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REMARKS ON GRASSMANNIAN SUPERMANIFOLDS

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ABSTRACT. This paper studies some aspects of a particular class of examples of supermanifolds; the *supergrassmannians*, introduced in [**Manin**]. Their definition, in terms of local data and glueing isomorphisms, is reviewed. Explicit formulas in local coordinates are given for the Lie group action they come equipped with. It is proved that, for those supergrassmannians whose underlying manifold is an ordinary grassmannian, their structural sheaf can be realized as the sheaf of sections of the exterior algebra bundle of some canonical vector bundle. This realization holds true equivariantly for the Lie group action in question, thus making natural in these cases the identification given in [**Batchelor**]. The proof depends on the computation of the transition functions of the *supercotangent bundle* as defined in a previous work [**OASV 2**]. Finally, it is shown that there is a natural *supergroup action* involved (in the sense of [**OASV 3**]) and hence, the supergrassmannians may be regarded as examples of *superhomogeneous spaces*—a notion first introduced in [**Kostant**]. The corresponding *Lie superalgebra* action can be realized as superderivations of the structural sheaf; explicit formulas are included for those supergrassmannians identifiable with exterior algebra vector bundles.

Introduction. There are various definitions of supermanifolds in use. Mathematicians define a C^∞ (resp., holomorphic) supermanifold as a pair, $(M, \mathscr{A}_M)$, consisting of an ordinary C^∞ (resp., holomorphic) manifold M, together with a sheaf $\mathscr{A}_M$ of supercommutative superalgebras over $\mathbf{R}$ (resp., $\mathbf{C}$) satisfying certain conditions. The various definitions in the literature correspond to the various conditions imposed on the structural sheaf. There seem to be at least two streams of approach, which one may vaguely refer to as *algebraic* and *differential*, respectively (see, for example, [**Kostant**], [**Leĭtes**] and [**Manin**] for the first and [**Rogers**], [**Boyer and Gitler**] and [**Jadczyk and Pilch**] for the second; for a unified view and a generalization that uses a nontrivial underlying super ring, see [**Rothstein 2**]).

CONVENTION. For the purposes of this paper, we shall be concrete and shall understand the definition of supermanifold as in [**Manin**].

As far as the first approach is concerned, it is a theorem of Batchelor [**Batchelor**] that, in the C^∞ case, the sheaf $\mathscr{A}_M$ can be realized as the sheaf of sections, $\Gamma(\cdot, \bigwedge E)$, of the exterior algebra bundle, $\bigwedge E$, of some vector bundle $E \to M$ over M. This identification is not categorical, however: it depends on choices; besides, supermanifold morphisms are more general than vector bundle maps, a fact pointed out in [**Leĭtes**], and it is known that in the holomorphic case there are obstructions to such an identification (see, for example, [**Manin**] and [**Rothstein 1**]).

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In the physics literature, on the other hand, one frequently finds supermanifolds defined by the specification that *the odd variables are spinors.* What this means, presumably, is the following: Let G be some group acting on the manifold M and as bundle automorphisms of a vector bundle $E \to M$. Then G has an induced action on $\Gamma(\cdot, \bigwedge E)$ and hence acts as a group of automorphisms of the corresponding supermanifold. If M is a spin manifold and $E \to M$ is the spin bundle, this then specifies the supermanifold structure as a G-space.

The case of the conformally compactified and complexified Minkowski space, M, provides us with an important example (see [**Guillemin and Sternberg**] for details): M itself can be regarded as the space $G_2(\mathbf{C}^4)$ of two-dimensional (complex) subspaces of $\mathbf{C}^4$. The spin bundle then becomes identified with the tautological bundle $E \to G_2(\mathbf{C}^4)$ whose fiber at $u \in G_2(\mathbf{C}^4)$ is just the two-dimensional space E_u represented by u. The group $G = \mathrm{GL}(4, \mathbf{C})$ acts as bundle automorphisms, and the picture is now the one of the preceding paragraph. This obviously generalizes to $M = G_k(V^m)$, the Grassmannian manifold of k-planes of some m-dimensional vector space V^m, and $G = \mathrm{GL}(m)$ acting as bundle automorphisms of the rank k tautological bundle $E \to G_k(V^m)$.

In the supermanifold setting we may consider the class of *supergrassmannians* introduced in [**Manin**]: the supergrassmannian $\mathbf{G}_{k|h}(V^{m|n})$ arises as the supermanifold associated to the set of (k,h)-dimensional supervector subspaces, u, of a given (m,n)-dimensional supervector space $V^{m|n}$ $(= V_0^m \oplus V_1^n)$; that is,

$$u \subset V^{m|n}; \qquad \dim(u \cap V_0^m) = k \quad \text{and} \quad \dim(u \cap V_1^n) = h. \tag{1}$$

It turns out that the underlying manifold of $\mathbf{G}_{k|h}(V^{m|n})$ is just $G_k(V_0^m) \times G_h(V_1^n)$, and its odd dimension is $h(m-k) + k(n-h)$. Furthermore, since the Lie group $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$ clearly operates on the set of (k,h)-dimensional supervector subspaces of $V^{m|n}$, one obtains a group homomorphism

$$\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n) \to \mathrm{Aut}\,\mathscr{A}_{\mathbf{G}_{k|h}(V^{m|n})}. \tag{2}$$

On suitable open superdomains $\mathscr{U} \subset \mathbf{G}_{k|h}(V^{m|n})$, this action is nothing but the generalization of

$$z \mapsto (Az+B)(Cz+D)^{-1}; \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(V_0^m), \tag{3}$$

i.e., the action of $\mathrm{GL}(V_0^m)$ on (suitable open sets of) the ordinary grassmannian $G_k(V_0^m)$.

Just as in ordinary differential geometry, the supergrassmannians come equipped with two canonical supervector bundles, constructed via transition functions as in [**OASV 2**]: $\mathbf{E}_{k|h} = (E, \mathscr{A}_E)$ of type $(k|h)$ and $\mathbf{F}_{m-k|n-h} = (F, \mathscr{A}_F)$ of type $(m-k|n-h)$. As a pleasant result, *there is a canonical identification between the supercotangent bundle* $\mathbf{ST}^*\mathbf{G}_{k|h}(V^{m|n})$ *and the supermanifold of all the supervector bundle morphisms* $\mathbf{F}_{m-k|n-h} \to \mathbf{E}_{k|h}$; i.e.,

$$\mathbf{ST}^*\mathbf{G}_{k|h}(V^{m|n}) \simeq \mathscr{H}\!om(\mathbf{F}_{m-k|n-h}, \mathbf{E}_{k|h}). \tag{4}$$

In particular, the transition functions of this supervector bundle can be used to prove that, with respect to the $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$-action on $\mathbf{G}_{k|h}(V^{m|n})$ above and within the spirit of the theorem of Batchelor, the following is true.

THEOREM. (a) $\mathbf{G}_{k|0}(V^{m|n}) = (G_k(V_0^m), \mathscr{A}_{G_k(V_0^m)})$ *is a supermanifold of dimension* $(k(m-k), nk)$ *for which an equivariant identification*

$$\mathscr{A}_{G_k(V_0^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_0^m)}) \simeq \Gamma(\cdot, \bigwedge(V_1 \otimes E_1))$$

exists. (*Here* $E_0 \to G_k(V_0^m)$ *denotes the rank-k tautological vector bundle over the grassmannian* $G_k(V_0^m)$; $V_1^n \to G_k(V_0^m)$ *is the rank-n trivial bundle, and* $V_1^n \otimes E_0 \to G_k(V_0^m)$ *is the tensor product bundle.*)

REMARK. A similar result holds for the $(k(m-k), nk)$-dimensional supermanifold $\mathbf{G}_{0|k}(V^{n|m}) = (G_k(V_1^m), \mathscr{A}_{G_k(V_1^m)})$, in which case we have $\mathscr{A}_{G_k(V_1^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_1^m)}) \simeq \Gamma(\cdot, \bigwedge(V_0 \otimes E_1))$, equivariantly.

(b) $\mathbf{G}_{k|n}(V^{m|n}) = (G_k(V_0^m), \mathscr{A}_{G_k(V_0^m)})$ *is a supermanifold of dimension* $(k(m-k), n(m-k))$ *for which an equivariant identification*

$$\mathscr{A}_{G_k(V_0^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_0^m)}) \simeq \Gamma(\cdot, \bigwedge(V_1 \otimes F_0^*))$$

exists. (*Here* $F_0^* \to G_k(V_0^m)$ *denotes the rank-*$(m-k)$ *tautological vector bundle over the grassmannian* $G_k(V_0^m)$.)

