon (M_j, b_j) \to (V_{j-1}, b'_{j-1})$ and $f_j \colon (S^2, a) \to (M_j, b_j)$, respectively.

3. If f_n is an embedding, then f_0 has a simple closed double curve. (Hence the conclusion holds by step 1.) We can prove the statement by the same argument in the proof of the sphere theorem of 3-manifolds. Refer to Hempel [**1**, p. 52].

4. If f $(= f_0)$ has no simple closed double curve, then there exists a proper transversal OR-map f' from a bad sphere $(S^{2'}, a')$ to (M, b) such that (1) $t(f') < t(f)$, (2) $\Gamma_1 f' = \varnothing$. (Hence, the conclusion holds by the inductive hypothesis.) The proof is as follows: By step 3, we may assume that f_n is singular. Since $\pi_1(V_n)$ is finite, each component of ∂V_n is S^2. Clearly there exists a bad sphere (S^2, a'). Put $j = i_0 \circ (p_1 \circ i_1) \circ \cdots \circ (p_n \circ i_n)$. By the same way as the sphere theorem of 3-manifolds, we can show that $f' = j \circ s \colon (S^{2'}, a') \to (M, b)$ has desired properties (1) and (2). Thus the proof of step 4 has finished.

Next, we must consider the case when $\Gamma_1 f \neq \varnothing$. We also use the induction on $c(f)$. We can similarly proceed to steps 1, 2, and 3. We have only to assume that f_0 has no simple closed double curve. We can show that there exists an OR-embedding $s \colon (S^{2'}, a') \to (M, b)$, similar to step 4. Since p_j is an immersion, $f' = i_0 \circ (p_1 \circ i_1) \circ \cdots \circ (p_n \circ i_n)$ has no branch points. f' is transversal. So the proof is completed by the first case. Q.E.D.

DEFINITION 4.3. An OR-map $f \colon (M, b) \to (N, c)$ is called normal if, for any normal loop $\mu \in \Omega(M, b)$, there exists a normal loop $\nu \in \Omega(N, c)$ such that $f(\mu)$ and ν are freely homotopic in $N - \Sigma_N$.

THEOREM 4.4. *Let (S^2, a) be an elliptic sphere, (M, b) be a 3-branchfold which does not contain bad spheres, and $f \colon (S^2, a) \to (M, b)$ be a proper normal OR-map which is not extendable to an OR-map from $C(S^2, a)$. Then there exists an elliptic sphere $(S^{2'}, a')$ and a normal OR-embedding $g \colon (S^{2'}, a') \to (M, b)$ which is not extendable to an OR-map from $C(S^{2'}, a')$.*

PROOF. Since M is a manifold, we may assume that $f \colon S^2 \to M$ is general positive as a map. Moreover, by relation of the dimensions of S^2, M, and Σ_M, and by the normality of the OR-map f, we may assume that $\Gamma(f) \cap \Sigma_M = \varnothing$ and f is transversal.

If $c(f) = 0$, then f is clearly an OR-embedding. As the next step of the induction, assuming that $\Gamma_1(f) = \varnothing$ and that the conclusion holds for all maps f' (orbifolds (M', b'), etc.) such that $c(f') < c(f)$ and $\Sigma_1 f' = \varnothing$. We show that for f the conclusion holds.

1. If f has a simple closed double curve, then the conclusion holds. In fact, among such simple closed double curves of f, take the innermost one in S^2. The two components of the preimage of the curve, J_1 and J_2, bound disks D_1 and D_2, giving rise to suborbifolds (D_1, a_1) and (D_2, a_2), respectively.

Case 1. $a_1 = a_2 = 1$.

We define an OR-map $f_1\colon S^2 \to (M, b)$ by using $f(D_1) \cup f(D_2)$, and define an OR-map $f_2\colon (S^2, a) \to (M, b)$ by using $f((S^2, a) - \operatorname{Int} D_2) \cup f(D_1)$. We can separate f into f_1 and f_2. Since $\pi_2(M - \Sigma_M) = 0$, f_1 is extendable to CS^2. Hence f_2 is not extendable to $C(S^2, a)$. Since $c(f_2) < c(f)$, the conclusion holds from the inductive hypothesis.

Case 2. $a_1 = 1$ and $(D_2, a_2) = D^2(n)$.

Then $(f(D_1) \cup f(D_2), a_1 \cup a_2)$ is a bad sphere in (M, b). This contradicts the hypothesis that there is no bad sphere in (M, b).

Case 3. $a_1 = 1$ and $(D_2, a_2) = D^2(n, m)$.

Since there is no bad sphere in (M, b), $n = m$. Define an OR-map f_2 as in Case 1. f_2 is a proper OR-map from a bad sphere to (M, b). By Proposition 4.2, this contradicts the hypothesis that there is no bad sphere in (M, b).

Case 4. $a_1 = 1$ and $(D_2, a_2) = D^2(n_1, n_2, n_3)$.

Define OR-maps f_1 and f_2 as in Case 1. If both $f_1: S^2(n_1, n_2, n_3) \to (M, b)$ and $f_2: S^2 \to (M, b)$ are extendable to the cones, then f is extendable to the cone, since f_1 and f_2 consistent on $D_1 = D^2$. Hence at least one of f_1 and f_2 is not extendable to the cone. By using this fact, we can construct an OR-map from an elliptic sphere to (M, b) which is not extendable to the cone and of which complexity is smaller than $c(f)$.

Case 5. $(D_1, a_1) = D^2(m)$ and $(D_2, a_2) = D^2(n)$.

Since there is no bad sphere in (M, b), $m = n$. Define OR-maps f_1 and f_2 as in Case 1. If both $f_1\colon S^2(n, n) \to (M, b)$ and $f_2\colon S^2(n, n, k) \to (M, b)$ are extendable to the cones, then f is extendable to the cone, since f_1 and f_2 consistent on $(D_1, a_1) = D^2(n)$. We can show the rest of this proof in a similar manner to Case 4.

Case 6. $(D_1, a_1) = D^2(m)$ and $(D_2, a_2) = D^2(n_1, n_2)$.

Define OR-maps f_1 and f_2 as in Case 1. If both $f_1: S^2(m, n_1, n_2) \to (M, b)$ and $f_2\colon S^2(m, m) \to (M, b)$ are extendable to the cones, then f is extendable to the cone, since f_1 and f_2 consistent on $(D_1, a_1) = D^2(m)$. We can show the rest of this proof in a manner similar to Case 4.

We proceed to steps 2 and 3 quite similar to the proof of Proposition 4.2.

4. If f $(= f_0)$ has no simple closed double curve, then there exists a proper normal OR-map f' from an elliptic sphere (S_2', a') to (M, b) such that

(1) f' is not extendable to the cone.

(2) $t(f') < t(f)$.

(3) $\Gamma_1 f' = \varnothing$.

(Hence, the conclusion holds by the inductive hypothesis.)

The proof is as follows: By step 3, we may assume that f_n is singular. Since $\pi_1(V_n)$ is finite, the universal cover of $\hat{V}_n$ (the capping of V_n) is a homotopy 3-sphere. Hence, $\pi_2(\hat{V}_n) \cong \pi_2(\text{a universal cover of } \hat{V}_n) = 0$. Therefore $\pi_2(V_n)$ is generated by S^2 components of ∂V_n.

Claim $f_n\colon (S^2, a) \to (M_n, b_n)$ is OR-homotopic to the sum (in the sense of π_2) of S^2 components of ∂V_n.

PROOF. Let R be a regular neighborhood of Γf_n and $C_1, \dots, C_k$ be the components of $f_n(S^2) - \operatorname{Int} R$, respectively. Then $V_n = R \bigcup_{l=1}^{k} (C_l \times (-1, 1))$.

For any $x \in \Sigma_{S^2}$, there exists a C_l such that $f(x) \in C_l$. Hence $\Sigma_{V_n} = \bigcup_{i=1}^{3}(f_n(x_i)\times(-1,1))$, where $\{x_1, x_2, x_3\} = \Sigma_{S^2}$. Thus $\partial V_n \cap \Sigma_{V_n} = \{f_n(x_i)\times(\pm 1), i = 1,2,3\}$.

