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CAUCHY PROBLEM OF HYPERBOLIC CONSERVATION LAWS IN MULTIDIMENSIONAL SPACE WITH INTERSECTING JUMP INITIAL DATA

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ABSTRACT. Cauchy problem of hyperbolic conservation laws in multidimensional space is considered, where the initial data have several jump discontinuity surfaces which develop into shock fronts intersecting at a common submanifold. Local existence is proved, assuming compatible conditions and uniform stability. For isentropic flow in 2-dimensional space, the interaction of two shock fronts and the nonexistence of three intersecting shock fronts are discussed.

Introduction. In [**1**], the Cauchy problem was discussed for hyperbolic conservation laws in multidimensional space with initial data which have jump discontinuity on a smooth hypersurface and this initial jump develops into two or more shock fronts in $t > 0$. In discussing the interaction of two shock fronts, because of the fact that at fixed time $t = t_0$, part of the bumping shock fronts remains unchanged while the other part has already produced new shock fronts, it is necessary to consider the Cauchy problem with initial data which have jump discontinuity on more than one hypersurfaces that intersect with each other.

In [**7**], Metivier considered the interaction of two shock fronts for 2 conservation laws in 2-dimensional space. He reduced the problem of interaction to the problem of double shock fronts emanating from one discontinuity surface. For the stability analysis, we can follow the similar approach to discuss the stability of interaction of shock fronts for m conservation laws in n-dimensional space. But for the problem of existence, the situation in the general case becomes different from the one discussed in [**7**]. Certain additional conditions are necessary to get the existence result.

In this paper, the general m conservation laws in n-dimensional space will be discussed, not only for the problem of interaction of shock fronts, but also for the problem of other features. In particular, we discuss the isentropic hydrodynamic equations in 2-dimensional space and get the interesting result of the interaction of two shock fronts and that it is always impossible to have three stable shock fronts intersecting at one common curve.

1. Problem and result. As in [**1, 4, 5**], we discuss the following hyperbolic conservation laws

$$D_t(F_0(u)) + \sum_1^n D_{x_j}(F_j(u)) = 0 \tag{1.1}$$

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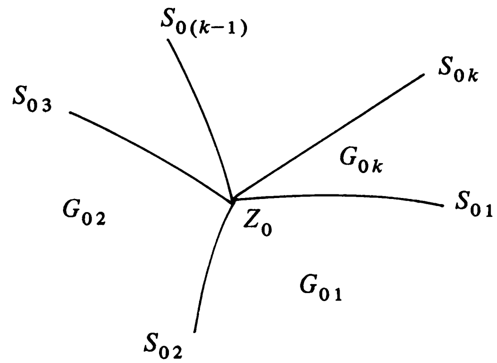

FIGURE 1

which can be written as a quasilinear symmetric hyperbolic system

$$D_t u + \sum_1^n A_j(u) D_{x_j} u + B(u)u = F(x,t). \tag{1.2}$$

Besides, we always assume that the linearization of (1.2) satisfies the block structural condition in [**4**].

We consider the Cauchy problem of (1.1) with the initial data

$$u(x,0) = u_0(x). \tag{1.3}$$

Here $u_0(x)$ is a piecewise smooth function having discontinuities of the first kind on k smooth hypersurfaces S_{0i} $(i = 1,\dots,k)$, which intersect transversally at a submanifold Z_0 of dimension $(n-2)$; see Figure 1.

Denote the domain between S_{0i} and $S_{0(i+1)}$ by G_{0i} $(i = 1,\dots,k)$, where $S_{0(k+1)} = S_{01}$. We will always assume in the following that every G_{0i} is diffeomorphic to a quarter space in R^n. Denote the value of $u_0(x)$ in G_{0i} by $u_{0i}(x)$. We assume $u_{0i}(x) \in C^\infty(\overline{G_{0i}})$ and $u_{0i}(x) \neq u_{0(i-1)}$ on S_{0i}, for $i = 2,\dots,k+1$.

In this paper, we are going to prove that, under certain assumptions on u_{0i} $(i = 1,\dots,k)$, there exists a $t_0 > 0$, such that for $0 < t < t_0$, the Cauchy problem (1.2), (1.3) has piecewise smooth solution $u(x,t)$ with discontinuity surfaces S_i $(i = 1,\dots,k)$ which intersect each other on a common submanifold Z of dimension $(n-1)$ in $R^n \times R^1_+$ and $S_i|_{t=0} = S_{0i}$, $Z|_{t=0} = Z_0$. And on each S_i, u has a shock wave discontinuity, satisfying Rankine-Hugoniot conditions

$$D_t p_i [F_0(u)]^i + \sum_1^n D_{x_j} p_i [F_j(u)]^i = 0, \quad \text{on } S_i,\ i = 1,\dots,k. \tag{1.4}$$

Here $[f]^i$ denotes the jump difference of f across S_i, and $S_i = \{(x,t);\ p_i(x,t) = 0\}$.

Denote the unit normal vector on S_{0i} by $\vec{n}_i = (n_{i1},\dots,n_{in})$, $i = 1,\dots,k$. We make the following assumptions:

(H1) *On every S_{0i}, there exists a scalar function n_{i0}, sufficiently smooth on S_{0i}, such that*

$$n_{i0}[F_0(u)]^i + \sum_1^n n_{ij}[F_j(u)]^i = 0, \quad \textit{on } S_{0i},\ i = 1,\dots,k. \tag{1.5}$$

When (H1) is satisfied, then by changing the sign of $(n_{i1}, \dots, n_{in})$ if necessary, we may always take $n_{i0} \geq 0$.

(H2) *On every S_{0i}, the vector $(n_{i0}, n_{i1}, \dots, n_{in})$ and the state $(u_{0i}, u_{0(i-1)})$ satisfy the uniform stability condition for shock front of Majda* [**4**].

(H3) *At Z_0, k vectors $(n_{i0}, n_{i1}, \dots, n_{in})$ $(i = 1, \dots, k)$ lie in a common hyperplane of dimension* 2.

The assumption (H3) is in fact equivalent to the hypothesis that the hyperplanes with normal vector $(n_{i0}, n_{i1}, \dots, n_{in})$ intersect at a common affine submanifold of dimension $(n-1)$, which will be denoted as Z_t in the following.

Obviously, the assumptions (H1)–(H3) are necessary for the existence of the intersecting stable shock fronts.

