AUTOMORPHISMS OF HYPERBOLIC DYNAMICAL SYSTEMS AND K_2

FRANK ZIZZA

ABSTRACT. Let $\sigma\colon \Sigma \to \Sigma$ be a subshift of finite type and $\mathrm{Aut}(\sigma)$ be the group of homeomorphisms of Σ which commute with σ. In [**W1**], Wagoner constructs an invariant for the group $\mathrm{Aut}(\sigma)$ using K-theoretic methods. Smooth hyperbolic dynamical systems can be modeled by subshifts of finite type over the nonwandering sets. In this paper we extend Wagoner's construction to produce an invariant on the group of homeomorphisms of a smooth manifold which commute with a fixed hyperbolic diffeomorphism. We then proceed to show that this dynamical invariant can be calculated (at least mod 2) from the homology groups of the manifold and the action of the diffeomorphism and the homeomorphisms on the homology groups.

In [**W1**], Wagoner introduces new methods for constructing invariants on $\mathrm{Aut}(\sigma_A)$, the group of homeomorphisms which commute with a subshift of finite type $\sigma_A\colon \Sigma_A \to \Sigma_A$. The methods are developed using the action of $\mathrm{Aut}(\sigma_A)$ on the collection of Markov partitions of Σ_A. If f is a Smale diffeomorphism of a closed manifold M, it is well known that there is a finite collection of subsets $\{\Omega_i\}$ of M, and $f|\Omega_i$ is topologically conjugate to a subshift of finite type. In analogy with $\mathrm{Aut}(\sigma_A)$ we can consider $\mathrm{Aut}(f)$, the group of *homeomorphisms* of M which commute with f. By explicitly constructing elements Wagoner has shown this group has a very rich structure.

THEOREM (WAGONER [**W2**]). *Let $q > 4$ and $1 < e < q - 2$. Then there is a Smale diffeomorphism $F_A\colon S^q \to S^q$ with a basic set Ω_e of index e together with a topological conjugacy between σ_A and $F_A|\Omega_e$ so that given any symmetry g in $\mathrm{Aut}(\sigma_A)$, there is a homeomorphism $G\colon S^q \to S^q$ satisfying*
(A) *G commutes with F_A on S^q.*
(B) *$G|\Omega_e = g$ under the identification of $\mathrm{Aut}(G|\Omega_e)$ and $\mathrm{Aut}(\sigma_A)$.*

Boyle, Lind and Rudolph [**BLR**] have recently shown that $\mathrm{Aut}(\sigma_A)$ contains a free nonabelian group on infinitely many generators. There are also examples where $\mathrm{Aut}(\sigma_A)$ contains every finite group.

In this paper we extend Wagoner's constructions to produce an invariant on the group $\mathrm{Aut}(f)$ by studying its action on the nonwandering set of f. Our main result is that, with certain hypotheses on f, this invariant can be calculated from the actions of f and elements of $\mathrm{Aut}(f)$ on the singular homology groups of M. Precisely, there are two homomorphisms, ϕ defined using the dynamics of the system and Φ defined by homological methods, from the group $\mathrm{Aut}(f)$ to the Whitehead group $\mathrm{Wh}_2(F(t))$. If f is a fitted Smale diffeomorphism and $F = \mathbb{Z}/2$, then the

two invariants are equal. For a field F, the group $\mathrm{Wh}_2(F(t))$ is isomorphic to $\bigoplus_{\wp}(F_{\wp})^{*}$ for $\wp$ a prime ideal in the ring of Laurent polynomials $F[t, t^{-1}]$ and $F_{\wp} = F[t, t^{-1}]/\wp$. The summation extends over all prime ideals $\wp$.

This paper contains results of my dissertation obtained under John Wagoner at Berkeley. I thank him for his guidance and friendship over the past years.

1. THE INVARIANTS Φ AND ϕ

1.1. Algebraic preliminaries. Let F be a field. If G is a finitely generated abelian group and σ is an endomorphism of G, then we can define $T(G, \sigma)$ to be $\mathrm{coker}(1 \otimes I - t \otimes \sigma)$ on $F[t, t^{-1}] \otimes_{\mathbb{Z}} G$. Observe that since G is finitely generated, $T(G, \sigma)$ is a torsion $F[t, t^{-1}]$ module since $1 \otimes \sigma$ must satisfy some polynomial on $F \otimes_{\mathbb{Z}} G \cong F^{n}$. If ρ is another endomorphism of G that commutes with σ, then $1 \otimes \rho$ commutes with $1 \otimes I - t \otimes \sigma$ and we let $T\rho$ denote the induced map on $T(G, \sigma)$. If we let $\mathcal{C}$ denote the category whose objects are pairs (G, σ) with morphisms $\rho \in \mathrm{hom}((G, \sigma), (G', \sigma'))$ if $\sigma \circ \rho = \rho \circ \sigma'$, then T can be considered a functor from $\mathcal{C}$ to the category of torsion $F[t, t^{-1}]$-modules.

We record some facts concerning the functor T for future reference.

PROPOSITION 1.1.1. *$T(G, \sigma)$ is naturally isomorphic to the direct limit of the system $F \otimes G \xrightarrow{1 \otimes \sigma} F \otimes G \xrightarrow{1 \otimes \sigma} F \otimes G \rightarrow \cdots$.*

PROOF. Let $N_i = F \otimes G \times \{i\}$ for $i \geq 0$. Let g denote an element of $F \otimes G$ and denote elements of N_i by (g, i). Also let $1 \otimes \sigma$ be denoted by σ. The directed system is defined by giving maps $\sigma_{ij}: N_i \rightarrow N_j$ for $i < j$. Let $\sigma_{ij}(g, i) = (\sigma^{j-i}g, j)$. Define $\varphi: \varinjlim_{i} N_i \rightarrow T(G, \sigma)$ by $\varphi(g, i) = t^{i} \cdot g$ (we are now considering $g \in F \otimes G$ as in $F[t, t^{-1}] \otimes G$). Notice that if $(g, i) \equiv (g', j)$ in the direct limit, then if $j > i$ we must have $\sigma^{k}g' = \sigma^{k+j-i}g$ for some $k \geq 0$ and it follows

$$\varphi(g, i) = t^{i}g \equiv t^{i}t^{k+j-i}\sigma^{k+j-i}g = t^{k+j}\sigma^{k+j-i}g = t^{k+j}\sigma^{k}g' \equiv t^{j}g' = \varphi(g', j).$$

Thus φ is well defined. If $t^{i}g \in F[t, t^{-1}] \otimes G$ there is a positive integer j so that $i + j \geq 0$. In $T(G, \sigma)$, $t^{i}g \equiv t^{i+j}\sigma^{j}g$. Let $\theta(t^{i}g) = (\sigma^{j}g, i+j)$. Different choices for j give congruence elements of $\varinjlim_{i} N_i$. Also $\theta(t^{i} \cdot t\sigma g) = (\sigma^{i+1}g, i+1+j) \equiv (\sigma^{j}g, i+1+j)$ which shows θ defines a map from $T(G, \sigma)$ to the direct limit. It is now trivial to check the induced maps are inverses. $\square$

Since direct limits commute with the formation of homology we have

COROLLARY 1.1.2. *The torsion functor T is an exact functor.*

PROOF. Let $0 \rightarrow (A, \alpha) \xrightarrow{i} (B, \beta) \xrightarrow{\pi} (C, \gamma) \rightarrow 0$ be an exact sequence of abelian groups $A, B,$ and C and $\alpha, \beta,$ and γ are endomorphisms of the corresponding groups so that $i\alpha = \beta i$ and $\pi\beta = \gamma\pi$. The homology of this sequence is zero since it is exact. Applying T, the homology of the resulting sequence is also zero since T is isomorphic to a direct limit. Therefore the sequence $0 \rightarrow T(A, \alpha) \rightarrow T(B, \beta) \rightarrow T(C, \gamma) \rightarrow 0$ is exact. $\square$

Now let M be a finitely generated torsion $F[t, t^{-1}]$ module and let g be an $F[t, t^{-1}]$ module automorphism of M. Let $\wp$ be a prime ideal in the principal ideal domain $F[t, t^{-1}]$, and let $M_{\wp}$ denote the $\wp$-primary component of M. Then $M \cong \bigoplus_{\wp} M_{\wp}$. Each $M_{\wp}$ is canonically filtered by $M_{\wp}^{i} = \wp^{i} M_{\wp}$ and $M_{\wp}^{i}/M_{\wp}^{i+1}$ is isomorphic to a vector space over the field $F_{\wp} = F[t, t^{-1}]/\wp$. Since g is an

automorphism, $gM_\wp^i \subset M_\wp^i$ and g induces a linear automorphism of the $F_\wp$-vector spaces $M_\wp^i/M_\wp^{i+1}$. Let

$$\det_\wp(g) = \prod_i \{\text{determinant of } g \text{ on } M_\wp^i/M_\wp^{i+1}\} \in F_\wp^*,$$

and let

$$\det(g) = \bigoplus_\wp \det_\wp(g) \in \bigoplus_\wp F_\wp^* \cong \mathrm{Wh}_2(F(t)).$$

$\mathrm{Wh}_2(F(t))$ arises in algebraic K-theory as a quotient

$$0 \to K_2(F[t,t^{-1}]) \to K_2(F(t)) \to \mathrm{Wh}_2(F(t)) \to 0.$$

1.2. The homological invariant Φ. Using the preliminaries from §1.1 we are able to define the homological invariant Φ. Let M be a compact n-dimensional smooth manifold and let f be a diffeomorphism of M. Let $g \in \mathrm{Aut}(f)$ be a homeomorphism of M which commutes with f. f and g induce automorphisms f_{*k} and g_{*k} of the groups $H_k(M;\mathbb{Z})$. Let Tg_{*k} be the induced automorphism of $T(H_k(M;\mathbb{Z}), f_{*k})$. Define $\Phi(g) = \prod_{k=0}^m \det(Tg_{*k})^{(-1)^k}$ in $\mathrm{Wh}_2(F(t))$, where we are writing each $F_\wp^*$ as a multiplicative group. The functorality of the construction shows Φ is a group homomorphism.

The image of Φ is contained in the finite set of $(F_\wp)^*$ where $\wp$ is a prime dividing $\det(I - t \otimes f_*)$, this determinant as an endomorphism of $F[t,t^{-1}] \otimes H_*(M;\mathbb{Z})$.

In §3.2 we will demonstrate that $\det(\cdot)$ is a Euler-Poincaré mapping [**L**, p. 98]. From this it follows the product defining Φ is a natural extension of $\det(\cdot)$ to define a Euler-Poincaré characteristic for complexes of torsion modules.

1.3. The space of Markov partitions. Following Wagoner's construction in [**W1**] of $\psi_A \colon \mathrm{Aut}(\sigma_A) \to \mathrm{Wh}_2(F(t))$, in the next two sections we outline the construction of $\phi \colon \mathrm{Aut}(f) \to \mathrm{Wh}_2(F(t))$ making a few minor changes and demonstrating statements only when the proofs differ from the corresponding statements found in [**W1**].

The central tool is the notion of a Markov partition of a basic set. We recall some definitions and basic facts from smooth dynamical systems theory. Let f be a smooth diffeomorphism of a closed manifold M. We say $x \in M$ is nonwandering if for every neighborhood U of x, $(\bigcup_{|n|>0} f^n U) \cap U \neq \varnothing$. The set of all nonwandering points will be denoted by Ω. We assume Ω has a hyperbolic structure. Briefly this means there is a splitting of $T_\Omega M$, the tangent bundle M over Ω, into two summands E^u and E^s so that $Df_x \colon E_x^u \to E_{f(x)}^u$ and $Df_x \colon E_x^s \to E_{f(x)}^s$ are expansions and contractions respectively relative to some norm on TM. Under this hypothesis Ω can be characterized as the closure of the periodic points of f. Furthermore the compact invariant set Ω decomposes as a finite disjoint union of compact invariant sets Ω_i. Each Ω_i contains an orbit of f which is dense in Ω_i. These Ω_i are called basic sets.

Hyperbolicity also guarantees the existence of smooth stable and unstable manifolds $W^s(x)$ and $W^u(x)$ through each point $x \in \Omega$. If

$$W_\varepsilon^s(x) = \{y \in M | d(f^n x, f^n y) < \varepsilon \text{ for } n \geq 0\}$$

and

$$W_\varepsilon^u(x) = \{y \in M | d(f^n x, f^n y) < \varepsilon \text{ for } n \leq 0\}$$

then $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ are small discs centered at x contained in $W^s(x)$ and $W^u(x)$ respectively. Assuming the transversality condition—that is $W^s(x)$ intersects $W^u(y)$ transversally for every x and y in Ω—there is a continuous map $[\ ,\ ]$ defined on some neighborhood of the diagonal in $\Omega \times \Omega$ (i.e. $d(x,y) < \varepsilon$ for some ε and ε depends only on f) by $(x,y) \mapsto [x,y] = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$. Let $R \subset \Omega$, we say R is a rectangle if the diameter of R is less than ε and for every $x, y \in R$, $[x,y] \in R$. We denote $W_\varepsilon^u(x) \cap R$ by $W^u(x,R)$ and $W_\varepsilon^s(x) \cap R$ by $W^s(x,R)$.