By Proposition 4.2, $f_n(x_1)$, $f_n(x_2)$ and $f_n(x_3)$ must be included in the same component of $f_n(S^2) - \Gamma f_n$; otherwise we can construct a proper OR-map from a bad sphere to (M, b) by restricting $p_1 \circ i_1 \circ \cdots \circ i_{n-1} \circ p_n$ on a component of ∂V_n. This contradicts the hypothesis. Hence there exists a C_l such that $f_n(x_1)$, $f_n(x_2)$, $f_n(x_3) \in C_l$. Therefore there exists a disk D in C_l such that $f_n(x_1)$, $f_n(x_2)$, $f_n(x_3) \in \operatorname{Int} D$. Let S_1 be a component of ∂V_n which includes $\operatorname{Int} D \times 1$ and S_2 be a component of ∂V_n which includes $\operatorname{Int} D \times (-1)$, respectively. Then, by the same way as the sphere theorem of 3-manifolds, we can show that no component of ∂V_n intersects $C_l \times 1$ and $C_l \times (-1)$ simultaneously. Thus $S_1 \neq S_2$.

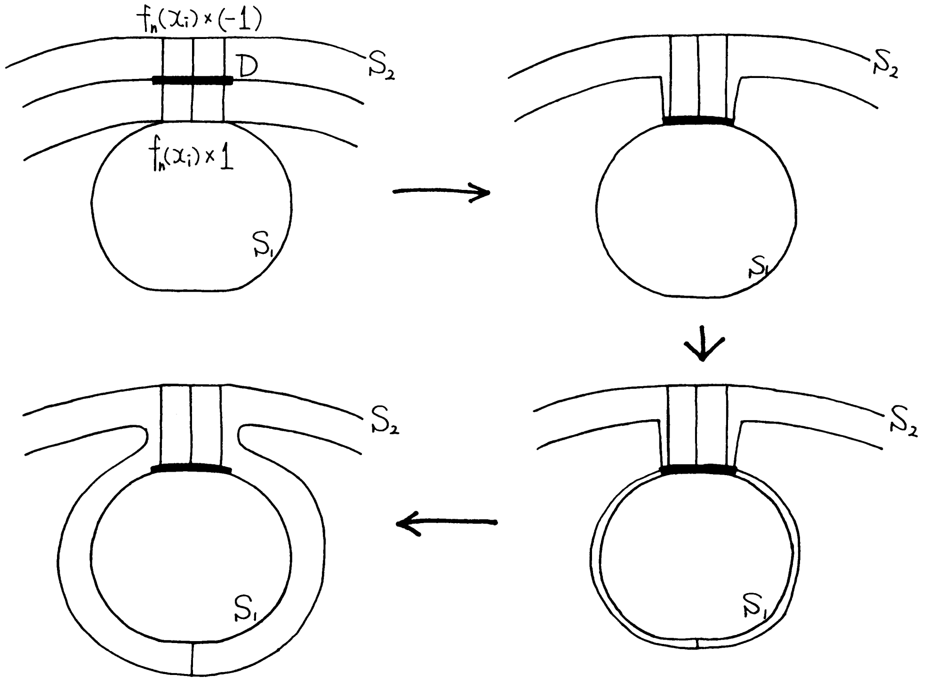

FIGURE 4.1

Hence f is OR-homotopic to the sum of S_1 and a map

$$f' \colon S^2 \to V_n - (\operatorname{Int} D \times (-1,1))$$

(cf. Figure 4.1). Since the capping of $(V_n - \operatorname{Int} D \times (-1,1))$ is an homotopy sphere, $(f') \in \pi_1(V_n - \operatorname{Int} D \times (-1,1))$ is homotopic to the sum of S^2 components of $\partial(V_n - \operatorname{Int} D \times (-1,1))$. When those S^2's include

$$(S_1 - \operatorname{Int} D \times 1) \cup (\partial D \times (-1,1)) \cup (S_2 - \operatorname{Int} D \times (-1)) \cong S^2,$$

f is OR-homotopic to the sum of S_1 and the S^2 components of ∂V_n. Though we have proven in the case of $\Sigma_{S^2} = 3$ points, we can do similarly in the case of $\Sigma_{S^2} = 2$ points. This completes the proof of the claim.

Put $j = i_0 \circ (p_1 \circ i_1) \circ \cdots \circ (p_n \circ i_n)$. Since $j \circ f_n = f\colon (S^2, a) \to (M, b)$ is not extendable to the OR-map from $C(S^2, a)$, there exists a component $(S^{2'}, a')$ of $\partial(V_n, b'_n)$ such that $j \circ s\colon (S^{2'}, a') \to (M, b)$ is not extendable to the OR-map from $C(S^{2'}, a')$, where $s\colon (S^{2'} a') \to (V_n, b'_n)$ is an OR-embedding naturally constructed by $(S^{2'}, a')$, otherwise the preceding claims show that $j \circ f_n = f$ is extendable to the OR-map from $C(S^2, a)$. This contradicts the hypothesis. By the same way as the sphere theorem of 3-manifolds, we can show that $f' = j \circ s\colon (S^{2'}, a') \to (M, b)$ has desired properties (2) and (3). The proof of step 4 is finished.

Next, we must consider the case in which $\Gamma_1 \neq \emptyset$. We also use the induction on $c(f)$. We can similarly proceed using steps 1, 2 and 3. We have only to think of the case where f_0 has no simple closed double curve. Since $f(\Sigma_{S^2}) \cap \Gamma_1 f = \emptyset$ from the normality of f, we can show that f is the sum of the S^2 components of ∂V_n and that there is an OR-embedding $s\colon (S^{2'}, a') \to (M, b)$, similar to step 4. Since p_j is an immersion, $f' = i_0 \circ (p_1 \circ i_1) \circ \cdots \circ (p_n \circ i_n)$ has no branch points. f' is normal and proper, and is not extendable to the cone. So the proof is completed by the first case. Q.E.D.

Notice that the normality of the OR-map is not necessary in the hypothesis of Proposition 4.2. At step 4 of the proof in Theorem 4.4, it is necessary that f is OR-homotopic to the sum of ∂V_n, otherwise it is not ensured that f' inherits the property that it is not extendable to the cone. But in Proposition 4.2, some components of $\partial(V_n, b'_n)$ must be bad spheres, so we can find an OR-map from a bad sphere with less complexity. We can show the next proposition by the same way as the proof of Proposition 4.2.

PROPOSITION 4.5. *Let (S^2, a) be an elliptic sphere, (M, b) be a 3-branchfold which does not contain bad spheres, and $f\colon (S^2, a) \to (M, b)$ be a proper and transversal OR-map, then there exists a transversal OR-embedding $g\colon (S^2, a) \to (M, b)$.*

5. Some applications of sphere theorems.

COROLLARY 5.1. *Let (M, b) be a 3-branchfold which does not contain bad spheres, then $[\mu]^k \neq 1$ in $\pi_1(M - \Sigma_M)$ for any normal loop μ and any $k \in \mathbf{N}$.*

PROOF. If there exists a normal loop μ and $k \in \mathbf{N}$ such that $[\mu]^k = 1$ in $\pi_1(M - \Sigma_M)$, then we can construct a map f' from a disk D^2 to $M - \Sigma_M$ such that $[f'|\partial D^2] = [\mu]^k$. Here, let $D^2(n)$ be the normal disk bounded by μ. We get a proper and transversal OR-map f' from a bad sphere $S^2(n)$ to (M, b), by attaching D^2 and $D^2(n)$ with those boundaries. By Proposition 4.2, this contradicts the fact that (M, b) has no bad spheres. Q.E.D.

COROLLARY 5.2. *Let (M, b) be a 3-branchfold which does not contain bad spheres, then μ and μ^k are not free homotopic in $M - \Sigma_M$ for any normal loop μ and any $k \in \mathbf{Z} - \{\pm 1\}$.*

PROOF. If $k = 0$, the statement is trivial by Corollary 5.1. We assume $k \neq 0$. If we negate the conclusion, then we may assume that there exists a proper and transversal OR-map f from an elliptic sphere $S^2(n, n)$ to (M, b) such that $f(p_1) \notin \Gamma f$ and $f(p_2) \in \Gamma_1 f$ for $\Sigma_{S^2} = \{p_1, p_2\}$. Hence, there exist l_1, l_2, a pair of preimages of a double curve, starting from $p_2 \in S^2(n, n)$ and ending in a point in