Another assumption we are going to make is connected with the concept of extreme shock. As in the case of one dimensional space, every shock front is associated to one genuinely nonlinear characteristic of the system (1.2), cf. e.g. J. Smoller [**8**]. We are here interested only in the extreme shock front which is associated to an extreme characteristic, i.e., the characteristic of the largest or smallest eigenvalue. In this situation, the direction $(n_{i0}, n_{i1}, \dots, n_{in})$ is space-like with respect to the value u_0 in one and only one side of S_{i0}. And consequently, we get $n_{i0} > 0$. The following is our fourth assumption

(H4) *There is a domain G_{0i}, such that $(n_{i0}, n_{i1}, \dots, n_{in})$ and $(n_{(i+1)0}, n_{(i+1)1}, \dots, n_{(i+1)n})$ are both space-like with respect to u_{i0}; and these two vectors are both pointing outward from G_{0i}.*

REMARK. The second statement in (H4) is equivalent to the requirement that the intersection Z_t of the hyperplanes with normals $(n_{i0}, n_{i1}, \dots, n_{in})$ and $(n_{(i+1)0}, n_{(i+1)1}, \dots, n_{(i+1)n})$ has its projection into the space of $x = (x_1, \dots, x_n)$ contained in the domain G_{0i}.

In particular, for Euler equations in gas dynamics, we could have at most two shock fronts developing from one discontinuity, and these two shock fronts are associated with the largest and the smallest eigenvalues respectively [**4**]. Since the uniformly stable shock front in gas dynamics with convex state function must be compressive, i.e., the pressure of the gas behind the shock front should be higher than the pressure ahead of the shock front, it is easy to see that there is always one and only one domain G_{i0} such that the pressures in the adjacent domains are higher than the one in G_{i0}. So for Euler equations of gas dynamics, hypothesis (H4) is automatically satisfied.

To fix the idea, we will always denote the domain in (H4) as G_{01}.

To have a piecewise sufficiently smooth solution with smooth discontinuity surface, it is necessary to have compatibility conditions which are derived from equation (1.1) and Rankine-Hugoniot conditions (1.4). Hence, we make the following assumption:

(H5) *On every S_{0i}, the compatibility conditions are satisfied up to sufficiently high order.*

REMARK. (H1) is nothing but the compatibility condition of order zero. The high order compatibility conditions are the requirements upon u_0 to guarantee the existence of compatible $D_t^h p_i$, $i = 1, \dots, k$; $h = 1, \dots, L$, for sufficiently large L.

In order to get the solution with similar discontinuity picture as Figure 1 in $t > 0$, we also need the higher order forms of (H3):

(H6) *There exists a submanifold Z of dimension $(n-1)$ containing Z_0 such that Z is tangent up to high order at Z_0 to every hypersurface S_i which passes through S_{0i} and has the same higher order curvatures at Z_0 determined by $D_t^h p_i$, $i = 1, \dots, k$; $h = 1, \dots, L$.*

In particular, if the initial discontinuity S_{0i}'s are all hyperplanes and u_0 in every G_{0i} is constant state, then (H6) is automatically satisfied.

Now, we are going to make our last assumption. Since in G_{01}, two directions $(n_{10}, n_{11}, \dots, n_{1n})$ and $(n_{20}, n_{21}, \dots, n_{2n})$ are both space-like with respect to u_0, so by the uniform stability assumption (H2), $(n_{10}, \dots, n_{1n})$ and $(n_{20}, \dots, n_{2n})$ could not be space-like with respect to u_{0k} and u_{02}, respectively. Examining the possibility in the adjacent domains G_{02} and G_{0k}, we have three different cases:

(1.6)(i) $(n_{30}, \dots, n_{3n})$ and $(n_{k0}, \dots, n_{kn})$ *are not space-like with respect to* u_{02} *and* u_{0k}, *respectively;*

(ii) $(n_{30}, \dots, n_{3n})$ *and* $(n_{k0}, \dots, n_{kn})$ *are both space-like with respect to* u_{02} *and* u_{0k}, *respectively;*

(iii) *One of* $(n_{30}, \dots, n_{3n})$ *and* $(n_{k0}, \dots, n_{kn})$ *is space-like, another is not.*

To fix the idea in case (iii), we will always take $(n_{k0}, \dots, n_{kn})$ being space-like with respect to u_{0k}, and $(n_{30}, \dots, n_{3n})$ not space-like with respect to u_{02}.

REMARK. For the three cases cited above, the case (i) can never happen in gas dynamics, because the stable shock front must be associated with extreme characteristics and hence we have one and only one domain G_{0i} which has two non-space-like boundaries.

Now let $\tilde{G}_0$ be the subset of the domains G_{0i}:

$\tilde{G}_0 = \{G_{0i}; i = 3, \dots, k-1$ or $i = 2, k$ and G_{0i} has two non-space-like boundaries$\}$. Let q be the number of G_{0i} in the set $\tilde{G}_0$. We have $k-3 \le q \le k-1$.

On all $q+1$ boundaries which are adjacent to at least one domain $G_{0i} \in \tilde{G}_0$, we have the Rankine-Hugoniot conditions

$$D_t p_i [F_0(u_{i+1}) - F_0(u_i)] + \sum_1^n D_{x_j} p_i [F_j(u_{i+1}) - F_j(u_i)] = 0, \quad \text{on } S_i. \tag{1.7}$$

These are $m(q+1)$ relations. Besides, the $(q+1)(n+1)$ components of $(D_t p_i, D_x p_i)$ are not independent of each other. The normalizations

$$|D_t p_i|^2 + \sum_1^n |D_{x_j} p_i|^2 = 1, \qquad G_{0i} \in \tilde{G}_0 \text{ or } G_{0(i+1)} \in \tilde{G}_0 \tag{1.8}$$

are $(q+1)$ relations. And as in (H3), they should lie in a common hyperplane of dimension 2, i.e., there exist $(n-1)$ unit vectors $\vec{a}_j$ in $R^n \times R^1_+$, $i = 1, \dots, (n-1)$, such that

$$(\vec{a}_i, \vec{a}_j) = \delta_{ij}, \qquad i, j = 1, \dots, (n-1) \tag{1.9}$$

and

$$((D_t p_i, D_x p_i), \vec{a}_j) = 0, \qquad j = 1, \dots, (n-1); \ G_{0i} \text{ or } G_{0\,(i+1)} \in \tilde{G}_0. \tag{1.10}$$

Once the intersecting submanifold Z is given, then the $\vec{a}_j$'s in (1.9), (1.10) are determined uniquely up to orthogonal transformations within themselves, while (1.10) is invariant for such transformations. Hence, we will have $(q+1)(n-1)$ relations for $(D_t p_i, D_x p_i)$ in (1.10).