For zero-dimensional basic sets Ω_i, a result of Bowen implies $f|\Omega_i$ is topologically conjugate to a subshift of finite type. The proof of Bowen's theorem uses the concept of a Markov partition of Ω_i. A Markov partition for a zero-dimensional basic set is a finite cover by rectangles $\{R_1, \ldots, R_n\}$ such that

(1) Each R_i is the closure of its interior.

(2) $R_i \cap R_j = \varnothing$ for $i \neq j$.

(3) If $x \in R_i$ and $f(x) \in R_j$, then $fW^u(x,R_i) \supset W^u(fx,R_j)$ and $fW^s(x,R_i) \subset W^s(fx,R_j)$.

We will denote the collection of all Markov partitions of Ω_i by $\mathcal{P}_i$, or if no confusion is possible by $\mathcal{P}$. We denote elements of $\mathcal{P}$ by $\mathcal{U}$, $\mathcal{V}$, etc. where each $\mathcal{U}$ is a collection $\{U_1, \ldots, U_n\}$ of rectangles. We say $\mathcal{U}$ is refined by $\mathcal{V}$ and write $\mathcal{U} < \mathcal{V}$ if for every $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ and $V \subset U$. We consider $\mathcal{P}$ as a simplicial complex where an n-simplex is an ordered $(n+1)$-tuple $[\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n]$ such that $\mathcal{U}_i < \mathcal{U}_j$ if $i < j$. It is easy to show that for $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{P}$, $\mathcal{U} \cap \mathcal{V}$ defined as $\{U \cap V | U \in \mathcal{U}$ and $V \in \mathcal{V}\}$ is also a Markov partition and $\mathcal{U} < \mathcal{U} \cap \mathcal{V} > \mathcal{V}$. A consequence of this is the simplicial complex $\mathcal{P}$ is contractible. If g is a homeomorphism of M and $g\mathcal{V}$ is defined as $\{gV_1, \ldots, gV_n\}$ one can show $\mathcal{U} \cap f\mathcal{V}$ and $\mathcal{U} \cap f^{-1}\mathcal{V}$ are also Markov partitions. We define $\mathcal{U}(-n,m) = f^{-n}\mathcal{U} \cap \cdots \cap \mathcal{U} \cap \cdots \cap f^m\mathcal{U}$ for positive integers n and m. The previous comment and induction show $\mathcal{U}(-n,m)$ is also a Markov partition. As n and $m \to +\infty$, the diffeomorphism f contracts rectangles in the stable direction and f^{-1} contracts rectangles in the unstable direction. It follows that as $n, m \to +\infty$, the diameters of the rectangles in $\mathcal{U}(-n,m)$ tend to 0, consequently given any $\mathcal{U}$ and $\mathcal{V}$ it follows from the Lebesgue covering lemma that there exist integers N and M so that $\mathcal{U} < \mathcal{V}(-N,M)$. We introduce something like a length function $l(\mathcal{U}, \mathcal{V})$ defined only if $\mathcal{U} < \mathcal{V}$ by

$$l(\mathcal{U}, \mathcal{V}) = \min\{n + m | \mathcal{V} < \mathcal{U}(-n,m)\}.$$

We say $[\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n]$ is a *tight* n-simplex if $l(\mathcal{U}_i, \mathcal{U}_{i+1}) \leq 1$ for all $0 \leq i < n$.

PROPOSITION 1.3.1. *Given* $\mathcal{U}$ *and* $\mathcal{V}$ *vertices in* $\mathcal{P}$*, there is a path in* $\mathcal{P}$ *which we denote as*

$$\mathcal{U} = \mathcal{U}_0 - \mathcal{U}_1 - \cdots - \mathcal{U}_{n-1} - \mathcal{U}_n = \mathcal{V}$$

such that $(\mathcal{U}_i, \mathcal{U}_{i+1})$ *is a tight edge for* $0 \leq i < n$.

PROOF. [**W1**]. □

Given a tight edge $(\mathcal{U}, \mathcal{V})$, there are four possibilities:

(1) $\mathcal{U} < \mathcal{V} < \mathcal{U}(0,1)$,

(2) $\mathcal{V} < \mathcal{U} < \mathcal{V}(0,1)$,

(3) $\mathcal{U} < \mathcal{V} < \mathcal{U}(-1,0)$,

(4) $\mathcal{V} < \mathcal{U} < \mathcal{V}(-1,0)$.

We denote the four cases by $\mathcal{U} \xrightarrow{+} \mathcal{V}$, $\mathcal{U} \xleftarrow{+} \mathcal{V}$, $\mathcal{U} \xrightarrow{-} \mathcal{V}$, and $\mathcal{U} \xleftarrow{-} \mathcal{V}$ respectively.

Tight edges are important because they define *explicit* isomorphisms between certain finitely generated torsion modules which we define now. Let $\mathcal{U}$ be a Markov partition. Define $\mathbb{Z}\mathcal{U}$ to be the free abelian group on generators $U_i \in \mathcal{U}$. We define an endomorphism of $\mathbb{Z}\mathcal{U}$ on generators U_j by

$$A(\mathcal{U})U_j = \sum_{U_i \in \mathcal{U}} A_{ij}U_i$$

where $A_{ij} = 1$ if $U_i \cap fU_j \neq \varnothing$ and $A_{ij} = 0$ otherwise. The finitely generated torsion module of interest is $T(\mathbb{Z}\mathcal{U}, A(\mathcal{U}))$ which we shall frequently abbreviate as $T\mathcal{U}$.

Different Markov partitions give rise to different endomorphisms $A(\mathcal{U})$; however, they are all related by the notion of shift equivalence. Two endomorphisms $A \colon G \to G$ and $B \colon H \to H$ are *shift equivalent* if there are morphisms $R \colon G \to H$ and $S \colon H \to G$ so that $RA = BR$, $AS = SB$ and $SR = A^p$, $RS = B^p$ for some natural number p.

PROPOSITION 1.3.2. *Two morphisms* $A = A(\mathcal{U})$ *and* $B = A(\mathcal{V})$ *that arise from Markov partitions* $\mathcal{U}$ *and* $\mathcal{V}$ *in* $\mathcal{P}$ *that form a tight edge are shift equivalent. In fact we can take* $p = 1$.

The proof is originally due to Williams and is well known. We recall it to set down notations and conventions to be used later.

PROOF OF 1.3.2. There are two cases to consider.

Case 1. $\mathcal{U} \xrightarrow{+} \mathcal{V}$. We want to exhibit two morphisms R and S that fit into a commutative diagram as below

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xrightarrow{A(\mathcal{U})} & \mathbb{Z}\mathcal{U} \\
S \downarrow & \nearrow{\scriptstyle R} & \downarrow S \\
\mathbb{Z}\mathcal{V} & \xrightarrow[A(\mathcal{V})]{} & \mathbb{Z}\mathcal{V}.
\end{array}
$$

Let $A = A(\mathcal{U})$, $B = A(\mathcal{V})$. For a fixed $V_r \in \mathcal{V}$, there is exactly one $U_i \in \mathcal{U}$ so that $V_r \subset U_i$. Define a zero-one matrix R_{ir} by $R_{ir} = 1$ iff $V_r \subset U_i$. Let R be the map defined by $R(V_r) = \sum_i R_{ir}U_i$. Let S_{ri} be the zero-one matrix where $S_{ri} = 1$ iff $V_r \cap fU_i \neq \varnothing$. Define $S(U_i) = \sum_r S_{ri}V_r$.

First we show $RS = A$, or equivalently that $A_{ij} = 1$ iff $(RS)_{ij} = \sum_r R_{ir}S_{rj} = 1$. If $\sum_r R_{ir}S_{rj} = 1$, then at least one term, say $R_{ir_0} \cdot S_{r_0j} = 1$. This implies $V_{r_0} \subset U_i$ and $V_{r_0} \cap fU_j \neq \varnothing$, it follows $\varnothing \neq V_r \cap fU_j \subset U_i \cap fU_j$ and $A_{ij} = 1$. There can be at most one r_0 so that $R_{ir_0}S_{r_0j} = 1$ because by hypothesis $\mathcal{V} < \mathcal{U}(0,1)$, and $V_{r_0} \cap fU_j \subset V_{r_0}$ determines r_0 uniquely. On the other hand if $A_{ij} = 0$, then $U_i \cap fU_j = \varnothing$ and there can be no V_r so that $V_r \subset U_i$ and $V_r \cap fU_j \neq \varnothing$. So $R_{ir}S_{rj} = 0$ for all r.

It remains now to show $SR = B$. Assume $B_{rs} = 1$, then $V_r \cap fV_s \neq \varnothing$. Let U_j be the unique element of $\mathcal{U}$ so that $V_s \subset U_j$. Then $R_{js} = 1$ and $R_{is} = 0$ for $i \neq j$. So $\sum_i S_{ri}R_{is} = S_{rj}R_{js}$. $S_{rj} = 1$ because $\varnothing \neq V_r \cap fV_s \subset V_r \cap fU_j$. The sum $\sum_i S_{ri}R_{is}$ can at most be one since for a fixed s, R_{is} can be nonzero for at most one value of i. To conclude we show $(SR)_{rs} \neq 0$ implies $B_{rs} \neq 0$. If $(SR)_{rs} \neq 0$ then there is a j so that $S_{rj}R_{js} = 1$, and so $V_r \cap fU_j \neq \varnothing$. By Lemma 2.1.9, Case

1, for $x \in V_s$, $W^u(x, V_s) = W^u(x, U_j)$ so $fW^u(x, V_s) = fW^u(x, U_j)$. Intersecting with V_r gives $fW^u_\varepsilon(x) \cap fV_s \cap V_r = fW^u_\varepsilon(x) \cap fU_j \cap V_r \neq \varnothing$. Thus $fV_s \cap V_r \neq \varnothing$ and $B_{rs} \neq 0$.

Case 2. $\mathcal{U} \overset{-}{\longrightarrow} \mathcal{V}$. We define morphisms P and Q so that the following diagram is commutative

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xrightarrow{\;A(\mathcal{U})\;} & \mathbb{Z}\mathcal{U} \\
{\scriptstyle P}\big\downarrow & \overset{Q}{\nearrow} & \big\downarrow{\scriptstyle P} \\
\mathbb{Z}\mathcal{V} & \xrightarrow[\;A(\mathcal{V})\;]{} & \mathbb{Z}\mathcal{V}.
\end{array}
$$

Let $P_{ri} = 1$ if $V_r \subset U_i$ and 0 otherwise, $Q_{js} = 1$ if $fV_s \cap U_j \neq \varnothing$ and 0 otherwise. Define $P(U_i) = \sum_r P_{ri} V_r$ and $Q(V_s) = \sum_j Q_{js} U_j$. The demonstration that the diagram commutes is similar to Case 1. $\quad\square$

The morphisms constructed above generate isomorphisms of $T\mathcal{U}$ to $T\mathcal{V}$. If $\mathcal{U} \overset{+}{\longrightarrow} \mathcal{V}$, then the morphisms R and S give rise to maps TR and TS which fit into a commutative diagram as a consequence of the functorality of T

$$
\begin{array}{ccc}
T\mathcal{U} & \xrightarrow{\;TA(\mathcal{U})\;} & T\mathcal{U} \\
{\scriptstyle TS}\big\downarrow & \overset{TR}{\nearrow} & \big\downarrow{\scriptstyle TS} \\
T\mathcal{V} & \xrightarrow[\;TA(\mathcal{V})\;]{} & T\mathcal{V}.
\end{array}
$$

From the definition of $T\mathcal{U}$ it follows $TA(\mathcal{U})$ corresponds to multiplication by t^{-1}. Thus TS and TR are isomorphisms since both $TS \circ TR$ and $TR \circ TS$ are isomorphisms. If $\mathcal{U} \overset{-}{\longrightarrow} \mathcal{V}$, then TP and TQ are isomorphisms and $(TP)^{-1} = t \cdot (TQ)$.

For arbitrary Markov partitions $\mathcal{U}$ and $\mathcal{V}$, by connecting them in $\mathcal{P}$ with a tight path we can construct an isomorphism of $T\mathcal{U}$ to $T\mathcal{V}$. Let

$$
\mathcal{U} = \mathcal{U}_0 - \mathcal{U}_1 - \mathcal{U}_2 - \cdots - \mathcal{U}_{n-1} - \mathcal{U}_n = \mathcal{V}
$$

be a tight path. We choose isomorphisms from $T\mathcal{U}_i$ to $T\mathcal{U}_{i+1}$ for $0 \le i < n$ in the following fashion:

(1) If $\mathcal{U}_i \overset{+}{\longrightarrow} \mathcal{U}_{i+1}$ the isomorphism is $(TR_i)^{-1}$.

(2) If $\mathcal{U}_i \overset{+}{\longleftarrow} \mathcal{U}_{i+1}$ the isomorphism is TR_{i+1}.

(3) If $\mathcal{U}_i \overset{-}{\longrightarrow} \mathcal{U}_{i+1}$ the isomorphism is TP_i.

(4) If $\mathcal{U}_i \overset{-}{\longleftarrow} \mathcal{U}_{i+1}$ the isomorphism is $(TP_{i+1})^{-1}$.