REMARK. Similarly, for $\mathbf{G}_{n|k}(V^{n|m}) = (G_k(V_1^m), \mathscr{A}_{G_k(V_1^m)})$, we have $\mathscr{A}_{G_k(V_1^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_1^m)}) \simeq \Gamma(\cdot, \bigwedge(V_0 \otimes F_1^*))$, equivariantly.

In particular, these results already cover the following special cases:

(i) All the supergrassmannians whose odd dimension is 1.

(ii) All the *superprojective spaces* (i.e., those supergrassmannians whose underlying manifold is a projective space), regardless of their odd dimension.

However, as soon as one considers supergrassmannians $\mathbf{G}_{k|h}(V^{m|n})$ with nontrivial underlying factors (i.e., for which neither $G_k(V_0^m)$, nor $G_h(V_1^n)$ reduce to a single point) there is no way of identifying $\mathscr{A}_{\mathbf{G}_{k|h}(V^{m|n})}$ with $\mathscr{G}r\,\mathscr{A}_{\mathbf{G}_{k|h}(V^{m|n})}$ in a $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$-equivariant manner. An explicit counterexample is provided by the supergrassmannian $\mathbf{G}_{1|1}(V^{2|2})$. This can be proved by means of a pedestrian use of the formulas developed in this paper. We remark, however, that there is a more satisfactory approach to this point due to M. Rothstein via the computation of certain Lie algebra cohomology classes [**Rothstein 3**]. At any rate, this result shows that under a Lie group action, Batchelor's Theorem does not reduce the theory of supermanifolds to that of vector bundles, even in the C^∞ category.

Finally, let us note that (2) is just an ordinary Lie group action. It would be desirable, however, to prove that the supergrassmannians are very natural examples of *superhomogeneous supermanifolds* (a notion already defined in [**Kostant**]). Based on some of our considerations in [**OASV 3**] on the one hand, and on the work of Kostant on the other, we may prove that the infinitesimal version of this action is what it should be; more precisely,

PROPOSITION. *There is a graded Lie algebra homomorphism* $\mathfrak{gl}(V_0^m|V_1^n) \to \mathrm{Der}\,\mathscr{A}_{\mathbf{G}_{k|h}(V^{m|n})}$ *whose restriction to* $\mathfrak{gl}(V_0^m) \oplus \mathfrak{gl}(V_1^n) = (\mathfrak{gl}(V_0^m|V_1^n))_0$ *coincides with the* (*classical*) *infinitesimal action obtained from* (2) *above.*

The Lie supergroup that acts on the supergrassmannian $\mathbf{G}_{k|h}(V^{m|n})$ is

$$\{S(\mathrm{End}\,V^{m|n})\}^*,$$

the subsupermanifold of $S(\operatorname{End} V^{m|n})$ described in local coordinates by the condition of having a nonzero Berezinian. The underlying Lie group is just $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$ and its odd dimension is $2 \dim V_0^m \dim V_1^n$, as pointed out in [**OASV 3**]. The computations in this work suggest there is indeed a notion of maximal parabolic subsupergroup, $\mathscr{P}_{k|h}(V^{m|n})$, satisfying the correct dimensionality relations; namely,

$$\text{even-dim}\, \mathbf{G}_{k|h}(V^{m|n}) = \text{even-dim}\{S(\operatorname{End} V^{m|n})\}^* - \text{even-dim}\ \mathscr{P}_{k|h}(V^{m|n})$$

and

$$\text{odd-dim}\, \mathbf{G}_{k|h}(V^{m|n}) = \text{odd-dim}\{S(\operatorname{End} V^{m|n})\}^* - \text{odd-dim}\ \mathscr{P}_{k|h}(V^{m|n}).$$

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1. Supergrassmannians. Let $V^{m|n} = V_0^m \oplus V_1^n$ be a given (m,n)-dimensional supervector space over $\mathbf{R}$ (or $\mathbf{C}$), to be kept fixed throughout this section. Let us assume that we are given some $(m-k, n-h)$-dimensional supervector subspace, say $W = W_0 \oplus W_1$, of $V^{m|n}$. This means that $W_\mu = W \cap V_\mu \subset V_\mu$ $(\mu = 0,1)$ and the ordinary vector space exact sequence

$$0 \to W \to V^{m|n} \to V^{m|n}/W \to 0 \tag{1.1}$$

takes the form

$$0 \to W_0 \oplus W_1 \underset{j}{\to} V_0^m \oplus V_1^n \underset{\pi}{\to} (V_0^m/W_0) \oplus (V_1^n/W_1) \to 0 \tag{1.2}$$

where both the injection j and the projection π are *even homomorphisms* (cf. [**Corwin et al.**] and [**OASV 1**]).

It is well known that if $\hat{\mathscr{U}}_W$ denotes the set of all vector subspaces of $V^{m|n}$ complementary to W and a choice $\hat{u} \in \hat{\mathscr{U}}_W$ is made, a bijection

$$\hat{\mathscr{U}}_W \to \operatorname{Hom}(V^{m|n}/W, W) \tag{1.3}$$

exists under which $\hat{u}$ corresponds to the zero map. Even though we can always choose $\hat{u}$ to be a (k,h)-dimensional supervector subspace of $V^{m|n}$, it is clear that not every element of $\hat{\mathscr{U}}_W$ has a direct sum decomposition $u_0 \oplus u_1$ with $u_0 \subset V_0^m$ and $u_1 \subset V_1^n$. However, we can detect precisely what the *complementary supervector subspaces* to W are, by just looking at the supervector space structure of $\operatorname{Hom}(V^{m|n}/W, W)$. More precisely, we have the following rather obvious fact:

1.1 PROPOSITION. *If $\hat{u}$ is chosen to be a (k,h)-dimensional supervector subspace of $V^{m|n}$, then* (1.3) *yields a one-to-one correspondence between the set of all (k,h)-dimensional supervector subspaces of $V^{m|n}$ and the even subspace* $(\operatorname{Hom}(V^{m|n}/W, W))_0$. □

Now, according to the theory developed in [**Kostant**] and [**Leĭtes**], one can associate to the supervector space

$$\operatorname{Hom}(V^{m|n}/W, W) = (\operatorname{Hom}(V^{m|n}/W, W))_0 \oplus (\operatorname{Hom}(V^{m|n}/W, W))_1 \tag{1.4}$$

a (*super-*)*affine supermanifold*, $((\mathrm{Hom}(V^{m|n}/W,W))_0,\mathscr{A})$, by letting the sheaf $\mathscr{A}$ be given by

$$\mathscr{A} = \mathscr{C}^\infty|_{(\mathrm{Hom}(V^{m|n}/W,W))_0} \otimes \bigwedge([(\mathrm{Hom}(V^{m|n}/W,W))_1]^*). \tag{1.5}$$

Note that

$$\begin{aligned}(\mathrm{Hom}(V^{m|n}/W,W))_0 &\simeq \mathrm{Hom}(V_0^m/W_0,W_0)\oplus \mathrm{Hom}(V_1^n/W_1,W_1)\\ &\simeq \mathscr{U}_{W_0}\times\mathscr{U}_{W_1}\end{aligned} \tag{1.6}$$

where $\mathscr{U}_{W_0}$ (resp., $\mathscr{U}_{W_1}$) is the open subset of $G_k(V_0^m)$ (resp., $G_h(V_1^n)$) consisting of those k (resp., h) dimensional subspaces complementary to W_0 (resp., W_1). That is, $\mathscr{U}_{W_0}\times\mathscr{U}_{W_1}\subset G_k(V_0^m)\times G_h(V_1^n)$ and this observation suggests to try to cover $G_k(V_0^m)\times G_h(V_1^n)$ by open sets of the form $\mathscr{U}_{I_0}\times\mathscr{U}_{I_1}$ over each of which we will have a sheaf $\mathscr{A}_{(I_0,I_1)}$ of supercommutative superalgebras given by (1.5) and then provide appropriate glueing isomorphisms $\varphi_{(I_0,I_1)(J_0,J_1)}$ so as to end up with a supermanifold $(G_k(V_0^m)\times G_h(V_1^n),\mathscr{A})$ whose underlying C^∞ (resp., holomorphic) manifold is precisely the space of (k,h)-dimensional supervector subspaces of $V^{m|n}$, namely, the product $G_k(V_0^m)\times G_h(V_1^n)$.