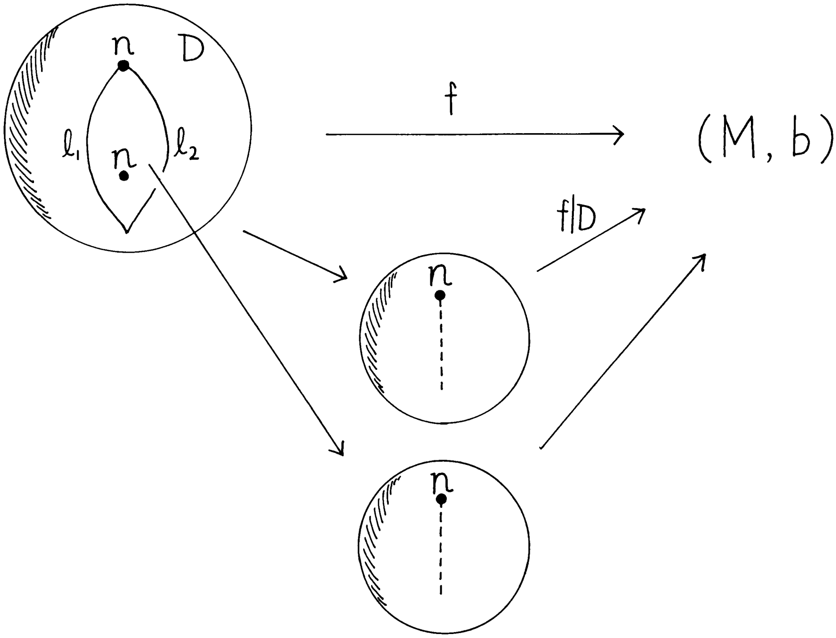

FIGURE 5.1

$S^2(n,n) - \{p_1, p_2\}$. l_1 and l_2 separate $S^2(n,n)$ into two disks. Let D be the side which does not contain p_2. Then, $(f|D)$ is a proper and transversal OR-map from $S^2(n)$ to (M,b). This contradicts the hypothesis from Proposition 4.2 (cf. Figure 5.1). Q.E.D.

COROLLARY 5.3. *Let (M,b) and (N,c) be 3-branchfolds and (N,c) be irreducible. Suppose that (N,c) does not contain bad spheres, $f\colon (M,b) \to (N,c)$ is a normal OR-map, and that there exist $l \in S_M^{(1)}$, $l' \in S_N^{(1)}$ and $x \in \dot{l}$, $x' \in \dot{l}'$ such that $f(x) = x'$. Then there exists a normal loop μ in a regular neighborhood $B(x,(M,b))$ for any regular neighborhood $B'(f(x),(N,c))$ such that $[f(\mu)] = [\nu]^\varepsilon$ in $\pi_1(B'(f(x),(N,c)) - \Sigma_N)$, where ν is the normal loop of l' and $\varepsilon = \pm 1$.*

PROOF. Since $f(x) \in \dot{l}'$, $B'(f(x),(N,c)) = CS^2(n,n)$. By the continuity of f, $f(B(x,(M,b))) \subset B'(f(x),(N,c))$ for a regular neighborhood $B(x,(M,b))$. Since $x \in \dot{l}$, we may assume $B(x,(M,b)) = CS^2(m,m)$, where $n|m$. Put $B_0 = B(x,(M,b)) - \Sigma_M$, $B_0' = B'(f(x),(N,c)) - \Sigma_N$. Since we define a continuous map $(f|B_0)\colon B_0 \to B_0'$ by the property of an OR-map, $[f(\mu)] = [\nu]^k$ in $\pi_1(B_0')$, where $k \in \mathbf{N}$. Hence, $f(\mu)$ and ν^k are freely homotopic in B_0'. On the other hand, $f(\mu)$ is freely homotopic to a normal loop ν' in N_0 $(= N - \Sigma_N)$, since f is normal. Put the branch points of the normal disks bounded by ν and ν' to p_1 and p_2, respectively. From the above, there exists a transversal OR-map from $S^2(n_1,n_2)$ to (N,c), where $n_1 = c(p_1)$, $n_2 = c(p_2)$. By the hypothesis that (N,c) contains no bad spheres, and Proposition 4.2, $n_1 = n_2$. Moreover, by the hypothesis that (N,c) is irreducible, and Proposition 4.5, ν and ν' must be the normal loops of an $l' \in S_N^{(1)}$. Hence, ν and ν' are freely homotopic in N_0. Thus we conclude $k = 1$ by Corollary 5.2. Q.E.D.

COROLLARY 5.4. *Let (M,b) and (N,c) be 3-branchfolds and (N,c) be irreducible. Suppose that (N,c) does not contain bad spheres, $f\colon (M,b) \to (N,c)$ is a normal OR-map. Then, for any $\bar{l} \in S_M$, there exists an $l' \in \mathcal{L}_N^{(1)}$ such that $f(\bar{l}) \subset \bar{l}'$.*

PROOF. At first, $f(x) \in \Sigma_N$ for any $x \in \bar{l}$. Otherwise, there exists a regular neighborhood $B(x,(M,b))$ (which is OR-isomorphic to $CS^2(n,n)$) for the regular neighborhood $B'(f(x),(N,c))$ (which is OR-isomorphic to a 3-ball) such that $f(B(x,(M,b))) \subset B'(f(x),(N,c))$. Hence, $[f(\mu)] = 1$ in $\pi_1(N_0)$ for the normal loop μ of l in $B(x,(M,b))$. On the other hand, $f(\mu)$ is freely homotopic to a normal loop ν by the normality of f. Thus, $[\nu] = 1$ in $\pi_1(N_0)$. By Corollary 5.1, this contradicts the fact that (N,c) contains no bad spheres.

Next, we have only to prove that there is no subarc l_1 of l such that $f(l_1) \cap l_1' \neq \emptyset$ and $f(l_1) \cap l_2' \neq \emptyset$ for some l_1' and $l_2' \in S_N^{(1)}$. Suppose there exists such a subarc l_1. Then, by the continuity of f, there exist points $p \in l_1$ and $q \in S_N^{(0)}$ such that there exists a regular neighborhood $B(p,(M,b))$ for any regular neighborhood $B'(q,(N,c))$ such that $f(B(p,(M,b))) \subset B'(q,(N,c))$. Put $B_0 = B(p,(M,b)) - \Sigma_M$ and $B_0' = B'(q,(N,c))$. We can define a continuous map $(f|B_0)\colon B_0 \to B_0'$, since f is an OR-map. Let μ be a normal loop of l_1 in B_0, ν_1 and ν_2 be normal loops of l_1' and l_2' in B_0, respectively. Note that we can choose $B(p,(M,b))$ arbitrarily small. By Corollary 5.3, for suitable normal loops μ_1 and μ_2 which homotopic to μ in B_0, we may assume $f(\mu_1) = \nu_1^{\varepsilon_1}$ and $f(\mu_2) = \nu_2^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = \pm 1$. Though μ_1 and μ_2 are freely homotopic in B_0, $\nu_1^{\varepsilon_1}$ and $\nu_2^{\varepsilon_2}$ cannot be freely homotopic in B_0. This is a contradiction. Q.E.D.

DEFINITION 5.5. We say 1-dimensional strata l_1, l_2 and l_3 are concentrating to $x \in S^{(0)}$ if $\bar{l}_1 \cap \bar{l}_2 \cap \bar{l}_3 = x$ and $l_1 \cap l_2 \cap l_3 = \emptyset$. In this condition, we also say the normal loops μ_i's of l_i's are concentrating to x.

COROLLARY 5.6. *Let (M,b) and (N,c) be 3-branchfolds, and (N,c) be irreducible. Suppose that (N,c) does not contain bad spheres, and $f\colon (M,b) \to (N,c)$ is a normal OR-map. Then, for any point $x \in S_M^{(0)}$, it holds that $f(x) \in S_N^{(0)}$. Moreover, for any normal loops μ_1, μ_2 and μ_3 which are concentrating to x, $f(\mu_1)$, $f(\mu_2)$ and $f(\mu_3)$ are concentrating normal loops to $f(x)$.*

PROOF. For any point $x \in \Sigma_M$, it holds that $f(x) \in \Sigma_N$, by a way similar to Corollary 5.4. Suppose that there exists a point $x \in S_M^{(0)}$ such that $f(x) \in S_N^{(1)}$. Then, by the continuity of f, for any regular neighborhood $B'(f(x),(N,c))$, there exists a regular neighborhood $B(x,(M,b))$, such that

$$f(B(x,(M,b))) \subset B'(f(x),(N,c)).$$

Put $B_0 = B(x,(M,b)) - \Sigma_M$ and $B_0' = B'(f(x),(N,c)) - \Sigma_N$. We can define a continuous map $(f|B_0)\colon B_0 \to B_0'$, since f is an OR-map. Let l_1, l_2 and l_3 be 1-dimensional strata which are concentrating to x and l' a 1-dimensional stratum which includes $f(x)$. Let ν be a normal loop of l' in B_0', μ_1, μ_2 and μ_3 normal loops of l_1, l_2 and l_3 in B_0, respectively. Note that we can choose $B(x,(M,b))$ arbitrarily small. By Corollary 5.3, we may assume that

$$[f(\mu_1)] = [f(\mu_2)] = [f(\mu_3)] = [\mu] \quad \text{in } \pi_1(B_0'), \tag{1}$$

$$[\mu_1]^{\varepsilon_1}[\mu_2]^{\varepsilon_2} = [\mu_3] \quad \text{in } \pi_1(B_0) \tag{2}$$

for a suitable orientation of μ_1, μ_2 and μ_3.

