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INFINITELY MANY PERIODIC SOLUTIONS FOR THE EQUATION: $u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t)$. II

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ABSTRACT. Existence of forced vibrations of nonlinear wave equation:

$$\begin{aligned} &u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t), && (x,t) \in (0,\pi) \times \mathbf{R}, \\ &u(0,t) = u(\pi,t) = 0, && t \in \mathbf{R}, \\ &u(x,t+2\pi) = u(x,t), && (x,t) \in (0,\pi) \times \mathbf{R}, \end{aligned}$$

is considered. For all $p \in (1,\infty)$ and $f(x,t) \in L^{(p+1)/p}$, existence of infinitely many periodic solutions is proved. This improves the results of the author [**29**, **30**].

We use variational methods to show the existence result. Minimax arguments and energy estimates for the corresponding functional play an essential role in the proof.

0. Introduction and statement of result. The main purpose of this paper is to show the existence of infinitely many periodic solutions of the following nonlinear vibrating string equation:

$$u_{tt} - u_{xx} \pm |u|^{p-1}u = f(x,t), \qquad (x,t) \in (0,\pi) \times \mathbf{R}, \tag{0.1$_\pm$}$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbf{R}, \tag{0.2}$$

$$u(x,t+2\pi) = u(x,t), \qquad (x,t) \in (0,\pi) \times R. \tag{0.3}$$

Here, $p > 1$ is a constant and $f(x,t)$ is a 2π-periodic function of t.

In case f *is a function of* x *alone*, the existence of nontrivial solutions of $(0.1)_\pm$–(0.3) has been established by Brezis-Coron-Nirenberg [**11**], Coron [**13**] and Rabinowitz [**20**, **23**]. See also Benci-Fortunato [**7**] and Sattinger [**24**]. But in case f *depends on* t, it seems that the existence of at least one solution of $(0.1)_\pm$–(0.3) is not obtained for all $p \in (1,\infty)$ and the existence of infinitely many solutions for all f has been obtained merely in the case:

$$1 < p < 1 + \sqrt{2}. \tag{0.4}$$

See Tanaka [**29**, **30**] and Ollivry [**17**]. This paper is a continuation of [**29**, **30**] and we will show the existence of infinitely many periodic solutions for all $f(x,t)$ without restriction (0.4). More precisely our main result is the following.

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THEOREM 0.1. *Assume that $p \in (1,\infty)$ and $f(x,t) \in L^{(p+1)/p}_{\mathrm{loc}}([0,\pi] \times \mathbf{R})$ is 2π-periodic in t. Then $(0.1)_\pm$–(0.3) possesses an unbounded sequence of weak solutions in $L^{p+1}_{\mathrm{loc}}([0,\pi] \times \mathbf{R})$.*

REMARK. (a) By a weak solution of $(0.1)_\pm$–(0.3) we mean a function $u(x,t)$ satisfying

$$\int_0^{2\pi}\int_0^{\pi}[u(\phi_{tt}-\phi_{xx}) \pm |u|^{p-1}u\phi - f\phi]\,dx\,dt = 0$$

for all smooth ϕ which satisfy (0.2) and (0.3).

(b) If $p = 2n+1$ ($n \in \mathbf{N}$) and f is smooth, it is known that any corresponding solution is smooth (cf. Brezis-Nirenberg [**12**]).

In case $f = 0$, the problem $(0.1)_\pm$–(0.3) possesses a natural symmetry, that is, the equation is equivariant under the Z_2 symmetry $u \to -u$. We shall treat the case $f \neq 0$ as a perturbation from a symmetric equation. A question for $(0.1)_\pm$–(0.3) is the effect of destroying the symmetry by adding an inhomogeneous term $f(x,t)$ to the right-hand side of $(0.1)_\pm$.

In several recent papers, similar questions have been studied for the problems of elliptic type and of Hamiltonian systems of ordinary differential equations. Bahri-Berestycki [**3**], Struwe [**28**] and Rabinowitz [**21**] considered the following problem of elliptic type:

$$-\Delta u = |u|^{p-1}u + f(x), \qquad x \in D, \tag{0.5}$$

$$u = 0, \qquad x \in \partial D, \tag{0.6}$$

where $D \subset \mathbf{R}^N$ ($N \geq 2$) is a bounded domain with a smooth boundary ∂D and $f(x) \in L^2(D)$. For all $f(x) \in L^2(D)$, they showed the existence of infinitely many solutions of (0.5)–(0.6) under the condition

$$1 < p < \frac{N+2+\sqrt{9N^2-4N+4}}{4(N-1)}.$$

They considered the functional

$$F(u) = \frac{1}{2}\int_D |\nabla u|^2\,dx - \frac{1}{p+1}\int_D |u|^{p+1}\,dx - \int_D fu\,dx$$

on $H_0^1(D)$ and sought for critical points of this functional. Restricted Lusternik Schnirelman theory and energy estimates for $F(u)$ played an essential role in their arguments.

Very recently, Bahri-Lions [**6**] has improved the results of [**3**, **21**, **28**] and showed the existence of infinitely many solutions under the condition $1 < p < N/(N-2)$. To get this existence result, they used a general result giving *a lower bound of the Morse index at critical points obtained through dual minimax variational principles* together with a suitable estimate for the eigenvalues of the Dirichlet problem on a bounded domain D. See also Bahri [**2**].

The existence of periodic solutions of the following forced Hamiltonian systems of O.D.E. is considered by Bahri-Berestycki [**4**].

$$dz/dt = JH'(z) + f(t), \qquad z(0) = z(T). \tag{0.7}$$

Here,

$$J = \begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix}$$

is the standard skewsymmetric matrix, $z = z(t) = (p, q)$: $\mathbf{R} \to \mathbf{R}^{2N}$, H: $\mathbf{R}^{2N} \to \mathbf{R}$ is a given Hamiltonian and f: $\mathbf{R} \to \mathbf{R}^{2N}$ is a given T-periodic function. Under some growth condition on H (in particular H is of *superquadratic* growth), they show that (0.7) has infinitely many periodic solutions for all $f(t)$. See also Bahri-Berestycki [**5**] and Pisani-Tucci [**18**].

Theorem 0.1 will be proved in §§1–6 via *variational methods.* An outline of this paper is as follows: In §1, we introduce *new* variational formulation of the problem $(0.1)_\pm$–(0.3). That is, we introduce a functional $I(u^+ + u^-)$, whose critical points and weak solutions of $(0.1)_\pm$–(0.3) possess one-to-one correspondence. Our functional $I(u^+ + u^-)$ takes a form

$$I(u^+ + u^-) = \tfrac{1}{2}\|u^+\|_E^2 - \tfrac{1}{2}\|u^-\|_E^2 - Q(u^+ + u^-) \in C^1(E^+ \oplus E^-, \mathbf{R}), \tag{0.8}$$

where $E^\pm$ are Hilbert spaces and $Q(u^+ + u^-)$ is a convex functional such that $Q'(u^+ + u^-)$ is compact. In §2, we apply the methods of Rabinowitz [**21**, **22**, **23**] to $I(u^+ + u^-)$ and obtain the existence of infinitely many periodic solutions of $(0.1)_\pm$–(0.3) under some assumption on the growth of minimax values:

$$b_n = \inf_{\gamma \in \Gamma_n} \sup_{u \in D_n} I(\gamma(u)) \quad \text{as } n \to \infty.$$

In the second part of §2, we introduce a comparison functional $K(u^+)$ on E^+

$$K(u^+) = \frac{1}{2}\|u^+\|_E^2 - \frac{a_0}{p+1}\|u^+\|_{p+1}^{p+1}, \tag{0.9}$$

which satisfies $K(u^+) \leq I(u^+) + C$ on E^+ for some constant $C > 0$. We deal with $K(u^+)$ to get an estimate of b_n. In §3, critical points of $K(u^+)$ are constructed in a similar way to Bahri-Berestycki [**5**]; max-min value β_n, which is a critical value of K, is defined as in [**5**]. Using the ideas from Ambrosetti-Rabinowitz [**1**], we find that $\beta_n \leq b_n + C$. In §§4–6, we use the ideas from Bahri-Lions [**6**] and get an estimate of the growth of the values β_n as $n \to \infty$; first we establish a lower bound of the *Morse index* at a critical point corresponding to β_n (§4). Next we develop an estimate of eigenvalues of $K''(u^+)$. Here, the notion of *trace ideals* plays an essential role (§5). Lastly in §6, combining the results in §§4–5, we obtain estimates from below of the values β_n and we complete the proof of Theorem 0.1.

Thus this paper is organized as follows:

0. Introduction and statement of result
1. Variational formulation and functional frame work
2. Minimax methods and existence theorem
3. Critical value β_n of $K \in C^2(E^+, \mathbf{R})$ and its relation to b_n
4. Morse index and β_n
5. Estimate for eigenvalues of $K''(u^+)$
6. Proof of Theorem 0.1

1. Variational formulation and functional frame work.

(a) *A new variational formulation.* We deal with the problem $(0.1)_+$–(0.3). The problem $(0.1)_-$–(0.3) is treated similarly. Let $\Omega = (0, \pi) \times (0, 2\pi)$ and $|\Omega| = 2\pi^2$.

For $q \in [1, \infty)$ we denote by L^q the space of 2π-periodic functions of t whose qth powers are integrable, that is,

$$\|u\|_q = \left(\int_\Omega |u(x,t)|^q \,dx\,dt\right)^{1/q} < \infty.$$

We also use the notation

$$(u, v) = \int_\Omega uv \,dx\,dt.$$

Solutions of $(0.1)_+$–(0.3) are obtained as *critical points* of

$$F(u) = \int_\Omega \left[\frac{1}{2}(u_t^2 - u_x^2) - \frac{1}{p+1}|u|^{p+1} + fu\right] dx\,dt.$$

The quadratic wave form and the term $|u|^{p+1}$ suggest a natural space in which to treat $F(u)$. Any smooth function u satisfying (0.2) and (0.3) has a Fourier expansion of the form

$$u = \sum_{j=1}^\infty \sum_{k=-\infty}^\infty a_{jk} \sin jx\, e^{ikt}, \qquad a_{j,-k} = \bar{a}_{jk}.$$

We define

$$\langle u, v\rangle = \frac{1}{4}|\Omega| \sum_{j,k} |k^2 - j^2| a_{jk}\bar{b}_{jk}, \qquad \|u\|_E^2 = \langle u, u\rangle,$$

for $u = \sum a_{jk} \sin jx\, e^{ikt}$ and $v = \sum b_{jk} \sin jx\, e^{ikt}$. We observe that $\|\cdot\|_E$ is a norm on the set $\{u; a_{jk} = 0 \text{ if } j = |k|\}$. Set

$$\begin{aligned} E^+ &= \overline{\text{span}}\{\sin jx\, e^{ikt};\ j < |k|\}, \\ E^- &= \overline{\text{span}}\{\sin jx\, e^{ikt};\ j > |k|\}, \\ E &= E^+ \oplus E^- \end{aligned}$$

where the closures are taken under the norm $\|\cdot\|_E$. Note that $(E, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Further set

$$\begin{aligned} N &= \left\{\varsigma(x+t) - \varsigma(t-x);\ \varsigma \in L^{p+1}(S^1), \int_0^{2\pi} \varsigma = 0\right\} \\ &= L^{p+1}\text{-closure of } \overline{\text{span}}\{\sin jx\, e^{\pm ijt};\ j \in \mathbf{N}\} \end{aligned}$$

with L^{p+1}-norm $\|\cdot\|_{p+1}$.

Then E^+, E^-, N are complementary subspaces of the space of functions satisfying (0.2)–(0.3). Moreover the wave form is positive definite, negative definite and null on E^+, E^- and N respectively. We will treat $F(u)$ in the space $E^+ \oplus E^- \oplus N \equiv E \oplus N$. The space E has the following property (cf. [**11**, **24**]):

(1.1) $$\|u\|_q \le c_q \|u\|_E \quad \text{for all } u \in E \text{ and } q \in [1, \infty),$$

(1.2) the embedding $E \to L^q$ is compact for all $q \in [1, \infty)$.

Note that

(1.3) $$\begin{aligned} F(u+v) &= \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{1}{p+1}\|u^+ + u^- + v\|_{p+1}^{p+1} \\ &\quad + (f, u^+ + u^- + v) \in C^2(E^+ \oplus E^- \oplus N, \mathbf{R}) \\ &\qquad \text{for } u = u^+ + u^- \in E = E^+ \oplus E^- \text{ and } v \in N. \end{aligned}$$

Observe that for fixed u^+, u^-, the functional $F(u^+ + u^- + v)$ is a strictly concave function of $v \in N$. So there is a one-to-one correspondence between critical points of F and those of I. Here, $I\colon E \to \mathbf{R}$ is a functional defined by

$$\begin{aligned} I(u) &= \max_{v\in N} F(u+v) \\ &= \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - Q(u) \quad \text{for } u = u^+ + u^- \in E, \end{aligned} \tag{1.4}$$

where

$$Q(u) = \min_{v\in N}\left[\frac{1}{p+1}\|u+v\|_{p+1}^{p+1} - (f, u+v)\right] \quad \text{for } u \in E. \tag{1.5}$$

Hence, in what follows we will seek critical points of $I(u)$. Remark that $Q(u)$ can be also defined for all $u \in L^{p+1}$ by (1.5). So we treat $Q(u)$ as a function from L^{p+1} to $\mathbf{R}$.