We are thus led to Manin's prescription [**Manin**]: The indexing set for the basic superdomains $\mathscr{U}_I$ will now consist of pairs of disjoint ordered sequences

$$I = I_0\hat{\cup}I_1 \qquad (\hat{\cup} := \text{disjoint union})$$

of length $m-k$ and $n-h$, respectively. (*Note.* The interpretation of such an indexing can be given in terms of some definite choice of homogeneous (ordered) basis $\{\mathbf{e}_i\}$ of $V_0^m\oplus V_1^n$ by letting W_I be the $(m-k,n-h)$-dimensional supervector subspace $\mathrm{Span}\{\mathbf{e}_i : i\in I = I_0\hat{\cup}I_1\}$.) Then we put

$$\mathscr{U}_I := \mathscr{U}_{I_0}\times\mathscr{U}_{I_1} \tag{1.7}$$

and let

$$\mathscr{A}_I := \mathscr{C}^\infty|\mathscr{U}_{I_0}\times\mathscr{U}_{I_1}\otimes\bigwedge((\mathrm{Hom}(V_0^m/W_{I_0},W_{I_1})\oplus\mathrm{Hom}(V_1^n/W_{I_1},W_{I_0}))^*). \tag{1.8}$$

Following [**Manin**] we shall arrange the coordinate functions of the superdomain $(\mathscr{U}_I,\mathscr{A}_I)$ into a $[(m-k)+(n-h)]\times[m+n]$ matrix, $p_I=(p_I^{ia})$, where, if $I^c = I_0^c\hat{\cup}I_1^c$ denotes the complementary ordered sequence of length (k,h),

$$p_I^{ia} = \begin{cases}\delta^{ia}; & i\in I,\ a\in I,\\ z_I^{ia}; & i\in I,\ a\in I^c,\end{cases} \tag{1.9}$$

and additionally,

$$z_I^{ia} = \begin{cases} x_I^{ia}; & i\in I_0,\ a\in I_0^c,\\ y_I^{ia}; & i\in I_1,\ a\in I_1^c,\\ \varsigma_I^{ia}; & i\in I_1,\ a\in I_0^c,\\ \xi_I^{ia}; & i\in I_0,\ a\in I_1^c,\end{cases} \tag{1.10}$$

where the x_I^{ia} (resp., y_I^{ia}) are the standard coordinate functions on $\mathscr{U}_{I_0}$ (resp., $\mathscr{U}_{I_1}$) and the ς_I^{ia} (resp., ξ_I^{ia}) are the generators of $\bigwedge((\mathrm{Hom}(V_0^m/W_{I_0},W_{I_1}))^*)$ [resp., $\bigwedge((\mathrm{Hom}(V_1^n/W_{I_1},W_{I_0}))^*)$]. We shall also write the matrix (p_I^{ia}) in the symbolic form

$$p_I = \begin{array}{c} \begin{pmatrix}\mathrm{id} & x_I & 0 & \xi_I\\ 0 & \varsigma_I & \mathrm{id} & y_I\end{pmatrix}\begin{array}{l}\}I_0\\ \}I_1\end{array}\\ \begin{array}{cccc} I_0 & I_0^c & I_1 & I_1^c\end{array}\end{array} \tag{1.11}$$

If $J = J_0 \hat{\cup} J_1$ is some other sequence, we define g_{IJ} as the submatrix of p_I obtained after deleting the columns indexed by $J^c = J_0^c \hat{\cup} J_1^c$. This yields an $(m-k)+(n-h)$ square *even* matrix

$$g_{IJ} = \begin{array}{c} \begin{pmatrix} g_{I_0 J_0} & g_{I_0 J_1} \\ g_{I_1 J_0} & g_{I_1 J_1} \end{pmatrix} \begin{array}{l} \}I_0 \\ \}I_1 \end{array} \\ \begin{array}{cc} J_0 & J_1 \end{array} \end{array} \tag{1.12}$$

As we know from [**Kostant**] and [**Leĭtes**], this matrix is invertible if and only if $g_{I_0 J_0}$ and $g_{I_1 J_1}$ are, and this fact guarantees that the glueing of sheaves below takes place precisely over $\mathscr{U}_{I_0} \cap \mathscr{U}_{J_0} \times \mathscr{U}_{I_1} \cap \mathscr{U}_{J_1} = \mathscr{U}_I \cap \mathscr{U}_J$. So define

$$\varphi_{IJ} \colon \mathscr{A}_I | \mathscr{U}_I \cap \mathscr{U}_J \xrightarrow{\sim} \mathscr{A}_J | \mathscr{U}_I \cap \mathscr{U}_J \tag{1.13}$$

by letting

$$\varphi_{IJ} z_I^{ia} = ((g_{JI})^{-1} \cdot p_J)^{ia}. \tag{1.14}$$

It is not difficult to check that the φ_{IJ}'s are sheaf isomorphisms with $(\varphi_{IJ})^{-1} = \varphi_{JI}$, $\varphi_{II} = \mathrm{id}$ and $\varphi_{JK}\varphi_{IJ} = \varphi_{IK}$ on $\mathscr{U}_I \cap \mathscr{U}_J \cap \mathscr{U}_K$, so that there exists a unique sheaf $\mathscr{A}$ of supercommutative superalgebras on $G_k(V_0^m) \times G_h(V_1^n)$, such that

$$\mathbf{G}_{k|h}(V^{m|n}) = (G_k(V_0^m) \times G_h(V_1^n), \mathscr{A}) \tag{1.15}$$

becomes a $(k(m-k)+h(n-h), h(m-k)+k(n-h))$-dimensional C^∞ (resp., holomorphic) supermanifold. Furthermore, it is clear from this construction that

$$\mathscr{G}r^0 \mathscr{A} = \mathscr{A}/\mathscr{J} \cong \mathscr{C}^\infty | G_k(V_0^m) \times G_h(V_1^n) \tag{1.16}$$

where $\mathscr{J}$ denotes the ideal generated by the odd elements. On the other hand, it has been proved in [**Manin**] that

$$\mathscr{G}r \mathscr{A} \simeq \Gamma(\cdot, \bigwedge([(F_1)^* \otimes E_0] \oplus [(F_0)^* \otimes E_1])) \tag{1.17}$$

where E_0 (resp., E_1) denotes the total space of the rank k (resp., h) tautological bundle over $G_k(V_0^m)$ (resp., $G_h(V_1^n)$), while $(F_0)^*$ (resp., $(F_1)^*$) denotes that of the dual of the rank $m-k$ (resp., $n-h$) tautological bundle over $G_k(V_0^m)$ (resp., $G_h(V_1^n)$). The proof consists of using the local data in terms of which the supergrassmannians have been defined in conjunction with the identifications

$$\begin{aligned} (\mathrm{Hom}(V_0^m/W_{I_0}, W_{I_1}))^* &\simeq (W_{I_1})^* \otimes (V_0^m/W_{I_0}), \\ (\mathrm{Hom}(V_1^n/W_{I_1}, W_{I_0}))^* &\simeq (W_{I_0})^* \otimes (V_1^n/W_{I_1}), \end{aligned} \tag{1.18}$$

although the details will be omitted here (see §3, however).

2. Homogeneous space structure. From the geometric interpretation given in §1, it is clear that the group $\mathrm{GL}(m) \times \mathrm{GL}(n)$, acting on $V^{m|n}$ in the usual way, will transform (k,h)-dimensional supervector subspaces into (k,h)-dimensional supervector subspaces, so that it is only natural to try to obtain a representation

$$\rho \colon \mathrm{GL}(m) \times \mathrm{GL}(n) \to \mathrm{Aut}\,\mathscr{A} \tag{2.1}$$

for which

$$\mathrm{Gr}(\rho) \colon \mathrm{GL}(m) \times \mathrm{GL}(n) \to \mathrm{Aut}(\mathscr{G}r \mathscr{A}) \tag{2.2}$$

reduces to the usual action. The prescription for this effect goes as follows (cf. [**Manin**]): given $g \in \mathrm{GL}(m) \times \mathrm{GL}(n)$ we consider $p_I g^{-1}$ and look for the invertible submatrix $\psi_{I\tilde{I}}$ obtained from the columns of $p_I g^{-1}$ indexed by $\tilde{I} = \tilde{I}_0 \hat{\cup} \tilde{I}_1$ (one may regard this as the definition of $\tilde{I}$). Then we set

$$\rho_{I\tilde{I}}(g) \colon \mathscr{A}|\mathscr{V}_I \to \mathscr{A}|\mathscr{V}_{\tilde{I}} \tag{2.3}$$

where $\mathscr{V}_I$ and $\mathscr{V}_{\tilde{I}}$ are the open subsets of $\mathscr{U}_I$ and $\mathscr{U}_{\tilde{I}}$ determined by the condition of $\psi_{I\tilde{I}}$ being invertible (see below). Explicitly, if we write p_I as in (11) and for $g \in \mathrm{GL}(m) \times \mathrm{GL}(n)$ we put

$$g^{-1} = \begin{array}{c} \begin{pmatrix} U & W & 0 & 0 \\ V & Z & 0 & 0 \\ 0 & 0 & E & F \\ 0 & 0 & G & H \end{pmatrix} \begin{array}{l} \}I_0 \\ \}I_0^c \\ \}I_1 \\ \}I_1^c \end{array} \\ \begin{array}{cccc} \tilde{I}_0 & \tilde{I}_0^c & \tilde{I}_1 & \tilde{I}_1^c \end{array} \end{array} \tag{2.4}$$