If Z is partly given, i.e., if Z is required to lie within a given surface S_2, then the a_j's in (1.10) should satisfy another $(n-1)$ supplementary conditions

$$(\vec{b}, \vec{a}_j) = 0, \qquad j = 1, \ldots, n-1, \tag{1.11}$$

where $\vec{b}$ is a given vector (the normal vector of S_2). Eliminating part of a_j's in (1.10) by (1.11), we get $(n-1)q$ relations for $(D_t p_i, D_x p_i)$, again denoted as (1.10), when there is no confusion possible.

If Z is completely not given, then we will have $(q-1)(n-1)$ relations in (1.10) for $(D_t p_i, D_x p_i)$.

From the examples discussed in §§3 and 4, we will see that for $n = 2$, the condition (1.10) is extremely simple. It consists of the determinant of a 3-order matrix and the introduction of $\vec{a}$'s is not necessary.

Now, we will view u_i in G_i with $G_{0i} \in \tilde{G}_0$ and $(D_t p_i, D_x p_i)$, $(D_t p_{i+1}, D_x p_{i+1})$ as unknowns. For these $q_1 = mq + (q+1)(n+1)$ unknowns, denoted by U in the following, we have $m(q+1)$ Rankine-Hugoniot conditions (1.7) on $(q+1)$ S_i, which is adjacent to at least one $G_{oi} \in \tilde{G}_0$. Besides, we have $(q+1)$ normalizing conditions (1.8) and certain number of conditions in (1.10) to determine the position of every S_i. The exact number of relations in (1.10) will depend on the determination state of Z.

Denote all the relations in (1.7), (1.8) and (1.10) as

$$Q(U) = 0. \tag{1.12}$$

Consider the three cases in (1.6).

In case (i), we have $q = k-1$. Since now all $(q+1)$ S_i's are taken unknowns, Z is not given beforehand. So we have in fact $(q-1)(n-1)$ relations in (1.10). Hence for q_1 unknowns U, we have $q_2 = m(q-1) + (q+1) + (q-1)(n-1) = (m+n)(q+1) - 2(n-1)$ relations in (1.12).

In case (ii), we have $q = k-3$. Now Z is determined by S_1 and S_2, so we have $(q+1)(n-1)$ relations in (1.10). Therefore, for q_1 U, we have $q_2 = m(q+1) + (q+1) + (q+1)(n-1) = (m+n)(q+1)$ relations in (1.12).

In case (iii), $q = k-2$. Now Z is required to be on a given surface S_1. So we have in fact $q(n-1)$ relations in (1.10). And for q_1 unknowns U, we have $q_2 = m(q+1) + (q+1) + q(n-1) = (m+n)(q+1) - (n-1)$ relations in (1.12).

For all these three cases where q_2 takes different values, we impose the following requirement

(H7) *Jacobian DQ/DU has rank q_2 at $t = 0$ and $x \in Z_0$.*

COROLLARY 1. *From* (H7), *we get at once that $q_2 \le q_1$. In case* (i), *it means $q \ge m - 2n + 1$. In case* (ii), *it means $q \ge m - 1$. In case* (iii), *it means $q \ge m - n$.*

REMARK 2. Since only one shock front could be associated with one genuinely nonlinear eigenvalue of the characteristic matrix of the system, we should always have $q+1 \leq m$. This is an absolute restriction on the number of initial discontinuities. In particular, in case (ii), this implies by Corollary 1 that $q = m-1$.

REMARK 3. Consider the particular problem of interaction of shock fronts in 2-dimensional space for two hyperbolic laws considered by Metivier in [**7**]. It corresponds to the case (ii) considered here, with $m = 2$, $q = 1$, exactly the only possible situation when (H7) can be satisfied.

Generally speaking, we cannot get rid of (H7) in order to get our desired result. Nevertheless, it is only a sufficient condition. In some special cases, e.g., in the examples of §3, where the condition in Corollary 1 is not satisfied, we still get the existence result.

Now, the main result of this paper can be stated as follows.

THEOREM. *Suppose that the Cauchy problem* (1.1), (1.3) *with intersecting discontinuity data satisfy all the conditions* (H1)–(H7). *Then there exists a* $t_0 > 0$, *such that* (1.1), (1.3) *has a piecewise differentiable solution* u *in* $0 < t < t_0$, *which has jump discontinuity at differentiable surfaces* S_i*'s, intersecting with each other at a common submanifold* Z *of dimension* $(n-1)$, *and on* S_i, *Rankine-Hugoniot conditions* (1.4) *are satisfied.*

2. Proof of the Theorem. The basic idea of the proof consists of two steps. First, we try to determine the position of Z. Then, by a transformation, we reduce the given problem to a problem of multishock fronts we discussed in [**1**].

First of all, by (H4), we know that the $(n-1)$-dimensional hyperplane Z_t, perpendicular to $(n_{10}, n_{11}, \ldots, n_{1n})$ and $(n_{20}, n_{21}, \ldots, n_{2n})$, has its projection in x-space contained in G_{01} for $t > 0$. We now discuss the three cases in (1.6).

Case (i). In this case, for q_1 unknown variables $(Dp_1, \ldots, Dp_k)$ and $(u_2, \ldots, u_k)$, we have q_2 independent relations $Q(U) = 0$. But at $t = 0$, according to our assumptions (H1) and (H3), we know $U_0 = (\vec{n}_1, \ldots, \vec{n}_k, u_{02}, \ldots, u_{0k})$ satisfies all these relations, i.e., $Q(U_0) = 0$. Thus, from (H7) and the implicit function theorem, we know that near $t = 0$, $x \in Z_0$, there is a vector function $U(t,x)$ satisfying $Q(U(t,x)) = 0$ and $U(0,x)|_{x \in Z_0} = U_0$. It is worth pointing out here that the function $U(t,x)$ is determined uniquely only when $q_2 = q_1$. Otherwise $U(t,x)$ may not be unique. (Similar argument also applies to the discussion of cases (ii) and (iii).) Now $(Dp_1, \ldots, Dp_k)$ satisfies (1.10) near $t = 0$, $x \in Z_0$, and it determines a submanifold Z of dimension $(n-1)$, with $Z|_{t=0} = Z_0$.