By (2), $[f(\mu_1)]^{\varepsilon_1}[f(\mu_2)]^{\varepsilon_2} = [f(\mu_3)]$ in $\pi_1(B_0')$. Thus, by (1), $[\nu]^{\varepsilon_1}[\nu]^{\varepsilon_2} = [\nu]$ in $\pi_1(B_0')$. Hence, $(\nu) = 1$ or $(\nu)^3 = 1$ in $\pi_1(B_0')$. By Corollary 5.1, this contradicts the fact that (N, c) does not contain bad spheres. So we can conclude $f(x) \in S_N^{(0)}$. Let ν_1, ν_2 and ν_3 be the concentrating normal loops to $f(x)$. By the property of the OR-map (the index of the image divides the index of the preimage), at most one of l_1, l_2 and l_3 can be mapped to $f(x)$. So we may assume $f(\mu_1)$ and $f(\mu_2)$ are one of ν_1, ν_2 and ν_3, by Corollary 5.3. Suppose

$$[f(\mu_1)] = [f(\mu_2)] = [\nu_1] \quad \text{in } \pi_1(B_0')$$

By the above, $[\nu_1]^{\varepsilon_1+\varepsilon_2} = [f(\mu_3)]$ in $\pi_1(B_0')$. By the same way as in the proof of Corollary 5.3, $[f(\mu_3)] = [\nu_1]^{\varepsilon}$ in $\pi_1(B_0')$, $\varepsilon = \pm 1$. Since $\varepsilon_1 + \varepsilon_2 = 0, \pm 2$, this is a contradiction. So we may assume that $[f(\mu_1)] = [\nu_1]$, $[f(\mu_2)] = [\nu_2]$ in $\pi_1(B_0')$. Thus, $[f(\mu_3)] = [\nu_1]^{\varepsilon_1}[\nu_2]^{\varepsilon_2}$ in $\pi_1(B_0')$. Hence, $[f(\mu_3)] = [\nu_3]^{\varepsilon_3}$ in $\pi_1(B_0')$, $\varepsilon_3 = \pm 1$. This proves that $f(\mu_1)$, $f(\mu_2)$, and $f(\mu_3)$ are concentrating normal loops to $f(x)$. Q.E.D.

From Theorem 4.4, we get the following corollary directly.

COROLLARY 5.7. *Let (M, b) be an irreducible 3-branchfold. Then any proper normal OR-map $f\colon (S^2, a) \to (M, b)$ must be extendable to the cone.*

DEFINITION 5.8. Let (M, b) and (N, c) be 3-branchfolds. We call a homomorphism $\phi\colon \pi_1(M - \Sigma_M) \to \pi_1(N - \Sigma_N)$ normal if there exists a normal loop $\nu \in \Omega(N, c)$ for any normal loop $\mu \in \Omega(M, b)$ such that $\phi([\mu])$ conjugates to $[\nu]$ in $\pi_1(N - \Sigma_N)$.

DEFINITION 5.9. Let (M, b) and (N, c) be 3-branchfolds. We call a homomorphism $\phi\colon \pi_1(M - \Sigma_M) \to \pi_1(N - \Sigma_N)$ proper if, for any normal loop $\mu \in \Omega(M, b)$, the order of $\phi([\mu])$ in $\pi_1(N, c)$ is equal to the order of $[\mu]$ in $\pi_1(M, b)$.

REMARK 5.10. Let (M, b), (N, c) be 3-branchfolds, $H = \pi_1(M - \Sigma_M)$ and $H' = \pi_1(N - \Sigma_N)$. If a homomorphism $\phi\colon H \to H'$ is proper, then ϕ induces a canonical homomorphism $\overline{\phi}\colon \pi_1(M, b) \to \pi_1(N, c)$ under the mapping $\overline{\phi}(\overline{\sigma}) = \overline{\phi \circ \sigma}$ for $[\sigma] \in \pi_1(M - \Sigma_M)$, where $^-$ implies the equivalent class of the quotient group of $\pi_1(M, b) = H/H\langle \mu^b \rangle$ and $\pi_1(N, c) = H'/H'\langle \nu^c \rangle$.

PROPOSITION 5.11. *Let (M, b) and (N, c) be uniformizable 3-branchfolds and $\phi\colon \pi_1(M - \Sigma_M) \to \pi_1(N - \Sigma_N)$ be a normal and proper homomorphism. If (N, c) is irreducible, then there exists a normal OR-map $f\colon (M, b) \to (N, c)$ such that $(f|(M - \Sigma_M))_{\#} = \phi$ and $f_* = \overline{\phi}$.*

PROOF. Let $M_0 = M - \mathrm{Int}$(a regular neighborhood of Σ_M) and $N_0 = N - \mathrm{Int}$(a regular neighborhood of Σ_N). Since (N, c) is irreducible, N_0 is irreducible, too. Hence, we construct a map $f'\colon M_0 \to N_0$ such that $f'_{\#} = \phi$. We assume that, for any $\mu \in \Omega(M, b)$, $f'(\mu) \in \Omega(N, c)$, by modifying f' with homotopy in N_0.

Let μ be a normal loop of $l' \in S_M^{(1)}$ and $f'(\mu)$ be a normal loop of $l' \in S_N^{(1)}$, respectively. Then, by the uniformizability of (M, b) and (N, c), the orders of (μ) in $\pi_1(M, b)$ and $(f'(\mu))$ in $\pi_1(N, c)$ are $b(l)$ and $c(l')$, respectively (cf. M. Kato (3)). Since ϕ is proper, $c(l')$ divides $b(l)$. Thus, we can extend f' to an OR-map $f''\colon M_0 \cup (\bigcup_i D(m_i)) \to N_0 \cup (\bigcup_j D(n_j))$, where $D(m_i)$ is a normal disk of (M, b)

and $D(n_j)$ is a normal disk of (N,c). We can show that we can extend this f'' to an OR-map $f\colon (M,b) \to (N,c)$, as follows.

Let μ_1, μ_2, and μ_3 be concentrated normal loops of (M,b) and $D(m_1)$, $D(m_2)$ and $D(m_3)$ be their bounding normal disks. Let $f''(\mu_1)$, $f''(\mu_2)$, and $f''(\mu_3)$ be also bounding normal disks in (N,c). Let them be $D(n_1)$, $D(n_2)$, and $D(n_3)$, respectively. Remark that $n_i | m_i$, $i = 1,2,3$. Under this relation, the elliptic triples $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, and $(2,3,4)$ must be mapped to elliptic triples. Thus, $S^2(n_1,n_2,n_3)$ is an elliptic sphere. By Corollary 5.7, a normal and proper OR-map from $S^2(n_1,n_2,n_3)$ to (N,c) is extendable to an OR-map from $CS^2(n_1,n_2,n_3)$ to (N,c). Remark that we can regard the normal OR-map from $S^2(m_1,m_2,m_3)$ to (N,c) as the composition of the proper OR-map from $S_2(m_1,m_2,m_3)$ to $S^2(n_1,n_2,n_3)$, which is homotopic with respect to the underlying space, and the proper and normal OR-map from $S^2(n_1,n_2,n_3)$ to (N,c). From the above, we can extend the normal OR-map from $S^2(m_1,m_2,m_3)$ to (N,c) to the normal OR-map from $CS^2(m_1,m_2,m_3)$ to (N,c). Thus, we can extend $f''\colon M_0 \cup (\bigcup_i D(m_i)) \to N_0 \cup (\bigcup_j D(n_j))$ to an OR-map $f\colon (M,b) \to (N,c)$. It is trivial that $(f|(M-\Sigma_M))_\# = \phi$, $f_* = \overline{\phi}$ and f is normal. Q.E.D.

6. A classification of a class of 3-branchfolds. Throughout this chapter, we assume that the underlying spaces of orbifolds are manifolds.