Then

$$p_I g^{-1} = \begin{array}{c} \begin{pmatrix} U + x_I V & W + x_I Z & \xi_I G & \xi_I H \\ \varsigma_I V & \varsigma_I Z & E + y_I G & F + y_I H \end{pmatrix} \\ \begin{array}{cccc} \tilde{I}_0 & \tilde{I}_0^c & \tilde{I}_1 & \tilde{I}_1^c \end{array} \end{array} \tag{2.5}$$

$\mathscr{V}_I$ and $\mathscr{V}_{\tilde{I}}$ are now determined by the conditions $\det(U + x_I V) \neq 0$ and $\det(E + y_I G) \neq 0$, so that

$$\psi_{I\tilde{I}} = \begin{pmatrix} U + x_I V & \xi_I G \\ \varsigma_I V & E + y_I G \end{pmatrix} \tag{2.6}$$

is invertible. Now, writing $(\psi_{I\tilde{I}})^{-1}$ in the form

$$(\psi_{I\tilde{I}})^{-1} = \begin{array}{c} \begin{pmatrix} \chi_{\tilde{I}_0 I_0} & \chi_{\tilde{I}_0 I_1} \\ \chi_{\tilde{I}_1 I_0} & \chi_{\tilde{I}_1 I_1} \end{pmatrix} \begin{array}{l} \}\tilde{I}_0 \\ \}\tilde{I}_1 \end{array} \\ \begin{array}{cc} I_0 & I_1 \end{array} \end{array} \tag{2.7}$$

and using Gauss's decomposition on $\psi_{I\tilde{I}}$ (cf. [**Leĭtes**] and §5 below), one explicitly finds

$$\begin{aligned} \chi_{\tilde{I}_0 I_0} &= [\mathrm{id} + (U + x_I V)^{-1} \xi_I G (E + y_I G)^{-1} \varsigma_I V]^{-1} (U + x_I V)^{-1}, \\ \chi_{\tilde{I}_0 I_1} &= -(U + x_I V)^{-1} \xi_I G [\mathrm{id} + (E + y_I G)^{-1} \varsigma_I V (U + x_I V)^{-1} \xi_I G]^{-1} \\ &\qquad \times (E + y_I G)^{-1}, \\ \chi_{\tilde{I}_1 I_0} &= -(E + y_I G)^{-1} \varsigma_I V [\mathrm{id} + (U + x_I V)^{-1} \xi_I G (E + y_I G)^{-1} \varsigma_I V]^{-1} \\ &\qquad \times (U + x_I V)^{-1}, \\ \chi_{\tilde{I}_1 I_1} &= [\mathrm{id} + (E + y_I G)^{-1} \varsigma_I V (U + x_I V)^{-1} \xi_I G]^{-1} (E + y_I G)^{-1}; \end{aligned} \tag{2.8}$$

and therefore $\rho(g)$ is given in local coordinates by

$$
\begin{aligned}
x_I &\mapsto [\mathrm{id}+(U+x_IV)^{-1}\xi_IG(E+y_IG)^{-1}\varsigma_IV]^{-1}(U+x_IV)^{-1}(W+x_IZ)\\
&\quad+(U+x_IV)^{-1}\xi_IG[\mathrm{id}+(E+y_IG)^{-1}\varsigma_IV(U+x_IV)^{-1}\xi_IG]^{-1}\\
&\quad\times(E+y_IG)^{-1}\varsigma_IZ,\\
\varsigma_I &\mapsto -(E+y_IG)^{-1}\varsigma_IV[\mathrm{id}+(U+x_IV)^{-1}\xi_IG(E+y_IG)^{-1}\varsigma_IV]^{-1}\\
&\quad\times(U+x_IV)^{-1}(W+x_IZ)\\
&\quad+[\mathrm{id}+(E+y_IG)^{-1}\varsigma_IV(U+x_IV)^{-1}\xi_IG]^{-1}(E+y_IG)^{-1}\varsigma_IZ,\\
\xi_I &\mapsto [\mathrm{id}+(U+x_IV)^{-1}\xi_IG(E+y_IG)^{-1}\varsigma_IV]^{-1}(U+x_IV)^{-1}\xi_IH\\
&\quad-(U+x_IV)^{-1}\xi_IG[\mathrm{id}+(E+y_IG)^{-1}\varsigma_IV(U+x_IV)^{-1}\xi_IG]^{-1}\\
&\quad\times(E+z_IG)^{-1}(F+y_IH),\\
y_I &\mapsto [\mathrm{id}+(E+y_IG)^{-1}\varsigma_IV(U+x_IV)^{-1}\xi_IG]^{-1}(E+y_IG)^{-1}(F+y_IH)\\
&\quad+(E+y_IG)^{-1}\varsigma_IV[\mathrm{id}+(U+x_IV)^{-1}\xi_IG(E+y_IG)^{-1}\varsigma_IV]^{-1}\\
&\quad\times(U+x_IV)^{-1}\xi_IH,
\end{aligned}
\tag{2.9}
$$

from which we deduce that $\operatorname{Gr}\rho(g)$ is given by

$$
\begin{aligned}
x_I &\mapsto (U+x_IV)^{-1}(W+x_IZ)=-(Ax_I-B)(Cx_I-D)^{-1},\\
\varsigma_I &\mapsto -(E+y_IG)^{-1}\varsigma_I[V(U+x_IV)^{-1}(W+x_IZ)-Z]\\
&\quad=-(E+y_IG)^{-1}\varsigma_I(Cx_I-D)^{-1},\\
\xi_I &\mapsto -(U+x_IV)^{-1}\xi_I[G(E+y_IG)^{-1}(F+y_IH)-H]\\
&\quad=-(U+x_IV)^{-1}\xi_I(Ny_I-P)^{-1},\\
y_I &\mapsto (E+y_IG)^{-1}(F+y_IH)=-(Ly_I-M)(Ny_I-P)^{-1},
\end{aligned}
\tag{2.10}
$$

where

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} L & M\\ N & P\end{pmatrix}$$

are the inverses of

$$\begin{pmatrix} U & W\\ V & Z\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} E & F\\ G & H\end{pmatrix}$$

respectively.

These formulas show explicitly that $\operatorname{Gr}\rho(g)$ gives indeed the usual action on the base $G_k(V_0^m)\times G_h(V_1^n)$ and on the odd coordinates when regarded as sections of the corresponding tautological bundles. (We shall come back to the group action in §5.)

3. Tautological supervector bundles. It is well known from classical differential geometry that the ordinary grassmannian manifold of k-planes in V_0^m, $G_k(V_0^m)$, can be coordinatized by means of the matrices (cf. (1.9) and assume for the moment that $I_1=\varnothing$; hence, $I=I_0$)

$$p_I=(\delta_{ia}\mid x_I^{ia}),\quad i\in I, \tag{3.1}$$

with $a\in I$ in the first column and $a\in I^c$ in the second. The geometrical interpretation goes as follows: p_I is the matrix (with respect to some fixed basis $\{\mathbf{e}_\alpha\}$ of V_0^m) of the projection map

$$p_I\colon V_0^m\to W_I \tag{3.2}$$

onto the $(m-k)$-dimensional subspace $W_I := \mathrm{Span}\{\mathbf{e}_i : i \in I\}$ (see the note in §1). The coordinates (x_I^{ia}) appearing in (3.1) correspond to the k-dimensional subspace $\mathrm{Ker}\, p_I$. A basis for this subspace can be taken as

$$\sigma_a^I = \sum_{i \in I} x_I^{ia} \mathbf{e}_i - \mathbf{e}_a, \qquad a \in I^c, \tag{3.3}$$

and it is understood that $\mathrm{Ker}\, p_I$ belongs to the coordinate chart

$$\mathscr{U}_I \simeq \mathrm{Hom}(V_0^m / W_I, W_I).$$

It is then clear that the rank-k tautological vector bundle E_k over the grassmannian $G_k(V_0^m)$ corresponds to the locally free sheaf of $\mathscr{C}^\infty | G_k(V_0^m)$-modules defined by letting $\Gamma(\mathscr{U}_I, E_k)$ be the $\mathscr{C}^\infty(\mathscr{U}_I)$-module freely generated by the k linearly independent sections (3.3).