LEMMA 1.1. (i) *For all $u \in L^{p+1}$, there exists a unique $v(u) \in N$ such that*

$$Q(u) = \frac{1}{p+1}\|u+v(u)\|_{p+1}^{p+1} - (f, u+v(u)). \tag{1.6}$$

(ii) *$v(u)\colon L^{p+1} \to N$ is continuous.*
(iii) *$Q(u)$ is of class C^1 on E and*

$$\langle Q'(u), h\rangle = (|u+v(u)|^{p-1}(u+v(u)) - f, h) \quad \textit{for all } u, h \in E. \tag{1.7}$$

In particular, $Q'(u)\colon E \to E^*$ is compact and there are $C_1 = C_1(\|f\|_{(p+1)/p}) > 0$ and $C_2 = C_2(\|f\|_{(p+1)/p}) > 0$ constants such that for all $u \in E$,

$$\|Q'(u)\|_{E^*} \le C_1(|Q(u)|^{p/(p+1)} + 1), \tag{1.8}$$

$$|\langle Q'(u), u\rangle - (p+1)Q(u)| \le C_2(|Q(u)|^{1/(p+1)} + 1). \tag{1.9}$$

From now on we denote by C various constants which depend on $\|f\|_{(p+1)/p}$ but are independent of $u \in E$.

PROOF. (i) We can easily deduce assertion (i) from the fact that

$$v \to \frac{1}{p+1}\|u+v\|_{p+1}^{p+1} - (f, u+v) \tag{1.10}$$

is a strictly convex, coercive functional on N.

(ii) Suppose that $u_j \to u$ in L^{p+1}. We will show that $v(u_j) \to v(u)$ strongly in N. By the definition of $v(u_j)$, we have

$$\begin{aligned} &\frac{1}{p+1}\|u_j + v(u)\|_{p+1}^{p+1} - (f, u_j + v(u)) \\ &\quad \ge \frac{1}{p+1}\|u_j + v(u_j)\|_{p+1}^{p+1} - (f, u_j + v(u_j)). \end{aligned} \tag{1.11}$$

We find that $\{v(u_j)\}_{j=1}^{\infty}$ is bounded in N (i.e., in L^{p+1}). We extract a subsequence—still denoted by u_j—such that $v(u_j)$ converges weakly to $\bar{v}$ in N. Letting

$j \to \infty$ in (1.11), we get

$$\begin{aligned}
&\frac{1}{p+1}\|u+v(u)\|_{p+1}^{p+1} - (f, u+v(u)) \\
&\quad \geq \varliminf_{j\to\infty} \left(\frac{1}{p+1}\|u_j + v(u_j)\|_{p+1}^{p+1} - (f, u_j + v(u_j)) \right) \\
&\quad \geq \frac{1}{p+1}\|u+\bar{v}\|_{p+1}^{p+1} - (f, u+\bar{v}).
\end{aligned}$$

By the uniqueness of $v(u)$, we observe $\bar{v} = v(u)$ and $\overline{\lim}\,\|u + v(u_j)\|_{p+1} = \|u+v(u)\|_{p+1}$. Thus we obtain $v(u_j) \to v(u)$ strongly in N.

(iii) By the convexity of (1.10), we find that for $w \in N$,

$$w = v(u) \quad \text{iff} \quad (|u+w|^{p-1}(u+w) - f, \varsigma) = 0 \quad \text{for all } \varsigma \in N. \tag{1.12}$$

By the convexity of the function $|\xi|^{p+1}/(p+1) - f\xi$, we have for all $u, h \in E$ and $\tau > 0$,

$$\begin{aligned}
Q(u+\tau h) - Q(u) &= \frac{1}{p+1}\left(\|u+\tau h + v(u+\tau h)\|_{p+1}^{p+1} - \|u+v(u)\|_{p+1}^{p+1} \right) \\
&\quad + (f, \tau h + v(u+\tau h) - v(u)) \\
&\geq (|u+v(u)|^{p-1}(u+v(u)) - f, \tau h + v(u+\tau h) - v(u)).
\end{aligned}$$

Since $v(u+\tau h) - v(u) \in N$, we get by (1.12)

$$Q(u+\tau h) - Q(u) \geq \tau(|u+v(u)|^{p-1}(u+v(u)) - f, h). \tag{1.13}$$

Similarly we have,

$$\begin{aligned}
&Q(u+\tau h) - Q(u) \\
&\quad \leq \tau(|u+\tau h + v(u+\tau h)|^{p-1}(u+\tau h + v(u+\tau h)) - f, h).
\end{aligned} \tag{1.14}$$

Letting $\tau \to 0$ in (1.13) and (1.14), we obtain (1.7). Thus $Q(u) \in C^1(E, \mathbf{R})$. Moreover from (1.2) and the continuity of $v(u)\colon L^{p+1} \to N$, we deduce $Q'(u)\colon E \to E^*$ is compact. Using (1.7), we have

$$\begin{aligned}
\|Q'(u)\|_{E^*} &= \sup_{\|h\|_E = 1} (|u+v(u)|^{p-1}(u+v(u)) - f, h) \\
&\leq \sup_{\|h\|_E = 1} \|\,|u+v(u)|^{p-1}(u+v(u)) - f\|_{(p+1)/p}\|h\|_{p+1} \\
&\leq c_{p+1}\|\,|u+v(u)|^{p-1}(u+v(u)) - f\|_{(p+1)/p}.
\end{aligned}$$

By the Hölder inequality and (1.6),

$$\|Q'(u)\|_{E^*} \leq C\left(\frac{1}{p+1}\|u+v(u)\|_{p+1}^{p} + 1 \right) \leq C(|Q(u)|^{p/(p+1)} + 1).$$

Inequality (1.9) can be easily obtained from (1.6), (1.7) and Hölder's inequality. Thus we have obtained the desired results. □

Now we can verify the Palais-Smale compactness condition (P.S.) for $I(u)$. This condition is required when we apply minimax methods to $I(u)$.

PROPOSITION 1.1. *$I(u) \in C^1(E, \mathbf{R})$ satisfies the following Palais-Smale compactness condition* (P.S.):

(P.S.) *Whenever a sequence $(u_j)_{j=1}^{\infty}$ in E satisfies for some $M > 0$,*

$$I(u_j) \le M \quad \text{for all } j, \tag{1.15}$$

$$I'(u_j) \to 0 \quad \text{in } E^* \text{ as } j \to \infty, \tag{1.16}$$

there is a subsequence of (u_j)which converges in E.

PROOF. We have for $u_j = u_j^+ + u_j^- \in E^+ \oplus E^- = E$,

$$\langle I'(u_j), h\rangle = \langle u_j^+ - u_j^-, h\rangle - \langle Q'(u_j), h\rangle \quad \text{for } h \in E.$$

Setting $h = u_j$ or $h = u_j^+ - u_j^-$, we get

$$\left| \|u_j^+\|_E^2 - \|u_j^-\|_E^2 - \langle Q'(u_j), u_j\rangle \right| \le m\|u_j\|_E, \tag{1.17}$$

$$\left| \|u_j\|_E^2 - \langle Q'(u_j), u_j^+ - u_j^-\rangle \right| \le m\|u_j\|_E, \tag{1.18}$$

where $m = \sup \|I'(u_j)\|_{E^*}$.

By assumption (1.15),

$$\tfrac{1}{2}\|u_j^+\|_E^2 - \tfrac{1}{2}\|u_j^-\|_E^2 - Q(u_j) \le M. \tag{1.19}$$

It follows from (1.17) and (1.19) that

$$\tfrac{1}{2}\langle Q'(u_j), u_j\rangle - Q(u_j) \le M + m\|u_j\|_E.$$

By (1.19), we get

$$\left(\frac{p+1}{2} - 1\right) Q(u_j) - C_2(|Q(u_j)|^{1/(p+1)} + 1) \le M + m\|u_j\|_E.$$

Hence we have

$$Q(u_j) \le C(\|u_j\|_E + 1) \quad \text{for all } j, \tag{1.20}$$

where $C > 0$ is independent of j.

On the other hand by (1.8), (1.20)

$$\begin{aligned} |\langle Q'(u_j), u_j^+ - u_j^-\rangle| &\le \|Q'(u_j)\|_{E^*}\|u_j\|_E \\ &\le C(|Q(u_j)|^{p/(p+1)} + 1)\|u_j\|_E \\ &\le C(\|u_j\|_E^{p/(p+1)} + 1)\|u_j\|_E. \end{aligned}$$

By (1.18), we have

$$\begin{aligned} \|u_j\|_E^2 &\le m\|u_j\|_E + \langle Q'(u_j), u_j^+ - u_j^-\rangle \\ &\le m\|u_j\|_E + C(\|u_j\|_E^{p/(p+1)} + 1)\|u_j\|_E. \end{aligned}$$

Thus we find that (u_j) is bounded in E.

Observe that $I'(u_j) = u_j^+ - u_j^- - Q'(u_j)$ where $Q'\colon E \to E^*$ is a compact operator and $I'(u_j) \to 0$ as $j \to \infty$. Hence $u_j^+ - u_j^-$ is precompact in E, that is, $u_j = u_j^+ + u_j^-$ is precompact in E. Thus the proof is completed. □

REMARK. We can verify that $F \in C^2(E \oplus N, \mathbf{R})$ satisfies (P.S.) in a similar way to the proof of Proposition 1.1. The reason for introducing $I(u)$ is as follows:

Since $Q'(u)$ is compact, the situation of the problem is analogous to the problem of periodic solutions of Hamiltonian systems of O.D.E. (cf. [**4**, **13**, **22**]) and the methods, which are used to find solutions of Hamiltonian systems, are applicable to $I(u)$ after a simple modification.

(b) *Modified functional.* Next, as in [**18**, **21**, **29**, **30**], we replace $I(u)$ by a modified functional $J(u) \in C^1(E, \mathbf{R})$.

Let $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\chi(\tau) = 1$ for $\tau \leq 1$, $\chi(\tau) = 0$ for $\tau \geq 2$ and $-2 \leq \chi'(\tau) \leq 0$, $0 \leq \chi(\tau) \leq 1$ for $\tau \in \mathbf{R}$. For $u = u^+ + u^- \in E^+ \oplus E^- = E$ we set

$$\Phi(u) = a(I(u)^2 + 1)^{1/2}, \quad \psi(u) = \chi(\Phi(u)^{-1} Q_0(u)),$$
$$J(u) = \tfrac{1}{2}\|u^+\|_E^2 - \tfrac{1}{2}\|u^-\|_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)),$$

where $a = \max\{1, 12/(p-1)\}$ and $Q_0(u) \in C^1(E, \mathbf{R})$ is a functional defined by

$$Q_0(u) = \min_{v \in N} \frac{1}{p+1} \|u + v\|_{p+1}^{p+1} \quad \text{for } u \in E. \tag{1.21}$$

We remark that as in Lemma 1.1 there is a unique $v_0(u) \in N$ such that

$$Q_0(u) = \frac{1}{p+1} \|u + v_0(u)\|_{p+1}^{p+1}.$$

The reason for introducing $J(u)$ is that the first assertion of the following proposition holds for $J(u)$ but not for $I(u)$.

PROPOSITION 1.2. *The functional $J(u) \in C^1(E, \mathbf{R})$ satisfies*
(i) *there is a constant $\alpha = \alpha(\|f\|_{(p+1)/p}) > 0$ such that for $u \in E$,*

$$|J(u) - J(-u)| \leq \alpha(|J(u)|^{1/(p+1)} + 1). \tag{1.22}$$

(ii) *There is a constant $M_0 = M_0(\|f\|_{(p+1)/p}) > 0$ such that $J(u) \geq M_0$ and $\|J'(u)\|_{E^*} \leq 1$ imply that $J(u) = I(u)$.*

The proof of Proposition 1.2 is rather technical and independent of further arguments, so we prove it in Appendix A. As immediate corollaries to Propositions 1.1, 1.2, we have

COROLLARY 1.1. *Whenever $u \in E$ satisfy $J'(u) = 0$ and $J(u) \geq M_0$, then $I(u) = J(u)$ and $I'(u) = 0$.*

COROLLARY 1.2. *$J(u)$ satisfies the Palais-Smale compactness condition* (P.S.) *on $A_{M_0} = \{u \in E; J(u) \geq M_0\}$.*

By Corollary 1.1, we see that large critical values of $J(u)$ are also critical values of $I(u)$. Hence we seek large critical values of $J(u)$ in the following sections.