The composition of these yields an isomorphism $\psi(\mathcal{U}, \mathcal{V}): T\mathcal{U} \to T\mathcal{V}$.

A fundamental fact is that this isomorphism does not depend on the particular choice of tight path from $\mathcal{U}$ to $\mathcal{V}$. An important ingredient of the proof of this fact (found in [**W**1]) is that any loop in $\mathcal{P}$ of tight edges can be bound by a complex consisting of tight two simplices.

1.4. The dynamical invariant ϕ. Elements of $\mathrm{Aut}(f)$ must preserve the nonwandering set. If $g \in \mathrm{Aut}(f)$ and Ω_i is a basic set in Ω, then $g\Omega_i$ must also be a basic set although it may be different from Ω_i. Thus the elements of $\mathrm{Aut}(f)$ permute the basic sets of f. We do have some control of this. Since each basic set contains a dense orbit, the dimension of $E^u_x \subset T_x M$ is independent of $x \in \Omega_i$. We define the *index* of Ω_i to be $\dim(E^u_x)$. Since g commutes with f, $g\Omega_i$ must have

the same index as Ω_i. We let Ω^k denote the union of basic sets of index k and Ω_i^k denotes a particular basic set of index k.

We will study $\text{Aut}(f)$ by considering the action of its elements on Markov partitions of the various Ω_i^k and using the ideas from the previous section. First we need a technical lemma.

LEMMA 1.4.1. *Let $\mathcal{U}$ be a Markov partition of Ω_i. Then there are Markov partitions $\mathcal{V}$ refining $\mathcal{U}$ so that $g\mathcal{V}$ is also a Markov partition of $g\Omega_i$.*

PROOF. We will construct $\mathcal{V}$ of the form $\mathcal{U}(-n, m)$. It is clear $\mathcal{U}(-n, m) < \mathcal{U}(-N, M)$ if $N \geq n$ and $M \geq m$. Let ε be as in the definition of rectangle. Since g is uniformly continuous we can choose n and m so large that the diameter of gU is less than ε for every $U \in \mathcal{U}(-n, m)$. Observe $gW^u(x) = W^u(g(x))$ and $gW^s(x) = W^s(g(x))$, recall $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ contain small discs about x in $W^s(x)$ and $W^u(x)$ respectively. Also, from [**B1**] we know $W_\varepsilon^u(x) \cap B_{\varepsilon'}(x) = W_{\varepsilon'}^u(x)$ where $B_{\varepsilon'}(x)$ denotes the ε'-ball at x and $\varepsilon' < \varepsilon$. From these facts and the uniform continuity of g and g^{-1} it follows there are small numbers ε' and ε'' so that

$$W_{\varepsilon'}^s(g(x)) \subset gW_\varepsilon^s(x) \subset W_{\varepsilon''}^s(g(x)),$$
$$W_{\varepsilon'}^u(g(x)) \subset gW_\varepsilon^u(x) \subset W_{\varepsilon''}^u(g(x)).$$

Now choose n and m so that the diameters of gU for $U \in \mathcal{U}(-n, m)$ are less than the minimum of the set $\{\varepsilon, \varepsilon', \varepsilon''\}$. If $x, y \in gU$, then $[x, y] = W_\varepsilon^s(x) \cap W_\varepsilon^u(y) = W_{\varepsilon'}^s(x) \cap W_{\varepsilon'}^u(y) \subset gW_\varepsilon^s(g^{-1}x) \cap gW_\varepsilon^u(g^{-1}x) = g[g^{-1}x, g^{-1}y]$. Thus gU is a rectangle. Properties 1 and 2 for partitions are clear because g is a homeomorphism. For Property 3 let $x \in gU_i \cap f^{-1}gU_j$ for $U_i, U_j \in \mathcal{U}(-n, m)$. We want to show $fW^u(x, gU_i) \supset W^u(fx, gU_j)$. Now $g^{-1}x \in U_i \cap fU_j$ and since $\mathcal{U}(-n, m)$ is a Markov partition $fW^u(g^{-1}x, U_i) \supset W^u(fg^{-1}x, U_j)$. We rewrite this as $fW_\varepsilon^u(g^{-1}x) \cap fU_i \supset W_\varepsilon^u(fg^{-1}x) \cap U_j$. Applying g

$$fgW_\varepsilon^u(g^{-1}x) \cap fgU_i \supset gW_\varepsilon^u(fg^{-1}x) \cap gU_j \supset W_{\varepsilon'}^u(fx) \cap gU_j = W_\varepsilon^u(fx) \cap gU_j$$

since the diameter of gU_j is smaller than ε and ε'. On the left

$$fgW_\varepsilon^u(g^{-1}x) \cap fgU_i = f(gW_\varepsilon^u(g^{-1}x) \cap gU_i) \subset f(W_{\varepsilon''}^u(x) \cap gU_i) = f(W_\varepsilon^u(x) \cap gU_i)$$

since the diameter of gU_i is less than ε and ε'. This shows

$$fW^u(x, gU_i) \supset W^u(fx, gU_j).$$

The demonstration that $fW^s(x, gU_i) \subset W^s(fx, gU_j)$ is similar. $\square$

For each basic set Ω_i^k, $i = 1, 2, \ldots, j(k)$, of index k let $\mathcal{U}_i$ be a Markov partition. Let $\mathcal{V}_i$ be a refinement of $\mathcal{U}_i$ so that $g\mathcal{V}_i$ is also a Markov partition of $g\Omega_i^k$. We let g also denote the permutation of the index set, $g\Omega_i^k = \Omega_{g(i)}^k$. From §1.3 there are isomorphisms $\psi(\mathcal{U}_i, \mathcal{V}_i): T\mathcal{U}_i \to T\mathcal{V}_i$ and $\psi(g\mathcal{V}_i, \mathcal{U}_{g(i)}): Tg\mathcal{V}_i \to T\mathcal{U}_{g(i)}$. There is also a canonical identification of $T\mathcal{V}_i$ and $Tg\mathcal{V}_i$ given by $V \in \mathcal{V} \mapsto gV \in g\mathcal{V}$. Under the map induced by this from $\mathbb{Z}\mathcal{V}$ to $\mathbb{Z}g\mathcal{V}$ we have the identification $A(\mathcal{V}_i) = A(g\mathcal{V}_i)$. We let $TG(\mathcal{V}_i)$ denote this isomorphism. Altogether we have a composite isomorphism $\psi(\mathcal{U}_i, \mathcal{V}_i) \circ TG(\mathcal{V}_i) \circ \psi(g\mathcal{V}_i, \mathcal{U}_{g(i)}): T\mathcal{U}_i \to T\mathcal{U}_{g(i)}$. (Read composition from left to right here.) By taking direct sums over all basic sets of index k we obtain an isomorphism

$$\bigoplus_i [\psi(\mathcal{U}_i, \mathcal{V}_i) \circ TG(\mathcal{V}_i) \circ \psi(g\mathcal{V}_i, \mathcal{U}_{g(i)})]: \bigoplus_i T\mathcal{U}_i \to \bigoplus_i T\mathcal{U}_{g(i)}.$$

Under the rearrangement of the indices $\bigoplus_i TU_{g(i)} \cong \bigoplus_i TU_i$. Call this rearrangement isomorphism R. Define an automorphism $T(g,k)$ of $\bigoplus_i TU_i$ by $T(g,k) = \bigoplus_i [\psi(U_i, V_i) \circ TG(V_i) \circ \psi(gV_i, U_{g(i)})] \circ R$. We have already remarked that this definition does not depend on the tight paths chosen. It is also independent of the choice of $\{V_i\}$. This is a consequence of the following lemma.

LEMMA 1.4.2. $TG(V') = \psi(V', V) \circ TG(V) \circ \psi(gV, gV')$.

PROOF. It is possible to choose a tight path from V to V' so that the image under g of each partition in the path is also a Markov partition. Thus it suffices to check the equation for (V, V') a tight edge. Assume $V \xrightarrow{+} V'$. Then $gV \xrightarrow{+} gV'$ and the matrices R and S for these two tight edges are identical if we use the natural order on gV coming from V, i.e. if $V = \{V_1, \ldots, V_n\}$ then $gV = \{U_1, \ldots, U_n\}$ where $U_i = gV_i$ for each i. The case $V \xrightarrow{-} V'$ is similar. □

The dynamical invariant $\phi(g)$ is defined as $\prod_k \det(T(g,k))^{(-1)^k}$ where again we are using multiplicative notation in each $F_\wp^*$. The product is from $k = 0$ to $k = \dim M$.

We remark that the definition of $\phi(g)$ is also indepdent of $\{U_i\}$. In fact if $\{U_i'\}$ is another choice of partitions, then $\bigoplus_i \psi(U_i, U_i')$ defines a conjugacy from $T(g,k)$ and $T(g,k)'$ defined using the $\{U_i'\}$ partitions. That is

$$T(g,k)' = \bigoplus_i \psi(U_i, U_i') \circ T(g,k) \circ \left[\bigoplus_i \psi(U_i, U_i')\right]^{-1}.$$

It follows $\det(T(g,k)') = \det(T(g,k))$.

We also would like to note that our order of composition of the isomorphisms defining $T(g,k)$ is reversed to Wagoner's original definition and so our automorphisms are the inverses of the ones he defines. Of course they contain the same information and we choose this order to avoid an inverse appearing in the statement of our main theorem.

The argument to show ϕ is a group homomorphism follows essentially verbatim from [W1] and is a consequence of $\psi(U, V)$ being independent of path.

2. THE MAIN RESULT

THEOREM. *Let f be a fitted Smale diffeomorphism of an oriented closed smooth manifold M. If we take $F = \mathbb{Z}/2$ in the definition of the torsion functor T, then as homomorphisms from $\mathrm{Aut}(f)$ to $\mathrm{Wh}_2(\mathbb{Z}/2(t))$, ϕ and Φ are equal.*

REMARKS. The theorem is true for any field F provided f preserves the orientations assigned E_Ω^u; however, this is an exceptional case as most examples show. The theorem is probably also valid for nonfitted diffeomorphisms. As the proof stands the only problem extending the theorem is 2.1.2 and concerns technical matters on extending the stable and unstable manifolds of f.

The proof divides naturally into two parts. Part one demonstrates that the mod 2 dynamical invariant can be calculated in terms of the homolopgy groups of certain pairs in M. The second part relates this information to the absolute homology groups of M and the homological invariant Φ.

2.1. Filtrations and filtration pairs. Given a manifold M and a diffeomorphism f there are finite sequences of compact submanifolds of M of the same

dimension as $M: \varnothing \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$, with the properties that $f(M_i) \subset \operatorname{int} M_i$ and that for each $1 \leq i \leq n$, exactly one basic set Ω_i is contained in $M_i - M_{i-1}$. Furthermore $\Omega_i = \bigcap_{n \in \mathbb{Z}} f^n(M_i - M_{i-1})$. These *filtrations* also have the property that the basic sets in $M_i - M_{i-1}$ occur with nondecreasing index. We use the following notation to reflect this.

$$\varnothing = M_1^0 \subset M_2^0 \subset \cdots \subset M_{j(0)}^0 = M^0$$
$$= M_0^1 \subset M_1^1 \subset \cdots \subset M_{j(1)}^1 \subset M^1 \subset M_0^2 \subset \cdots \subset M_0^n \subset M_1^n \subset \cdots \subset M_{j(n)}^n$$
$$= M^n = M,$$

where $M_i^k - M_{i-1}^k$ contains the basic set of index k denoted by Ω_i^k for $0 \leq k \leq n$ and $1 \leq i \leq j(k)$.

(M_i^k, M_{i-1}^k) is an example of what is called a filtration pair for the basic set Ω_i^k.

DEFINITION. A pair (X, A) of compact submanifolds of M with boundary and of the same dimension as M is a *filtration pair* of a basic set Ω if

(1) $f(X) \subset \operatorname{int} X$ and $f(A) \subset \operatorname{int} A$.

(2) $\Omega = \bigcap_{n \in \mathbb{Z}} f^n(X - A)$.

For a sufficiently fine Markov partition of a basic set Ω there is a relation between the endomorphisms $A(\mathcal{U})$ and f_* acting on the relative homology of a filtration pair for Ω. To make the relationship precise we assign an orientation to E_Ω^u (possible because Ω is zero dimensional and hence totally disconnected). Let $\Delta(x) = \pm 1$, depending on whether Df_x preserves or reverses the orientations. By taking a sufficiently fine Markov partition $\mathcal{U}$ we can assume Δ is constant on each rectangle in $\mathcal{U}$. Define $B_{ij} = \Delta(U_i)$ if $U_i \cap fU_j \neq \varnothing$ and 0 otherwise. Let $B(\mathcal{U})$ be the endomorphism of $\mathbb{Z}\mathcal{U}$ defined on the generators $U_j \in \mathcal{U}$ by $B(\mathcal{U})U_j = \sum B_{ij}U_i$. Since $|B_{ij}| = A_{ij}$ it is clear that as endomorphisms of $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}\mathcal{U}$, $A(\mathcal{U})$ and $B(\mathcal{U})$ are equal, and that $T(\mathbb{Z}\mathcal{U}; A(\mathcal{U})) \cong T(\mathbb{Z}\mathcal{U}, B(\mathcal{U}))$ for T defined using $F = \mathbb{Z}/2$, which we assume from now on.