The transition functions of this bundle are easy to obtain: if $\mathrm{Ker}\, p_I = u \in \mathscr{U}_I \cap \mathscr{U}_J$, we can use the versions I and J of (3.3) to find a $k \times k$ invertible matrix, $\gamma_{IJ} = (\gamma_{IJ}^{bc})$, such that

$$\sigma_c^J(u) = \sum_{b \in I^c} \gamma_{IJ}^{bc}(u) \sigma_b^I(u), \qquad c \in J^c, \tag{3.4}$$

and in fact,

$$\gamma_{IJ} = \begin{pmatrix} -x_J^{bc} & -x_J^{bc} \\ 0 & \delta_{bc} \end{pmatrix} \begin{matrix} \}b \in I^c \cap J \\ \}b \in I^c \cap J^c \end{matrix} \tag{3.5}$$

with $c \in J^c \cap I$ in the first column and $c \in J^c \cap I^c$ in the second, and

$$\gamma_{IJ}^{-1} = \begin{pmatrix} -x_I^{bc} & -x_I^{bc} \\ 0 & \delta_{bc} \end{pmatrix} \begin{matrix} \}b \in J^c \cap I \\ \}b \in J^c \cap I^c \end{matrix}. \tag{3.6}$$

with $c \in J^c \cap I$ in the first column and $c \in J^c \cap I^c$ in the second. We can similarly describe the other tautological bundle; namely, the rank-$(m-k)$ bundle, $F_{m-k} \to G_k(V^m)$, whose fiber at the point $u \in G_k(V^m)$ is given by

$$F_u = (V^m/u)^* = \mathrm{Hom}(V^m/u, \mathbf{R}). \tag{3.7}$$

In this case the map $p_I : V^m \to W_I$ yields an injection $(p_I)^* : (W_I)^* \to (V^m)^*$ whose image we identify with F_u ($u = \mathrm{Ker}\, p_I$). Explicitly, let $\{\boldsymbol{\theta}^a\}$ be the dual basis of $\{\mathbf{e}_a\}$ above. Then

$$\lambda_I^i := (p_I)^* \boldsymbol{\theta}^i = \boldsymbol{\theta}^i + \sum_{a \in I^c} \boldsymbol{\theta}^a x_I^{ia}, \qquad i \in I, \tag{3.8}$$

gives $m-k$ linearly independent sections of F_{m-k} over $\mathscr{U}_I$. The transition functions for this bundle are also easy to obtain: this time we let $B(u) = (B_{ji}(u))$ be the matrix such that, for $u \in \mathscr{U}_I \cap \mathscr{U}_J$,

$$\lambda_J^j(u) = \sum_{i \in I} \lambda_I^i(u) B_{ji}(u). \tag{3.9}$$

Again, using (3.8) for I and J, one is led to the conclusion that B must satisfy the matrix equation

$$p_J = B \cdot p_I. \tag{3.10}$$

In other words,

$$B = (g_{IJ})^{-1} \tag{3.11}$$

where g_{IJ} is as in (1.12) (under the assumptions $I_1 = J_1 = \varnothing$; hence, $I = I_0$ and $J = J_0$).

Let us now go back to the grassmannian supermanifolds

$$\mathbf{G}_{k|h}(V^{m|n}) = (G_k(V_0^m) \times G_h(V_1^n), \mathscr{A}).$$

According to the observations above, it is natural to define the tautological supervector bundle $\mathbf{E}_{k|h}$ (resp., $\mathbf{F}_{m-k|n-h}$) by means of the locally free sheaf of $\mathscr{A}$-modules, $\mathscr{E}_{k|h}$ (resp., $\mathscr{F}_{m-k|n-h}$), obtained upon the specification that $\mathscr{E}_{k|h}(\mathscr{U}_I)$ (resp., $\mathscr{F}_{m-k|n-h}(\mathscr{U}_I)$) is the free $\mathscr{A}(\mathscr{U}_I)$-bundle generated by the k *even sections*

$$\sigma_a^I = \sum_{i \in I} z_I^{ia} \mathbf{e}_i - \mathbf{e}_a, \qquad a \in I_0^c, \tag{3.12}$$

and the h *odd sections*

$$\tau_a^I = \sum_{i \in I} z_I^{ia} \mathbf{e}_i - \mathbf{e}_a, \qquad a \in I_1^c \tag{3.13}$$

(resp., the $m-k$ *even sections*

$$\lambda_I^i = \boldsymbol{\theta}^i + \sum_{a \in I^c} \boldsymbol{\theta}^a z_I^{ia}, \quad i \in I_0, \tag{3.14}$$

and the $n-h$ *odd sections*

$$\mu_I^i = \boldsymbol{\theta}^i + \sum_{a \in I^c} \boldsymbol{\theta}^a z_I^{ia}, \qquad i \in I_1). \tag{3.15}$$

It is a straightforward matter now to verify that the transition functions for these bundles are given by exactly the same expressions as their classical counterparts, namely,

$$\gamma_{IJ} = \begin{array}{c} \begin{pmatrix} -z_J & -z_J \\ 0 & \mathrm{id} \end{pmatrix} \begin{array}{l} \}I^c \cap J \\ \}I^c \cap J^c \end{array} \\ \begin{array}{cc} J^c \cap I & J^c \cap I^c \end{array} \end{array} \quad \text{and} \quad g_{IJ}^{-1} \tag{3.16}$$

respectively, where z_J has the meaning of (1.10) and g_{IJ} that of (1.12).

On the other hand, proceeding as in [**OASV 2**], one can compute the transition functions of the *supercotangent bundle sheaf*, $\mathscr{ST}^*\mathscr{A}$, of $G_{k|h}(V^{m|n})$. The computation is tedious but straightforward and we obtain, for $j \in J = J_0 \hat{\cup} J_1$ and $i \in I = I_0 \hat{\cup} I_1$,

$$\begin{aligned} &\frac{\partial}{\partial z_I^{ia}} (g_{JI}^{-1} p_I)_{jb} \\ &\quad = (-1)^{(|a|+|b|)(|a|+|j|)} (g_{IJ}^{-1})_{ji} \begin{cases} -z_J^{ab}, & a \in I^c \cap J,\ b \in I^c \cap J^c, \\ -z_J^{ab}, & a \in I^c \cap J,\ b \in I \cap J^c, \\ \delta_{ab}, & a \in I^c \cap J^c,\ b \in I^c \cap J^c, \\ 0, & a \in I^c \cap J^c,\ b \in I \cap J^c. \end{cases} \end{aligned} \tag{3.17}$$

In particular, one obtains the following analog of the well-known classical result (for another proof of this fact without using explicitly the transition functions of the supercotangent bundle sheaf, see [**Manin**]).

PROPOSITION. *There is a natural identification* $\mathscr{ST}^*\mathscr{A} \simeq \mathscr{F}_{m-k|n-h} \otimes \mathscr{E}_{k|h}$. □

4. Equivariant trivialization theorem. We shall now prove that *some* supergrassmannians admit the type of *global trivialization* that makes their structural sheaf look as the sheaf of sections of the exterior algebra bundle of some canonical vector bundle attached to the base manifold. Furthermore, it turns out that for this particular source of examples, such a trivialization is equivariant for the $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$-action on $\mathbf{G}_{k|h}(V^{m|n})$ (given as in §2) and the induced $\mathrm{GL}(V_0^m) \times \mathrm{GL}(V_1^n)$-action on the space of sections of the bundle in question. Thus, Batchelor's theorem becomes *natural* for certain examples of supermanifolds (see [**Batchelor**]), namely, at least those described in the statement of the following

4.1 THEOREM. (a) $\mathbf{G}_{k|0}(V^{m|n}) = (G_k(V_0^m), \mathscr{A}_{G_k(V_0^m)})$ *is a supermanifold of dimension* $(k(m-k), nk)$ *for which an equivariant identification*

$$\mathscr{A}_{G_k(V_0^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_0^m)}) \simeq \Gamma(\cdot, \bigwedge(V_1 \otimes E_1))$$

exists. (*Here,* $E_0 \to G_k(V_0^m)$ *denotes the rank-k tautological vector bundle over the grassmannian* $G_k(V_0^m)$*;* $V_1^n \to G(V_0^m)$ *is the rank-n trivial bundle and* $V_1^n \otimes E_0 \to G_k(V_0^m)$ *is the tensor product bundle.*)

REMARK. A similar result holds true for the $(k(m-k), nk)$-dimensional supermanifold $\mathbf{G}_{0|k}(V^{n|m}) = (G_k(V_1^m), \mathscr{A}_{G_k(V_1^m)})$, in which case we have $\mathscr{A}_{G_k(V_1^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_1^m)}) \simeq \Gamma(\cdot, \bigwedge(V_0 \otimes E_1))$, equivariantly.