2. Minimax methods and existence theorem.

(a) *Construction of critical points.* In this section we construct critical points of $J(u)$ via minimax methods.

We observe that the eigenvalues of the wave operator $\partial_t^2 - \partial_x^2$ under periodic-Dirichlet conditions (0.2)–(0.3) are $\{j^2 - k^2; j \in \mathbf{N}, k \in \mathbf{Z}\}$ and corresponding eigenfunctions are $\sin jx \cos kt$ and $\sin jx \sin kt$. We rearrange the negative eigenvalues in the following order, denoted by $0 > -\mu_1 \geq -\mu_2 \geq -\mu_3 \geq \cdots$ with repetitions according to the multiplicity of each eigenvalue and denote by e_j the eigenfunctions which correspond to $-\mu_j$. We assume $\langle e_i, e_j \rangle = \delta_{ij}$ for $i, j \in \mathbf{N}$. We find that

$$E^+ = \overline{\mathrm{span}}\{e_j; j \in \mathbf{N}\}.$$

We define

$$E_n^+ = \mathrm{span}\{e_j; 1 \le j \le n\}.$$

Note that

$$\|u\|_E \le \mu_n^{1/2} \|u\|_2 \quad \text{for } u \in E_n^+. \tag{2.1}$$

For all $u = u^+ + u^- \in E_n^+ \oplus E^-$, we have by Lemma 1.1, (A.2) (in Appendix A) and (2.1)

$$\begin{aligned}
J(u) &= \tfrac{1}{2}\|u^+\|_E^2 - \tfrac{1}{2}\|u^-\|_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)) \\
&\le \tfrac{1}{2}\|u^+\|_E^2 - \tfrac{1}{2}\|u^-\|_E^2 - Q_0(u) + C(Q_0(u)^{1/(p+1)} + 1) \\
&\le \tfrac{1}{2}\|u^+\|_E^2 - \tfrac{1}{2}Q_0(u) - \tfrac{1}{2}\|u^-\|_E^2 + C \\
&= \tfrac{1}{2}\|u^+\|_E^2 - 2(p+1)^{-1}\|u + v_0(u)\|_{p+1}^{p+1} - \tfrac{1}{2}\|u^-\|_E^2 + C \\
&\le \tfrac{1}{2}\|u^+\|_E^2 - c\|u^+ + u^- + v_0(u)\|_2^{p+1} - \tfrac{1}{2}\|u^-\|_E^2 + C \\
&\le \tfrac{1}{2}\|u^+\|_E^2 - c\|u^+\|_2^{p+1} - \tfrac{1}{2}\|u^-\|_E^2 + C \\
&\le \tfrac{1}{2}\|u^+\|_E^2 - c\mu_n^{-(p+1)/2}\|u^+\|_E^{p+1} - \tfrac{1}{2}\|u^-\|_E^2 + C.
\end{aligned}$$

Hence there is a constant $R_n > 0$ such that

$$J(u) \le 0 \quad \text{for all } u \in E_n^+ \oplus E^- \text{ with } \|u\|_E \ge R_n. \tag{2.2}$$

We may assume that $R_n < R_{n+1}$ for all n.

Let

$$\begin{aligned}
B_R &= \{u \in E; \|u\|_E \le R\} \quad \text{for } R \ge 0, \\
D_n &= B_{R_n} \cap (E_n^+ \oplus E^-), \\
\Gamma_n &= \{\gamma \in C(D_n, E);\ \gamma \text{ satisfies } (\gamma_1) - (\gamma_3)\},
\end{aligned}$$

where

- (γ_1) γ is odd, i.e., $\gamma(-u) = -\gamma(u)$ for all $u \in D_n$,
- (γ_2) $\gamma(u) = u$ for all $u \in \partial D_n$,
- (γ_3) for $u = u^+ + u^- \in D_n$, $\gamma(u) = \alpha(u)u + \kappa(u)$ where $\alpha \in C(D_n, [1, \bar\alpha])$ is an even functional ($\bar\alpha \ge 1$ depends on γ) and κ is a compact operator such that $\alpha(u) = 1$ and $\kappa(u) = 0$ on ∂D_n.

Moreover, set

$$\begin{aligned}
U_n &= D_{n+1} \cap \{u \in E; \langle u, e_{n+1} \rangle \ge 0\}, \\
\Lambda_n &= \{\lambda \in C(U_n, E);\ \lambda \text{ satisfies } (\lambda_1)\text{–}(\lambda_3)\},
\end{aligned}$$

where

- (λ_1) $\lambda|_{D_n} \in \Gamma_n$,
- (λ_2) $\lambda(u) = u$ on $\partial U_n \backslash D_n$,
- (λ_3) for $u = u^+ + u^- \in U_n$, $\lambda(u) = \tilde\alpha(u)u + \tilde\kappa(u)$ where $\tilde\alpha \in C(U_n, [1, \bar\alpha])$ ($\bar\alpha \ge 1$ depends on λ) and $\tilde\kappa$ is a compact operator such that $\tilde\alpha(u)$ is even on D_n, $\tilde\alpha(u) = 1$, $\tilde\kappa(u) = 0$ on $\partial U_n \backslash D_n$.

Define for $n \in \mathbf{N}$,

$$b_n = \inf_{\gamma\in\Gamma_n} \sup_{u\in D_n} J(\gamma(u)), \quad c_n = \inf_{\lambda\in\Lambda_n} \sup_{u\in U_n} J(\lambda(u)). \tag{2.3}$$

The above definitions are analogous to those of Rabinowitz [**21**, **23**] and Pisani-Tucci [**18**], which are used to prove the existence of multiple critical points of perturbed symmetric functionals. By the definitions it is clear that $c_n \geq b_n$. In case $c_n > b_n$ we have the following existence result for critical points of $I(u)$.

PROPOSITION 2.1 (cf. Lemma 1.57 of Rabinowitz [**21**]). *Suppose that $c_n > b_n \geq M_0$. Let $d \in (0, c_n - b_n)$ and*

$$\Lambda_n(d) = \{\lambda \in \Lambda_n;\ J(\lambda) \leq b_n + d \text{ on } D_n\}.$$

Define

$$c_n(d) = \inf_{\lambda\in\Lambda_n(d)} \sup_{u\in U_n} J(\lambda(u)) \quad (\geq c_n). \tag{2.4}$$

Then $c_n(d)$ is a critical value of $I(u)$.

SKETCH OF THE PROOF. Since Corollary 1.1 holds, we shall show that $c_n(d)$ is a critical value of $J(u)$. Remark that $J(u)$ satisfies (P.S.) condition (Corollary 1.2) and $J'(u)$ is an operator of the form:

$$J'(u) = (1 + T_1(u))(u^+ - u^-) + (\text{compact})$$

where $|T_1(u)| \leq \frac{1}{2}$ on $\{u \in E; J(u) \geq M_0\}$ (see Lemma A.3). Hence we can use the following Lemma 2.1. Using this lemma, we can prove Proposition 2.1 as in [**21**]. □

LEMMA 2.1 (cf. Lemma 1.36 of [**22**], Proposition 2.33 of [**23**]). *Suppose that $c > M_0$ is a regular value of $J(u)$, that is, $J'(u) \neq 0$ when $J(u) = c$. Then for any $\bar{\varepsilon} > 0$ there exist an $\varepsilon \in (0, \bar{\varepsilon}]$ and $\eta \in C([0,1] \times E, E)$ such that*

1° *$\eta(t, \cdot)$ is odd for all $t \in [0,1]$ if $f(x,t) \equiv 0$.*
2° *$\eta(t, \cdot)$ is a homeomorphism of E onto E for all t.*
3° *$\eta(0, u) = u$ for all $u \in E$.*
4° *$\eta(t, u) = u$ if $J(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$.*
5° *$J(\eta(1, u)) \leq c - \varepsilon$ if $J(u) \leq c + \varepsilon$.*
6° *For $u = u^+ + u^- \in E^+ \oplus E^- = E$, $\eta(1, u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u)$ where $\alpha^+ \in C(E, [0,1])$, $\alpha^- \in C(E, [1, \bar{\alpha}])$ ($\bar{\alpha} \geq 1$ is a constant) and κ is a compact operator.* □

Therefore, the existence of subsequence of c_n's which satisfy $c_n > b_n \geq M_0$ guarantees the existence of critical values. In what follows, we will show the existence of subsequence (n_j) such that

$$c_{n_j} > b_{n_j} \geq M_0 \quad \text{for } j \in \mathbf{N}, \tag{2.5}$$

$$b_{n_j} \to \infty \quad \text{as } j \to \infty. \tag{2.6}$$

Arguing indirectly, we have the following proposition.

PROPOSITION 2.2. *If $c_n = b_n$ for all $n \geq n_0$, then there is a constant $\overline{C} > 0$ such that*

$$b_n \leq \overline{C} n^{(p+1)/p} \quad \textit{for all } n \in \mathbf{N}. \tag{2.7}$$

PROOF. Using (i) of Proposition 1.2, the proof is as in Lemma 1.64 of [**21**]. □

To show the existence of subsequence (n_j) satisfying (2.5), (2.6), we will prove the existence of a sequence (n_j) such that for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ satisfying

$$b_{n_j} \geq C_\varepsilon n_j^{(p+1)/(p-1)-\varepsilon} \quad \text{for all } j \in \mathbf{N}, \tag{2.8}$$

which contradicts (2.7).

(b) *Comparison functional.* To verify (2.8) we need some comparison functional. By (A.2) and the definition of $Q_0(u)$, we have for $u = u^+ + u^- \in E = E^+ \oplus E^-$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)) \\ &\geq \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - 2Q_0(u) - a_1 \\ &= \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{2}{p+1}\min_{v \in N}\|u^+ + u^- + v\|_{p+1}^{p+1} - a_1 \\ &\geq \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{2}{p+1}\|u^+ + u^-\|_{p+1}^{p+1} - a_1 \\ &\geq \frac{1}{2}\|u^+\|_E^2 - \frac{1}{2}\|u^-\|_E^2 - \frac{a_0}{p+1}\|u^+\|_{p+1}^{p+1} - \frac{a_0}{p+1}\|u^-\|_{p+1}^{p+1} - a_1, \end{aligned} \tag{2.9}$$

where $a_0 > 0$, $a_1 > 0$ are constants independent of u. We set

$$K(u^+) = \frac{1}{2}\|u^+\|_E^2 - \frac{a_0}{p+1}\|u^+\|_{p+1}^{p+1} \in C^2(E^+, \mathbf{R}). \tag{2.10}$$

Then we have

LEMMA 2.2.
(i) *$J(u^+) \geq K(u^+) - a_1$ for all $u^+ \in E^+$.*
(ii) *$K(u^+)$ satisfies the Palais-Smale condition on E^+.*

PROOF. (i) By (2.9), the first assertion is obvious.

(ii) Since the embedding $E^+ \to L^{p+1}$ is compact, the proof is done in the standard way (cf. Proposition 1.1). □

In the next section, critical values β_n of $K(u^+)$ satisfying $\beta_n \leq b_n + a_1$ will be constructed and we will prove (2.8) for β_n instead of b_n in §§4–6.

3. Critical value β_n of $K \in C^2(E^+, \mathbf{R})$ and its relation to b_n.

(a) *Bahri-Berestycki's max-min value β_n.* In this section, we are concerned with the functional $K(u^+) \in C^2(E^+, \mathbf{R})$. We define family of mappings and max-min values β_n as follows. These definitions are analogous to those of Bahri-Berestycki [**4,5**], which are used to prove the existence of forced oscillations for superquadratic Hamiltonian systems. In the later sections, we state *index property of these max-min values*, which play an important role in verifying (2.8).

For $m > n$, $n, m \in \mathbf{N}$, set

$$A_n^m = \{\sigma \in C(S^{m-n}, E_m^+); \sigma(-x) = -\sigma(x) \text{ for all } x\}, \tag{3.1}$$

$$\beta_n^m = \sup_{\sigma \in A_n^m} \min_{x \in S^{m-n}} K(\sigma(x)). \tag{3.2}$$

Some properties of these numbers β_n^m are listed in the following proposition.

PROPOSITION 3.1.
(i) $0 \le \beta_n^m \le \beta_{n+1}^m < \infty$ *for all* m, n;
(ii) *for all* $n \in \mathbf{N}$, *there exist* $\nu(n)$ *and* $\tilde{\nu}(n)$ *such that*

$$0 \le \nu(n) \le \beta_n^m \le \tilde{\nu}(n) < \infty \quad \text{for all } m \ge n+1; \tag{3.3}$$

(iii) *moreover*, $\nu(n) \to \infty$ *as* $n \to \infty$.