Let G be an abelian group and let $B(\mathcal{U})$ define a map from $G^n \to G^n$ by

$$B(\mathcal{U})(g_1, \ldots, g_n) = \left(\sum_{i=1}^n B_{i1}g_i, \ldots, \sum_{i=1}^n B_{in}g_i \right).$$

Bowen and Frank [**BF**] demonstrated that if M is orientable and if (X, A) is a filtration pair for the basic set of index k then $B(\mathcal{U}): G^n \to G^n$ and $f_*: H_k(X, A; G) \to H_k(X, A; G)$ are shift equivalent. This implies

$$T(\mathbb{Z}\mathcal{U}, B(\mathcal{U})) \cong T(H_k(X, A; \mathbb{Z}), f_*).$$

For our purposes we need to make this isomorphism explicit. We shall construct a map $\alpha: \mathbb{Z}\mathcal{U} \to H_k(X, A; \mathbb{Z})$ for a suitable filtration pair (X, A). This map shall have several important properties and to establish them we must assume f is a *fitted* diffeomorphism. Diffeomorphisms are fitted with respect to some handle decomposition of M. We recall the definitions (see [**F**, p. 28]). A handle set $H(k)$ is a union $\bigcup_i h_i(k)$ where each $h_i(k)$ is diffeomorphic to $D_i^k \times D_i^{n-k}$, a product of discs. The handles are constructed using local coordinates around critical points of index k for some self-indexing Morse function. For $x \in h_i(k)$ corresponding to $(p, q) \in D_i^k \times D_i^{n-k}$, we let $W_i^u(x)$ be the image of $D_i^k \times q$ and $W_i^s(x)$ the image of $p \times D_i^{n-k}$. A diffeomorphism is *fitted* with respect to handle sets $H(k)$ if f is

hyperbolic over the handle sets and if $k < l$ and $x \in h_i(l)$ and $y = f^n x \in h_j(k)$ for $n \geq 0$, then $f^n W_i^u(x) \supset W_j^u(y)$ and $f^{-n} W_j^s(y) \subset W_i^s(x)$. It follows $W_i^u(x) \subset W^u(x)$ and $W_i^s(x) \supset W^s(x)$ if $x \in \Omega$. Notice that $W_i^u(x)$ and $W_i^s(x)$ are defined even if x is *not* in the nonwandering set. Thus we can define an extension of the canonical coordinates $(x, y) \mapsto [x, y] = W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ from some neighborhood of $\Delta\Omega$, the diagonal in $\Omega \times \Omega$, to some neighborhood of $\Delta\Omega$ in $M \times M$. That is, if x and y are in the same handle $h_i(k)$ and $d(x, y) < \varepsilon$, let $(x, y) \mapsto [x, y] = W_i^s(x) \cap W_i^u(y)$. This extension is used to construct $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A)$. We remark that the hypothesis that f be fitted can be weakened slightly using the ideas in [**HPPS**] of semi-invariant disc families with the hypothesis that the families can be chosen to foliate a neighborhood of the nonwandering set. This is clearly the case for fitted diffeomorphisms although it is still unknown whether such a choice of families is always possible for an arbitrary diffeomorphism f.

To define $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A; \mathbb{Z})$ we let $\mathcal{U}$ be a Markov partition as in the definition of $B(\mathcal{U})$. For each rectangle $U_i \in \mathcal{U}$, let W_i be a neighborhood of U_i in M so that $W_i \cap W_j = \varnothing$ for $i \neq j$. This is possible because the rectangles are open and closed in Ω. For each rectangle U_i choose a periodic point x_i of f in U_i. $x_i \in W^u(x_i, U_i) \subset W_\varepsilon^u(x_i) \subset W^u(x_i)$. Let $\alpha(x_i, U_i)$ be a neighborhood of $W^u(x_i, U_i)$ contained in $W_\varepsilon^u(x_i) \cap W_i$. In [**BF**] these neighborhoods are assumed to be k-dimensional submanifolds of $W^u(x_i)$ with boundary. This will not be preserved under our setting; we assume each $\alpha(x_i, U_i)$ is homeomorphic to some pseudomanifold with boundary (see [**Sp**, p. 150]). This can be accomplished by choosing the neighborhood to be a manifold as in [**BF**] and then triangulating. We call such neighborhoods *nice*. Similarly we can construct nice neighborhoods $\beta(x_i, U_i)$ of $W^s(x_i, U_i)$ contained in $W_\varepsilon^s(x_i) \cap W_i$. By taking $\mathcal{U}$ fine enough and $\alpha(x_i, U_i)$ and $\beta(x_i, U_i)$ small enough we can assume $[\alpha(x_i, U_i), \beta(x_i, U_i)]$ is defined using the extended canonical coordinates. The local product structure of the handles guarantees this is a neighborhood of U_i in M. $U_i \subset \operatorname{int}[\alpha(x_i, U_i), \beta(x_i, U_i)] \subset W_i$. Therefore $\bigcup_i [\alpha(x_i, U_i), \beta(x_i, U_i)]$ is a neighborhood of the basic set Ω in M.

LEMMA 2.1.1. *Let W be a neighborhood of a basic set Ω in M. Then there is a filtration pair (X, A) so that $X - A \subset \operatorname{int} W$.*

PROOF. Let (Y, B) be any filtration pair. Since $f(Y) \subset Y$ and $f(B) \subset B$ and $\Omega = \bigcap_{n \in \mathbb{Z}} f^n(X - A)$, it follows there is an integer N so that $f^N Y - f^{-N} B \subset \operatorname{int} W$. The desired filtration pair is $(f^N Y, f^{-N} B)$. $\square$

Choose a filtration pair (X, A) so that $X - A \subset \operatorname{int} \bigcup_i [\alpha(x_i, U_i), \beta(x_i, U_i)]$. It follows $\alpha(x_i, U_i) \subset X$ and $\partial\alpha(x_i, U_i) \subset A$. ($\partial$ as a k-dimensional pseudomanifold contained in $W^u(x_i)$.) Also $\beta(x_i, U_i) \subset M - A$ with $\partial\beta(x_i, U_i) \subset M - X$. Since the $\alpha(x_i, U_i)$ and $\beta(x_i, U_i)$ where chosen as nice, they represent homology classes $[\alpha(x_i, U_i)]$ and $[\beta(x_i, U_i)]$ in $H_k(X, A; \mathbb{Z})$ and $H_{n-k}(M - A, M - X; \mathbb{Z})$ respectively. Until further notice we will drop the coefficients in the homology groups; they are assumed to be the integers. Define $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A)$ on the generators $U_i \in \mathcal{U}$ by $\alpha(U_i) = [\alpha(x_i, U_i)]$ and extend linearly. Since the W_i are pairwise disjoint, it follows the $\alpha(U_i)$ represent independent elements in $H_i(X, A)$. By orienting the $\alpha(x_i, U_i)$ and $\beta(x_i, U_i)$ we can assume that the cohomology classes $[\beta(x_i, U_i)]^*$ in $H^k(X, A)$ (which are the images of $[\beta(x_i, U_i)]$ under the duality isomorphism

$$H_{n-k}(M - A, M - X) \cong H^k(X, A))$$

when paired with $[\alpha(x_j, U_j)]$ give δ_{ij}, the Kronecker delta. Since the pairing is linear, it follows that it is zero on the torsion subgroup of $H_k(X, A)$ and therefore the images of the $[\alpha(x_i, U_i)]$ remain independent in F, the quotient group $H_k(X, A)/\mathrm{Tor}(H_k(X, A))$. By abuse of notation we shall denote the images of $[\alpha(x_i, U_i)]$ in F also as $[\alpha(x_i, U_i)]$.

We use the extended canonical coordinates to define $\alpha(y_i, U_i)$ for other choices of $y_i \in U_i$. Let $\alpha(y_i, U_i) = [\alpha(x_i, U_i), y_i]$. The extended canonical coordinates still satisfy the formula $[[x, y], z] = [x, z]$ and so for any two $y_i, y_i' \in U_i$, $[\alpha(y_i, U_i), y_i'] = \alpha(y_i', U_i)$. Similarly, let $\beta(y_i, U_i) = [y_i, \beta(x_i, U_i)]$. The construction of (X, A) shows these sets also represent homology classes. Unfortunately, in $H_k(X, A)$, $[\alpha(y_i, U_i)] \neq [\alpha(x_i, U_i)]$ for all choices of y_i. However, the ambiguity is removed when we consider their images in $T(H_k(X, A), f_*)$.

PROPOSITION 2.1.2. *The image of the class $[\alpha(y_i, U_i)]$ in $T(H_k(X, A), f_*)$ is independent of $y_i \in U_i$.*

PROOF. It suffices to show the images of $[\alpha(x_i, U_i)]$ and $[\alpha(y_i, U_i)]$ are equal. There is a homotopy in M from the set $\alpha(x_i, U_i)$ to $\alpha(y_i, U_i)$ constructed by choosing a path $\gamma \colon I \to W^s(x_i)$ with $\gamma(0) = x_i$ and $\gamma(1) = [x_i, y_i]$ and letting $H \colon \alpha(x_i, U_i) \times I \to M$ by $H(z, t) = [z, \gamma(t)]$. $H(z, 0) = [z, x_i] = z$ and $H(z, 1) = [z, [x_i, y_i]] = [z, y_i] \in \alpha(y_i, U_i)$. This homotopy does not show

$$[\alpha(x_i, U_i)] = [\alpha(y_i, U_i)]$$

because the homotopy may not remain in X. Since x_i is a periodic point of f, there is an integer p so that $f^p x_i = x_i$, and it follows $f^p(W^u(x_i)) = W^u(x_i)$. Given $z \in \alpha(x_i, U_i)$, z and $[z, y_i]$ both lie in $W^s(z)$, so we can choose n such that the distance between $f^{pn} z$ and $f^{pn}[z, y_i]$ is arbitrarily small provided these images remain in the handles. By compactness of $\alpha(x_i, U_i)$ we can do this uniformly over $\alpha(x_i, U_i)$. Since $fX \subset \mathrm{int}\, X$ and $f^p W^u(x_i) = W^u(x_i)$, $d(W^u(x_i), \overline{M - X}) = \varepsilon > 0$. Therefore $B_\varepsilon(W^u(x_i)) \subset X$. By choosing n so large that $d(f^{np} z, f^{np}[z, y_i]) < \varepsilon$ for all $z \in \alpha(x_i, U_i)$ such that the images remain in the handle set implies $d(f^{pn} z, f^{pn}[z, \gamma(t)]) < \varepsilon$ for these z. Points z where $f^{pn} z$ does not tend to $f^{pn}[z, y_i]$ imply $f^{pn} z \in A$. Therefore $f^{pn} H(\cdot, \cdot) \subset X$ and this gives a homotopy from $f^{pn} \alpha(x_i, U_i)$ to $f^{pn} \alpha(y_i, U_i)$ over $\overline{X - A}$. In $T(H_k(X, A), f_*)$,

$$[\alpha(y_i, U_i)] \equiv t^{np} f_*^{np}[\alpha(y_i, U_i)] = t^{np}[f^{np}\alpha(y_i, U_i)]$$
$$= t^{np}[f^{np}\alpha(x_i, U_i)] = t^{np} f_*^{np}[\alpha(x_i, U_i)] \equiv [\alpha(x_i, U_i)]. \quad \square$$

THEOREM 2.1.3. *The endomorphism $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A)$ induces an isomorphism $T\alpha \colon T(\mathbb{Z}\mathcal{U}, B(\mathcal{U})) \to T(H_k(X, A), f_*)$ of $F[t, t^{-1}]$ modules.*

LEMMA 2.1.4. *Let $\mathrm{Tor}(H_k(X, A))$ denote the torsion subgroup of $H_k(X, A)$ and let F denote the free group $H_k(X, A)/\mathrm{Tor}(H_k(X, A))$. If $\overline{f}_*$ is the map induced on F by f_*, then the canonical quotient map from $H_k(X, A)$ to F induces an isomorphism $T(H_k(X, A), f_*) \cong T(F, \overline{f}_*)$.*

PROOF OF 2.1.4. Apply T to the short exact sequence

$$0 \to \mathrm{Tor}(H_k(X, A)) \to H_k(X, A) \to F \to 0.$$

Assuming M is orientable we can apply a result from [**BF**], $f_*|\mathrm{Tor}(H_k(X, A))$ is nilpotent. This implies $T(\mathrm{Tor}(H_k(X, A)), f_*) = 0$ and the result follows since T is an exact functor. $\square$

PROOF OF 2.1.3. Comments after the construction of α show that its image injects in F. So by the lemma, it is sufficient to show α induces an isomorphism from $T(\mathbb{Z}\mathcal{U}, B(\mathcal{U}))$ to $T(F, f_*|F)$.