(b) $\mathbf{G}_{k|n}(V^{m|n}) = (G_k(V_0^m), \mathscr{A}_{G_k(V_0^m)})$ *is a supermanifold of dimension* $(k(m-k), n(m-k))$ *for which an equivariant identification*

$$\mathscr{A}_{G_k(V_0^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_0^m)}) \simeq \Gamma(\cdot, \bigwedge(V_1 \otimes F_0^*))$$

exists. (*Here,* $F_0^* \to G_k(V_0^m)$ *denotes the rank-*$(m-k)$ *tautological vector bundle over the grassmannian* $G_k(V_0^m)$.)

REMARK. Similarly, for $\mathbf{G}_{n|k}(V^{n|m}) = (G_k(V_1^m), \mathscr{A}_{G_k(V_1^m)})$, we have $\mathscr{A}_{G_k(V_1^m)} \xrightarrow{\sim} \mathscr{G}r(\mathscr{A}_{G_k(V_1^m)}) \simeq \Gamma(\cdot, \bigwedge(V_0 \otimes F_1^*))$, equivariantly.

PROOF. We shall prove (a) and (b) simultaneously. The proof itself is divided into two steps; the first consists of convincing ourselves that the sheaves $\mathscr{A}$ and $\mathscr{G}r\mathscr{A}$ of the supergrassmannians in question can be identified indeed. To this end, we first use the following lemma.

4.2 LEMMA. *For any supermanifold* $(M, \mathscr{A})$, *we have*

$$\mathscr{G}r\mathscr{A} \simeq \Gamma(\cdot, \bigwedge((\mathrm{ST}^*M)_1)),$$

where $(\mathrm{ST}^*M)_1$ *is the* ***odd*** (*Whitney*) ***summand*** *of the underlying manifold of the supercotangent bundle* (*cf.* [**OASV 2**]).

PROOF OF THE LEMMA. This follows essentially from the definition of a supermanifold (cf. [**Manin**] for details). □

We now use the transition functions (3.17) that define the supercotangent bundle sheaf of the supergrassmannians and note that, for $\mathbf{G}_{k|0}(V^{m|n})$ (resp., $\mathbf{G}_{k|n}(V^{m|n})$), the indexing sequences $I = I_0 \hat{\cup} I_1$ of the covering $\{\mathscr{U}_I\}$ have the following special property:

(4.1a) $\qquad (\forall I, J) \qquad I_1 = \{1, 2, \ldots, n\} = J_1 \quad \text{and} \quad I_1^c = \varnothing = J_1^c$

(4.1b) $$[\text{resp., } (\forall I, J) \quad I_1 = \varnothing = J_1 \quad \text{and} \quad I_1^c = \{1, 2, \ldots, n\} = J_1^c].$$

Therefore, the corresponding matrices p_I in (1.11) respectively reduce to

(4.2a) $$p_I = \begin{array}{c} \begin{pmatrix} \mathrm{id} & x_I & 0 \\ 0 & \varsigma_I & \mathrm{id} \end{pmatrix} \begin{array}{l} \}I_0 \\ \}I_1 \end{array} \\ \begin{array}{ccc} I_0 & I_0^c & I_1 \end{array} \end{array}$$

and

(4.2b) $$p_I = \begin{array}{c} (\mathrm{id} \quad x_I \quad \xi_I)\}I_0 \\ \begin{array}{ccc} I_0 & I_0^c & I_1^c \end{array} \end{array}$$

Hence, the matrices g_{IJ} in (1.12) will take the form

(4.3a) $$g_{IJ} = \begin{array}{c} \begin{pmatrix} g_{I_0 J_0} & 0 \\ * & \mathrm{id} \end{pmatrix} \begin{array}{l} \}I_0 \\ \}I_1 \end{array} \\ \begin{array}{cc} J_0 & J_1 \end{array} \end{array}$$

and

(4.3b) $$g_{IJ} = (g_{I_0 J_0})$$

respectively. In particular,

(4.4a) $$g_{IJ}^{-1} = \begin{array}{c} \begin{pmatrix} g_{I_0 J_0}^{-1} & 0 \\ * & \mathrm{id} \end{pmatrix} \begin{array}{l} \}J_0 \\ \}J_1 \end{array} \\ \begin{array}{cc} I_0 & I_1 \end{array} \end{array}$$

and

(4.4b) $$g_{IJ}^{-1} = (g_{I_0 J_0}^{-1}).$$

Now, we can use these expressions in conjunction with (1.10) and (3.17) to conclude that

(4.5a) $$\frac{\partial z_J^{jb}}{\partial z_I^{ia}} = \frac{\partial x_J^{ib}}{\partial \varsigma_I^{ia}} \Leftrightarrow a \in I_0,\ b \in J_0,\ i \in I_1,\ j \in J_0$$

and

(4.5b) $$\frac{\partial z_J^{jb}}{\partial z_I^{ia}} = \frac{\partial x_J^{jb}}{\partial \xi_I^{ia}} \Leftrightarrow a \in I_0,\ b \in J_0,\ i \in I_0,\ j \in J_0$$

respectively. Note right away that (3.17) and (4.4a) immediately imply that

(4.6a) $$\frac{\partial x_J^{jb}}{\partial \varsigma_I^{ia}} = 0 \quad \text{because } (g_{IJ}^{-1})_{ji} = 0 \quad \text{for all } i \in I_1,\ j \in J_0.$$

On the other hand, for case (b), the only nonvacuous condition on Lemma 4.2 is $a \in I_1^c \cap J_1^c$ and $b \in I_0^c \cap J_0^c$, in which case $\delta_{ab} = 0$. Therefore,

(4.6b) $$\frac{\partial x_J^{jb}}{\partial \xi_I^{ia}} = 0.$$

Similarly, one notes that

$$\frac{\partial z_J^{jb}}{\partial z_I^{ia}} = \frac{\partial \varsigma_J^{jb}}{\partial \varsigma_I^{ia}} \Leftrightarrow a \in I_0,\ b \in J_0,\ i \in I_1,\ j \in J_1 \tag{4.7a}$$

and

$$\frac{\partial z_J^{jb}}{\partial z_I^{ia}} = \frac{\partial \xi_J^{jb}}{\partial \xi_I^{ia}} \Leftrightarrow a \in I_1,\ b \in J_1,\ i \in I_0,\ j \in J_0. \tag{4.7b}$$

For case (a) one sees from (4.4a) that

$$(g_{IJ}^{-1})_{ji} = \delta_{ij} \quad \text{for } i \in I_1,\ j \in J_1.$$

Therefore, (3.17) implies that

$$\frac{\partial \varsigma_J^{jb}}{\partial \varsigma_I^{ia}} = \delta_{ij} \times \{\text{Transition functions of the rank-}k\text{ bundle, } E_0^* \to G_k(V_0^m)\}. \tag{4.8a}$$

On the other hand, for case (b), the only nonvacuous condition in Lemma 4.2 is $a \in I_1^c \cap J_1^c$ and $b \in I_1^c \cap J_1^c$, in which case,

$$\frac{\partial \xi_J^{jb}}{\partial \xi_I^{ia}} = \delta_{ab} \times \{\text{Transition functions of the rank-}(m-k)\text{ bundle, } F_0^* \to G_k(V_0^m)\}. \tag{4.8b}$$

But now we are in the situation of having a covering $\{\mathscr{U}_I\}$ for a supermanifold $(M, \mathscr{A})$ with sheaf isomorphisms $\mathscr{A}|\mathscr{U}_I \to \mathscr{G}r\mathscr{A}|\mathscr{U}_I$ satisfying the additional property that the transition functions

$$\begin{tikzcd}
(\mathscr{G}r\mathscr{A}|\mathscr{U}_J)|\mathscr{U}_I \cap \mathscr{U}_J \arrow[rr, "\varphi_{IJ}"] & & (\mathscr{G}r\mathscr{A}|\mathscr{U}_I)|\mathscr{U}_I \cap \mathscr{U}_J \\
& \mathscr{A}|\mathscr{U}_I \cap \mathscr{U}_J \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

are already of the form $\mathrm{Gr}(\varphi_{IJ})$ for all I and J. (This is precisely the content of equations (4.7) and (4.8) above, as no higher-order terms appear in the transition functions for x_J^{jb} and ξ_J^{jb} which, in principle, should be polynomials in the odd variables ξ_I^{ib}.) As the sheaf $\mathscr{A}$ used in the construction of the supergrassmannians was uniquely determined by the transition functions φ_{IJ}, the conclusion is then that we have a global identification $\mathscr{A} \xrightarrow{\sim} \mathscr{G}r\mathscr{A}$ for both cases (a) and (b). In other words, equations (4.8), together with Lemma 1, imply that

$$\mathscr{A} \xrightarrow{\sim} \Gamma(\cdot, \bigwedge(V_1^n \otimes E)) \tag{4.9a}$$

and

$$\mathscr{A} \xrightarrow{\sim} \Gamma(\cdot, \bigwedge(V_1^n \otimes F^*)) \tag{4.9b}$$

hold true globally for the sheaves $\mathscr{A}$ that define the supergrassmannians $\mathbf{G}_{k|0}(V^{m|n})$ and $\mathbf{G}_{k|n}(V^{m|n})$ respectively.