PROOF. Note that for any $\theta \in (0, 1/(p+1))$ there exists $C_\theta > 0$ such that

$$\|u\|_{p+1} \le C_\theta \mu_n^{-\theta} \|u\|_E \quad \text{for all } u \in (E_n^+)^\perp. \tag{3.4}$$

Using (3.4), the proof is essentially as in Proposition 3.1 of [**5**]. We prove it in Appendix B. □

As in Proposition 1.1, we can verify the following compactness conditions $(\text{P.S.})_*$, $(\text{P.S.})_m$ ($m \in \mathbf{N}$) for $K(u^+)$.

$(\text{P.S.})_*$ If $(u_m) \subset E^+$ satisfies $u_m \in E_m^+$, $K(u_m) \le C$ and $\|(K|_{E_m^+})'(u_m)\|_{(E_m^+)^*} \to 0$ as $m \to \infty$, then (u_m) is relatively compact in E^+;

$(\text{P.S.})_m$ If $(u_j) \subset E_m^+$ satisfies $K(u_j) \le C$ and $(K|_{E_m^+})'(u_j) \to 0$ as $j \to \infty$, then (u_j) is relatively compact in E_m^+.

We have the following result via standard argument. (Remark that K is an even functional.)

PROPOSITION 3.2. *Suppose that* $\nu(n) > 0$. *Then* β_n^m *is a critical value of the restriction of* K *to* E_m^+. *Furthermore, the limit of any convergent subsequence of* β_n^m *as* $m \to \infty$ *is a critical value of* K. □

By (3.3), we can choose a sequence (m_j) such that $m_j \to \infty$ as $j \to \infty$,

$$\beta_n = \lim_{j \to \infty} \beta_n^{m_j} \text{ exists for all } n \in \mathbf{N}. \tag{3.5}$$

We find by Proposition 3.1 that
1° β_n is a critical value of $K \in C^2(E^+, \mathbf{R})$ for each n;
2° $\beta_n \le \beta_{n+1}$ for all n;
3° $\beta_n \to \infty$ as $n \to \infty$.
Next we state the relation between b_n and β_n.

(b) *The relation between* b_n *and* β_n. The main result in this section is as follows:

PROPOSITION 3.3. *For all* $n \in \mathbf{N}$,

$$b_n \ge \beta_n - a_1, \tag{3.6}$$

where a_1 *is the number which appeared in* (2.9).

To prove the above proposition, we need several lemmas. First we state a version of the Borsuk-Ulam theorem.

LEMMA 3.1. *Let $a, b \in \mathbf{N}$. Suppose that $h \in C(S^a, \mathbf{R}^{a+b})$ and $g \in C(\mathbf{R}^b, \mathbf{R}^{a+b})$ are continuous mappings such that*

$$h(-x) = -h(x) \quad \text{for all } x \in S^a, \tag{3.7}$$

$$g(-y) = -g(y) \quad \text{for all } y \in \mathbf{R}^b, \tag{3.8}$$

$$\text{there is a } r_0 > 0 \text{ such that } g(y) = y \text{ for } |y| \geq r_0. \tag{3.9}$$

Then $h(S^a) \cap g(\mathbf{R}^b) \neq \emptyset$.

PROOF. We choose $R \geq r_0$ such that $R > \max_{x \in S^a} |h(x)|$. Write

$$D^{a+1} = \{tx \in \mathbf{R}^{a+1};\ t \in [0,1],\ x \in S^a\},$$
$$D^b = \{y \in \mathbf{R}^b;\ |y| \leq R\}.$$

Define $F \in C(\partial(D^{a+1} \times D^b), \mathbf{R}^{a+b})$ by

$$F(tx, y) = th(x) - g(y).$$

This is well defined and odd on $\partial(D^{a+1} \times D^b)$. Remark that $\partial(D^{a+1} \times D^b) \simeq S^{a+b}$ (odd homeomorphic). Thus by the Borsuk-Ulam theorem, there is a $(t_0x_0, y_0) \in \partial(D^{a+1} \times D^b)$ such that

$$F(t_0x_0, y_0) = 0, \quad \text{i.e.,} \quad t_0h(x_0) = g(y_0).$$

Since $\partial(D^{a+1} \times D^b) = S^a \times D^b \cup D^{a+1} \times \partial D^b$, the following two cases should be considered:

1° $t_0 = 1$, $x_0 \in S^a$ and $y_0 \in D^b$;

2° $t_0 \in [0,1)$, $x_0 \in S^a$ and $y_0 \in \partial D^b$.

Case 1. We have $h(x_0) = g(y_0)$. So we have $h(S^a) \cap g(\mathbf{R}^b) \neq \emptyset$. This is the desired result.

Case 2. Since $g(y) = y$ on ∂D^b, we have $|g(y_0)| = |y_0| = R$. On the other hand, by the choice of R, we get $|t_0h(x_0)| < R$. These are incompatible with $t_0h(x_0) = g(y_0)$. So this case cannot take place. □

From the above lemma, we can deduce the following

LEMMA 3.2. *For all $\gamma \in \Gamma_n$ and $\sigma \in A_n^m$,*

$$((P_m\gamma)(D_n) \cup \{u \in E_n^+ \oplus E^-; \|u\|_E \geq R_n\}) \cap \sigma(S^{m-n}) \neq \emptyset,$$

where $P_m \colon E = E^+ \oplus E^- \to E_m^+ \oplus E^-$ is the orthogonal projection.

PROOF. We extend γ to $\tilde{\gamma} \in C(E_n^+ \oplus E^-, E)$ by

$$\tilde{\gamma}(u) = \gamma(u) \quad \text{if } \|u\|_E \leq R_n,$$

$$\tilde{\gamma}(u) = u \quad \text{if } \|u\|_E \geq R_n.$$

Obviously, $\tilde{\gamma}(u)$ is well defined and odd in $E_n^+ \oplus E^-$ and

$$P_m\tilde{\gamma}(E_n^+ \oplus E^-) = P_m\gamma(D_n) \cup \{u \in E_n^+ \oplus E^-; \|u\|_E \geq R_n\}.$$

Therefore, it suffices to prove $P_m\tilde{\gamma}(E_n^+ \oplus E^-) \cap \sigma(S^{m-n}) \neq \emptyset$. We rearrange $\{\sin jx \cos kt, \sin jx \sin kt;\ j > |k|\}$ as follows, denoted by $f_1, f_2, f_3, \ldots$. We set for $l \in \mathbf{N}$

$$E_l^- = \operatorname{span}\{f_j;\ 1 \leq j \leq l\}$$

and let $P_{m,l}\colon E = E^+ \oplus E^- \to E_m^+ \oplus E_l^-$ be the orthogonal projection. Consider the operators

$$\sigma\colon S^{m-n} \to E_m^+ \subset E_m^+ \oplus E_l^-, \quad P_{m,l}\tilde{\gamma}\colon E_n^+ \oplus E_l^- \to E_m^+ \oplus E_l^- .$$

Applying Lemma 3.1 for $h = \sigma$ and $g = P_{m,l}\tilde{\gamma}$ (obviously (3.7)–(3.9) are satisfied), we get for some $x_l \in s^{m-n}$ and $u_l \in E_n^+ \oplus E_l^-$,

$$\sigma(x_l) = P_{m,l}\tilde{\gamma}(u_l). \tag{3.10}$$

Since S^{m-n} is compact, there is a subsequence x_{l_j} such that

$$x_{l_j} \to x \quad \text{in } S^{m-n}, \tag{3.11}$$

$$\sigma(x_{l_j}) \to \sigma(x) \quad \text{in } E_m^+. \tag{3.12}$$

On the other hand, by (γ_3)

$$P_{m,l}\tilde{\gamma}(u_l) = P_{m,l}[\alpha(u_l)u_l + \kappa(u_l)] = \alpha(u_l)u_l + P_{m,l}\kappa(u_l),$$

where $\alpha(u_l) \geq 1$ on $E_n^+ \oplus E^-$ and $\overline{\kappa(E_n^+ \oplus E^-)} = \overline{\kappa(D_n)}$ is compact. Hence we have

$$u_l = \frac{1}{\alpha(u_l)} P_{m,l}[\tilde{\gamma}(u_l) - \kappa(u_l)] = \frac{1}{\alpha(u_l)} P_{m,l}[\sigma(x_l) - \kappa(u_l)].$$

By (3.12), (u_l) has a convergent subsequence (u_{l_j}), that is,

$$u_{l_j} \to u \quad \text{in } E_n^+ \oplus E^-. \tag{3.13}$$

Passing to the limit in (3.10), we obtain from (3.11), (3.13)

$$P_m\tilde{\gamma}(u) = \sigma(x), \quad \text{i.e.,} \quad P_m\tilde{\gamma}(E_n^+ \oplus E^-) \cap \sigma(S^{m-n}) \neq \emptyset.$$

Thus the proof is completed. □

PROOF OF PROPOSITION 3.3. Since $J(u) \leq 0$ on $\{u \in E_n^+ \oplus E^-;\ \|u\|_E \geq R_n\}$, we have from Lemma 3.2

$$\min_{x \in S^{m-n}} J(\sigma(x)) \leq \sup_{u \in D_n} J(P_m\gamma(u))$$

for all $\gamma \in \Gamma_n$ and $\sigma \in A_n^m$.

By (i) of Lemma 2.2,

$$\min_{x \in S^{m-n}} K(\sigma(x)) - a_1 \leq \sup_{u \in D_n} J(P_m\gamma(u)).$$

Hence we obtain

$$\sup_{\sigma \in A_n^m} \min_{x \in S^{m-n}} K(\sigma(x)) - a_1 \leq \inf_{\gamma \in \Gamma_n} \sup_{u \in D_n} J(P_m\gamma(u)),$$

i.e.,

$$\beta_n^m - a_1 \leq b_n^m \equiv \inf_{\gamma \in \Gamma_n} \sup_{u \in D_n} J(P_m\gamma(u)). \tag{3.14}$$

Letting $m = m_j \to \infty$, we get

$$\beta_n - a_1 \leq \limsup_{m \to \infty} b_n^m. \tag{3.15}$$

Thus, to get (3.6), it suffices to show the following lemma. □

LEMMA 3.3. *For $n \in \mathbf{N}$, $b_n = \lim_{m\to\infty} b_n^m$.*

PROOF. Since $P_m\Gamma_n = \{P_m\gamma; \gamma \in \Gamma_n\} \subset \Gamma_n$, it is clear that $b_n \le b_n^m$ for $m > n$. Let us prove

$$b_n \ge \limsup_{m\to\infty} b_n^m \quad \text{for } n \in \mathbf{N}.$$

From the definition of b_n, for any $\varepsilon > 0$ there is a $\gamma \in \Gamma_n$ such that

$$\sup_{u\in D_n} J(\gamma(u)) \le b_n + \varepsilon. \tag{3.16}$$

By (γ_3), $\gamma(u)$ takes a form: $\gamma(u) = \alpha(u)u + \kappa(u)$, where $\alpha(u) \in C(D_n, [1, \bar{\alpha}])$ and $\overline{\kappa(D_n)}$ is compact. Since

$$P_m\kappa(u) \to \kappa(u) \quad \text{as } m \to \infty \text{ uniformly in } D_n,$$

we have

$$P_m\gamma(u) = \alpha(u)u + P_m\kappa(u) \to \alpha(u)u + \kappa(u) = \gamma(u) \quad \text{uniformly in } D_n.$$

Hence

$$\sup_{u\in D_n} J(P_m\gamma(u)) \to \sup_{u\in D_n} J(\gamma(u)) \quad \text{as } m \to \infty. \tag{3.17}$$

By (3.16), (3.17), we obtain

$$\begin{aligned} \limsup_{m\to\infty} b_n^m &\le \limsup_{m\to\infty} \sup_{u\in D_n} J(P_m\gamma(u)) \\ &= \sup_{u\in D_n} J(\gamma(u)) \le b_n + \varepsilon. \end{aligned}$$

Since the above inequality holds for any $\varepsilon > 0$, we get the desired result. □

Thus, combining (3.15) and Lemma 3.3, the proof of Proposition 3.3 is completed.

REMARK. In this section the idea from Ambrosetti-Rabinowitz [**1**] is used to get Proposition 3.3. More precisely, let us consider

$$\begin{aligned} \tilde{A}_n^m &= \{\sigma \in C(S^{m-n}, E_m^+ \oplus E^-); \sigma(-x) = -\sigma(x) \text{ for all } x \in S^{m-n}\}, \\ \tilde{\beta}_n^m &= \sup_{\sigma\in\tilde{A}_n^m} \min_{x\in S^{m-n}} J(\sigma(x)). \end{aligned}$$

This is a *dual version* (in the sense of [**1**]) of minimax value:

$$b_n^m = \inf_{P_m\gamma\in P_m\Gamma_n} \sup_{u\in D_n} J(P_m\gamma(u)).$$

(Compare with Theorem 2.8 and Theorem 2.13 of [**1**].) Moreover we have $b_n^m \ge \tilde{\beta}_n^m$. Since $A_n^m \subset \tilde{A}_n^m$, we deduce (3.14) from (i) of Lemma 2.2.