The morphisms $B(\mathcal{U})$ and f_* define morphisms of the free $\mathbb{Z}$-modules $\mathbb{Z}\mathcal{U}$ and F. Let B be the matrix of $B(\mathcal{U})$ relative to the basis $U_i \in \mathcal{U}$ and let M be the matrix of f_* obtained by extending the independent set $\{\alpha(x_i, U_i)\}_{i=1,\ldots,n}$ to the basis for F. B and M are integer matrices and M has the form $\begin{pmatrix} B & C \\ D & N \end{pmatrix}$. Relative to these bases the matrix of the linear map α is $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$. The first task is to show α defines a map on the cokernels. Let $\langle 1 \otimes I - t \otimes B \rangle = \mathrm{image}\, 1 \otimes I - t \otimes B$ and write $t \otimes B$ as tB and $t \otimes M$ as tM. We must show $\alpha\langle I - tB \rangle \subset \langle I - tM \rangle$. Let v be the coordinates of $\overline{v} \in \mathbb{Z}\mathcal{U}$. $\alpha\overline{v} \in F$ has components

$$\begin{pmatrix} I_n \\ 0 \end{pmatrix} v = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

$$\alpha\langle I - tB \rangle = \{\alpha(I - tB)v | \overline{v} \in V\} = \left\{ \begin{pmatrix} v - tBv \\ 0 \end{pmatrix} \middle| \overline{v} \in V \right\}.$$

To show

$$\begin{pmatrix} v - tBv \\ 0 \end{pmatrix} \in \langle I - tM \rangle$$

we must solve the equations

$$(I - tM) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} v - tBv \\ 0 \end{pmatrix}$$

(a and b are column vectors with entries from $F[t, t^{-1}]$).

$$(I - tM) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - tBa - tCb \\ b - tDa - tNb \end{pmatrix}.$$

Let $a = v$. We want b so that $Cb = 0$ and $b - tDv - tNb = 0$. From the second equation $(I - tN)b = tDv$. In [**BF**] it is shown the matrix N is nilpotent (Lemma 3.6). Therefore $I - tN$ is invertible and if p is such that $N^{p+1} = 0$, then

$$b = (I - tN)^{-1}tDv = (I + tN + \cdots + t^p N^p)tDv.$$

Bowen and Franks also show $CN^j D = 0$ for all $j \geq 0$. It follows that $Cb = 0$. (Remark: It is crucial to be working with $\mathbb{Z}$ coefficients at this stage.)

From [**BF**] we know B and M are shift equivalent. Therefore as vector spaces over $\mathbb{Z}/2$ we know $T(\mathbb{Z}\mathcal{U}, B(\mathcal{U})) \cong T(F, f_*|F)$. Thus to show $T\alpha$ is an isomorphism it suffices to show $T\alpha$ is an injection.

$$\alpha(\overline{v}) = \begin{pmatrix} v \\ 0 \end{pmatrix} \equiv 0 \quad \text{iff} \quad \begin{pmatrix} v \\ 0 \end{pmatrix} = (I - tM) \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} a \\ b \end{pmatrix}.$$

$$(I - tM) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a - tBa - tCb \\ b - tDa - tNb \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

gives the equations $b = (I - tN)^{-1}tDa$ and so as before $Cb = 0$. We get $v = a - tBa - tCb = a - tBa = (I - tB)a$ and so $v \equiv 0$ module $\langle I - tB \rangle$. $\square$

Recall now the construction of the automorphisms $T(g, k)$ of $\bigoplus_i TU_i$ used to define the dynamical invariant $\phi(g)$. For simplicity, let us assume there is one basic set Ω of index k. We observed the value of $\phi(g)$ is independent of the initial choice of Markov partition $\mathcal{U}$ since the determinant function $\det(\cdot)$ is independent of conjugacy. So for our convenience we may assume out initial Markov partition $\mathcal{U}$ is chosen so that $g\mathcal{U}$ is also a Markov partition. This reduces $T(g, k)$ to $TG(\mathcal{U}) \circ \psi(g\mathcal{U}, \mathcal{U})$ since the initial term $\psi(\mathcal{U}, \mathcal{U})$ is the identity. Let $\mathcal{U}_0 = g\mathcal{U}$, $\mathcal{U}_n = \mathcal{U}$ and let

$$\mathcal{U}_0 - \mathcal{U}_1 - \cdots - \mathcal{U}_{n-1} - \mathcal{U}_n$$

be a tight path connecting $g\mathcal{U}$ and $\mathcal{U}$. The isomorphisms defining $T(g, k)$ arise from the following sequence after T is applied

$$\mathbb{Z}\mathcal{U} \xrightarrow{G(\mathcal{U})} \mathbb{Z}g\mathcal{U} = \mathbb{Z}\mathcal{U}_0 - \mathbb{Z}\mathcal{U}_1 - \cdots - \mathbb{Z}\mathcal{U}_n = \mathbb{Z}\mathcal{U}.$$

The morphisms from $\mathbb{Z}\mathcal{U}_i$ to $\mathbb{Z}\mathcal{U}_{i+1}$ for $0 \leq i < n$ are obtained from Proposition 1.3.2.

The point is to use the morphisms α to translate this information to information determined by homology groups of filtration pairs for Ω in M and the maps induced by f and g. This is accomplished in the following three propositions.

Before the construction of the α morphisms we need a lemma to ensure the Markov partitions appearing in our tight path defining $T(g, k)$ can be chosen arbitrarily fine.

LEMMA 2.1.5. *Given a tight path from $\mathcal{U}$ to $\mathcal{V}$*

$$\mathcal{U} = \mathcal{U}_1 - \mathcal{U}_2 - \cdots - \mathcal{U}_n = \mathcal{V}$$

the path

$$\mathcal{U}_1(-N, M) - \mathcal{U}_2(-N, M) - \cdots - \mathcal{U}_n(-N, M)$$

is also a tight path connection $\mathcal{U}(-N, M)$ to $\mathcal{V}(-N, M)$.

PROOF. It is sufficient to show $(\mathcal{U}(-N, M), \mathcal{V}(-N, M))$ is a tight edge if $(\mathcal{U}, \mathcal{V})$ is tight. This follows from induction on N and M after the observations:
  (1) If $\mathcal{U} < \mathcal{V}$ then $\mathcal{U}(-1, 0) < \mathcal{V}(-1, 0)$ and $\mathcal{U}(0, 1) < \mathcal{V}(0, 1)$.
  (2) $\mathcal{U}(-i, j)(0, 1) = \mathcal{U}(-i, j + 1)$ and $\mathcal{U}(-i, j)(-1, 0) = \mathcal{U}(-i - 1, j)$. $\square$

Thus we may assume all the partitions defining $T(g, k)$ can be chosen so that $B(\mathcal{U}_i)$ is defined for each $0 \leq i \leq n$.

PROPOSITION 2.1.6. *Let $\mathcal{U}$ be a Markov partition of a basic set Ω_i^k of index k so fine that $g\mathcal{U}$ is also a Markov partition of $g\Omega_i^k$ and so fine that the morphisms $B(\mathcal{U})$ and $B(g\mathcal{U})$ are both defined. Let (X, A) be a filtration pair of Ω_i^k as in the construction of a morphism $\alpha: \mathbb{Z}\mathcal{U} \to H_k(X, A)$. Then there is a filtration pair (Y, B) of $g\Omega_i^k$ and a morphism $\alpha': \mathbb{Z}g\mathcal{U} \to H_k(Y, B)$ so that $T\alpha'$ is an isomorphism and the following diagram commutes*

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xrightarrow{G(\mathcal{U})} & \mathbb{Z}g\mathcal{U} \\
\alpha \downarrow & & \downarrow \alpha' \\
H_i(X, A) & \xrightarrow[g_*]{} & H_k(Y, B).
\end{array}
$$

PROOF. Let $(Y, B) = g(X, A) = (gX, gA)$ and let $\alpha'(gU_i) = [g\alpha(x_i, U_i)]$ where $\alpha(x_i, U_i)$ are the neighborhoods of $W^u(x_i, U_i)$ chosen to construct α. From the lemma that $\mathcal{U}$ can be chosen so that $g\mathcal{U}$ is also a partition we see that $g\alpha(x_i, U_i)$ is a nice neighborhood of $W^u(gx_i, gU_i)$. Commutativity of the diagram is trivial. When T is applied $T\alpha'$ is also an isomorphism. To show this one checks the steps in the construction of α. These steps are:

(1) Periodic points $x_i \in U_i$ are chosen and neighborhoods W_i of U_i are chosen to be pairwise disjoint,

(2) Nice neighborhoods $\alpha(x_i, U_i)$ of $W^u(x_i, U_i)$ are chosen that are contained in $W^u(x_i) \cap W_i$. Also neighborhoods $\beta(x_i, U_i)$ of $W^s(x_i, U_i)$ are chosen.

(3) Using extended canonical coordinates $[,]$ the set $W = \bigcup_i[\alpha(x_i, U_i), \beta(x_i, U_i)]$ is a neighborhood of the basic set and (X, A) is a filtration pair of Ω_i^k chosen so that $X - A \subset \operatorname{int} W$.

(4) For all $y_i \in U_i$, $\alpha(y_i, U_i)$ defined as $[\alpha(x_i, U_i), y_i]$ and $\beta(y_i, U_i) = [y_i, \beta(x_u, U_i)]$ are neighborhoods of $W^u(y_i, U_i)$ and $W^s(y_i, U_i)$ respectively with $\partial\alpha(y_i, U_i) \subset A$ and $\partial\beta(y_i, U_i) \subset M - X$.

In these steps to demonstrate $T\alpha'$ is an isomorphism, x_i is replaced by gx_i, $\alpha(x_i, U_i)$ is replaced by $g\alpha(x_i, U_i)$ which is a nice neighborhood of $W^u(gx_i, gU_i)$ and $\beta(x_i, U_i)$ is replaced by $g\beta(x_i, U_i)$. It may be necessary to reorient the $g\beta(x_i, U_i)$ to show $T\alpha'$ is an isomorphism but this does not affect the construction of α'. It is clear (gX, gA) is a filtration pair of $g\Omega_i$ and that $\partial g\alpha(x_i, U_i) \subset gA$. □

PROPOSITION 2.1.7. *Let $(\mathcal{U}, \mathcal{V})$ be a tight edge. Let $\alpha: \mathbb{Z}\mathcal{U} \to H_k(X, A)$ as in Proposition 2.1.6. Then there are a filtration pair (Y, B) and a map $\alpha': \mathbb{Z}\mathcal{V} \to H_k(Y, B)$ which induce an isomorphism after T is applied. Furthermore (Y, B) can be chosen so that there is an inclusion from (X, A) to (Y, B) or vice versa (depending on the morphism $\mathbb{Z}\mathcal{U} - \mathbb{Z}\mathcal{V}$ constructed for the tight edge as in Proposition 1.3.2) so that the following diagram commutes after T is applied*

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \longrightarrow & \mathbb{Z}\mathcal{V} \\
\alpha \downarrow & & \downarrow \alpha' \\
H_k(X, A) & \xrightarrow[\text{inclusion}]{\text{induced by}} & H_k(Y, B).
\end{array}
$$

To prove Proposition 2.1.7 it is necessary to known how the various stable and unstable sets $W^s(u_r, V_r)$ and $W^u(v_r, V_r)$ are related to $W^s(v_r, U_i)$ and $W^u(v_r, U_i)$ for a tight edge $(\mathcal{U}, \mathcal{V})$.

LEMMA 2.1.8. *If $\mathcal{U}$ is a Markov partition of a basic set, then so is $\mathcal{U} \cap f\mathcal{U}$ and for sufficiently fine $\mathcal{U}$, if $x \in U_i \cap fU_j$ then*
(i) $W^u(x, U_i \cap fU_j) = W^u(x, U_i)$,
(ii) $W^s(x, U_i \cap fU_j) = fW^s(f^{-1}x, U_j)$.

COROLLARY. *Since the stable manifold of f is the unstable manifold of f^{-1} and the unstable manifold of f is the stable manifold of f^{-1} we have a dual statement for the partition $\mathcal{U} \cap f^{-1}\mathcal{U}$:*
(i) $W^s(x, U_i \cap f^{-1}U_j) = W^s(x, U_i)$.
(ii) $W^u(x, U_i \cap f^{-1}U_j) = f^{-1}W^u(fx, U_j)$.

These are used to prove the following lemma.

LEMMA 2.1.9. *Let $(\mathcal{U}, \mathcal{V})$ be a tight edge. There are four cases to consider. All unions are disjoint unions.*

Case 1. $\mathcal{U} \xrightarrow{+} \mathcal{V}$. Let $v_r \in V_r \in \mathcal{V}$ and $V_r \subset U_i \in \mathcal{U}$. Let J be the set of indices so that $V_r = \bigcup_{j \in J}(U_i \cap fU_j)$ and let v_{ij} be an arbitrary point in $W^s(v_r, V_r) \cap U_i \cap fU_j$ for each $j \in J$. Then

$$W^u(v_r, V_r) = W^u(v_r, U_i),$$
$$W^s(v_r, V_r) = \bigcup_{j \in J} fW^s(f^{-1}v_{ij}, U_j).$$

Case 2. $\mathcal{U} \xleftarrow{+} \mathcal{V}$. Let $v_r \in U_i \subset V_r$ and let I be the set of indices so that $V_r = \bigcup_{i \in I} U_i$. Choose $x_i \in U_i \cap V_r \cap W_\varepsilon^s(v_r)$ for each $i \in I$. Then

$$W^u(v_r, V_r) = W^u(v_r, U_i),$$
$$W^s(v_r, V_r) = \bigcup_{i \in I} W^s(x_i, U_i).$$

Case 3. $\mathcal{U} \xrightarrow{-} \mathcal{V}$. Let $v_r \in V_r \in \mathcal{V}$ and $V_r \subset U_i$. Let J be the set of indices so that $V_r = \bigcup_{j \in J} U_i \cap f^{-1}U_j$. Choose $x_{ij} \in U_i \cap f^{-1}U_j \cap W_\varepsilon^u(v_r)$ arbitrarily. Then

$$W^u(v_r, V_r) = \bigcup_{j \in J} f^{-1}W^u(fx_{ij}, U_j),$$
$$W^s(v_r, V_r) = W^s(v_r, U_i).$$

Case 4. $\mathcal{U} \xleftarrow{-} \mathcal{V}$. Let $v_r \in V_r$. Let I be the set of indices so that $V_r = \bigcup_{i \in I} U_i$. Choose $x_i \in U_i \cap W_\varepsilon^u(v_r)$. Then

$$W^u(v_r, V_r) = \bigcup_{i \in I} W^u(x_i, U_i),$$
$$W^s(v_r, V_r) = W^s(v_r, U_i).$$

The proofs are straightforward and omitted.