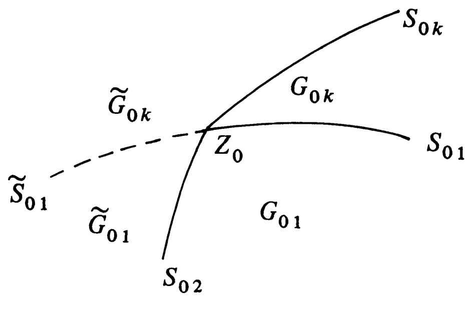

FIGURE 2

Case (ii). In this case, $(n_{30}, \dots, n_{3n})$ and $(n_{k0}, \dots, n_{kn})$ are both space-like with respect to u_{02} and u_{0k}. We are now to follow the approach of G. Metivier. First, we extend the hypersurface S_{01} beyond Z_0 into a smooth hypersurface $\tilde{S}_{01}$ without boundary. $\tilde{S}_{01}$ will divide R^n into two new domains, denoted as $\tilde{G}_{01}$ and $\tilde{G}_{0k}$, with G_{01} contained in $\tilde{G}_{01}$ and G_{0k} in $\tilde{G}_{0k}$. See Figure 2.

And also, we extend u_{01} and u_{0k} smoothly into $\tilde{G}_{01}$ and $\tilde{G}_{0k}$ such that the extended values $\tilde{u}_{01}$ and $\tilde{u}_{0k}$ satisfy all Rankine-Hugoniot conditions and the uniform stability condition of Majda in [**4**]. By the result of Majda [**5**], we know the following Cauchy problem

$$
\left\{
\begin{array}{l}
D_t(F_0(u)) + \sum\limits_{1}^{n} D_{x_j}(F_j(u)) = 0, \quad \text{in } t > 0,\ x \in R^n, \\
u(x,0) = \begin{cases} u_{01}(x), & x \in \tilde{G}_{01}, \\ u_{0k}(x), & x \in \tilde{G}_{0k}, \end{cases}
\end{array}
\right.
\tag{2.1}
$$

has a local shock wave solution $u(x,t)$ which is sufficiently smooth on either side of a sufficiently smooth hypersurface $\tilde{S}_1$. Denote two domains separated by $\tilde{S}_1$ as $\tilde{G}_1$ and $\tilde{G}_k$, with $\tilde{G}_{01}$ contained in $\tilde{G}_1$, and $\tilde{G}_{0k}$ in $\tilde{G}_k$. The value of $u(x,t)$ in $\tilde{G}_1$ and $\tilde{G}_k$ will be denoted as $\tilde{u}_1(x,t)$ and $\tilde{u}_k(x,t)$.

Similarly, we can extend hypersurface S_{02} across Z_0 into $\hat{S}_{02}$ which separates $\hat{G}_{01}$ and $\hat{G}_{02}$. Also, we extend u_{01} and u_{02} into $\hat{u}_{01}$ and $\hat{u}_{02}$, and then solve the Cauchy problem with initial data $(\hat{u}_{01}, \hat{u}_{02})$, having one jump discontinuity. The resulted shock wave solution will be denoted by $\hat{u}_1(x,t)$, $\hat{u}_2(x,t)$ and $\hat{S}_2$ which separates $\hat{G}_1$ and $\hat{G}_2$.

Now in the domain $G_1 = \tilde{G}_1 \cap \hat{G}_1$, we have $\tilde{u}_1(x,t) = \hat{u}_1(x,t)$. The conclusion follows from the fact that the directions $(n_{10}, \dots, n_{1n})$ and $(n_{20}, \dots, n_{2n})$ are both space-like with respect to u_{01}. At $t = 0$, the hypersurfaces $\hat{S}_1$ and $\hat{S}_2$ have their normal vectors equal to $(n_{10}, \dots, n_{1n})$ and $(n_{20}, \dots, n_{2n})$, respectively. Hence, by continuity, for small t, the hypersurface $\tilde{S}_1$ and $\hat{S}_2$ would be space-like with respect to $\tilde{u}_1$ and $\hat{u}_1$. Hence the values of $\tilde{u}_1$ and $\hat{u}_1$ in the domain $\tilde{G}_1 \cap \hat{G}_1$ will depend only on the initial values in G_{01} where $\tilde{u}_1(x,0) = \hat{u}_1(x,0) = u_{01}(x)$. Consequently, we have $\tilde{u}_1(x,t) = \hat{u}_1(x,t)$ in $G_1 = \tilde{G}_1 \cap \hat{G}_1$.

Together with the value $u_1(x,t)$ in $\tilde{G}_1 \cap \hat{G}_1$, we also get the values of $\tilde{u}_k(x,t)$ and $\hat{u}_2(x,t)$ in $\tilde{G}_k$ and $\hat{G}_2$. In particular, we get the intersecting submanifold Z of the hypersurfaces $\tilde{S}_1$ and $\hat{S}_2$.

Case (iii). This is the intermediate case between case (i) and case (ii). As in case (ii), we extend S_{01} beyond Z_0 and extend u_{01}, u_{0k} into $\tilde{u}_{01}$ and $\tilde{u}_{0k}$. By solving the corresponding Cauchy problem, we get the solution $\tilde{u}_1(x,t)$, $\tilde{u}_k(x,t)$ and the separating shock front $\tilde{S}_1$, for small t.

By assumption (H7), as we have done in case (i), we can solve q_1 functions $(Dp_2, \dots, Dp_k)$ and $(u_2, \dots, u_{k-1})$ from q_2 relations $Q(U) = 0$. Thus we get a submanifold Z of dimension $(n-1)$ on $\tilde{S}_1$.

Summing up, in all three cases in (1.6), we have constructed submanifold Z on which are defined q_1 functions satisfying (1.7), (1.8) and (1.10). This concludes our first step in the proof of the Theorem.