4. Morse index and β_n. In this section, some index property of max-min value β_n is discussed. Combining the estimates of eigenvalues, which will be studied in §5, we will get the growth estimate $\beta_{n_j} \ge C_\varepsilon n_j^{(p+1)/(p-1)-\varepsilon}$ for a suitable subsequence (β_{n_j}). By Proposition 3.3, we obtain (2.8).

For $u \in E^+$, we define a *index* of $K''(u)$ by

$$\begin{aligned} &\text{index } K''(u) = \text{the number of eigenvalues of } K''(u) \\ &\qquad\qquad \text{which are nonpositive.} \end{aligned}$$

That is,

$$\text{index } K''(u) = \max\{\dim H;\ H \subset E^+ \text{ subspace such that } \langle K''(u)h, h\rangle \le 0 \text{ for } h \in H\}.$$

This is a generalization of a Morse index (cf. Bahri-Lions [**6**]).

The main result in this section is the following

PROPOSITION 4.1. *Suppose that $\beta_n < \beta_{n+1}$. Then there exists a $u_n \in E^+$ such that*

$$K(u_n) \le \beta_n, \tag{4.1}$$

$$K'(u_n) = 0, \tag{4.2}$$

$$\text{index } K''(u_n) \ge n. \tag{4.3}$$

Since β_n is a critical value of $K(u)$, the result without assertion (4.3) is obvious. To get (4.3), we first consider finite dimensional case.

PROPOSITION 4.2. *Suppose that $\beta_n^m < \beta_{n+1}^m$, $m > n+1$. Then there exists a $u_n^m \in E_m^+$ such that*

$$K(u_n^m) \le \beta_n^m, \tag{4.4}$$

$$(K|_{E_m^+})'(u_n^m) = 0, \tag{4.5}$$

$$\text{index}(K|_{E_m^+})''(u_n^m) \ge n. \tag{4.6}$$

To prove the above proposition, we adapt a classical theorem from Morse theory, i.e., a result concerning the relationship between certain homotopy groups of level sets of a functional and its critical points. Since we must treat the case where critical points may be *degenerate*, we use the following approximation result due to Marino-Prodi [**16**].

PROPOSITION 4.3 (Marino-Prodi [**16**], cf. Proposition 2.3 of Bahri-Berestycki [**5**]). *Let U be a C^2 open subset of some Hilbert space H and let $\phi \in C^2(U, \mathbf{R})$. Assume that ϕ'' is a Fredholm operator* (*of null index*) *on the critical set $Z(\phi) = \{x \in U; \phi'(x) = 0\}$. Lastly, suppose that ϕ satisfies the condition* (P.S.) *and that $Z(\phi)$ is compact. Then, for any $\varepsilon > 0$, there exists $\psi \in C^2(U, \mathbf{R})$ satisfying* (P.S.) *with the following properties*:

(i) *$\psi(x) = \phi(x)$ if distance $\{x, Z(\phi)\} \ge \varepsilon$*;

(ii) *$|\psi(x) - \phi(x)|$, $\|\psi'(x) - \phi'(x)\|$, $\|\psi''(x) - \phi''(x)\| \le \varepsilon$ for all $x \in U$*;

(iii) *the critical points of ψ are finite in number and nondegenerate.* □

We remark that $K|_{E_m^+} \in C^2(E_m^+, \mathbf{R})$ satisfies (P.S.) and that all critical values of $K|_{E_m^+}$ are nonnegative, in fact, suppose that $u \in E_m^+$ is a critical point of $K|_{E_m^+}$, then we have

$$K(u) = K(u) - \frac{1}{2}\langle (K|_{E_m^+})'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{p+1}\right) a_0 \|u\|_{p+1}^{p+1} \ge 0.$$

On the other hand, there is a constant $\tilde{R}_m > 0$ such that $K(u) < 0$ for $u \in E_m^+$ with $\|u\|_E \geq \tilde{R}_m$. Therefore $Z(K|_{E_m^+})$ is compact. Applying Proposition 4.3 to $K|_{E_m^+}$, for all $\varepsilon > 0$ there exists a $\phi_\varepsilon \in C^2(E_m^+, \mathbf{R})$ satisfying (P.S.) with the following properties:

$$|\phi_\varepsilon(u) - K(u)|,\ \|\phi_\varepsilon'(u) - (K|_{E_m^+})'(u)\|, \|\phi_\varepsilon''(u) - (K|_{E_m^+})''(u)\| \leq \varepsilon \quad \text{for all } u \in E_m^+; \tag{4.7}$$

(4.8) the critical points of ϕ_ε are finite in number and *nondegenerate.*

We set for $m > n$ and $\varepsilon > 0$

$$\beta_n^m(\varepsilon) = \sup_{\sigma \in A_n^m} \min_{x \in S^{m-n}} \phi_\varepsilon(\sigma(x)).$$

By (4.7)

$$\beta_n^m - \varepsilon \leq \beta_n^m(\varepsilon) \leq \beta_n^m + \varepsilon.$$

Moreover, we have

LEMMA 4.1. *Suppose that $a_\varepsilon \in \mathbf{R}$ satisfies*

$$\beta_n^m(\varepsilon) < a_\varepsilon - 2\varepsilon < a_\varepsilon < \beta_{n+1}^m(\varepsilon).$$

Then

$$\pi_{m-n-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) \neq 0 \quad \textit{for some } p \in [\phi_\varepsilon \geq a_\varepsilon]_m, \tag{4.9}$$

where

$$[\phi_\varepsilon \geq a_\varepsilon]_m = \{u \in E_m^+; \phi_\varepsilon(u) \geq a_\varepsilon\}.$$

PROOF. We argue by contradiction and suppose that

$$\pi_{m-n-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) = 0 \quad \text{for all } p \in [\phi_\varepsilon \geq a_\varepsilon]_m.$$

By the definition of $\beta_{n+1}^m(\varepsilon)$, there is a $\sigma \in A_{n+1}^m$ such that $\sigma(S^{m-n-1}) \subset [\phi_\varepsilon \geq a_\varepsilon]_m$. Since $\pi_{m-n-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) = 0$, there is a homotopy

$$H\colon [0,1] \times S^{m-n-1} \to [\phi_\varepsilon \geq a_\varepsilon]_m \tag{4.10}$$

such that

$$H(0,x) = \sigma(x), \quad H(1,x) = p \quad \text{for all } x \in S^{m-n-1}.$$

Write

$$S^{m-n} = \{(t,x); x \in \mathbf{R}^{m-n}, t \in \mathbf{R}, |x|^2 + t^2 = 1\}.$$

Define $\tilde{\sigma}\colon S^{m-n} \to E_m^+$ by

$$\tilde{\sigma}(t,x) = \begin{cases} p & \text{for } t = 1,\ x = 0, \\ H(t, x/|x|) & \text{for } 0 \leq t < 1, \\ -H(-t, -x/|x|) & \text{for } -1 < t \leq 0, \\ -p & \text{for } t = -1,\ x = 0. \end{cases} \tag{4.11}$$

By (4.10), we get $\tilde{\sigma}(S_+^{m-n}) \subset [\phi_\varepsilon \geq a_\varepsilon]_m$, where we denote $S_\pm^{m-n} = \{(t,x) \in S^{m-n}; t \gtrless 0\}$.

On the other hand, we obtain from (4.7) and evenness of $K(u)$ that

$$|\phi_\varepsilon(-u) - \phi_\varepsilon(u)| \leq 2\varepsilon \quad \text{for } u \in E_m^+.$$

So we have $\tilde{\sigma}(S_-^{m-n}) \subset [\phi_\varepsilon \geq a_\varepsilon - 2\varepsilon]_m$. Thus we get $\tilde{\sigma}(S^{m-n}) \subset [\phi_\varepsilon \geq a_\varepsilon - 2\varepsilon]_m$. From the definition of $\beta_n^m(\varepsilon)$,

$$\beta_n^m(\varepsilon) \geq \min_{x \in S^{m-n}} \phi_\varepsilon(\tilde{\sigma}(x)) \geq a_\varepsilon - 2\varepsilon.$$

But this contradicts with the assumption. Thus the proof is completed. □

By the property (4.8) we can apply a classical theorem from Morse theory to ϕ_ε. Applying it, we obtain

LEMMA 4.2 (Proposition 2.2 of Bahri-Berestycki [**5**]). *For a regular value $a \in \mathbf{R}$ of ϕ_ε, set*

$$L(\varepsilon; a) = \max\{\text{index } \phi_\varepsilon''(x); \phi_\varepsilon(x) \leq a, \phi_\varepsilon'(x) = 0\}.$$

Then

$$\pi_l([\phi_\varepsilon \geq a]_m, p) = 0 \quad \textit{for all } p \in [\phi_\varepsilon \geq a]_m \textit{ and } l \leq m - L(\varepsilon; a) - 2. \tag{4.12}$$

PROOF. Let $b \in \mathbf{R}$, $b < a$ be such that ϕ_ε has no critical values in $(-\infty, b]$. By the "noncritical neck principle" (cf. Theorem 4.67 of Schwartz [**25**]), $[\phi_\varepsilon \geq b]_m$ is a deformation retract of E_m^+. Hence

$$\pi_l([\phi_\varepsilon \geq b]_m, p) = 0 \quad \text{for all } l \in \mathbf{N} \text{ and for all } p.$$

Using Theorem 7.3 in Schwartz [**25**], we obtain

$$\pi_l([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) = 0 \quad \text{for } l \leq m - L(\varepsilon; a) - 1.$$

Using the homotopy exact sequence:

$$\begin{aligned} \to \pi_{l+1}([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) \to \pi_l([\phi_\varepsilon \geq a]_m, p) \to \pi_l([\phi_\varepsilon \geq b]_m, p) \\ \to \pi_l([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) \to \cdots \end{aligned}$$

we obtain (4.12). □

PROOF OF PROPOSITION 4.2. Since $\beta_n^m < \beta_{n+1}^m$ and (4.8) holds, there is a sequence $a_\varepsilon \in \mathbf{R}$ $(0 < \varepsilon \leq \varepsilon_0)$ such that

$$a_\varepsilon \text{ is a regular value of } \phi_\varepsilon, \tag{4.13}$$

$$\beta_n^m(\varepsilon) < a_\varepsilon - 2\varepsilon < a_\varepsilon < \beta_{n+1}^m(\varepsilon), \tag{4.14}$$

$$a_\varepsilon \to \beta_n^m \quad \text{as } \varepsilon \to 0. \tag{4.15}$$

Apply Lemmas 4.1 and 4.2, compare (4.9) and (4.12), then we observe

$$L(\varepsilon; a_\varepsilon) \geq n \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Therefore there is a $u_\varepsilon \in E_m^+$ such that

$$\phi_\varepsilon(u_\varepsilon) \leq a_\varepsilon, \tag{4.16}$$

$$\phi_\varepsilon'(u_\varepsilon) = 0, \tag{4.17}$$

$$\text{index } \phi_\varepsilon''(u_\varepsilon) \geq n. \tag{4.18}$$

It follows from (4.7) that (u_ε) satisfies

$$\begin{aligned} &K(u_\varepsilon) \text{ is bounded} \quad \text{as } \varepsilon \to 0, \\ &(K|_{E_m^+})'(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Since $K|_{E_m^+}$ satisfies (P.S.) on E_m^+, we can choose a convergent subsequence $u_{\varepsilon_j} \to u_n^m$ ($\varepsilon_j \to 0$). Obviously (4.4)–(4.6) follow from (4.7), (4.15)–(4.18). □

PROOF OF PROPOSITION 4.1. Since $\beta_n < \beta_{n+1}$, we have $\beta_n^{m_j} < \beta_{n+1}^{m_j}$ for sufficiently large j. Hence there is a $u_n^{m_j} \in E_{m_j}^+$ satisfying (4.4)–(4.6) by Proposition 4.2. Since $K \in C^2(E^+, \mathbf{R})$ satisfies (P.S.)$_*$, $(u_n^{m_j})$ has a convergent subsequence $(u_n^{m_j'})$. Let $u_n = \lim u_n^{m_{j'}}$. Then (4.1), (4.2) follow from (4.4), (4.5) easily. Let us prove (4.3). First of all, we have

$$\text{index } K''(u_n^m) \geq \text{index}(K|_{E_m^+})''(u_n^m) \quad \text{for all } m \in \mathbf{N}. \tag{4.19}$$

On the other hand, we observe that $K''(u_n)$ is an operator of type: $K''(u_n) = \text{id} + (\text{compact})$. Hence there exists an $\varepsilon > 0$ such that for $h \in E^+$

$$\langle K''(u_n)h, h\rangle \leq 0 \quad \text{if and only if} \quad \langle K''(u_n)h, h\rangle \leq \varepsilon \|h\|_E^2,$$

i.e.,

$$\text{index } K''(u_n) = \text{index}(K''(u_n) - \varepsilon). \tag{4.20}$$

Since $K \in C^2(E^+, \mathbf{R})$, we have for some j_0',

$$\|K''(u_n^{m_{j'}}) - K''(u_n)\| < \varepsilon \quad \text{for } j' \geq j_0'.$$

Thus for $j' \geq j_0'$ and $h \in E^+$,

$$\langle K''(u_n)h, h\rangle - \varepsilon\|h\|_E^2 \leq \langle K''(u_n^{m_{j'}})h, h\rangle.$$

That is,

$$\text{index}(K''(u_n) - \varepsilon) \geq \text{index } K''(u_n^{m_{j'}}). \tag{4.21}$$

Therefore (4.3) follows from (4.6), (4.19)–(4.21). Thus the proof is completed. □

5. Estimate for eigenvalues of $K''(u^+)$. The aim of this section is to get the following estimate (5.1). Combining (5.1), (4.1)–(4.3) and (3.6), we will obtain the desired estimate (2.8) in the next section.