The second and last part of the proof consists of looking at the action of the group $G = \mathrm{GL}(m) \times \mathrm{GL}(n)$ according to the representation $\rho\colon G \to \mathrm{Aut}\,\mathscr{A}$ of (2.9).

But in view of the properties (4.1) of the indexing sequences I and J, the action by $\rho(g)$ with

$$g = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & E \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} U & W & 0 \\ V & Z & 0 \\ 0 & 0 & E^{-1} \end{pmatrix} \tag{4.10}$$

is reduced in each case to simply

$$\begin{aligned} &x_I \mapsto (U + x_I V)^{-1}(W + x_I Z) = -(Ax_I - B)(Cx_I - D)^{-1}, \\ &\varsigma_I \mapsto -E\varsigma_I(Cx_I - D)^{-1} \end{aligned} \tag{4.11a}$$

and

$$\begin{aligned} &x_I \mapsto (U + x_I V)^{-1}(W + x_I Z) = -(Ax_I - B)(Cx_I - D)^{-1}, \\ &\xi_I \mapsto (U + x_I V)^{-1}\xi_I E^{-1} \end{aligned} \tag{4.11b}$$

respectively, which is precisely the action under $\mathrm{Gr}(\rho(g))$. □

5. Supergroup action; infinitesimal version. We note that the group action discussed so far is just an ordinary Lie group action. It would be desirable, however, to prove that the supergrassmannians are natural examples of *superhomogeneous supermanifolds* (in the sense already defined in [**Kostant**]); that is, that there is a *supergroup action* defined on them. Theorems regarding superhomogeneous supermanifolds in general shall be postponed for a future work. Here, we shall restrict ourselves to the *infinitesimal action* for the examples at hand (the supergrassmannians) and get some insight into what to expect in the general case. Along these lines, we should be able to prove the following:

5.1 PROPOSITION. *There is a graded Lie algebra homomorphism*

$$\tilde{\rho}\colon \mathfrak{gl}(V_0^m|V_1^n) \to \mathrm{Der}\,\mathscr{A}_{\mathbf{G}_{k|h}(V^{m|n})}$$

whose restriction to $\mathfrak{gl}(V_0^m) \oplus \mathfrak{gl}(V_1^n) = (\mathfrak{gl}(V_0^m|V_1^n))_0$ *coincides with the (classical) infinitesimal action obtained from* §2.

The idea of the proof and the actual realization of the action $\tilde{\rho}$ as *supervector fields* (i.e., superderivations of the structural sheaf $\mathscr{A}$) on the supergrassmannian $\mathbf{G}_{k|h}(V^{m|n})$ are based entirely on classical-like arguments. The reason why such classical arguments work on the superhomogeneous setting follows, on the one hand, from Kostant's results [**Kostant**, §§3.9, 3.10] and, on the other, from our work in [**OASV 3**] which allows classical interpretations while computing with matrices (see, in particular, §5.3 below). We shall indicate how an explicit representation of $\mathfrak{gl}(V_0^m|V_1^n)$ may be obtained within a coordinate chart and shall give some formulas for the supergrassmannians $\mathbf{G}_{q|0}(V^{2q|n})$. A first step in this direction is given by the following rather trivial

5.2 OBSERVATION. Let $\rho\colon \mathfrak{gl}(V_0 \oplus V_1) \to \mathrm{End}(M_0 \oplus M_1)$ be a representation of the ordinary Lie algebra $\mathfrak{gl}(V_0 \oplus V_1) \simeq \mathfrak{gl}(m+n)$ into a supervector space $M = M_0 \oplus M_1$. Suppose that

$$\rho((\mathfrak{gl}(V_0 \oplus V_1))_0) \subset (\mathrm{End}(M_0 \oplus M_1))_0$$

and

$$\rho((\mathfrak{gl}(V_0 \oplus V_1))_1) \subset (\mathrm{End}(M_0 \oplus M_1))_1$$

where

$$(\mathfrak{gl}(V_0 \oplus V_1))_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A \in \mathfrak{gl}(V_0),\ D \in \mathfrak{gl}(V_1) \right\}$$

and

$$(\mathfrak{gl}(V_0 \oplus V_1))_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B \in \mathrm{Hom}(V_1, V_0),\ C \in \mathrm{Hom}(V_0, V_1) \right\}.$$

Then, ρ gives rise to a representation $\tilde{\rho}\colon \mathfrak{gl}(V_0|V_1) \to \mathrm{End}(M_0 \oplus M_1)$ of the graded Lie algebra $\mathfrak{gl}(V_0|V_1)$ into the same supervector space $M = M_0 \oplus M_1$.

PROOF OF 5.2. All that is involved in this assertion is the one-to-one correspondence between the representations ρ of $\mathfrak{gl}(V_0 \oplus V_1)$ and the representations $\boldsymbol{\rho}$ of its universal enveloping algebra $\mathfrak{U}(\mathfrak{gl}(V_0 \oplus V_1))$. In fact, taking into account the vector space isomorphisms

$$(\mathfrak{gl}(V_0 \oplus V_1))_0 \simeq \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1) \simeq (\mathfrak{gl}(V_0|V_1))_0,$$
$$(\mathfrak{gl}(V_0 \oplus V_1))_1 \simeq \mathrm{Hom}(V_1, V_0) \oplus \mathrm{Hom}(V_0, V_1) \simeq (\mathfrak{gl}(V_0|V_1))_1$$

and the fact that under the ordinary matrix multiplication (i.e., composition of linear maps)

$$(\mathfrak{gl}(V_0 \oplus V_1))_\mu \cdot (\mathfrak{gl}(V_0 \oplus V_1))_\nu \subset (\mathfrak{gl}(V_0 \oplus V_1))_{(\mu+\nu) \bmod (2)},$$

we can define a degree-zero linear mapping $\tilde{\rho}\colon \mathfrak{gl}(V_0|V_1) \to \mathrm{End}(M_0 \oplus M_1)$ by letting

$$(\forall x \in \mathfrak{gl}(V_0|V_1),\ \textit{homogeneous}) \qquad \tilde{\rho}(x) := \boldsymbol{\rho}(x) \quad (= \rho(x))$$

and extending it linearly. We now claim that for homogeneous $x, y \in \mathfrak{gl}(V_0|V_1)$ we have

$$\tilde{\rho}(x \cdot y - (-1)^{|x|\,|y|} y \cdot x) = \tilde{\rho}(x)\tilde{\rho}(y) - (-1)^{|x|\,|y|}\tilde{\rho}(y)\tilde{\rho}(x).$$

Obviously, it suffices to check this equation for any two odd elements $x, y \in \mathfrak{gl}(V_0|V_1)_1$ (i.e., $|x| = |y| = 1$). But in this case we have

$$\begin{aligned} \tilde{\rho}(x \cdot y + y \cdot x) &= \boldsymbol{\rho}(x \cdot y + y \cdot x) = \boldsymbol{\rho}(x \cdot y) + \boldsymbol{\rho}(y \cdot x) \\ &= \boldsymbol{\rho}(y)\boldsymbol{\rho}(x) + \boldsymbol{\rho}([x, y]) + \boldsymbol{\rho}(x)\boldsymbol{\rho}(y) + \boldsymbol{\rho}([y, x]) \\ &= \boldsymbol{\rho}(y)\boldsymbol{\rho}(x) + \boldsymbol{\rho}(x)\boldsymbol{\rho}(y) = \tilde{\rho}(x)\tilde{\rho}(y) + \tilde{\rho}(y)\tilde{\rho}(x). \quad \square \end{aligned}$$