In our next step, we begin by constructing two smooth hypersurfaces Y_1 and Y_2 such that $Z = Y_1 \cap Y_2$ and the following conditions are satisfied:

(1) Y_1, Y_2 are not tangent, at $t = 0$, to any S_{0i} at Z_0;

(2) In case (i), $Y_1|_{t=0} \subset G_{01}$, $Y_2|_{t=0} \subset G_{01}$ and Y_1, Y_2 are both space-like with respect to u_{01}. In fact, we will choose Y_1 near the direction $(n_{20}, \dots, n_{2n})$ and Y_2 near the direction $(n_{10}, \dots, n_{1n})$ in the neighborhood of Z_0. Since $(n_{20}, \dots, n_{2n})$ and $(n_{10}, \dots, n_{1n})$ are space-like with respect to u_{01}, so are the hypersurfaces Y_1 and Y_2.

In case (ii), $Y_1|_{t=0} \subset G_{02}$, $Y_2|_{t=0} \subset G_{0k}$. Also for Y_1 near $(n_{30}, \dots, n_{3n})$, Y_2 near $(n_{k0}, \dots, n_{kn})$, they are space-like with respect to u_{02} and u_{0k}, respectively.

In case (iii), $Y_1|_{t=0} \subset G_{01}$, $Y_2|_{t=0} \subset G_{0k}$. And Y_1, Y_2 are space-like with respect to u_{01} and u_{0k} respectively.

Having constructed the hypersurfaces Y_1 and Y_2, we are ready to perform the necessary transformation of variables.

First, we perform a transformation in R^n such that in new coordinates, $Z_0 = \{x; x_1 = x_2 = 0\}$ and the projections of Y_1 and Y_2 onto the hyperplane $x_1 = 0$ are the domains $x_2 \le 0$ and $x_2 \ge 0$, respectively. Let $Y_2 = \{(t,x); t = r(x), x_2 \ge 0\}$. Extending $r(x)$ smoothly into $x_2 < 0$, denoted by $\tilde{r}(x)$, and performing the transformation

$$x' = x, \qquad t' = t - \tilde{r}(x), \tag{2.2}$$

we get $Y_2 = \{(t', x');\ t' = 0,\ x_2' \ge 0\}$ in new coordinates. In order to simplify the notation, we will omit the prime in the following.

Now, the plane $t = 0$ becomes $t + \tilde{r}(x) = 0$. By another transformation, we can make this hypersurface become the hyperplane $x_1 = 0$, with the original domain $t > 0$ becoming $x_1 > 0$. Notice that by all these transformations, the solvability of the problems remains equivalent and the space-like hypersurfaces remain to be space-like.

Let $Y_1 = \{(t,x); t = s(x), x_2 \le 0\}$ in new coordinates. Let $r^2 = t^2 + x_2^2$, $y = \arctan(t/x_2)$, then Y_1 can be written as $y = K(r, x_1, x_3, \dots, x_n)$ where $\pi/2 < y < 3\pi/2$. Now perform the transform

$$\begin{cases} t' = r\sin(\pi/K)y, \\ x_2' = r\cos(\pi/K)y, \\ x_1' = x_1,\ x_3' = x_3, \dots, x_n' = x_n. \end{cases} \tag{2.3}$$

Then, in the new coordinates, Y_1 is a half plane $\{t' = 0,\ x_2' < 0\}$, while Y_2 remains to be $\{t' = 0,\ x_2' < 0\}$.

REMARK 1. Here we notice that transformation (2.2) has singularity at $t = x_2 = 0$ when taken as a transform in the whole space, because it transforms a broken surface into a smooth one. But it is a diffeomorphism in any domain $y_1 < y < y_2$, with $y_2 - y_1 < \pi$.

REMARK 2. If in particular, we can take Y_1 and Y_2 to be two parts of one smooth hypersurface $t = r(x)$, as in the physical examples we discuss in §3, then the transform changing Y_2 into $t = 0$ automatically changes Y_1 into $t = 0$.

After all these transformations, we denote the resulted quasilinear hyperbolic system as

$$A_0(u)D_t u + A_1(u)D_{x_1} u + \sum_2^n A_j(u)D_{x_j} u + B(u)u = F(x,t), \quad \text{in } x_1 > 0,\ t > 0. \tag{2.4}$$

Here we omit the prime in new coordinates.

On $x_1 = 0$, $t > 0$, we have the boundary condition

$$u = u_0(t, x'). \tag{2.5}$$

Here $u_0(t,x')$ is in fact part of the u_0 in (1.3), after the transformation (2.2) and (2.3). It is worth pointing out that the hyperplane $x_1 = 0$ is space-like in new coordinates, though we call (2.5) a boundary condition.

On $t = 0$, $x_1 > 0$, we impose the initial condition

$$u(x,0) = w_0(x) \tag{2.6}$$

where the value of $w_0(x)$ is determined as follows.

In fact, in all three cases of (1.6), we can always extend S_{01}, S_{02} and u_{01}, u_{02}, u_{0k} as we did before for the case (ii), and then construct the one shock wave solution separately. As in the case (ii), we denote by $\tilde{u}_1(x,t)$ and $\tilde{u}_k(x,t)$ the shock wave solution resulted from extending S_{01}, u_{01} and u_{0k}, denote by $\hat{u}_1(x,t)$ and $\hat{u}_2(x,t)$ the shock wave solution resulted from extending S_{02}, u_{01} and u_{02}.

As we pointed out in case (ii), $\tilde{u}_1(x,t) = \hat{u}_1(x,t)$ in their common domain of definition, independent of the way of the extension. Also in the cases (i) and (iii), by the same argument, we have $\tilde{u}_1(x,t) = \hat{u}_1(x,t)$ in their common domain of definition.

Now, we take $w_0(x)$ as follows: If $Y_1|_{t=0}$ (or $Y_2|_{t=0}$) lies in G_{01}, $w_0(x)$ takes the value of $\tilde{u}_1(x,t) = \hat{u}_1(x,t)$ (transformed by (2.2) and (2.3)). If $Y_1|_{t=0}$ ($Y_2|_{t=0}$) lies in G_{02} (G_{0k}), $w_0(x)$ takes the transformed value of $\tilde{u}_2(x,t)$ ($\hat{u}_k(x,t)$).

With $w_0(x)$ thus determined, we are to consider the initial-boundary value problem (2.4)–(2.6). Here the initial data $w_0(x)$ have jump discontinuity at $x_2 = 0$, the boundary $x_1 = 0$ is space-like and the boundary data $u_0(t,x')$ is piecewise smooth having jump discontinuity surfaces emitting from $x_1 = x_2 = 0$.