THEOREM 6.1. *Let (F,b) and (G,c) be uniformizable 2-branchfolds. If*

$$f\colon ((F,b),\partial(F,b)) \to ((G,c),\partial(G,c))$$

is a normal OR-map such that $f_*\colon \pi_1(F,b) \to \pi_1(G,c)$ and

$$(f|(F-\Sigma_F))_\#\colon \pi_1(F-\Sigma_F) \to \pi_1(G-\Sigma_G)$$

are monic, then either (a), (b) *or* (c) *holds.*

(a) *There exists an OR-homotopy $f_t\colon ((F,b),\partial(F,b)) \to ((G,c),\partial(G,c))$ such that $f_0 = f$ and f_1 is an OR-covering.*

(b) *(F,b) is an annulus.*

(c) *(F,b) is $S^2(n,n)$, where $n \in \mathbf{N}$.*

PROOF. Since $f_*\colon \pi_1(F,b) \to \pi_1(G,c)$ is monic, it follows that, for any $p \in S_F^{(0)}$, it holds that $f(p) \in S_G^{(0)}$. Since f is normal, for any normal loop μ_p of p, $f(\mu_p)$ is a normal loop of $f(p)$. Moreover, f is proper from the injectivity of f_*. Let F_0 and G_0 be surfaces which are obtained by deriving suitable normal disks of $S_F^{(0)}$ and $S_G^{(0)}$ from F and G, respectively. By modifying f under an OR-homotopy, we may assume that there exists a component ν of $\partial\overline{G}_0 - \partial G$ for each component μ of $\partial\overline{F}_0 - \partial F$ such that $(f|\mu)\colon \mu \to \nu$ is a homeomorphism. By the hypothesis and Remark 1.4, $(f|\overline{F}_0)\colon \pi_1(\overline{F}_0) \to \pi_1(\overline{G}_0)$ is a monic. Thus, by Nielsen's Theorem (cf. Waldhausen [**5**, 1.4.3]), there exists a homotopy $g_t\colon \overline{F}_0 \to \overline{G}_0$, $t \in I$, $g_0 = f|F_0$, $g_t|(\partial\overline{F}_0 - \partial F) = f|(\partial\overline{F}_0 - \partial F)$, for all t (hence, we can extend g_t to an OR-homotopy $f_t\colon (F,b) \to (G,c)$), such that either (1) or (2) holds.

(1) $g_1\colon \overline{F}_0 \to \overline{G}_0$ is a covering.

(2) $\overline{F}_0$ is an annulus and $g_1(\overline{F}_0) \subset \partial\overline{G}_0$.

Case 1. For any component S of ∂F, (S) is an infinite order element in $\pi_1(F,b)$. Thus, $f(S)$ must not be homotopic to any component of $\partial\overline{G}_0 - \partial G$. Hence, for any component ν of $\partial\overline{G}_0 - \partial G$, there exists a component μ of $\partial\overline{F}_0 - \partial F$, such that $f(\mu) = \nu$. Thus, (a) holds.

Case 2. Put $\partial\overline{F}_0 = S_1 \cup S_2$.

(1) $\partial\overline{F}_0 \subset \partial(F,b)$. Since (F,b) is an annulus, (b) holds.

(2) $S_1 \subset \partial(F,b)$ and $S_2 \subset \mathrm{Int}(F,b)$. In this case, $(F,b) = D^2(n)$. By the fact that $g_1(\overline{F}_0) \subset \partial\overline{G}_0$ and $g_1(\partial\overline{F}_0 - \partial F) \subset \partial\overline{G}_0 - \partial G$, $g_1(S_2)$ is a component of $\partial\overline{G}_0 - \partial G$. Thus, $(G,c) = D^2(m)$. Since $(g_1)_*$ is a monic, $n = m$. By modifying g_1 under an OR-homotopy, (a) holds (cf. Figure 6.1).

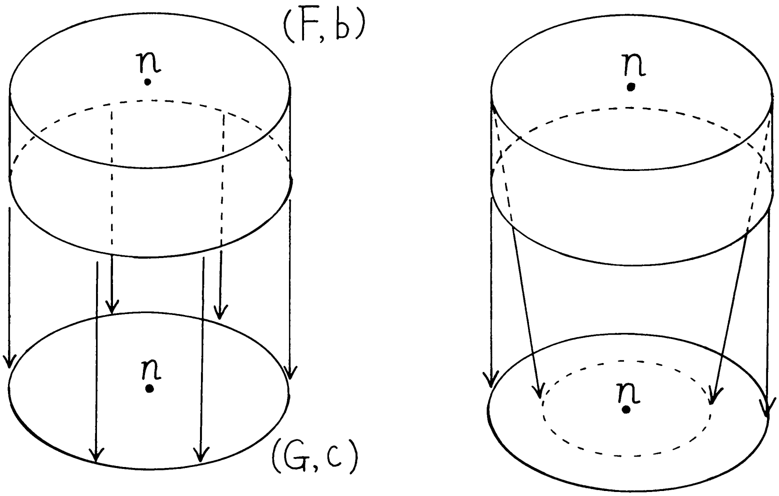

FIGURE 6.1

(3) $\partial F_0 \subset \mathrm{Int}(F,b)$. In this case, $(F,b) = S^2(n,n)$. Thus, (c) holds. Q.E.D.

DEFINITION 6.2. Let (M,b) be a 3-branchfold. We call a 2-suborbifold (F,b') in (M,b) homotopically incompressible (h-incompressible) if $\mathrm{Ker}(\pi_1(F,b') \to \pi_1(M,b)) = 1$.

DEFINITION 6.3. *ω is a class of 3-branchfolds whose element (M,b) satisfies*

(1) $S_M^{(1)} \neq \emptyset$.

(2) *(M,b) is uniformizable.*

(3) *(M,b) is irreducible.*

(4) *$\partial(M,b)$ is h-incompressible.*

(5) *$\partial(M - \mathring{U}(\Sigma_M))$ is incompressible in $M - \mathring{U}(\Sigma_M)$, where $U(\Sigma_M)$ is a regular neighborhood of Σ_M.*

LEMMA 6.4. *Let (M,b) be a uniformizable 3-branchfold. If $\partial(M,b)$ does not contain elliptic spheres and each of its components is h-incompressible, then a loop σ in (F,b') which is homotopic in $M - \Sigma_M$ to the boundary of a normal disk $D(n)$ of (M,b), is homotopic in $F - \Sigma_F$ to the boundary of a normal disk $D'(n)$ of (F,b'), where (F,b') is any component of $\partial(M,b)$.*

PROOF. Since σ is homotopic in $M - \Sigma_M$ to the boundary of a normal disk $D(n)$ of (M,b), we can construct a normal and proper OR-map $\theta\colon (D(n), \partial D(n)) \to ((M,b),(F,b'))$ such that $(\theta|\partial D(n)) = \sigma$. Since $[\sigma]$ has order n in $\pi_1(M,b)$ by the uniformizability of (M, b) and h-incompressibility of (F, b'), $[\sigma]$ has order n in $\pi_1(F, b')$. Since (F, b') is not an elliptic sphere, $|\pi_1(F, b')| = \infty$. Thus, by Proposition II, 3.6, of Jaco and Shalen [**2**], there exists a normal loop $\mu \in \Omega(F, b')$ of order rn in $\pi_1(F, b')$ such that

$$[\sigma] = x_1[\mu_1]^{b_1}x_1^{-1} \cdots x_1[\mu_1]^{b_1}x_1^{-1}x[\mu]^r x^{-1} \cdots x_k[\mu_k]^{b_k}x_k^{-1}$$

in $\pi_1(F - \Sigma_F)$, where $\mu_i \in \Omega(F, b')$, b_i is the order of $[\mu_i]$ in $\pi_1(F, b')$, and x_i, x are the elements of $\pi_1(F - \Sigma_F)$. By modifying σ under a homotopy in $F - \Sigma_F$, we may assume that, around each $p \in S_F^{(0)}$, the self-intersections of σ are transversal and the intersection number is minimum with respect to a modification under homotopies (cf. Figure 6.2).

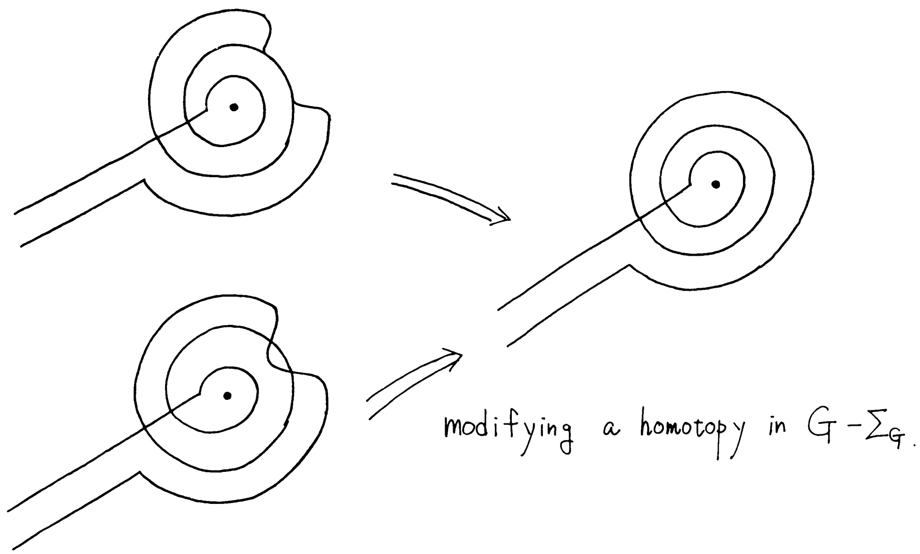

FIGURE 6.2

At first, we will show that $[\sigma] = x[\mu]x^{-1}$ in $\pi_1(F - \Sigma_F)$.