PROPOSITION 5.1. *For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for $u \in E^+$,*

$$\text{index } K''(u) \leq C_\varepsilon \|u\|_{(p-1)(1+\varepsilon)}^{(p-1)(1+\varepsilon)}. \tag{5.1}$$

Note that for $u, h \in E^+$,

$$\langle K''(u)h, h\rangle = \|h\|_E^2 - pa_0(|u|^{p-1}h, h). \tag{5.2}$$

From the definition of index $K''(u)$, it is clear that

$$\text{index } K''(u) = \max\{\dim H; H \subset E^+ \text{ subspace such that } pa_0(|u|^{p-1}h, h) \geq \|h\|_E^2 \text{ for } h \in H\}. \tag{5.3}$$

We define an operator $D\colon L^2 \to E^+$ by

$$(Dv)(x,t) = \sum_{j<|k|} (k^2 - j^2)^{-1/2} a_{jk} \sin jx\, e^{ikt}, \quad \text{for } v(x,t) = \sum_{j\in\mathbf{N}, k\in\mathbf{Z}} a_{jk} \sin jx\, e^{ikt}. \tag{5.4}$$

It is easily seen that D is an isometry from $L^2_+ = L^2$-closure of span $\{\sin jx\, e^{ikt}; j < |k|\}$ to E^+ and $D = 0$ on L^2-closure of span$\{\sin jx\, e^{ikt}; j \geq |k|\}$. Setting $h = Dv$ in (5.3), we get

$$\begin{aligned} \text{index } K''(u) = {} & \max\{\dim H;\ H \subset L^2 \text{ subspace such that} \\ & \qquad pa_0(|u|^{p-1}Dv, Dv) \geq \|v\|_2^2 \text{ for } v \in H\} \\ = {} & \text{the number of the eigenvalues of } D^*(pa_0|u|^{p-1})D \\ & \text{which are greater than or equal to 1.} \end{aligned} \tag{5.5}$$

For the above reason, we are concerned with an operator $T_{V,\theta}\colon L^2 \to L^2$ defined by

$$T_{V,\theta}v = V(x,t)\sum_{j,k}\theta_{jk}a_{jk}\sin jx\, e^{ikt} \quad \text{for } v = \sum_{j,k} a_{jk}\sin jx\, e^{ikt}, \tag{5.6}$$

where $V(x,t)$ is a function on Ω and $\theta = (\theta_{jk})$ is a sequence on $\mathbf{N} \times \mathbf{Z}$. If we set

$$\tilde{V}(x,t) = \sqrt{p\overline{a}_0}|u|^{(p-1)/2}, \tag{5.7}$$

$$\tilde{\theta}_{jk} = \begin{cases} (k^2 - j^2)^{-1/2} & \text{if } j < |k|, \\ 0 & \text{if } j \geq |k|, \end{cases} \tag{5.8}$$

then

$$D^*(pa_0|u|^{p-1})D = T^*_{\tilde{V},\tilde{\theta}}T_{\tilde{V},\tilde{\theta}}. \tag{5.9}$$

To analyze the operator $T_{V,\theta}$, we need notion of *trace ideals* (cf. Simon [**26**, **27**]).

DEFINITION. Let $A\colon L^2 \to L^2$ be a compact operator. The *singular values* of A, $s_n(A)$ are the eigenvalues of $|A| = \sqrt{A^*A}$ listed according to $s_1(A) \geq s_2(A) \geq \cdots$. For $1 \leq q < \infty$, A is said to lie in *trace ideal* I_q if and only if

$$\|A\|_{I_q} = \left(\sum_{n=1}^{\infty} s_n(A)^q\right)^{1/q} < \infty \quad \text{for } 1 \leq q < \infty.$$

For $q = \infty$, we set I_∞ =the set of bounded linear operators: $L^2 \to L^2$ and

$$\|A\|_{I_\infty} = \sup\{\|Au\|_2; \|u\|_2 \leq 1\} < \infty.$$

The following properties of trace ideals are known (cf. [**26**, **27**]):

1° I_2 is the *Hilbert-Schmidt class* on L^2;

2° Let $\mathcal{B}$ denote the family of orthogonal sequences in L^2, then

$$\|A\|_{I_q} = \sup_{\{\phi\},\{\psi\}\in\mathcal{B}} \left(\sum_n |(\phi_n, A\psi_n)|^q\right)^{1/q}. \tag{5.10}$$

When $q = 2$, for any *complete* orthogonal sequence $\{\psi\}$ in L^2;

$$\|A\|_{I_2} = \left(\sum_n \|A\psi_n\|_2^2\right)^{1/2}. \tag{5.11}$$

3° For $q \geq 2$, $A \in I_q$ if and only if $A^*A \in I_{q/2}$ and

$$\|A\|_{I_q}^2 = \|A^*A\|_{I_{q/2}}. \tag{5.12}$$

We denote by $l^q = l^q(\mathbf{N}\times\mathbf{Z})$ the space of sequences $\theta = (\theta_{jk})$ which satisfy

$$\|\theta\|_{l^q} = \Big(\textstyle\sum_{j,k} |\theta_{jk}|^q\Big)^{1/q} < \infty \qquad \text{for } q \in [1,\infty),$$

$$\|\theta\|_{l^\infty} = \sup_{j,k} |\theta_{jk}| < \infty \qquad \text{for } q = \infty.$$

The following estimate for $T_{V,\theta}$ is a consequence from the interpolation theory.

PROPOSITION 5.2. *Suppose that $V \in L^q$ and $\theta = (\theta_{jk}) \in l^q$ for $q \in [2,\infty]$. Then $T_{V,\theta} \in I_q$ and there exists a constant $C_q > 0$, which is independent of V and θ, such that*

$$\|T_{V,\theta}\|_{I_q} \le C_q \|V\|_q \, \|\theta\|_{l^q} \quad \textit{for all } V \textit{ and } \theta. \tag{5.13}$$

PROOF. First we deal with the case $q = 2$. Setting $\{\psi\} = \{(1/\pi)\sin jx\, e^{ikt}\}$ in (5.11), we get

$$\begin{aligned}\|T_{V,\theta}\|_{I_2}^2 &= \sum_{j,k} \frac{1}{\pi^2}\|T_{V,\theta}(\sin jx\, e^{ikt})\|_2^2 \\ &= \sum_{j,k} \frac{1}{\pi^2}\|V(x,t)\theta_{jk}\sin jx\, e^{ikt}\|_2^2 \\ &\le \sum_{j,k} \frac{1}{\pi^2}|\theta_{jk}|^2\, \|V\|_2^2 = \frac{1}{\pi^2}\|\theta\|_{l^2}^2 \|V\|_2^2.\end{aligned} \tag{5.14}$$

Next we deal with the case $q = \infty$. For $v = \sum a_{jk}\sin jx\, e^{ikt}$,

$$\begin{aligned}\|T_{V,\theta}v\|_2 &= \Big\| V(x,t)\sum \theta_{jk}a_{jk}\sin jx\, e^{ikt}\Big\|_2 \\ &\le \|V\|_\infty \Big\|\sum \theta_{jk}a_{jk}\sin jx\, e^{ikt}\Big\|_2 \\ &\le \|V\|_\infty \, \|\theta\|_{l^\infty}\|v\|_2.\end{aligned}$$

That is,

$$\|T_{V,\theta}\|_{I_\infty} = \sup_{\|u\|_2=1} \|T_{V,\theta}u\|_2 \le \|V\|_\infty \, \|\theta\|_{l^\infty}. \tag{5.15}$$

Lastly, we prove (5.13) for general $2 \le q \le \infty$. Fix $\{\phi\}$, $\{\psi\} \in \mathcal{B}$ and consider the operator: $L^q \times l^q \to l^q$ defined by $(V,\theta) \to \{(\phi_n, T_{V,\theta}\psi_n)\}_{n\in\mathbf{N}}$. By (5.14), (5.15) we get

$$\|(\phi_n, T_{V,\theta}\psi_n)\|_{l^2} \le \|T_{V,\theta}\|_{I_2} \le \pi^{-1}\|V\|_2\, \|\theta\|_{l^2},$$
$$\|(\phi_n, T_{V,\theta}\psi_n)\|_{l^\infty} \le \|T_{v,\theta}\|_{I_\infty} \le \|V\|_\infty\, \|\theta\|_{l^\infty}.$$

By the complex interpolation (cf. [**8**]), we get for $q \in (2,\infty)$

$$\|(\phi_n, T_{V,\theta}\psi_n)\|_{l^q} \le C_q\|V\|_q\, \|\theta\|_{l^q},$$

where C_q is a constant independent of $\{\phi\}$, $\{\psi\} \in \mathcal{B}$. By (5.10), we get the desired result. □

Now we can prove Proposition 5.1.

PROOF OF PROPOSITION 5.1. Since $T_{V,\theta}^* T_{V,\theta}$ is a positive selfadjoint operator,

$$\|T_{V,\theta}^* T_{V,\theta}\|_{I_{q/2}} = \Big(\sum_n \lambda_n^{q/2}\Big)^{2/q} \qquad \text{for } q \ge 2,$$

where λ_n are the eigenvalues of $T^*_{V,\theta}T_{V,\theta}$. Hence we have from the definition of I_q and (5.12)

$$\begin{aligned}&\text{the number of the eigenvalues of } T^*_{V,\theta}T_{V,\theta} \text{ which are}\\ &\text{greater than or equal to } 1\\ &\qquad \le \|T^*_{V,\theta}T_{V,\theta}\|^{q/2}_{I_{q/2}} = \|T_{V,\theta}\|^q_{I_q} \quad \text{for } q \ge 2.\end{aligned}$$

Set $\tilde{V}$ and $\tilde{\theta}$ as in (5.7) and (5.8). Then we have from (5.5), (5.9)

$$\text{index } K''(u) \le \|T_{\tilde{V},\tilde{\theta}}\|^q_{I_q} \quad \text{for } q \in (2,\infty]. \tag{5.16}$$

Note that for any $q \in (2,\infty]$

$$\begin{aligned}\|\tilde{\theta}\|^q_{l^q} &= \sum_{j<|k|}(k^2-j^2)^{-q/2} = 2\sum_{j,l\in\mathbf{N}}((j+l)^2-j^2)^{-q/2}\\ &= 2\sum_{j,l} l^{-q/2}(2j+l)^{-q/2} \le \sum_{j,l} l^{-q/2}j^{-q/2} < \infty.\end{aligned}$$

That is $\tilde{\theta} \in l^q$ for any $q \in (2,\infty]$. We deduce from (5.13) that

$$\begin{aligned}\text{index } K''(u) &\le \|T_{\tilde{V},\tilde{\theta}}\|^q_{I_q} \le C_q\|\tilde{\theta}\|^q_{l^q}\|\tilde{V}\|^q_q\\ &\le C'_q\|\,|u|^{(p-1)/2}\|^q_q \le C'_q\|u\|^{(p-1)q/2}_{(p-1)q/2}.\end{aligned}$$

Since $q > 2$ is arbitrary, we obtain the desired result. □

REMARK. The result developed in this section is a modification of the result concerning the elliptic eigenvalue problem:

$$\begin{cases} -\lambda\Delta u = V(x)u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain and $V(x) \in L^{N/2}$. The following estimate is obtained by Birman-Solomjak [**9**]:

$$\text{the number of eigenvalues } (\ge 1) \le C_{N,\Omega}\|V\|^{N/2}_{N/2}.$$

Roughly speaking, under the condition (5.8) Proposition 5.2 deals with the eigenvalue problem for the equation:

$$-\lambda\Box u = P_+(V(x,t)^2u) \quad \text{for } u \in L^2_+,$$

where $P_+\colon L^2 \to L^2_+$ is the orthogonal projection.