From the assumptions (H2) and (H7), we know by implicit function theorem that for small $x_1 > 0$, there exist sufficiently smooth functions (u_i, λ_j) on $x_2 = 0$ such that the transformed Rankine-Hugoniot conditions (1.5) are satisfied. Here $i = 2,\dots,k$; $j = 1,\dots,k$ (for case (i)) or $i = 3,\dots,(k-1)$; $j = 3,\dots,k$ (for case (ii)) or $i = 2,\dots,(k-1)$; $j = 2,\dots,k$ (for case (iii)). Since the set of coefficients for which uniform stability conditions are satisfied is open, we know that for small $x_1 > 0$ and for these (u_i, λ_j), the uniform stability conditions (H2) is satisfied. In particular, at $x_1 = 0$, u_i's are equal to the boundary value u_0 and λ_j coincide with the tangent directions of the discontinuity surfaces of u_0.

The problem (2.4)–(2.6) is very much like the multishock wave problem we discussed in [**1**], except that we now have the boundary condition (2.5).

As a matter of fact, for the multishock wave problem without boundary condition on $x_1 = 0$, we proved in [**1**] the following:

THEOREM FOR MULTISHOCK WAVES. *For the Cauchy problem of the hyperbolic conservation laws* (1.1) *with initial data* $u^0 = (u^0_+, u^0_-)$ *which have jump discontinuity along one smooth hypersurface* S_0, *if*

(i) *there are smooth functions* u^0_i, λ_j *on* S_0, $i = 2, \dots, k$; $j = 1, \dots, k$; *such that Rankine-Hugoniot conditions are satisfied for k sets of* $(\lambda_j, u^0_j, u^0_{j+1})$, $j = 1, \dots, k$; *with* $u^0_1 = u^0_-$, $u^0_{k+1} = u^0_+$;

(ii) *these k shock fronts are separately linear stable in the sense of Majda in* [**4**];

(iii) *high order compatibility conditions are satisfied*;

then, there exists a positive t_0, *such that in* $[0, t_0]$, *there is a piecewise smooth solution of* (1.1) *with k shock fronts.*

Now for the problem (2.4)–(2.6), we can proceed just as in [**1**] in proving the above theorem. First of all, we perform the transformation containing the unknown shock fronts so that the original free boundary problem is reduced to a fixed multiboundary problem with newly introduced unknown functions p_j, describing the position of the shock fronts. Second, by the compatibility hypothesis (H5), we can construct an approximating solution (u^0_i, p^0_j) such that the problem is further reduced to a problem for $(v_i, q_j) = (u^0_i - u_i, p^0_j - p_j)$, which satisfied the homogeneous initial and boundary conditions at $t = 0$ and $x_1 = 0$, with the right side of the equation having zero traces at $t = 0$ and $x_1 = 0$ up to sufficiently high order. Then let $t' = \log t$, $x'_2 = x_2/t$, we get a new problem without initial conditions and all the boundaries are uncoupled. Such a problem can be treated similarly as in [**1**] and the existence of the solution follows.

This concludes our proof of the theorem.

3. Interaction of isentropic shock fronts in 2-dimensional space. As the application of our Theorem, we consider isentropic hydrodynamic equations of polytropic gas in 2-dimensional space:

$$
\begin{cases} D_t f + D_{x_1}(f v_1) + D_{x_2}(f v_2) = 0, \\ D_t(f v_1) + D_{x_1}(f v_1^2 + p) + D_{x_2}(f v_1 v_2) = 0, \\ D_t(f v_2) + D_{x_1}(f v_1 v_2) + D_{x_2}(f v_2^2 + p) = 0. \end{cases} \tag{3.1}
$$

Here v_1, v_2 are velocities of the gas in x_1, x_2 directions; f is the density of the gas and $p = A f^r$ with A, r constants and $r > 1$.

The problem we are going to discuss comes from the interaction of two shock fronts, as G. Metivier discussed in [**7**] for 2×2 systems in 2-dimensional space. Here, for physical examples, (3.1) is a 3×3 system.

The problem of interaction of two shock fronts can be reduced to the problem in §1 with $k = 4$, provided that only two shock fronts are produced after the interaction.

First, we consider the special case of constant state. Suppose at $t = 0$, $u = (f, v_1, v_2)$ is piecewise constant in R^n with four discontinuity straight lines intersecting at $x_1 = x_2 = 0$. See Figure 3.

Denote the four discontinuity straightlines by $S_{\pm 1}$ and $S_{\pm 2}$, the four domains between them by $G_1, G_\pm, G_2$, and the values of u in four domains by $u_1, u_\pm, u_2$.

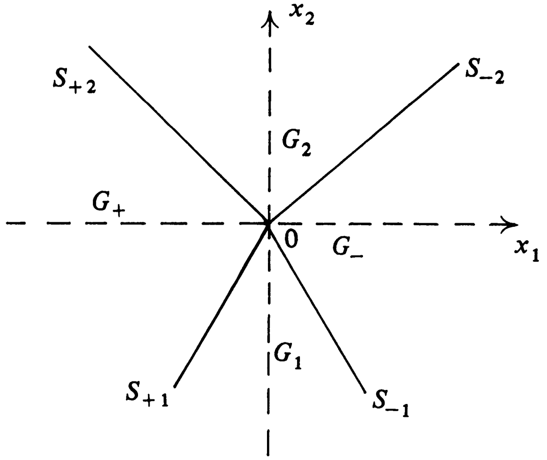

FIGURE 3

We will restrict ourselves to the discussion of the special case where the data are symmetric with respect to x_2 axis, i.e.

$$\left\{\begin{array}{l} S_{\pm 1} = \{x;\ x_2 = \pm k_1 x_1\}, \\ S_{\pm 2} = \{x;\ x_2 = \pm k_2 x_1\}, \end{array}\right. \qquad k_1, k_2 > 0, \tag{3.2}$$

and

$$v_{+1} = -v_{-1}, \quad v_{+2} = v_{-2} = v_2, \quad v_{11} = v_{12} = v_{21} = 0, \quad f_+ = f_-. \tag{3.3}$$