If $[\sigma]$ has a factor $x_i[\mu_i]^{b_i}x_i^{-1}$ different from $x[\mu]^r x^{-1}$ then there exists a subarc I such that $\theta(I) = \mu_i$ (cf. Figure 6.3).

Since ∂I are preimages of a double point of θ, there are preimages of a double curve in $D(n)$ which start from ∂I. The preimage of a double curve ends either at a point on $\partial D(n)$ or at a branch point in $\operatorname{Int} D(n)$. In either case $D(n)$ is separated into two parts by the preimages of a double curve. Let A be the part in which boundary I is included and B be another (cf. Figure 6.4).

Let q be a singular point of $D(n)$. By the generality of θ, we may assume that $q \in \operatorname{Int} P$ or $\operatorname{Int} Q$. In the case of $q \in \operatorname{Int} Q$, Q is $D(n)$. Thus we construct an OR-map $\theta'\colon D(n) \to (M, b)$ such that $(\theta'|\partial D)$ has a factor $x_i[\mu_i]^{b_i-1}x_i^{-1}$. By Proposition 3.6. of (2), such a $[\theta'|\partial D]$ has infinite order in $\pi_1(F, b')$. On the other hand, since $(\theta'|\partial D)$ is extendable in (M, b) to an OR-map from $D(n)$, $[\theta'|\partial D]$ has a finite order in $\pi_1(M, b)$. This contradicts the hypothesis that (F, b') is h-incompressible in (M, b). In the case of $q \in \operatorname{Int} P$, Q is D^2. We proceed similarly

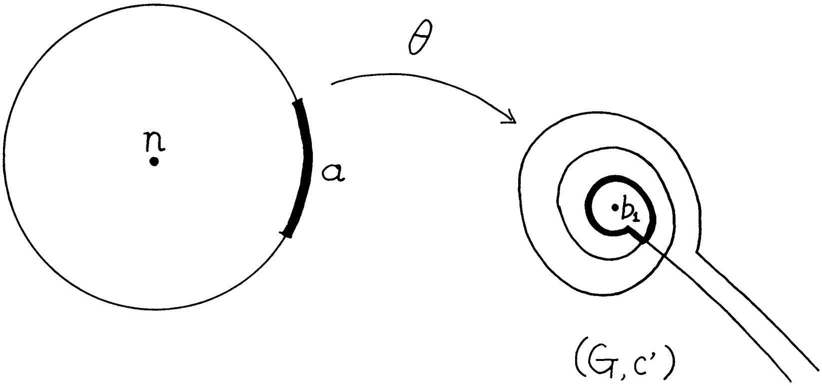

FIGURE 6.3

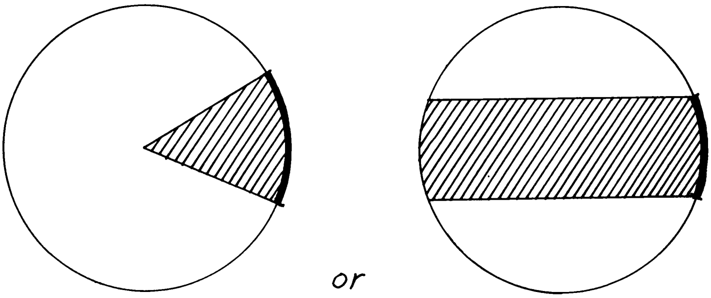

FIGURE 6.4

in the first case with respect to an OR-map $\theta'\colon D^2 \to (M,b)$. Next, we will show that $r=1$. If $r>1$, then, by carrying out the same surgery in the first step, we construct an OR-map $\theta'\colon D \to (M,b)$ such that $\theta'(\partial D)$ bounds $D^2(rn)$ in (F,b'), where $D = D^2(n)$ or D^2. Thus, we construct a transversal and proper OR-map either from $S^2(n,rn)$ to (M,b) or from $S^2(rn)$ to (M,b). Hence, $D = D^2(n)$ or $D = D^2$. By Proposition 4.2, this contradicts the hypothesis. Thus, $[\sigma] = x[\sigma]x^{-1}$ in $\pi_1(F-\Sigma_F)$ (cf. Figure 6.5). Q.E.D.

DEFINITION 6.5. We call a homomorphism $\phi\colon \pi_1(M,b) \to \pi_1(N,c)$ peripheral in $\pi_1(N-\Sigma_N)$ if, for any component (F,b') of $\partial(M,b)$, there exists a component (G,c) of $\partial(N,c)$, such that $\phi(i_*\pi_1(F,b'))$ is conjugate in $\pi_1(N-\Sigma_N)$ to a subgroup of $j_*\pi_1(G,c')$, where i and j are inclusions.

LEMMA 6.6. *Let (M,b) and (N,c) be 3-branchfolds. (N,c) is uniformizable and irreducible. Moreover, $\partial(N,c)$ is h-incompressible and does not contain bad spheres. If there exists a normal and proper homomorphism $\phi\colon \pi_1(M-\Sigma_M) \to \pi_1(N-\Sigma_N)$ such that $\overline{\phi}\colon \pi_1(M,b) \to \pi_1(N,c)$ is peripheral in $\pi_1(N-\Sigma_N)$, then there exists a normal OR-map $f\colon ((M,b),\partial(M,b)) \to ((N,c),\partial(N,c))$ such that $f_* = \overline{\phi}$ and $(f|(M-\Sigma_M))_\# = \phi$.*

PROOF. At first, by Proposition 5.11, we construct an OR-map $f'\colon (M,b) \to (N,c)$ such that $f'_* = \overline{\phi}$ and $(f'|(M-\Sigma_M))_\# = \phi$. Let (F,b') be a component

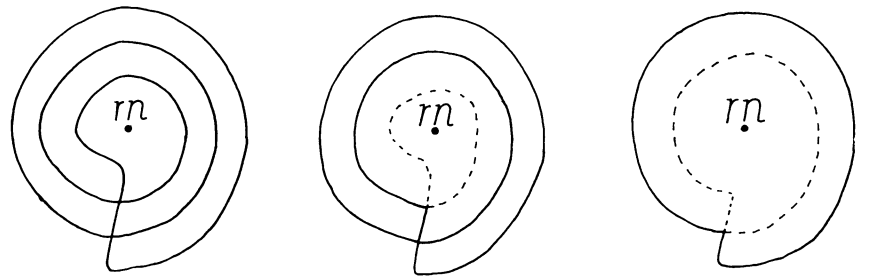

FIGURE 6.5

of $\partial(M, b')$. Take a simplicial division K_F of F such that each point of $S_F^{(0)}$ is included in an interior of a 2-simplex of K_F. Since $\overline{\phi}\colon \pi_1(M, b) \to \pi_1(N, c)$ is a peripheral in $\pi_1(N - \Sigma_N)$, we extend f' to a map $f^1\colon K_F \cup (K_F^{(1)} \times I) \to (N, c)$ such that $f^1(K_F^{(1)} \times 1) \subset \partial(N, c)$. Let (G, c') be a component of $\partial(N, c)$, which includes $f^1(K_F^{(1)} \times 1)$. By Lemma 6.4, we extend f^1 to a map $f^2\colon K_F \cup (K_F^{(1)} \times I) \cup (K_F \times 1) \to (N, c)$ such that $f^2(K_F \times 1) \subset (G, c')$. Hence, by Corollary 5.7, we extend f^2 to a map $f^3\colon K_F \times I \to (N, c)$. Thus, we have shown that, for any component (F, b') of $\partial(M, b)$, there exists an OR-map $H_{(F,b')}\colon (F, b') \times I \to (N, c)$ such that $H_{(F,b')}(x, 0) = f(x)$, $x \in (F, b')$ and $H_{(F,b')}((F, b'), 1) \subset (N, c)$. This implies that the statement is proved. Q.E.D.