5.3. Now, the classical-like argument by which we may arrive at the desired representation $\tilde{\rho}\colon \mathfrak{gl}(V_0|V_1) \to \mathrm{Der}\,\mathscr{A}$ is very simple: choose a local coordinate chart on the supergrassmannian so that (cf. §1)

$$p = \begin{pmatrix} \mathrm{id} & x & 0 & \xi \\ 0 & \varsigma & \mathrm{id} & y \end{pmatrix}.$$

Then we choose the *supercoset representative*

$$\sigma_p = \begin{pmatrix} \mathrm{id} & x & 0 & \xi \\ 0 & \mathrm{id} & 0 & 0 \\ 0 & \varsigma & \mathrm{id} & y \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

In order to see what the action of $\mathfrak{gl}(V_0|V_1)$ looks like in terms of these local coordinates, we first look at the equation

$$\begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \eta & \varphi & e & f \\ \rho & \tau & g & h \end{pmatrix} \begin{pmatrix} 1 & x & 0 & \xi \\ 0 & 1 & 0 & 0 \\ 0 & \varsigma & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & g\cdot x & 0 & g\cdot\xi \\ 0 & 1 & 0 & 0 \\ 0 & g\cdot\varsigma & 1 & g\cdot y \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} J_F & 0 & A & 0 \\ c & J_E & \gamma & \Delta \\ N & 0 & J_H & 0 \\ \rho & T & g & J_G \end{pmatrix}$$

where

$$g := \begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \eta & \varphi & e & f \\ \rho & \tau & g & h \end{pmatrix} \in \mathrm{GL}(V_0 \oplus V_1) \simeq \mathrm{GL}(m+n)$$

is assumed to be close to the identity and J_E, J_F, J_G and J_H are elements of $\mathrm{GL}(k)$, $\mathrm{GL}(m-k)$, $\mathrm{GL}(h)$ and $\mathrm{GL}(n-h)$, respectively. Thus, we find that

$$\begin{aligned} g\cdot x &= \{[(ax+b)+\alpha\varsigma] - [(\alpha y+\beta)+a\xi]J_G^{-1}T\}(I - J_E^{-1}\Delta J_G^{-1}T)^{-1}J_E^{-1}, \\ g\cdot\varsigma &= \{[(\eta x+\varphi)+e\varsigma] - [(ey+f)+\eta\xi]J_G^{-1}T\}(I - J_E^{-1}\Delta J_G^{-1}T)^{-1}J_E^{-1}, \\ g\cdot y &= \{-[(\eta x+\varphi)+e\varsigma]J_E^{-1}\Delta + [(ey+f)+\eta\xi]\}(I - J_G^{-1}TJ_E^{-1}\Delta)^{-1}J_G^{-1}, \\ g\cdot\xi &= \{-[(ax+b)+\alpha\varsigma]J_E^{-1}\Delta + [(\alpha y+\beta)+a\xi]\}(I - J_G^{-1}TJ_E^{-1}\Delta)^{-1}J_G^{-1}, \end{aligned}$$

where, furthermore,

$$\begin{aligned} J_E &= (cx+d)+\gamma\varsigma, & T &= (\rho x+\tau)+g\varsigma, \\ \Delta &= (\gamma y+\delta)+c\xi, & J_G &= (gy+h)+\rho\xi. \end{aligned}$$

Note that these calculations are valid under the assumption that $g \in \mathrm{GL}(V_0 \oplus V_1)$ stays close to the identity, for then we can make sense of $g\cdot x$, $g\cdot\varsigma$, $g\cdot y$ and $g\cdot\xi$ as coordinates within the same chart. This condition is certainly fulfilled by any smooth curve $t \mapsto g_t \in \mathrm{GL}(V_0 \oplus V_1)$ passing through the identity at $t=0$. In particular, if we set

$$(d/dt)(g_t)|_{t=0} =: \dot{g}_0 \in \mathfrak{gl}(V_0 \oplus V_1)$$

we can differentiate the above formulas for $g_t\cdot x$, $g_t\cdot\varsigma$, $g_t\cdot y$ and $g_t\cdot\xi$ to obtain an expression in local coordinates for the action of the Lie algebra $\mathfrak{gl}(V_0 \oplus V_1)$ and hence, for the action of the Lie superalgebra $\mathfrak{gl}(V_0|V_1)$, provided that the conditions of 5.2 are satisfied.

Let us illustrate this construction on the supergrassmannians $\mathbf{G}_{k|0}(V^{m|n})$. In this case the coordinates $\{y\}$ and $\{\xi\}$ are nonexistent and the above formulas for $g\cdot x$ and $g\cdot\varsigma$ reduce to

$$\begin{aligned} g\cdot x &= \{(ax+b)(cx+d)^{-1} + \alpha\varsigma(cx+d)^{-1}\}\{I+\gamma\varsigma(cx+d)^{-1}\}^{-1}, \\ g\cdot\varsigma &= \{(\eta x+\varphi)(cx+d)^{-1} + e\varsigma(cx+d)^{-1}\}\{I+\gamma\varsigma(cx+d)^{-1}\}^{-1} \end{aligned}$$

whenever

$$g := \begin{pmatrix} a & b & \alpha \\ c & d & \gamma \\ \eta & \varphi & e \end{pmatrix} \in \mathrm{GL}(V_0 \oplus V_1) \simeq \mathrm{GL}(m+n).$$

Therefore, the corresponding action of $\mathfrak{gl}(V_0 \oplus V_1)$ will be given by

$$X \cdot x = \dot{a}x - x\dot{d} + \dot{b} - x\dot{c}x + \dot{\alpha}\varsigma - x\dot{\gamma}\varsigma,$$
$$X \cdot \varsigma = \dot{e}\varsigma - \varsigma\dot{d} + \dot{\varphi} - \varsigma\dot{\gamma}\varsigma + \dot{\eta}\varsigma - \varsigma\dot{c}x$$

whenever

$$X := \begin{pmatrix} \dot{a} & \dot{b} & \dot{\alpha} \\ \dot{c} & \dot{d} & \dot{\gamma} \\ \dot{\eta} & \dot{\varphi} & \dot{e} \end{pmatrix} \in \mathfrak{gl}(V_0 \oplus V_1) \simeq \mathfrak{gl}(m+n).$$

One can now read off from these equations the corresponding action in terms of derivations (with respect to the local coordinates $\{x^{ij}\}$ and $\{\varsigma^{\mu j}\}$).

To write down some explicit formulas, let us further restrict ourselves to the case $m = 2q$ and $k = m/2$, so that the latin indices $i, j, \ldots$ will run through $\{1, 2, \ldots, q\}$ while the greek indices $\mu, \nu, \ldots$ will run through $\{1, 2, \ldots, n\}$. Thus,

$$\begin{pmatrix} \varepsilon_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \sum_k x^{jk}\partial_{x^{ik}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto -\sum_k x^{ki}\partial_{x^{kj}} - \sum_\mu \varsigma^{\mu i}\partial_{\varsigma^{\mu j}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ \varepsilon_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto -\sum_{k,m} x^{mi}x^{jk}\partial_{x^{mk}} - \sum_{k,\mu} \varsigma^{\mu i}x^{jk}\partial_{\varsigma^{\mu k}},$$
$$\begin{pmatrix} 0 & \varepsilon_{ij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \partial_{x^{ij}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{\nu\mu} \end{pmatrix} \mapsto \sum_k \varsigma^{\mu k}\partial_{\varsigma^{\nu k}},$$
$$\begin{pmatrix} 0 & 0 & \varepsilon_{i\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \sum_k \varsigma^{\mu k}\partial_{x^{ik}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{i\mu} \\ 0 & 0 & 0 \end{pmatrix} \mapsto -\sum_{k,m} x^{mi}\varsigma^{\mu k}\partial_{x^{mk}} - \sum_{k,\nu} \varsigma^{\nu i}\varsigma^{\mu k}\partial_{\varsigma^{\nu k}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon_{\mu i} & 0 & 0 \end{pmatrix} \mapsto \sum_k x^{ik}\partial_{\varsigma^{\mu k}},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varepsilon_{\mu i} & 0 \end{pmatrix} \mapsto \partial_{\varsigma^{\mu i}}.$$

Note that even (resp., odd) elements of $\mathfrak{gl}(V_0 \oplus V_1)$ are mapped into even (resp., odd) derivations of the sheaf of local coordinates. Therefore, our observation 5.2 applies and the same correspondences provide us with an explicit realization of the

superalgebra $\mathfrak{gl}(V_0|V_1)$ as superderivations of the structural sheaf of the supergrassmannian $\mathbf{G}_{q|0}(V^{2q|n})$.

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