Since shock fronts are not characteristic, from Rankine-Hugoniot conditions it follows that the tangential speed to $S_{\pm 1}$ in $G_\pm$ must be equal to the tangential speed in G_1, i.e., zero. So velocity $(v_{\pm 1}, v_{\pm 2})$ in $G_\pm$ should satisfy

$$v_{\pm 1} \pm k_1 v_{\pm 2} = 0. \tag{3.4}$$

Similarly, the tangential velocities on two sides of $S_{\pm 2}$ being equal leads to

$$v_{\mp 1} \pm k_2 v_{\mp 2} = v_{21} \pm k_2 v_{22} = \pm k_2 v_{22}. \tag{3.5}$$

Let the shock front direction of $S_{\pm 1}$ be $(h_{\pm 1}, \mp k_1, 1)$. Taking $h_{+1} = h_{-1} = h_1$ and by (3.3) and (3.4), we get from Rankine-Hugoniot condition:

$$\left\{\begin{array}{l} h_1(f_+ - f_1) + k_1^2 f_+ v_2 + f_+ v_2 = 0, \\ -h_1 f_+ v_2 - f_+ v_2^2 (1 + k_1^2) - (p_+ - p_-) = 0. \end{array}\right. \tag{3.6}$$

Similarly, let the shock front direction of $S_{\pm 2}$ be $(h_{\pm 2}, \pm k_2, 1)$. Taking $h_{+2} = h_{-2} = h_2$ and noticing (3.3), (3.5), we get

$$\left\{\begin{array}{l} f_+(h_2 + v_2 + k_2^2(v_2 - v_{22})) - f_2(h_2 + v_{22}) = 0, \\ f_+(h_2 + v_2 + k_2^2(v_2 - v_{22}))(v_2 - v_{22}) + (p_+ - p_2) = 0. \end{array}\right. \tag{3.7}$$

The hypothesis (H3) for $(h_1, \mp k_1, 1)$ and $(h_2, \pm k_2, 1)$ lying in a common plane becomes very explicit now. In 3-dimensional space, three vectors lie in a common plane if and only if the corresponding 3×3 matrix is degenerate. Thus we have

$$\det \begin{vmatrix} h_1 & k_1 & 1 \\ h_1 & -k_1 & 1 \\ h_2 & k_2 & 1 \end{vmatrix} = 0, \qquad \det \begin{vmatrix} h_1 & k_1 & 1 \\ h_1 & -k_1 & 1 \\ h_2 & -k_2 & 1 \end{vmatrix} = 0$$

or equivalently

$$h_1 = h_2. \tag{3.8}$$

Here, because $k_1 \neq k_2$, so (3.8) does not mean that S_{+1} and S_{+2} have the same slope with respect to $t = 0$.

In (3.6)–(3.8), we have five relations for nine variables $(h_1, h_2, f_+, f_1, f_2, v_2, v_{22}, k_1, k_2)$. We should choose these parameters such that (3.6)–(3.8) are satisfied and $h_1, h_2 > 0$, $f_2 > f_+ > f_1$. For the variables thus chosen, our hypotheses (H1)–(H3) are all satisfied.

Eliminating h_1, h_2, k_1, k_2 in (3.6)–(3.8), we get

$$f_2(v_2 - v_{22})(Af_1^\tau - Af_+^\tau + f_1 v_2 v_{22}) + A(f_+^\tau - f_2^\tau) f_1 v_2 = 0. \tag{3.9}$$

Now fixing $f_+ = 1$ and $f_2 > 1$, we denote the function on the left-hand side of (3.9) by $T(f_1)$,

$$T(f_1) = f_2(v_2 - v_{22})(Af_1^\tau - A + f_1 v_2 v_{22}) + A(1 - f_2^\tau) f_1 v_2.$$

For $v_{22} < v_2 < 0$, we have

$$\begin{cases} T(0) = f_2(v_2 - v_{22})(-A) < 0, \\ T(1) = f_2(v_2 - v_{22}) f_1 v_2 v_{22} + A(1 - f_2^\tau) v_2 > 0. \end{cases} \tag{3.10}$$

Consequently, there exists at least one $f_1 \in (0, 1)$ such that (3.9) is satisfied and $0 < f_1 < f_+ < f_2$.

Now for $\varepsilon << 1$, we take

$$v_2 = O(\varepsilon^4), \qquad v_{22} = O(\varepsilon^2), \qquad 1 - f_2 = O(\varepsilon). \tag{3.11}$$

Then from (3.6)–(3.8), we have

$$\begin{aligned} & h_1 > 0, \qquad 1 - f_1 = O(\varepsilon^3), \qquad h_1 = O(\varepsilon^{-1}), \text{ and} \\ & 1 + k_1^2 = \frac{A}{f_1 v_2^2}(f_1 - 1)(f_1^\tau - 1) = O(\varepsilon^{-2}). \end{aligned} \tag{3.12}$$

Therefore, for $\varepsilon << 1$, we have a real positive solution for k_1.

From (3.7), we have

$$k_2^2(v_2 - v_{22}) = f_2(h_1 + v_{22}) - (h_1 + v_2) = (f_2 - 1)h_1 + f_2 v_{22} - v_2. \tag{3.13}$$

Since $v_2 - v_{22} > 0$, $(f_2 - 1)h_1 > 0$, $(f_2 - 1)h_1 = O(1)$, $f_2 v_{22} - v_2 = O(\varepsilon^2)$, so (3.13) also has a real solution for k_2.

Thus, we can choose $(h_1, h_2, f_1, f_+, f_2, v_2, v_{22}, k_1, k_2)$ such that (3.6)–(3.8) are satisfied and $h_1, h_2 > 0$, $f_2 > f_+ > f_1$. Consequently (H1)–(H3) are satisfied.

(H4) is satisfied because $f_+ = f_- > f_1$, and so G_1 is the required domain. (H5), (H6) are automatically satisfied for constant initial data. For variable data, (H5), (H6) are also satisfied if the data are C^∞ tangent to the constant data at $x = 0$.