PROOF OF PROPOSITION 2.1.7. There are four cases to consider.

Case 1. $\mathcal{U} \xrightarrow{+} \mathcal{V}$. Given $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A)$ we want to choose (Y, B) and construct $\alpha' \colon \mathbb{Z}\mathcal{V} \to H_k(Y, B)$ so that the following diagram will commute when T is applied

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xleftarrow{\quad R \quad} & \mathbb{Z}\mathcal{V} \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\
H_k(X, A) & \xleftarrow{\quad i_* \quad} & H_k(Y, B).
\end{array}
$$

R is the morphism defined in Proposition 1.3.2 and i is inclusion. In fact let $(Y, B) = (fX, A) \subset (X, A)$. Choose periodic points $v_r \in V_r \in \mathcal{V}$. By hypothesis there is an integer i and an index set J so that $V_r \subset U_i$ and $V_r = \bigcup_{j \in J} U_i \cap fU_j$. Let W_i be the neighborhoods of U_i from the construction of α, then $W_r' = \bigcup_{j \in J} W_i \cap fW_j$ is an open neighborhood of V_r in Lemma 2.1.5 guarantees that the edges $(\mathcal{U}, \mathcal{V})$ in our tight path can be chosen so fine that our extended canonical coordinates are

defined. We use these coordinates to define $\alpha'(v_r, V_r)$ and $\beta'(v_r, V_r)$ in terms of $\alpha(x_i, U_i)$ and $\beta(x_i, U_i)$ that define α. Let

$$\alpha'(v_r, V_r) = \alpha(v_r, U_i) = [\alpha(x_i U_i), v_r],$$
$$\beta'(v_r, V_r) = \bigcup_{j \in J} f\beta(f^{-1}v_{ij}, U_j) = \bigcup_{j \in J} f[f^{-1}v_{ij}, \beta(x_i, U_j)].$$

The v_{ij} may be chosen arbitrarily in $W^s(v_r, V_r) \cap U_i \cap fU_j$. Also, note the union over J is a disjoint union. The previous lemma shows these two sets are neighborhoods of $W^i(v_r, V_r)$ and $W^s(v_r, V_r)$ respectively. The disjoint neighborhoods W'_r in M and the continuity of f show these sets are nice. For the filtration pair $(Y, B) = (fX, A)$, $fX - A \subset \text{int} \bigcup_r [\alpha'(v_r, V_r), \beta'(v_r, V_r)]$. It follows that $\alpha'(v_r, V_r) \subset fX$ and $\partial \alpha'(v_r, V_r) \subset A$, also $\beta'(v_r, V_r) \subset M - A$ and $\partial \beta'(v_r, V_r) \subset M - fX$. Therefore the sets $\alpha'(v_r, V_r)$ and $\beta'(v_r, V_r)$ represent homology class $[\alpha'(v_r, V_r)]$ and $\beta'(v_r, V_r)]$ in $H_k(fX, A)$ and $H_{n-k}(M - A, M - fX)$ respectively. If $[\beta'(v_r, V_r)]^*$ is the dual of $[\beta'(v_r, V_r)]$ in $H^k(X, A)$ under the isomorphism $H_{n-k}(M - A, M - fX) \cong H^k(fX, A)$ and if $\langle \ , \ \rangle$ is the pairing of k-dimensional cohomology classes with k-dimensional homology classes, then by reorienting $\beta_r(v_r, V_r)$ if necessary we can assume $\langle [\beta'(v_r, V_r)]^*, [\alpha'(v_r, V_r)] \rangle = \delta_{rs}$, the Kronecker delta.

Now define $\alpha' : \mathbb{Z}\mathcal{V} \to H_k(Y, B)$ on generators $V_r \in \mathcal{V}$ by $\alpha'(V_r) = [\alpha'(v_r, V_r)]$ and extend linearly.

As in the proof of Theorem 2.1.3, α' induces an isomorphism $T\alpha' : T(\mathbb{Z}\mathcal{V}, B(\mathcal{V})) \to T(H_k(fX, A), f_*)$.

If $V_r \in \mathcal{V}$, $R(V_r) = U_i$ and $\alpha(U_i) = [\alpha(x_i, U_i)]$. On the other hand $\alpha'(V_r) = [\alpha'(v_r, V_r)] = [\alpha(x_i, U_i), v_r]$. Proposition 2.1.2 now shows the diagram commutes after T is applied.

Case 2. $\mathcal{U} \xrightarrow{+} \mathcal{V}$. For $V_r \in \mathcal{V}$ the hypothesis gives $V_r = \bigcup_{i \in I} U_i$ for some set of indices I. Let W_i be the neighborhoods of U_i from the definition of α, and define $W'_r = \bigcup_{i \in I} W_i$. The W'_r are pairwise disjoint neighborhoods of the sets V_r. Let v_r be a periodic point in V_r and let $i \in I$ be defined by $v_r \in \mathcal{U}_i$.

$$\alpha'(v_r, V_r) = \alpha(v_r, U_i), \quad \beta'(v_r, V_r) = \bigcup_{i \in I} \beta(y_i, U_i).$$

The y_i may be chosen arbitrarily from $U_i \cap W^s(v_r, V_r)$ and the union is disjoint. Lemma 2.1.9 shows these sets are nice neighborhoods of $W^i(v_r, V_r)$ and $W^s(v_r, V_r)$ respectively. Let $(Y, B) = (X, A)$, then

$$X - A \subset \text{int} \bigcup_r [\alpha'(v_r, V_r), \beta'(v_r, V_r)] = \text{int} \bigcup_i [\alpha(v_r, U_i), \beta(y_i, U_i)]$$
$$= \text{int} \bigcup_i [\alpha(x_i, U_i), \beta(x_i, U_i)].$$

It follows $\partial \alpha'(v_r, V_r) \subset A$ and $\partial \beta'(v_r, V_r) \in M - X$. Define $\alpha' : \mathbb{Z}\mathcal{V} \to H_k(X, A)$ by $\alpha'(V_r) = [\alpha(v_r, V_r)]$, the construction results in an isomorphism $T\alpha'$.

For $U_i \in \mathcal{U}$, recall $R(U_i) = V_r$ where r is defined by $U_i \subset V_r$. $\alpha'(R(U_i)) = \alpha'(V_r) = [\alpha(v_r, V_r)]$ and $\alpha(U_i) = [\alpha(x_i, U_i)]$. Proposition 2.1.2 shows the following

diagram commutes after T is applied

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xrightarrow{\ R\ } & \mathbb{Z}\mathcal{V} \\
\alpha\downarrow & & \downarrow\alpha' \\
H_i(X,A) & \xrightarrow{\ =\ } & H_k(X,A).
\end{array}
$$

Case 3. $\mathcal{U} \xrightarrow{\ } \mathcal{V}$. For $V_r \in \mathcal{V}$, then $V_r \subset U_i$, for some i and $V_r = \bigcup_{j\in J} U_i \cap f^{-1}U_j$ for some set of indices J. Let $W_r' = \bigcup_{j\in J} W_i \cap f^{-1}W_j$ be the disjoint open neighborhoods of V_r in M. Choose $v_r \in V_r$ to be periodic and define

$$
\alpha'(v_r, V_r) = \bigcup_{j\in J} f^{-1}\alpha(fx_{ij}, U_j),
$$
$$
\beta'(v_r, V_r) = \beta(v_r, U_i).
$$

The $x_{ij} \in W^u(v_r, U_i \cap f^{-1}U_j)$ are arbitrary. Lemma 2.1.9, Case 3 shows these sets are nice neighborhoods of $W^u(v_r, V_r)$ and $W^s(v_r, V_r)$ respectively. The neighborhoods W_r' show the union defining $\alpha'(v_r, V_r)$ is a disjoint union. Let (Y, B) be the filtration pair $(X, f^{-1}A)$. By the construction

$$
X - f^{-1}A \subset \mathrm{int}\bigcup_r [\alpha'(v_r, V_r), \beta'(v_r, B_r)]
$$

and $\partial\alpha'(v_r, V_r) \subset f^{-1}A$. Define $\alpha': \mathbb{Z}\mathcal{V} \to H_k(X, f^{-1}A)$ by $\alpha'(V_r) = [\alpha'(v_r, V_r)]$. $T\alpha'$ is an isomorphism and if we let $i : (X, A) \subset (X, f^{-1}A)$ and P be the linear map defined in Proposition 1.3.2 we have a diagram

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \xrightarrow{\ P\ } & \mathbb{Z}\mathcal{V} \\
\alpha\downarrow & & \downarrow\alpha' \\
H_k(X,A) & \xrightarrow[i_*]{} & H_k(X, f^{-1}A).
\end{array}
$$

For $u_i \in \mathcal{U}$, let I be the set of indices such that $r \in I$ iff $V_r \subset U_i$, then $P(U_i) = \sum_{r\in I} V_r$.

$$
i_*\alpha(U_i) = [\alpha(x_i, U_i)],
$$
$$
\alpha'P(U_i) = \alpha'\left(\sum_r V_r\right) = \sum_r \alpha'(V_r) = \sum_r [\alpha'(v_r, V_r)].
$$

By Proposition 2.1.2 $\sum_r [\alpha'(v_r, V_r)] \equiv \sum_r [\alpha'(v_r', V_r)]$ for any choice of $v_r' \in V_r \cap W^u(x_i, U_i)$ ($\equiv$ means congruent modulo the image of $\langle 1 \otimes I - t \otimes f_* \rangle$)

$$
\sum_r [\alpha'(v_r', V_r)] = \sum \left[\bigcup_{j\in J} f^{-1}\alpha(fv_{ij}^r, U_j)\right]
$$

for any $v_{ij}^r \in W^u(v_r', V_r) \cap U_i \cap f^{-1}U_j$.

Let J_r be the set of indices such that $V_r = \bigcup_{j\in J_r}(U_i \cap f^{-1}U_j)$. Since $W^u(v_r, V_r)$ $= W^u(v_r', V_r) = W^u(x_i, U_i) \cap V_r$, the sets $\alpha(x_i, U_i)$ and $\bigcup_{r\in I}\bigcup_{j\in J_r} f^{-1}\alpha(fv_{ij}^r, U_j)$ are both nice neighborhoods of $W^u(x_i, U_i)$ in $W^u(x_i)$. Both neighborhoods

also have boundary in $f^{-1}A$. Therefore the homology classes they represent in $H_k(X, f^{-1}A)$ are equal.

$$[\alpha(x_i, U_i)] = \left[\bigcup_r \bigcup_{j \in J_r} f^{-1}\alpha(fv_{ij}^r, U_j) \right].$$

Since $W_r' \supset \bigcup_{j \in J_r} f^{-1}\alpha(fv_{ij}^r, U_j)$ and $\{W_r'\}$ is a disjoint family

$$\left[\bigcup_r \bigcup_{j \in J_r} f^{-1}\alpha(fv_{ij}^r, U_j) \right] = \sum_r \left[\bigcup_{j \in J_r} f^{-1}\alpha(fv_{ij}^r, U_j) \right] = \sum_r [\alpha'(v_r', V_r)].$$

This shows the diagram above commutes after T is applied.