LEMMA 6.7. *Let (M, b) and (N, c) be 3-branchfolds. Moreover, (N, c) is uniformizable and irreducible. If $f\colon ((M, b), \partial(M, b)) \to ((N, c), \partial(N, c))$ is a normal OR-map such that $f_*\colon \pi_1(M, b) \to \pi_1(N, c)$ is injective, then there exists an OR-map $g\colon (M, b) \to (N, c)$ such that*

(1) $g_* = f_*$, $(g|M - \Sigma_M)_\# = (f|M - \Sigma_M)_\#$, *and* $g|\partial(M, b) = f|\partial(M, b)$, *and*

(2) *for suitable regular neighborhoods $U(\Sigma_M)$ and $U(\Sigma_N)$, the following* (a), (b), *and* (c) *hold.*

(a) $g(M - \mathring{U}(\Sigma_M)) \subset N - \mathring{U}(\Sigma_N)$.

(b) $(g|(M - \mathring{U}(\Sigma_M)))_\# \circ i_\# = j_\# \circ (g|(M - \mathring{U}(\Sigma_M)))_\#$, *where i and j are inclusions.*

(c) $(g|\partial U(\Sigma_M))\colon \partial U(\Sigma_M) \to g(\partial U(\Sigma_M))$ $(\subset \partial U(\Sigma_M))$ *is a covering.*

PROOF. We may assume that, for a sufficiently small regular neighborhood (B_p, b') of p, $f((B_p, b'))$ is included in a regular neighborhood $(B_{f(p)}, c')$ of $f(p)$. By Corollary 5.6, for any point $p \in S_M^{(0)}$, $f(p) \in S_N^{(0)}$, and, since f_* is injective, (B_p, b') is OR-isomorphic to $CS^2(m_1, m_2, m_3)$ and $(B_{f(p)}, c')$ is OR-isomorphic to $CS^2(n_1, n_2, n_3)$ where $n_i = m_i$, $i = 1, 2, 3$. Since f is normal, we may assume that f is an OR-embedding near the points p_1, p_2, $p_3 \in \Sigma_{B_p} \cap \partial B_p$ and

$$f(\partial B_p - \{p_1, p_2, p_3\}) \subset B_{f(p)} - \Sigma_{B_{f(p)}}.$$

On the other hand, by the property of an OR-map, $f(M - \Sigma_M) \subset N - \Sigma_N$, hence, for a sufficiently small regular neighborhood $(B'_{f(p)}, c')$ of $f(p)$, $f(M - \mathring{B}_p) \subset N - \mathring{B}'_{f(p)}$. Take such a $(B'_{f(p)}, c')$ in the interior of $(B'_{f(p)}, c')$ (cf. Figure 6.6).

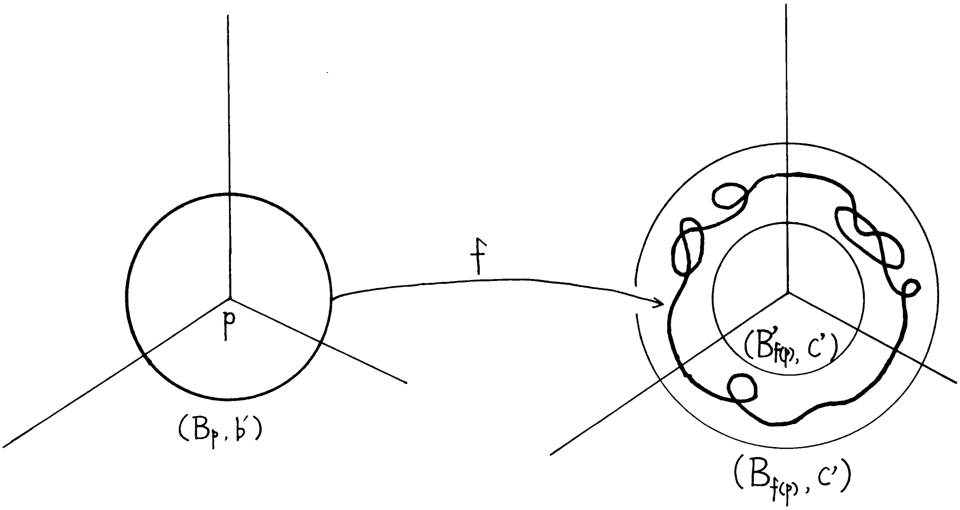

FIGURE 6.6

Notice that we get an OR-map $(f|(M - \mathring{B}_p))\colon (M,b) - (\mathring{B}_p, b') \to (N,c) - (\mathring{B}'_{f(p)}, c')$. Since $\pi_2(B_{f(p)} - (B'_{f(p)} \cup \Sigma_N)) = 0$, we can coincide $f(\partial(B_p, b'))$ to $\partial(B'_{f(p)}, c')$ by modifying f under an OR-homotopy (cf. Figure 6.7).

Hence, we can extend the OR-map $(f|(M - \mathring{B}_p))\colon (M,b) - (\mathring{B}_p, b') \to (N,c) - (\mathring{B}'_{f(p)}, c')$ to $\partial(B, b') \times I$ of which restriction to $\partial(B, b') \times 1$ is an OR-isomorphism to $\partial(B'_{f(p)}, c')$. By carrying out these operations around all $p \in S_M^{(0)}$, we get an OR-map

$$f'\colon (M,b) - \bigcup_{p \in S_M} (0)(\mathring{B}_p, b') \to (N,c) - \bigcup_{p \in S_M} (0)(\mathring{B}_{f(p)}, c')$$

of which restriction to each $\partial(B_p, b')$ is an OR-isomorphism to $\partial(B'_{f(p)}, c')$.

We can proceed with the same operation for regular neighborhoods (B_l, b')'s of $l \cap (M - \bigcup_{p \in S_M^{(0)}} (\mathring{B}_p, b'))$, where $l \in S_M^{(1)}$, fixing $\partial(B_p, b')$. Consequently, we get an OR-map

$$f''\colon \mathrm{cl}\left\{(M,b) - \left(\bigcup_{p \in S_M^{(0)}} (B_p, b') \cup \bigcup_{p \in S_M^{(1)}} (B_l, b')\right)\right\}$$
$$\to \mathrm{cl}\left\{(N,c) - \left(\bigcup_{p \in S_M^{(0)}} (B'_{f(p)}, b') \cup \bigcup_{l \in S_M^{(1)}} (B'_{f(l)}, c')\right)\right\}$$

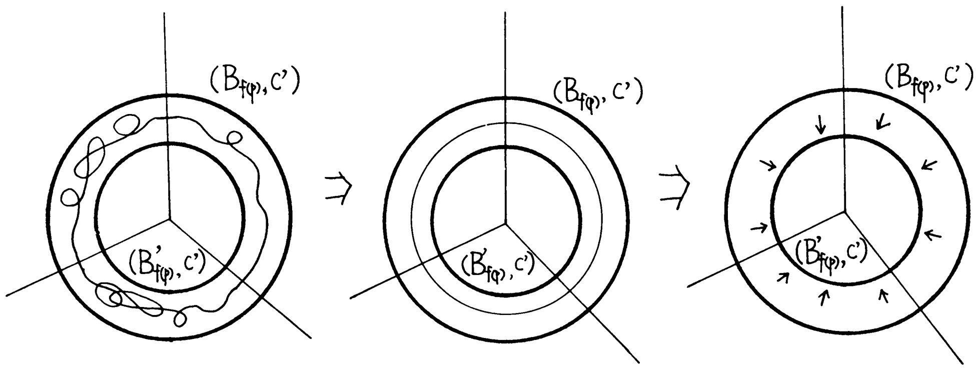

FIGURE 6.7

of which restriction to $\partial(\bigcup_{p\in S_M^{(0)}}(B_p, b') \cup \bigcup_{l\in S_M^{(1)}}(B_l, b'))$ is a covering to

$$\partial\left(\bigcup_{p\in S_M^{(0)}}(B'_{f(p)}, b') \cup \bigcup_{l\in S_M^{(1)}}(B'_{f(l)}, c')\right).$$

Put

$$U(\Sigma_M) = \bigcup_{p\in S_M^{(0)}}(B_p, b') \cup \bigcup_{l\in S_M^{(1)}}(B_l, b')$$

and

$$U(\Sigma_N) = \bigcup_{p\in S_M^{(0)}}(B'_{f(p)}, c') \cup \bigcup_{l\in S_M^{(1)}}(B'_{f(l)}, c').$$

We get the desired g by extending f'' naturally into $U(\Sigma_M)$. Q.E.D.

THEOREM 6.8. *Let (M, b) and (N, c) be 3-branchfolds which belong to ω. If there exists a proper and normal monomorphism $\phi\colon \pi_1(M-\Sigma_M) \to \pi_1(N-\Sigma_N)$ such that $\overline{\phi}\colon \pi_1(M, b) \to \pi_1(N, c)$ is a monomorphism and peripheral in $\pi_1(N-\Sigma_N)$, then either* (a) *or* (b) *holds.*

(a) *There exists an OR-covering $g\colon (M, b) \to (N, c)$ such that $g_* = \overline{\phi}$ and $(g|(M-\Sigma_M))_\# = \phi$.*

(b) $M - \mathrm{Int}$(*a regular neighborhood of* Σ_M) $=$ (*a closed surface*) $\times I$.