6. Proof of Theorem 0.1. Using results in previous sections, we complete the proof of Theorem 0.1.

PROOF OF THEOREM 0.1. By Propositions 2.1 and 2.2, we see that (2.8) ensures the existence of an unbounded sequence of critical values of $I(u)$. That is, there exists a sequence $(u_n) \subset E$ of critical points of $I(u)$ such that

$$\begin{aligned}I(u_n) = \frac{1}{2}\|u_n^+\|^2_E - \frac{1}{2}\|u_n^-\|^2_E - \frac{1}{p+1}\|u_n + v(u_n)\|^{p+1}_{p+1}\\ - (f, u_n + v(u_n)) \to \infty \quad \text{as } n \to \infty.\end{aligned} \tag{6.1}$$

Since $I'(u_n) = 0$, we have

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \|u_n^+\|_E^2 - \|u_n^-\|_E^2 \\ &\qquad - (|u_n + v(u_n)|^{p-1}(u_n + v(u_n)) - f, u_n + v(u_n)) \\ &= 0.\end{aligned} \tag{6.2}$$

By (6.1), (6.2), we obtain

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n + v(u_n)\|_{p+1}^{p+1} - \frac{1}{2}(f, u_n + v(u_n)) \to \infty \quad \text{as } n \to \infty. \tag{6.3}$$

We remark that the weak solution of $(0.1)_+$–(0.3), which corresponds to u_n, is $u_n + v(u_n)$. It follows from (6.3) that

$$\|u_n + v(u_n)\|_{p+1} \to \infty \quad \text{as } n \to \infty.$$

This is the desired result. So we shall show (2.8).

By Proposition 3.3, it suffices to show the existence of a sequence (n_j) with the following property: for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$\beta_{n_j} \geq C_\varepsilon n_j^{(p+1)/(p-1)-\varepsilon} \quad \text{for } j \in \mathbf{N}. \tag{6.4}$$

Since $\beta_n \to \infty$ as $n \to \infty$, there is a sequence (n_j) such that $\beta_{n_j} < \beta_{n_j+1}$. Applying Proposition 4.1, there are $u_j \in E^+$ such that

$$K(u_j) \leq \beta_{n_j}, \tag{6.5}$$

$$K'(u_j) = 0, \tag{6.6}$$

$$\operatorname{index} K''(u_j) \geq n_j \quad \text{for } j \in \mathbf{N}. \tag{6.7}$$

Next applying Proposition 5.1, we get

$$C_\varepsilon \|u_j\|_{(p-1)(1+\varepsilon)}^{(p-1)(1+\varepsilon)} \geq n_j.$$

Choosing $\varepsilon \in (0, 2/(p-1))$, we obtain

$$\|u_j\|_{p+1}^{p+1} \geq C\|u_j\|_{(p-1)(1+\varepsilon)}^{p+1} \geq C'_\varepsilon n_j^{(p+1)(p-1)^{-1}(1+\varepsilon)^{-1}} \quad \text{for } j \in \mathbf{N}. \tag{6.8}$$

On the other hand, we have by (6.6)

$$\langle K'(u_j), u_j\rangle = \|u_j\|_E^2 - a_0\|u_j\|_{p+1}^{p+1} = 0. \tag{6.9}$$

By (6.5), (6.9), we obtain

$$\begin{aligned}\beta_{n_j} \geq K(u_j) &= \frac{1}{2}\|u_j\|_E^2 - \frac{a_0}{p+1}\|u_j\|_{p+1}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) a_0\|u_j\|_{p+1}^{p+1}.\end{aligned}$$

Therefore by (6.8), we conclude

$$\beta_{n_j} \geq C_\varepsilon n_j^{(p+1)/(p-1)-\varepsilon} \quad \text{for all } j \in \mathbf{N}.$$

Thus the proof of Theorem 0.1 is completed. □

REMARK 6.1. After a slight modification, our method is applicable to more general equation:

$$(6.10_\pm) \qquad u_{tt} - u_{xx} \pm g(x,t,u) = f(x,t), \qquad (x,t) \in (0,\pi) \times \mathbf{R},$$
$$(6.11) \qquad u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbf{R},$$
$$(6.12) \qquad u(x,t+2\pi) = u(x,t), \qquad (x,t) \in (0,\pi) \times \mathbf{R},$$

where $g \in C([0,\pi] \times \mathbf{R} \times \mathbf{R})$ is a 2π-periodic function of t.

THEOREM 6.1. *Assume that $g(x,t,\xi)$ satisfies*

(g_1) $g(x,t,\xi)$ *is a strictly increasing function of* $\xi \in \mathbf{R}$,

(g_2) *there exist* $\mu > 2$ *and* $r \geq 0$ *such that* $0 < \mu \int_0^\xi g(x,t,\tau)d\tau \leq \xi g(x,t,\xi)$ *for all* $(x,t) \in \Omega$ *and* $|\xi| \geq r$,

(g_3) *there exist* $p > 1$, $C_1, C_2, C_3, C_4 > 0$ *such that* $C_1|\xi|^p - C_2 \leq |g(x,t,\xi)| \leq C_3|\xi|^p + C_4$ *for all* $(x,t) \in \Omega$ *and* $\xi \in \mathbf{R}$,

(g_4) $g(x,t,-\xi) = -g(x,t,\xi)$ *for all* $(x,t) \in \Omega$ *and* $\xi \in \mathbf{R}$.

Then, for all $f(x,t) \in L^{(p+1)/p}$, (6.10)$_\pm$–(6.12) *possesses an unbounded sequence of weak solutions in* L^{p+1}.

REMARK 6.2. In Theorems 0.1 and 6.1, we treat the problem (6.10)$_\pm$–(6.12) as a perturbation from a $\mathbf{Z}_2$-equivariant equation: $u_{tt} - u_{xx} \pm g(x,t,u) = 0$. In "+" case we may act on S^1-symmetry. That is, we assume that

(g_4') $\qquad g = g(x,\xi)$ is independent of t,

instead of (g_4) and we define S^1-action on E by

$$(T_\theta u)(x,t) = u(x,t+\theta) \quad \text{for } u \in E \text{ and } \theta \in [0,2\pi) \cong S^1.$$

Note that when $f = 0$ equation (6.10)$_+$–(6.12) is S^1-equivariant. We treat the case $f \neq 0$ as a perturbation from S^1-symmetry and we obtain

THEOREM 6.2. *Assume that g satisfies* (g_1), (g_2), (g_3), (g_4'). *Then, for all* $f(x,t) \in L^{(p+1)/p}$, (6.10)$_+$–(6.12) *possesses an unbounded sequence of weak solutions in* L^{p+1}.

The proof of the above theorem is done in a similar way to the previous sections but we act on S^1-symmetry. As to minimax arguments for a perturbation from S^1-symmetry, see Long [**32**] (cf. Rabinowitz [**22**, **23**] and Pisani-Tucci [**18**]). Using S^1-version of Borsuk-Ulam theorem (Fadell-Husseini-Rabinowitz [**14**] and Nirenberg [**33**]), analogous results to §3 can be obtained. As to S^1-version of the result of §4, see Bahri-Berestycki [**5**].

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Appendix A. The purpose of this appendix is to prove Proposition 1.2. To do so, we need the following lemma.

LEMMA A.1. *There is a constant $C = C(\|f\|_{(p+1)/p}) > 0$ such that for $u \in E$,*

$$|Q(u)| \leq C(Q_0(u) + 1), \tag{A.1}$$

$$|Q(u) - Q_0(u)| \leq C(Q_0(u)^{1/(p+1)} + 1). \tag{A.2}$$

PROOF. By the definition of $Q(u)$,

(A.3)

$$\begin{aligned} Q(u) - Q_0(u) &= \min_{v \in N}\left[\frac{1}{p+1}\|u+v\|_{p+1}^{p+1} - (f, u+v)\right] - \frac{1}{p+1}\|u+v_0(u)\|_{p+1}^{p+1} \\ &\leq -(f, u+v_0(u)) \leq \|f\|_{(p+1)/p}\|u+v_0(u)\|_{p+1} \\ &\leq CQ_0(u)^{1/(p+1)}. \end{aligned}$$

Similarly we have

$$Q(u) - Q_0(u) \geq -C(|Q(u)|^{1/(p+1)} + 1). \tag{A.4}$$

Obviously (A.3) implies (A.1). By (A.1), (A.4) we have

$$Q(u) - Q_0(u) \geq -C(Q_0(u)^{1/(p+1)} + 1).$$

Thus we get (A.2) from the above inequality and (A.3). □

Setting $f = 0$ in Lemma 1.1, we have for $u, h \in E$,

$$\langle Q_0'(u), h\rangle = (|u+v_0(u)|^{p-1}(u+v_0(u)), h), \tag{A.5}$$

$$\|Q_0'(u)\|_{E^*} \leq C(Q_0(u)^{p/(p+1)} + 1), \tag{A.6}$$

$$\langle Q_0'(u), u\rangle = (p+1)Q_0(u). \tag{A.7}$$

PROOF OF (i) OF PROPOSITION 1.2. From the definition of $J(u)$, we have

$$|J(-u) - J(u)| \leq \psi(u)|Q(u) - Q_0(u)| + \psi(-u)|Q(-u) - Q_0(-u)|. \tag{A.8}$$

Suppose that $-u \in \operatorname{supp}\psi$, i.e., $Q_0(u) \leq 2\Phi(-u) = 2a(I(-u)^2 + 1)^{1/2}$. From the definition of $J(u)$,

$$I(-u) = J(u) + (Q_0(u) - Q(-u)) - \psi(u)(Q(u) - Q_0(u)).$$

By Lemma A.1, we get

$$\begin{aligned} |I(-u)| &\leq |J(u)| + C(Q_0(u)^{1/(p+1)} + 1) \\ &\leq |J(u)| + C\Phi(-u)^{1/(p+1)}. \end{aligned}$$

Using Young's inequality, we deduce that

$$|I(-u)| \leq 2|J(u)| + C.$$

Hence we get

$$\begin{aligned} Q_0(u) &\leq 2\Phi(-u) = 2a(I(-u)^2 + 1)^{1/2} \\ &\leq C|J(u)| + C \quad \text{for } -u \in \operatorname{supp}\psi. \end{aligned} \tag{A.9}$$

Similarly we have

$$Q_0(u) \leq C|J(u)| + C \quad \text{for } u \in \operatorname{supp}\psi. \tag{A.10}$$

From (A.2), (A.8), (A.9), (A.10) we obtain for $u \in E$

$$\begin{aligned}|J(-u) - J(u)| &\le C(\psi(u) + \psi(-u))(Q_0(u)^{1/(p+1)} + 1)\\ &\le \beta(|J(u)|^{1/(p+1)} + 1).\end{aligned}$$

This is the desired result. □

To prove the second assertion of Proposition 1.2, we need the following

LEMMA A.2. *There is a constant* $M_1 = M_1(\|f\|_{(p+1)/p}) > 0$ *such that* $J(u) \ge M_1$ *and* $u \in \operatorname{supp}\psi$ *imply* $I(u) \ge \frac{1}{3}J(u)$.