Now consider (H7). For the problem with constant initial data, (H7) is not needed. For the general problem with variable initial data, noticing that our problem corresponds to case (ii) in (1.6), we have $k = 4$, $q = k - 3 = 1$, $m = 3$. Therefore, $q = 1 < 2 = m - 1$, contradictory to Corollary 1 in section 1. Nevertheless, if we consider the special case of interaction of two symmetric shock fronts, so the Rankine-Hugoniot conditions on $S_{\pm 2}$ are symmetric. We will have three

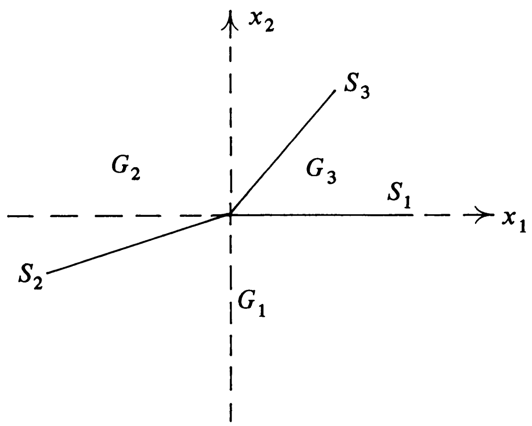

FIGURE 4

unknown functions h_2, v_2, f_2. Hence the condition in Corollary 1 is satisfied. Since (H7) is to exclude certain submanifold of lower dimension, by suitably choosing initial data, we can also make (H7) satisfied.

In summary, with the above chosen initial data, we get the local existence result for the problem of interaction of two symmetric isentropic shock fronts in 2-dimensional space, by the Theorem in §1.

4. An example of nonexistence. Again, we consider the isentropic hydrodynamic equations for polytropic gas (3.1). At $t = 0$, $u = (f, v_1, v_2)$ is piecewise smooth in R^2 with three discontinuity straight lines intersecting at $x_1 = x_2 = 0$. See Figure 4.

Without loss of generality, we assume $S_1 = \{x; x_2 = 0,\ x_1 > 0\}$, $S_2 = \{x;\ x_1 = k_2x_2,\ x_2 < 0\}$, $S_3 = \{x;\ x_1 = k_3x_2,\ x_2 > 0\}$, and $v_{11} = v_{12} = 0$. Let $(h_1, 0, 1)$, $(h_2, -1, k_2)$, $(h_3, -1, k_3)$ be the shock front directions on S_1, S_2 and S_3, then from the fact that the tangential speeds on two sides of a shock front should be equal, i.e.,

$$v_{31} = 0, \qquad k_2v_{21} + v_{22} = 0, \qquad v_{32} = k_3v_{21} + v_{22} = (k_3 - k_2)v_{21} = 0 \tag{4.1}$$

we deduce from Rankine-Hugoniot conditions

$$\begin{cases} h_1(f_3 - f_1) + f_3v_{32} = h_1(f_3 - f_1) + (k_3 - k_2)v_{21} = 0, \\ h_1f_3v_{32} + f_3v_{32}^2 + p_3 - p_1 = 0. \end{cases} \tag{4.2}$$

$$\begin{cases} h_2(f_2 - f_1) - f_2v_{21}(1 + k_2^2) = 0, \\ h_2f_2v_{21} - f_2v_{21}^2(1 + k_2^2) - (p_2 - p_1) = 0. \end{cases} \tag{4.3}$$

$$\begin{cases} h_3(f_3 - f_2) + v_{21}(f_2 + f_2k_2k_3 - f_3k_2k_3 + f_3k_3^2) = 0, \\ - h_3f_2v_{21} + (1 + k_2k_3)f_2v_{21}^2 - (p_3 - p_2) = 0. \end{cases} \tag{4.4}$$

In order to have three stable shock fronts intersecting on a common line, (4.2)–(4.4) must be satisfied together with the condition for $(h_1, 0, 1)$, $(h_2, -1, k_2)$ and $(h_3, -1, k_3)$ lying in a common plane

$$\det \begin{vmatrix} h_1 & 0 & 1 \\ h_2 & -1 & k_2 \\ h_3 & -1 & k_3 \end{vmatrix} = h_1(k_2 - k_3) - (h_2 - h_3) = 0. \tag{4.5}$$

We are going to show that there is no set of data $(v_{21}, f_1, f_2, f_3, k_2, k_3, h_1, h_2, h_3)$ satisfying (4.2)–(4.5) with $f_3 > f_2 > f_1$ for a convex state function $p = p(f)$.

Eliminating h_1, h_2, h_3 from (4.2)–(4.5), we get

$$\begin{cases} (f_3 v_{21})^2 (k_3 - k_2)^2 = (f_3 - f_1)(p_3 - p_1), \\ f_1 f_2 v_{21}^2 (1 + k_2^2) = (f_2 - f_1)(p_2 - p_1), \\ f_2 f_3 v_{21}^2 (1 + k_3^2) = (f_3 - f_2)(p_3 - p_2), \\ \dfrac{(k_2 - k_3)^2 f_3}{f_3 - f_1} - \dfrac{(1 + k_2^2) f_2}{f_2 - f_1} - \dfrac{f_2 + f_3 k_3^2}{f_3 - f_2} + k_2 k_3 = 0. \end{cases} \tag{4.6}$$

Further eliminating v_{21}^2 and k_2, k_3 from (4.6), we have the relation for (f_1, f_2, f_3), which after simplification can be written as

$$f_2 f_3 (p_2 - p_3) + f_1 f_3 (p_3 - p_1) - f_2 f_1 (p_2 - p_1) = 0. \tag{4.7}$$

Fixing f_1, f_2, we write the left-hand side of (4.7) as $M(f_3)$. Since $M(f_2) = 0$, we need only to show that $M'(f_3) < 0$ for $f_3 > f_2$.

$$\begin{aligned} M'(f_3) &= f_2 p(f_2) - f_1 p(f_1) + (f_1 - f_2) p(f_3) + (f_1 - f_2) f_3 p'(f_3) \\ &= (f_1 - f_2)(f_3 p'(f_3) + p(f_3)) + (f_2 - f_1)(f^* p'(f^*) + p(f^*)) \end{aligned}$$

where $f^* = f_1 + \theta(f_2 - f_1)$, $0 < \theta < 1$.

From $f_3 > f^*$, $p' > 0$, $p'' > 0$, we get $f^* p'(f^*) + p(f^*) < f_3 p'(f_3) + p(f_3)$, consequently $M'(f_3) < 0$. Thus we know (4.7) is never satisfied for $f_3 > f_2$. Therefore we get the conclusion that there can be no such shock fronts in $R^2 \times R_+$ for hydrodynamic gas equations with convex state function, which have three stable shock fronts intersecting at a common curve.

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