PROOF. There is no elliptic sphere component in boundaries of a 3-branchfold which belongs to ω. Then, we can apply Lemmas 6.6 and 6.7 to (M, b) and (N, c). Thus, we can construct an OR-map $f'\colon ((M, b), \partial(M, b)) \to ((N, c), \partial(N, c))$, which satisfies the following:

(1) $f'_* = \overline{\phi}$ and $(f'|(M-\Sigma_M))_\# = \phi$.

(2) There exist regular neighborhoods $U(\Sigma_M)$ and $U(\Sigma_N)$ such that the following (i), (ii), and (iii) hold.

(i) $f'(M - \mathring{U}(\Sigma_M)) \subset N - \mathring{U}(\Sigma_N)$

(ii) $(f'|(M - \mathring{U}(\Sigma_M)))_\# \circ i_\# = j_\# \circ (f'|(M - \mathring{U}(\Sigma_M)))_\#$, where i and j are inclusions.

(iii) $(f'|\partial U(\Sigma_M))\colon \partial U(\Sigma_M) \to f'(\partial U(\Sigma_M))$ is a covering.

By the h-incompressibility of $\partial(M,b)$ and $\partial(N,c)$, for any component (F,b') of $\partial(M,b)$, there exists a component (G,c') of $\partial(N,c)$, such that

$$(f'|(F,b'))_*\colon \pi_1(F,b') \to \pi_1(G,c')$$

is monic. Since $(f'|(M-\Sigma_M))_\#$ is monic, by Lemma 6.4,

$$(F|(F-\Sigma_F))_\#\colon \pi_1(F-\Sigma_F) \to \pi_1(G-\Sigma_G)$$

is monic. Thus, by Theorem 6.1, we can modify $f'|(F,b')$ to an OR-covering under an OR-homotopy. Let f'' be the OR-map which is constructed by carrying out such a modification on all boundary components of (M,b). From the construction of f', for any $\mu \in \Omega(F,b')$, $f'(\mu) \in \Omega(G,c')$. Hence, the above modification fixes μ. Hence, by (2), for any component T of ∂M_0, there exists a component S of ∂N_0, such that $(f''|T)\colon T \to S$ is a covering, where $M_0 = M - \mathring{U}(\Sigma_M)$ and $N_0 = N - \mathring{U}(\Sigma_N)$. Hence, by Waldhausen's Theorem (cf. [5, Theorem 6.1]), either (I) or (II) holds.

(I) $(f''|M_0)$ is homotopic to a covering $h\colon M_0 \to N_0$, fixing ∂M_0. (Hence, $(f''|\partial M_0)\colon \partial M_0 \to \partial N_0$ is already a covering.)

(II) $M_0 =$ (a closed surface) $\times I$.

Case (II). This is the conclusion (b).

Case (I). For any $\mu \in \Omega(M,b)$, it holds that $h(\mu) \in \Omega(N,c)$. By the injectivity of $h_* = \overline{\phi}$, $[\mu]$ and $[h(\mu)]$ have the same order in $\pi_1(M,b)$ and $\pi_1(N,c)$, respectively. By the uniformizability of (M,b) and (N,c), $b(l_\mu) = c(l_{h(\mu)})$, where l_μ and $l_{h(\mu)}$ are the strata of which μ and $h(\mu)$ rounds, respectively. From the construction of h, for any $\nu \in \Omega(N,c)$, there exists a $\mu \in \Omega(M,b)$, such that $h(\mu) = \nu$. Thus we get a surjective correspondence between $\Omega(M,b)$ and $\Omega(N,c)$, by corresponding $\mu \in \Omega(M,b)$ to $\nu \in \Omega(N,c)$.

Let $h_\mu\colon l_\mu \times D^2 \to l_{h(\mu)} \times D^2$, be an OR-isomorphism, where D^2 and D^2, are disks.

Put

$$M_0' = M_0 \cup \left(\bigcup_{\mu \in \Omega(M,b)} (l_\mu \times D^2) \right)$$

and

$$N_0' = N_0 \cup \left(\bigcup_{\nu \in \Omega(N,c)} (l_\nu \times D^{2'}) \right).$$

We can get an OR-covering $h'\colon M_0' \to N_0'$, by pasting h and h_μ's. Take a component A of $\partial M_0'$ which is an elliptic sphere. Since $\partial(M,b)$ does not contain elliptic spheres, A lies in $\operatorname{Int} M$. Since $h'(\partial M_0') = \partial N_0'$, $h'(A) = a$ component B of $\partial N_0'$. By way of the construction of h', $h'(A) \subset \operatorname{Int} N$. Thus, B is an elliptic sphere, since each component of $\partial N_0' \cap \operatorname{Int} N$ is an elliptic sphere. Since A is a component of $h'^{-1}(B)$, $(h'|A)\colon A \to B$ is a covering. Since B is simply connected, $(h'|A)\colon A \to B$ is a homeomorphism. Hence, it is an OR-isomorphism. We can extend the OR-isomorphism to an OR-isomorphism from CA to CB. Next, take a component B of $\partial N_0' \cap \operatorname{Int} N$. Since h' is a covering, there exists a component A such that $h'(X) = B$. We can show that we can extend $h'|A$ to an OR-isomorphism from CA to CB, by a way similar to the preceding. Thus, we can extend the OR-covering h' to the desired OR-covering $g\colon (M,b) \to (N,c)$. Q.E.D.

COROLLARY 6.9. *In Theorem* 6.8, *if* ϕ *and* $\overline{\phi}$ *are isomorphisms, then there exists an* OR*-isomorphism* $g\colon (M, b) \to (N, c)$.

PROOF. *Case* (a). Since $g_* = \overline{\phi}$ is an isomorphism, by Corollary 3.4, g is an OR-isomorphism.

Case (b). Since ϕ is an isomorphism and normal, ϕ^{-1} also satisfies the same hypothesis. Hence, there exists a covering $N_0 \to M_0$ or $N_0 =$ (a closed surface) $\times I$. In either case, $M_0 = N_0 =$ (a closed surface) $\times I$, since ϕ^{-1} is an isomorphism.

Put $M_0 = S_M \times I$ and $N_0 = S_N \times I$, where S_M and S_N are homeomorphic closed surfaces. Let f'' be the map which is constructed in the proof of Theorem 6.8. Since $f_0'' = (f''|M_0)\colon M_0 \to N_0$ is a covering on ∂M_0, $f_0''|(S_M \times 0)$ and $f_0''|(S_M \times 1)$ are also coverings. We may assume that $f_0''|(S_M \times 0)$ is a map to $S_N \times 0$, by changing 0 and 1, if necessary. We will show that $f''|(S_M \times 1)$ is a map to $S_N \times 1$. If, for a $\mu \in \Omega(M, b)$ on $S_M \times 1$, $f_0''(\mu) \in \Omega(N, c)$ on $S_N \times 0$, then there exists a $\mu' \in \Omega(M, b)$ on $S_M \times 0$, such that $f_0''(\mu') = f_0''(\mu)$, since $f_0''|(S_M \times 0)$ is a map to $S_N \times 0$. Since $f_{0\#}''\colon \pi_1(M_0) \to \pi_1(N_0)$ is an isomorphism, it holds that $[\mu] = [\mu']$ in $\pi_1(M_0)$.

On the other hand, μ and μ' are homotopic to a longitude and a meridian of S_M, respectively. This is a contradiction.

Let $i_k\colon S_M \times k \to M_0$ and $j_k\colon S_N \times k \to N_0$ be inclusions, where $k = 0, 1$. Since $(i_K)_\#\colon \pi_1(S_M \times k) \to \pi_1(M_0)$, $(j_k)_\#\colon \pi_1(S_N \times k) \to \pi_1(N_0)$, and $(f''|M_0)_\#\colon \pi_1(M_0) \to \pi_1(N_0)$ are isomorphisms, and $(j_k)_\# \circ (f''|S_M \times k) = (f''|M_0)_\# \circ (i_k)_\#$ and $(f''|S_M \times k)_\#$ are isomorphisms. Hence, $f''|(S_M \times k)$ is a homeomorphism. Since $f''\colon S_M \times I \to S_N \times I$ gives a homotopy between these homeomorphisms, these are isotopic. So we can construct a homeomorphism $g\colon M_0 \to N_0$ such that $g|(S_M \times k) = f''|(S_M \times k)$, $k = 0, 1$. Hence, we can reduce to Case (I) in the proof of Theorem 6.8. Q.E.D.

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