PROOF. From the definition of $J(u)$,

$$\begin{aligned}J(u) &= I(u) - (1 - \psi(u))(Q(u) - Q_0(u))\\ &\le I(u) + C(Q_0(u)^{1/(p+1)} + 1).\end{aligned}$$

By definition of ψ, we get for $u \in \operatorname{supp}\psi$

$$\begin{aligned}J(u) &\le I(u) + C(|I(u)|^{1/(p+1)} + 1)\\ &\le I(u) + \tfrac{1}{2}|I(u)| + C_1.\end{aligned}$$

Choosing $M_1 = 2C_1$, we get the desired result. □

LEMMA A.3. *For all* $u = u^+ + u^- \in E = E^+ \oplus E^-$ *and* $h \in E$,

$$\begin{aligned}\langle J'(u), h\rangle &= (1 + T_1(u))\langle u^+ - u^-, h\rangle - (1 + T_2(u))\langle Q_0'(u), h\rangle\\ &\quad - (\psi(u) + T_1(u))\langle Q'(u) - Q_0'(u), h\rangle,\end{aligned} \tag{A.11}$$

where $T_1(u)$, $T_2(u) \in C(E, \mathbf{R})$ *are functionals satisfying*

$$\sup\{|T_i(u)|; u \in E, J(u) \ge M_2,\ i = 1, 2\} \to 0 \quad \text{as } M_2 \to \infty. \tag{A.12}$$

PROOF. For all $u = u^+ + u^- \in E$ and $h \in E$, we have

$$\begin{aligned}\langle J'(u), h\rangle &= \langle u^+ - u^-, h\rangle - \langle Q_0'(u), h\rangle\\ &\quad - \langle \psi'(u), h\rangle(Q(u) - Q_0(u)) - \psi(u)\langle Q'(u) - Q_0'(u), h\rangle,\end{aligned}$$

where

$$\begin{aligned}\langle \psi'(u), h\rangle &= \chi'(\Phi(u)^{-1}Q_0(u))\Phi(u)^{-3}\\ &\quad \times [-a^2 I(u)\langle I'(u), h\rangle Q_0(u) + \Phi(u)^2\langle Q_0'(u), h\rangle],\\ \langle I'(u), h\rangle &= \langle u^+ - u^-, h\rangle - \langle Q_0'(u), h\rangle - \langle Q'(u) - Q_0'(u), h\rangle.\end{aligned}$$

By regrouping terms, we get (A.11) for

$$\begin{aligned}T_1(u) &= a^2\chi'(\cdot)\Phi(u)^{-3}I(u)Q_0(u)(Q(u) - Q_0(u)),\\ T_2(u) &= T_1(u) + \chi'(\cdot)\Phi(u)^{-1}(Q(u) - Q_0(u)).\end{aligned}$$

Let us prove (A.12). Suppose that $u \in E$ satisfies $J(u) \ge M_2$. Using (A.2), we get

$$|T_1(u)| \le C|\chi'(\cdot)|\Phi(u)^{-2}Q_0(u)(Q_0(u)^{1/(p+1)} + 1).$$

If $u \notin \operatorname{supp}\psi$, then $T_1(u) = 0$. Otherwise, by the definition of $\psi(u)$, we have $Q_0(u) \le 2\Phi(u)$. On the other hand, we get from Lemma A.2

$$\Phi(u) \ge I(u) \ge \tfrac{1}{3}J(u) \ge \tfrac{1}{3}M_2.$$

Hence we obtain

$$|T_1(u)| \le C\Phi(u)^{-p/(p+1)} \le CM_2^{-p/(p+1)} \to 0 \quad \text{as } M_2 \to \infty.$$

Similarly we have $T_2(u) \to 0$ as $M_2 \to \infty$. Thus we get (A.12). □

PROOF OF (ii) OF PROPOSITION 1.2. It suffices to show that $\psi(u) = 1$, that is, by the definition of $\psi(u)$, to show that

$$Q_0(u) \le \Phi(u) \tag{A.13}$$

for $u \in E$ such that $J(u) \ge M_0$ and $\|J'(u)\|_{E^*} \le 1$. For sufficiently large $M_0 > 0$, we can assume by (A.12) that $J(u) \ge M_0$ implies $|T_1(u)| \le \frac{1}{2}$, $|T_2(u)| \le 1$ and

$$\frac{(p+1)(1+T_2(u))}{2(1+T_1(u))} - 1 > \frac{p-1}{4} \equiv b.$$

From (A.11), we obtain

$$\begin{aligned} I(u) &- \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle \\ &= -Q(u) + \frac{1+T_2(u)}{2(1+T_1(u))}\langle Q_0'(u), u\rangle \\ &\quad + \frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\langle Q'(u) - Q_0'(u), u\rangle \\ &= \left(\frac{(p+1)(1+T_2(u))}{2(1+T_1(u))} - 1\right) Q_0(u) - (Q(u) - Q_0(u)) \\ &\quad + \frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\langle Q'(u) - Q_0'(u), u\rangle \\ &\equiv (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned} \tag{A.14}$$

By (A.2)

$$|(\mathrm{II})| \le C(Q_0(u)^{1/(p+1)} + 1). \tag{A.15}$$

Using (1.9), (A.1), (A.2), (A.7), we get

$$\begin{aligned} |\langle Q'(u) - Q_0'(u), u\rangle| &\le |(p+1)Q(u) - \langle Q'(u), u\rangle| + (p+1)|Q(u) - Q_0(u)| \\ &\le C(Q_0(u)^{1/(p+1)} + 1), \end{aligned}$$

i.e.,

$$|(\mathrm{III})| \le C(Q_0(u)^{1/(p+1)} + 1). \tag{A.16}$$

From (A.14), (A.15), (A.16), we deduce

$$\begin{aligned} I(u) &- \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle \\ &\ge \left(\frac{(p+1)(1+T_2(u))}{2(1+T_1(u))} - 1\right) Q_0(u) - C(Q_0(u)^{1/(p+1)} + 1) \\ &\ge bQ_0(u) - C. \end{aligned} \tag{A.17}$$

On the other hand, letting $h = u^+ - u^-$ in (A.11), we get

$$\begin{aligned} \langle J'(u), u^+ - u^-\rangle = (1+T_1(u))\|u\|_E^2 &- (1+T_2(u))\langle Q_0'(u), u^+ - u^-\rangle \\ &- (\psi(u) + T_1(u))\langle Q'(u) - Q_0'(u), u^+ - u^-\rangle. \end{aligned} \tag{A.18}$$

By (A.6), we have

$$\begin{aligned}|\langle Q_0'(u), u^+ - u^-\rangle| &\le \|Q_0'(u)\|_{E^*}\|u\|_E \\ &\le C(Q_0(u)^{p/(p+1)} + 1)\|u\|_E.\end{aligned}$$

Similarly by (1.8) and (A.1),

$$|\langle Q'(u), u^+ - u^-\rangle| \le C(Q_0(u)^{p/(p+1)} + 1)\|u\|_E.$$

Therefore we get from (A.18), $|T_1(u)| \le \frac{1}{2}$ and the assumption: $\|J'(u)\|_{E^*} \le 1$,

$$\begin{aligned}\tfrac{1}{2}\|u\|_E^2 &\le \|J'(u)\|_{E^*}\|u\|_E + C(Q_0(u)^{p/(p+1)} + 1)\|u\|_E \\ &\le C(Q_0(u)^{p/(p+1)} + 1)\|u\|_E,\end{aligned}$$

that is,

$$\|u\|_E \le C(Q_0(u)^{p/(p+1)} + 1). \tag{A.19}$$

Using (A.17), (A.19), we get

$$\begin{aligned}I(u) &\ge \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle + bQ_0(u) - C \\ &\ge -C\|J'(u)\|_{E^*}\|u\|_E + bQ_0(u) - C \\ &\ge bQ_0(u) - C(Q_0(u)^{p/(p+1)} + 1) \\ &\ge bQ_0(u)/2 - C_0.\end{aligned} \tag{A.20}$$

We remark that

$$\inf\{Q_0(u); \|J'(u)\|_{E^*} \le 1 \text{ and } J(u) \ge M\} \to \infty \quad \text{as } M \to \infty.$$

This follows from (A.19). In fact, $J(u) \to \infty$ implies $\|u\|_E \to \infty$. By (A.19), we get $Q_0(u) \to \infty$.

Now we may assume that $J(u) \ge M_0$ implies $bQ_0(u)/6 - C_0 \ge 0$, i.e., $I(u) \ge bQ_0(u)/3$. Thus

$$Q_0(u) \le aI(u) \le \Phi(u).$$

Thus the proof is completed. □

Appendix B. The aim of this appendix is to give a proof of Proposition 3.1. The proof is essentially as in Proposition 3.1 of [**5**] or as in Proposition 3.3 of [**4**]. But for the sake of completeness we explain this.

PROOF OF (i) OF PROPOSITION 3.1. For any $\sigma \in A_n^m$, it is clear that there is a $\tilde{\sigma} \in A_{n+1}^m$ with $\tilde{\sigma}(S^{m-n-1}) \subset \sigma(S^{m-n})$. Hence we have $\beta_n^m \le \beta_{n+1}^m$. □

To prove (ii) and (iii) of Proposition 3.1, we need the following lemmas.

LEMMA B.1. *For all* $\sigma \in A_n^m$,

$$\sigma(S^{m-n}) \cap E_n^+ \ne \emptyset. \tag{B.1}$$

PROOF. Apply Lemma 3.1 to $h = \sigma\colon S^{m-n} \to E_m^+$ and $g = \mathrm{id}\colon E_n^+ \to E_m^+$. Then we get (B.1). □

LEMMA B.2. *For all $\theta \in (0, 1/(p+1))$, there is a $C_\theta > 0$ independent of $n \in \mathbf{N}$ such that*

$$\|u\|_{p+1} \le C_\theta \mu_n^{-\theta} \|u\|_E \quad \text{for } u \in (E_n^+)^\perp, \tag{B.2}$$

where $(E_n^+)^\perp = \{v \in E^+; \langle v, e_i\rangle = 0$ for $i = 1, 2, \ldots, n\}$.

PROOF. We have by the definition of $\|\cdot\|_E$ and μ_n

$$\|u\|_2 \le \mu_n^{-1/2} \|u\|_E \quad \text{for } u \in (E_n^+)^\perp. \tag{B.3}$$

On the other hand, by (1.1)

$$\|u\|_q \le C_q \|u\|_E \quad \text{for all } u \in E \text{ and } q \in [1, \infty). \tag{B.4}$$

Using Hölder's inequality, we get for $q \in (p+1, \infty)$

$$\|u\|_{p+1} \le \|u\|_2^\tau \|u\|_q^{1-\tau} \quad \text{for } u \in E^+,$$

where

$$\tau = \frac{2(q-p-1)}{(p+1)(q-2)} \in \left(0, \frac{2}{p+1}\right).$$

It follows from (B.3), (B.4) that

$$\|u\|_{p+1} \le C_q^{1-\tau} \mu_n^{-\tau/2} \|u\|_E \quad \text{for } u \in (E_n^+)^\perp.$$

Thus we get the desired result. □

PROOF OF THE EXISTENCE OF $\tilde{\nu}(n)$. By Lemma B.1 we have for all $\sigma \in A_n^m$,

$$\min_{x \in S^{m-n}} K(\sigma(x)) \le \sup_{u \in E_n^+} K(u). \tag{B.5}$$

For $u \in E_n^+$, we have

$$\begin{aligned} K(u) &= \frac{1}{2}\|u\|_E^2 - \frac{a_0}{p+1}\|u\|_{p+1}^{p+1} \le \frac{1}{2}\|u\|_E^2 - C\|u\|_2^{p+1} \\ &\le \frac{1}{2}\|u\|_E^2 - C\mu_n^{-(p+1)/2}\|u\|_E^{p+1}. \end{aligned}$$

Thus the right-hand side of (B.5) is finite and independent of σ and m. Set

$$\tilde{\nu}(n) = \sup_{u \in E_n^+} K(u) < \infty,$$

then we obtain

$$\beta_n^m = \sup_{\sigma \in A_n^m} \min_{x \in S^{m-n}} K(\sigma(x)) \le \tilde{\nu}(n). \quad \square$$

PROOF OF THE EXISTENCE OF $\nu(n)$. We construct a special $\sigma \in A_n^m$ as follows: Write

$$S^{m-n} = \left\{ x = (x_n, \ldots, x_m) \in \mathbf{R}^{m-n+1}; \sum_{i=n}^m x_i^2 = 1 \right\}$$

and set $\sigma \colon S^{m-n} \to E_m^+ \backslash 0$ by

$$\sigma(x) = a_0^{-1/(p-1)} \|w(x)\|_{p+1}^{-(p+1)/(p-1)} w(x), \tag{B.6}$$

where $w(x)$ is defined by

$$w(x) = \sum_{i=n}^{m} x_i e_i. \tag{B.7}$$

Obviously we have $\sigma \in A_n^m$. Since $\|w(x)\|_E = 1$ on S^{m-n}, we have

$$K(\sigma(x)) = \left(\frac{1}{2} - \frac{1}{p+1}\right) a_0^{-2/(p-1)} \|w(x)\|_{p+1}^{-2(p+1)/(p-1)}. \tag{B.8}$$

Since

$$w(x) \in (E_{n-1}^+)^\perp, \quad \|w(x)\|_E = 1 \quad \text{for all } x \in S^{m-n}.$$

We get from (B.2) that

$$\|w(x)\|_{p+1} \le C_\theta \mu_{n-1}^{-\theta} \quad \text{for } x \in S^{m-n}, \tag{B.9}$$

where $\theta \in (0, 1/(p+1))$ and C_θ is a constant independent of n and x.

By (B.8) and (B.9)

$$K(\sigma(x)) \ge C'_\theta \mu_{n-1}^{2\theta(p+1)/(p-1)} \quad \text{for all } x \in S^{m-n}. \tag{B.10}$$

Remark that the right-hand side of (B.10) is independent of m. Set

$$\nu(n) = C'_\theta \mu_{n-1}^{2\theta(p+1)/(p-1)}.$$

Then we have

$$\beta_n^m \ge \min_{x \in S^{m-n}} K(\sigma(x)) \ge \nu(n) \quad \text{for } m > n.$$

Since $\mu_n \to \infty$ as $n \to \infty$, we obtain $\nu(n) \to \infty$ as $n \to \infty$. Therefore, the proof of (ii), (iii) of Proposition 3.1 is completed. □

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