Case 4. $\mathcal{U} \overset{-}{\longleftarrow} \mathcal{V}$. Fix $V_r \in \mathcal{V}$ and choose $v_r \in V_r$. We know $V_r = \bigcup_{i \in I} U_i$ for some set of indices I. Let $W_r' = \bigcup_{i \in I} W_i$ and let $\alpha'(v_r, V_r) = \bigcup_{i \in I} \alpha(x_i', U_i)$ for $x_i' \in W^u(v_r, V_r) \cap U_i$. Also let $\beta'(v_r, V_r) = \beta(v_r, U_{i_0})$ where i_0 is defined by $v_r \in U_{i_0} \subset V_r$. Let $(Y, B) = (X, A)$ and $\alpha': \mathbb{Z}\mathcal{V} \to H_k(X, A)$ by $\alpha(V_r) = [\alpha'(v_r, V_r)]$., $T\alpha'$ is an isomorphism. $P(V_r) = \sum_{i \in I} U_i$ and $\alpha'(V_r) = [\alpha'(v_r, V_r)] = [\bigcup_{i \in I} \alpha(x_i', U_i)]$. $[\bigcup_{i \in I} \alpha(x_i', U_i)] = \sum_{i \in I} [\alpha(x_i', U_i)]$ since $\alpha(x_i', U_i) \subset W_i$ and the W_i are disjoint. Each $[\alpha(x_i', U_i)] \equiv [\alpha(x_i, U_i)]$ for any other $x_i \in U_i$, thus

$$\alpha'(V_r) \equiv \sum_{i \in I} [\alpha(x_i, U_i)] = \alpha\left(\sum_{i \in I} U_i \right) = \alpha(P(V_r)).$$

This shows the diagram below commutes when T is applied

$$
\begin{array}{ccc}
\mathbb{Z}\mathcal{U} & \overset{P}{\longleftarrow} & \mathbb{Z}\mathcal{V} \\
\alpha \downarrow & & \downarrow \alpha' \\
H_k(X, A) & \overset{=}{\longrightarrow} & H_k(X, A). \quad \square
\end{array}
$$

This completes the proof of Proposition 2.1.7. At this point we have constructed a ladder of maps

$$
\begin{array}{ccccccccc}
\mathbb{Z}\mathcal{U} & \overset{G(\mathcal{U})}{\longrightarrow} & \mathbb{Z}_g\mathcal{U} & = & \mathbb{Z}\mathcal{U}_0 & -\cdots- & \mathbb{Z}\mathcal{U}_n \\
\alpha \downarrow & & \alpha' \downarrow & & \alpha_0 \downarrow & & \alpha_n \downarrow \\
H_k(X, A) & \underset{g_*}{\longrightarrow} & H_k(Y, B) & = & H_k(X_0, A_0) & -\cdots- & H_k(X_n, A_n)
\end{array}
$$

The horizontal maps from $\mathbb{Z}\mathcal{U}_i$ to $\mathbb{Z}\mathcal{U}_{i+1}$ are either R's or P's and the maps between the homology groups of the filtration pairs (X_i, A_i) are induced by inclusions. When T is applied all the maps induce isomorphisms and the diagram commutes. The composition of the isomorphism across the top row is $T(g, k)$. Unfortunately $(X_n, A_n) \neq (X, A)$ and $\alpha_n \neq \alpha$. The following proposition shows the ladder can be extended so that the additional isomorphisms on the top row are the identity and the added isomorphisms on the bottom row are induced by inclusions. Finally the last vertical map will be α.

From the constructions it follows $(Y, B) = g(X, A) = (X_0, A_0)$ and $(X_n, A_n) = (f^N X_0, f^{-M} A_0)$ for some large positive integers N and M.

LEMMA 2.1.10. *Let (X, A) and (X', A') be two filtration pairs of the same basic set. Define $(X^*, A^*) = (X \cup X, A \cup A')$. Let i and i' be the inclusions*

$$(X, A) \xrightarrow{i} (X^*, A^*) \xleftarrow{i'} (X', A').$$

Then i and i' induce isomorphisms after T is applied.

PROOF. The proof of this lemma can be found in [**BF**, p. 86].

PROPOSITION 2.1.11. *Let $(Z, C) = (X_n \cup X, A_n \cup A)$. Let $\alpha_n \colon \mathbb{Z}\mathcal{U} \to H_k(X_n, A_n)$ and $\alpha \colon \mathbb{Z}\mathcal{U} \to H_k(X, A)$.*

These maps fit into the following diagram

$$
\begin{array}{ccccc}
\mathbb{Z}\mathcal{U} & & \overset{=}{-\,-\,-\,-} & & \mathbb{Z}\mathcal{U} \\
\alpha_n \downarrow & & & & \downarrow \alpha \\
H_k(X_n, A_n) & \underset{i_{1_*}}{\longrightarrow} & H_k(Z, C) & \underset{i_{2_*}}{\longleftarrow} & H_k(X, Z)
\end{array}
$$

and this diagram commutes after T is applied.

PROOF. $i_{1_*}\alpha_n(U_i) = [\alpha_n(x_i, U_i)]$ and $i_{2_*}\alpha(U_i) = [\alpha(x_i, U_i)]$ in $H_k(Z, C)$. From Proposition 2.1.2, we know the images of these classes in $T(H_k(Z, C), f_*)$ are independent of x_i and x'_n. So we may assume $x_i = x'_i$. Then $\alpha(x_i, U_i)$ and $\alpha_n(x_i, U_i)$ are both neighborhoods of $W^u(x_i, U_i)$ contained in Z with boundary in C and it follows they represent the same homology class in $H_k(Z, C)$. $\square$

To sum up,

THEOREM 2.1.12. *In the constructed ladder*

$$
\begin{array}{ccccc}
\mathbb{Z}\mathcal{U} & \overset{G(u)}{\longrightarrow} & \mathbb{Z}_g\mathcal{U} & = & \mathbb{Z}\mathcal{U}_0 & -\cdots \\
\alpha \downarrow & & \alpha' \downarrow & & \alpha_0 \downarrow \\
H_k(X, A) & \underset{g}{\rule{1.2cm}{0.4pt}} & H_k(Y, B) & = & H_k(X_0, A_0) & -\cdots
\end{array}
$$

$$
\begin{array}{ccccc}
\cdots - & \mathbb{Z}\mathcal{U}_n & = & \mathbb{Z}\mathcal{U} & = & \mathbb{Z}\mathcal{U} \\
& \alpha_n \downarrow & & & & \downarrow \alpha \\
\cdots - & H_k(X_n, A_n) & \underset{i_{1_*}}{\longrightarrow} & H_k(Z, C) & \underset{i_{2_*}}{\longleftarrow} & H_k(X, A).
\end{array}
$$

The top row induces $T(g, k)$ when T is applied and the end vertical maps are α. $T\alpha$ gives us a conjugacy from $T(g, k)$ to a map on homology induced by g and some inclusions.

If there is more than one basic set of index k, we will have a collection of maps $\alpha_i \colon \mathbb{Z}\mathcal{U}_i \to H_k(X_i, A_i)$ where $\mathcal{U}_i$ are Markov partitions of the basic sets Ω_i^k and (X_i, A_i) are filtration pairs for Ω_i^k. The above ladder will have α_i on the left end and $\alpha_{g(i)}$ on the right. Taking direct sums over i and rearranging the indices on the right will produce a conjugacy $\bigoplus_i T\alpha_i$ of $T(g, k)$ with an isomorphism induced by maps of homology groups.

2.2. Putting the information together.

This section's aim is to translate the information from §2.1 obtained from the relative homology groups of filtration pairs on the bottom row of the ladder into information that is obtainable from the absolute homology groups of the manifold M.

We recall the filtration of M from the beginning of 2.1 composed of filtration pairs (M_i^k, M_{i-1}^k) for the individual basic sets

$$\emptyset \subset M_1^0 \subset M_2^0 \subset \cdots \subset M_{j(0)}^0 = M^0 M_1^1 \subset M_2^1 \subset \cdots \subset M^1 \subset \cdots \subset M^n = M.$$

Let (X_i, A_i) be the filtration pairs from the construction of the α. Proposition 2.1.10 shows that if (X_i, A_i) and (M_j^k, M_{j-1}^k) are filtration pairs of the same basic set then

$$T(H_k(X_i, A_i), f_*) \cong T(H_k(M_j^k, M_{j-1}^k), f_*)$$

and this isomorphism is induced by two inclusions. Taking direct sums we have isomorphisms

$$\bigoplus_i T(H_k(X_i, A_i), f_*) \cong \bigoplus_{j=1}^{j(k)} T(H_k(M_j^k, M_{j-1}^k), f_*).$$

The pair difference $M^k - M^{k-1}$ contains all the basic sets of index k; we regard (M^k, M^{k-1}) as a filtration pair for the portion of the nonwandering set of index k.

To this point the coefficient group of the homology functor has been the integers. Now we must reduce to $\mathbb{Z}/2$ coefficients. This makes no difference by virtue of the following lemma.

LEMMA 2.2.1. *If the torsion functor T is defined with field $F = \mathbb{Z}/2$, then for any filtration pair (X, A) of a basic set of index k*

$$T(H_i(X, A; \mathbb{Z}), f_*) \cong T(H_i(X, A; \mathbb{Z}/2), f_*) \quad \text{for all } i.$$

PROOF. From [**B2** and **BF**] we know f_* on $H_i(X, A; G)$ is nilpotent for $i \neq k$ and if $\text{Tor}(H_k(X, A; \mathbb{Z}))$ is torsion subgroup of $H_k(X, A; \mathbb{Z})$ then $f_*|\text{Tor}(H_k(X, A; \mathbb{Z}))$ is nilpotent. It follows $T(H_i(X, A; \mathbb{Z}), f_*)$ and $T(\text{Tor}(H_k(X, A; \mathbb{Z})), f_*)$ are the trivial module. Consider now the exact universal coefficient sequence

$$0 \to H_k(X, A) \otimes \mathbb{Z}/2 \to H_k(X, A; \mathbb{Z}/2) \to H_{k-1}(X, A) * \mathbb{Z}/2 \to 0$$

and the fact that $H_{k-1}(X, A) * \mathbb{Z}/2 \cong \text{Tor}(H_{k-1}(X, A)) * \mathbb{Z}/2$. Since T is an exact functor we obtain

$$0 \to T(H_k(X, A) \otimes \mathbb{Z}/2, f_*) \to T(H_k(X, A; \mathbb{Z}/2), f_*) \to 0 \to 0.$$

The result

$$T(H_k(X, A) \otimes \mathbb{Z}/2, f_*) \cong T(H_k(X, A), f_*)$$

follows using the identification $\mathbb{Z}/2 \otimes \mathbb{Z}/2[t, t^{-1}] \cong \mathbb{Z}/2[t, t^{-1}]$. $\square$

We observe that the modules in the statement of the proposition are zero unless $i = k$.

PROPOSITION 2.2.2. *From the portion of the filtration of M*

$$M^{k-1} = M_0^k \subset M_1^k \subset \cdots \subset M_j^k = M^k$$

we have

$$T(H_k(M^k, M^{k-1}; \mathbb{Z}/2), f_*) \cong \bigoplus_{i=1}^{j} T(H_k(M_i^k, M_{i-1}^k; \mathbb{Z}/2), f_*).$$

PROOF. The proof is by induction on i. For the duration of the proof let $M_i^k = M_i$. All coefficients are $\mathbb{Z}/2$. Consider the long exact Mayer-Vietoris sequence of the triple (M_2, M_1, M_0).

(2.2.3)
$$H_{k+1}(M_2, M_1) \xrightarrow{\partial_*} H_k(M_1, M_0) \rightarrow H_k(M_2, M_0)$$
$$\xrightarrow{j_*} H_k(M_2, M_1) \xrightarrow{\partial_*} H_{k-1}(M_1, M_0) \rightarrow \cdots .$$

Since (M_2, M_1) and (M_1, M_0) are filtration pairs for basic sets of index k, f_* on $H_{k+1}(M_2, M_1)$ and $H_{k-1}(M_1, M_0)$ are nilpotent. Since T is an exact functor the above long exact sequence gives the following short exact sequence

$$0 \rightarrow TH_k(M_1, M_0) \rightarrow TH_k(M_2, M_0) \rightarrow TH_k(M_2, M_1) \rightarrow 0.$$

(We have dropped f_* from each term.) This sequence is in fact split exact. If $\{z_i\} \subset H_k(M_2, M_1)$ is a basis over $\mathbb{Z}/2$ there is an integer N so that $f_*^N \partial_* z_i = 0$ for all i. From the exactness of (2.2.3) there are $y_i \in H_k(M_2, M_1)$ so that $j_* y_i = f_*^N z_i$. Define $s \colon H_k(M_2, M_1) \rightarrow H_k(M_2, M_0)$ by $s(z_i) = y_i$ and extend linearly. It follows $j_* s = f_*^N$ on $H_k(M_2, M_1)$. Apply T, $Tj_* \circ Ts = Tf_*^N \equiv \cdot t^{-N}$. Let $S = t^N \cdot Ts$. Then S splits the short exact sequence. It follows now that $TH_k(M_2, M_0) \cong TH_k(M_2, M_1) \oplus TH_k(M_1, M_0)$. To continue the induction it is necessary to know f_* is nilpotent on $H_{k-1}(M_2, M_0)$. This also follows from the Mayer-Vietoris sequence

$$\rightarrow H_{k-1}(M_1, M_0) \rightarrow H_{k-1}(M_2, M_0) \rightarrow H_{k-1}(M_2, M_1) \rightarrow$$

since f_* is nilpotent on each end. Applying the Mayer-Vietoris sequence to the triple (M_3, M_2, M_0) continues the induction. $\square$

The sets M_i^k can be taken to be closed submanifolds with boundary of the same dimension as M. We wish to consider maps of the filtrations of M, say M^k, $k = 0, 1, \ldots, n$, where M^k contains the basic sets of index $\leq k$ and such that $M^k \subset \bigcup_{i \leq k} W^s(\Omega_i)$. $W^s(\Omega_i)$ is the stable manifold of the basic sets Ω_i of index i, and is an open submanifold of M. The M^k do not form a dissection of M, that is $H_i(M^k, M^{k-1}) \neq 0$ for $i \neq k$ (however, the previous proposition shows $TH_i(H^k, M^{k-1}) = 0$ for $i \neq k$). Thus a spectral sequence argument is applied to relate the groups $H_i(M^k, M^{k-1})$ to $H_*(M)$. We first observe how the spectral sequence behaves with respect to maps between different filtrations for M.

Let $\{M^k\}$ be a filtration of M such that M^k contains all the basic sets of index less than or equal to k. If $g \in \mathrm{Aut}(f)$, then $N^k = gM^k$ is another filtration of M with the same property.

LEMMA 2.2.4. *The homeomorphism $g \colon M \rightarrow M$ induces a map $g \colon M^k \rightarrow N^k$. In the spectral sequences associated to the two filtrations, this map induces $g_* \colon E_{s,t}^1 \rightarrow E_{s,t}^1$ which can be identified with*

$$g_* \colon H_{s+t}(M^s, M^{s-1}) \rightarrow H_{s+t}(N^s, N^{s-1})$$

and the resulting morphism $g_* \colon E_{s,t}^\infty \rightarrow E_{s,t}^\infty$ induces $g_* \colon H_n(M) \rightarrow H_n(M)$ from the sequences
$$0 \rightarrow F_{s-1} H_n(M) \rightarrow F_s H_n(M) \rightarrow E_{s,n-s}^\infty \rightarrow 0,$$
$$F_s H_n(M) = \mathrm{image}\, H_n(M^s) \rightarrow H_n(M).$$

PROOF. Follows from the naturality of the spectral sequence of a filtration [**Sp**].
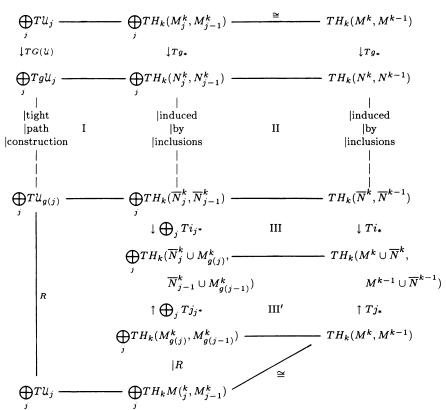
LEMMA 2.2.5. *If M^k and $\overline{M}^k$ are two filtrations of M and $i : M^k \subset \overline{M}^k$, then the induced maps on the spectral sequences are*

(1) *at the E^1 terms, $i_* : H_{s+t}(M^s, M^{s-1}) \to H_{s+t}(\overline{M}^s, \overline{M}^{s-1})$,*

(2) *at the E^∞ terms result in the identity map on $H_*(M)$.*

Using the results from 2.1 and Proposition 2.2.2 we can show the dynamical invariant ϕ can be calculated from the E^1 terms of spectral sequences. The conjugacies $T\alpha$ constructed in part one show $\det(T(g,k)) = \det(I_* g_{*k})$ where $g_{*k} : H_k(M^k, M^{k-1}) \to H_k(N^k, N^{k-1})$ and I_* is a composition of isomorphisms induced by inclusions between the many filtrations of M produced by the tight path.

We assemble this into a diagram and then prove the necessary commutativity.

$$
\begin{array}{ccccc}
\bigoplus_j TU_j & \longrightarrow & \bigoplus_j TH_k(M_j^k, M_{j-1}^k) & \xrightarrow{\cong} & TH_k(M^k, M^{k-1}) \\
\downarrow TG(u) & & \downarrow Tg_* & & \downarrow Tg_* \\
\bigoplus_j Tgu_j & \longrightarrow & \bigoplus_j TH_k(N_j^k, N_{j-1}^k) & \longrightarrow & TH_k(N^k, N^{k-1})
\end{array}
$$

$$
\begin{array}{ccc}
| \text{tight} & | \text{induced} & | \text{induced} \\
| \text{path} \quad\quad \mathrm{I} & | \text{by} \quad\quad \mathrm{II} & | \text{by} \\
| \text{construction} & | \text{inclusions} & | \text{inclusions}
\end{array}
$$

$$
\bigoplus_j TU_{g(j)} \longrightarrow \bigoplus_j TH_k(\overline{N}_j^k, \overline{N}_{j-1}^k) \longrightarrow TH_k(\overline{N}^k, \overline{N}^{k-1})
$$

$$
\downarrow \bigoplus_j Ti_{j*} \quad\quad \mathrm{III} \quad\quad \downarrow Ti_*
$$

$$
\bigoplus_j TH_k(\overline{N}_j^k \cup M_{g(j)}^k, \quad\quad\quad\quad TH_k(M^k \cup \overline{N}^k,
$$
$$
\overline{N}_{j-1}^k \cup M_{g(j-1)}^k) \quad\quad\quad\quad M^{k-1} \cup \overline{N}^{k-1})
$$

$$
\uparrow \bigoplus_j Tj_{j*} \quad\quad \mathrm{III}' \quad\quad \uparrow Tj_*
$$

$$
\bigoplus_j TH_k(M_{g(j)}^k, M_{g(j-1)}^k) \longrightarrow TH_k(M^k, M^{k-1})
$$

$$
| R \quad\quad\quad\quad\quad \cong
$$

$$
\bigoplus_j TU_j \longrightarrow \bigoplus_j TH_k M(_j^k, M_{j-1}^k)
$$

(with R on the left margin connecting $\bigoplus_j TU_{g(j)}$ to $\bigoplus_j TU_j$)

Commutativity in square I follows from 2.1.7. Commutativity in square II follows from Proposition 2.2.2 and the fact that every map is induced by an inclusion. It may be necessary to use Lemma 2.1.7 to fill out filtration pairs for basic sets into a filtration of M. Commutativity of square III is checked by demonstrating it on every summand $H_k(\overline{N}_j^k, \overline{N}_{j-1}^k)$ using the fact that the horizontal isomorphism of $\bigoplus_j(TH_k(\overline{N}_j^k, \overline{N}_{j-1}^k)$ with $TH_k(\overline{N}^k, \overline{N}^{k-1})$ can be reduced to some inclusion map on the individual $H_k(\overline{N}_j^k, \overline{N}_{j-1}^k)$. This argument would also show square III′ commutes. It remains to show Ti_* and Tj_* are isomorphisms on squares III and III′.

LEMMA 2.2.6. *Let $M^k, \overline{M}^k$ be closed subsets of M containing all the basic sets of index $\leq k$, and such that each is contained in $\bigcup_{i\leq k} W^s(\Omega_i)$. Then inclusions induce an isomorphism of $TH_k(M^k, M^{k-1})$ with $TH_k(\overline{M}^k, \overline{M}^{k-1})$.*

PROOF. Observe $M^k \cup \overline{M}^k$ is also a filtration of M with the properties mentioned in the statement of the lemma. So we can assume $i\colon M^k \subset \overline{M}^k$. Since M^k is a neighborhood of $\bigcup_{i\leq k}\Omega_i$ and $\overline{M}^k \subset \bigcup_{i\leq k} W^s(\Omega_i)$ is closed and hence compact, there is a positive integer N so that $f^N \overline{M}^k \subset M^k$ for each k. The commutative diagram shows Ti_* is an isomorphism.

$$
\begin{array}{ccc}
(M^k, M^{k-1}) & \xrightarrow{f^N} & (M^k, M^{k-1}) \\
{\scriptstyle i}\downarrow & {}^{f^N}\searrow & \downarrow{\scriptstyle i} \\
(\overline{M}^k, \overline{M}^{k-1}) & \xrightarrow[f^N]{} & (\overline{M}^k, \overline{M}^{k-1})
\end{array}
\qquad \square
$$

The composition of the vertical isomorphisms in the left-hand column yields $T(g,k)$; the composition of the maps in the right-hand column yields an automorphism $TH_k(M^k, M^{k-1})$. The horizontal isomorphisms show the determinant construction applied to both gives the same element of $\mathrm{Wh}_2(\mathbb{Z}/2(t))$. This right-hand column of isomorphisms gives an automorphism of TE^1 terms in the spectral sequence associated to the filtration $\{M^k\}$. (Recall $TH_i(M^k, M^{k-1}) = 0$ if $i \neq k$.) Furthermore, we can calculate $\phi(g)$ from the TE^1 terms.

Given an automorphism τ of a first quadrant E^1 spectral sequence consisting of finitely generated torsion $F[t,t^{-1}]$ modules we can form the following element of $\mathrm{Wh}_2(F(t))$

$$
\chi(\tau) = \prod_{k=0} (-1)^k \det(\tau\colon E_k^1 \to E_k^1), \qquad E_k^1 = \bigoplus_{i+j=k} E_{i,j}^1.
$$

PROPOSITION 2.2.7. *Let $E = \{E_i\}$ be a chain complex of finitely generated torsion $F[t,t^{-1}]$ modules and let $\tau = \{\tau_i\}$ be a chain map of automorphisms. τ induces the automorphism τ_* on the torsion modules $H_*(E)$. If*

$$
\chi(\theta) = \prod_i (-1)^i \det(\theta),
$$

then $\chi(\tau) = \chi(\tau_*)$.

PROOF. This follows from abstract nonsense [L] once we demonstrate the following property of $\det(\cdot)$.

LEMMA 2.2.8. *Let $0 \to B \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence of torsion $F[t,t^{-1}]$ modules. Let $\alpha, \beta,$ and γ be automorphisms so that $i\alpha = \beta i$ and $\pi\beta = \gamma\pi$. Then $\det(\beta) = \det(\alpha)\det(\gamma)$.*

PROOF. Let $\wp$ be a prime ideal in $F[t,t^{-1}]$. It suffices to verify the formula on $\wp$-primary torsion modules. Let $B^i = \wp^i B$, $C^i = \wp^i C$, and $A^i = i^{-1}B^i$. Then $0 \to A^i \xrightarrow{i} B^i \xrightarrow{\pi} C^i \to 0$ is exact. It follows from the snake lemma in homological algebra that $0 \to A^i/A^{i+1} \to B^i/B^{i+1} \to C^i/C^{i+1} \to 0$ is also exact. Let $\alpha_i, \beta_i,$

and γ_i be the morphisms induced by α, β, and γ on these quotient modules isomorphic to vector spaces over $F_\wp = F[t, t^{-1}]/\wp$. Then $\det(\beta_i) = \det(\alpha_i)\det(\gamma_i)$, for all elements of $F_\wp^*$. Taking products over i we obtain

$$\det_\wp(\beta) = \left(\prod_i \det(\alpha_i)\right) \cdot \det_\wp \gamma.$$

It remains to show $\prod_i \det(\alpha_i) = \det_\wp(\alpha)$ since the filtration used to define α_i is not the canonical filtration of A used to define det.

If $A = A_0 \supset A_1 \supset \cdots \supset A_{n+1} = 0$ is a filtration of A so that A_i/A_{i+1} is isomorphic to a vector space over $F_\wp$ and $A' \subset A$ is such that $A_i \supset A' \supset A_{i+1}$ and A_i/A and A'/A_{i+1} are also vector spaces over $F_\wp$, then $0 \to A'/A_{i+1} \to A_i/A_{i+1} \to A_i/A' \to 0$ shows $\det(\alpha_i) = \det(\alpha|A'/A_{i+1})\det(\alpha|A_i/A')$. It follows from the butterfly lemma and Schur's theorem for modules that $\bigoplus_i A_i/A_{i+1}$ is independent of the filtration up to isomorphism and consequently $\det(\alpha)$ is independent of the filtration of A. □

Since T commutes with homology, we can use 2.2.7 to compute $\phi(g)$ on the TE^2 terms of the spectral sequence. Since $TH_i(M^k, M^{k-1}) = 0$ if $i \neq k$, it follows $TE_{i,j}^2 = 0$ unless $i = j$ and that $TE_{i,j}^2 \cong TE_{i,j}^\infty$. Thus

$$T(H_n(M; \mathbb{Z}_2), f_*) \cong T(E_{n,0}^\infty, f_*) \cong T(E_{n,0}^2, f_*).$$

Since the automorphism of $TE_{i,j}^1$ was induced by g and inclusions, it follows from 2.2.4 and 2.2.5 that the automorphism of $T(H_n(M; \mathbb{Z}/2), f_*)$ is induced by Tg_* and identity maps. (Although the direction of the inclusions varies, the inverse of the induced identity map is still the identity.) But Tg_* defines $\Phi(g)$. Thus $\phi(g) = \Phi(g)$.

We conclude with a few remarks.

Since f_* is an isomorphism on the groups $H_k(M; \mathbb{Z}/2)$, the modules

$$T(H_k(M, \mathbb{Z}/2), f_*)$$

are all isomorphic to $H_k(M; \mathbb{Z}/2)$ as $\mathbb{Z}/2[t, t^{-1}]$ modules with $tu = f_*^{-1}u$.

Work is continuing at the present time to facilitate the computations. If $-$ denotes the involution of $\mathrm{Wh}_2(F(t))$ induced by $t \mapsto t^{-1}$, then we suspect Poincaré duality in the n-dimension manifold M will produce a formula for $\Phi(g)$ such as $\overline{\Phi(g)} = (-1)^{\dim M}\Phi(g)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON GN-50, SEATTLE, WASHINGTON 98195