TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 307, Number 2, June 1988

A VERY SINGULAR SOLUTION OF A QUASILINEAR DEGENERATE DIFFUSION EQUATION WITH ABSORPTION

L. A. PELETIER AND JUNYU WANG

ABSTRACT. The object of this paper is to study the existence of a nonnegative solution of the Cauchy problem

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q, \qquad u(x,0) = 0 \quad \text{if } x \neq 0,$$

which is more singular at $(0,0)$ than the fundamental solution of the corresponding equation without the absorption term.

1. Introduction. In a recent paper [**1**] Brezis, Peletier and Terman found a *very singular solution* of the heat equation with absorption

$$u_t = \Delta u - u^q \quad \text{in } S = \mathbf{R}^N \times (0,\infty) \tag{1.1}$$

when $1 < q < 1 + (2/N)$ and $N \geq 1$. By this was meant a solution $W(x,t)$ with the properties

(i) $W > 0$ in S;
(ii) W is smooth in $\bar{S}$, except at $(0,0)$;
(iii) $W(x,0) = 0$ for all $x \in \mathbf{R}^N$, except at $x = 0$;
(iv) W is more singular at the origin than the fundamental solution E of the heat equation, specifically

$$\int_{\mathbf{R}^n} W(x,t)\,dx \to \infty \quad \text{as } t \downarrow 0.$$

This very singular solution turned out to play an important role in the behavior of more general solutions of (1.1) as $t \to 0$ [**7**] and as $t \to \infty$ [**2, 6**].

A corresponding very singular solution for the Porous Media Equation with absorption

$$u_t = \Delta(u^m) - u^q \tag{1.2}$$

in which $m > 1$, was found to exist by Peletier and Terman [**8**] when $m < q < m + (2/N)$.

Motivated by these results we shall show in this paper that the quasilinear degenerate diffusion equation—involving the p-laplacian—with absorption

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q \quad \text{in } S \tag{1.3}$$

also has a very singular solution under suitable restrictions on p and q. Here we shall assume throughout that $p > 2$.

Received by the editors October 6, 1986 and, in revised form, April 27, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35K65; Secondary 35K15, 35K55.

By a very singular solution of equation (1.3) we mean a function $W(x,t)$ which satisfies (1.3) in some sense, and possesses the properties

$$W \geq 0 \text{ in } \bar{S}\backslash\{(0,0)\}; \tag{1.4}$$

$$W \text{ is continuous in } \bar{S}\backslash\{(0,0)\}; \tag{1.5}$$

$$W(x,0) = 0 \text{ for all } x \in \mathbf{R}^N\backslash\{0\}; \tag{1.6}$$

$$W \text{ is more singular than } E \text{ at } \{(0,0)\}. \tag{1.7}$$

In this definition E denotes the singular solution of equation (1.3) without absorption

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

with $p > 2$. It is given by

$$E(x,t) = t^{-1/\mu}\psi(\xi), \qquad \xi = |x|t^{-1/N\mu};$$

where

$$\mu = p - 2 + (p/N) \tag{1.8}$$

and ψ is given by

$$\psi(\xi) = c(p,N)\{[\xi_0^{p/(p-1)} - \xi^{p/(p-1)}]_+\}^{(p-1)/(p-2)}.$$

Here ξ_0 is an arbitrary positive constant,

$$c(p,N) = \left(\frac{p-2}{p}\right)^{(p-1)/(p-2)} (N\mu)^{-1/(p-2)}$$

and $[z]_+ = \max\{0,z\}$.

As in [**1** and **8**] we look for a very singular solution of the form

$$W(x,t) = t^{-1/(q-1)}f(\eta), \qquad \eta = |x|t^{-1/\beta}.$$

Such a function W will satisfy (1.3)–(1.6) if

$$\beta = p(q-1)/(q+1-p) \tag{1.9}$$

and if f is a solution of the problem

$$\text{(I)}\begin{cases} (|f'|^{p-2}f')' + \dfrac{N-1}{\eta}|f'|^{p-2}f' + \dfrac{1}{\beta}\eta f' + \dfrac{1}{q-1}f - f^q = 0 \quad \text{in } (0,\infty), & (1.10)\\ f(\eta) \geq 0 \quad \text{on } [0,\infty), & (1.11)\\ f'(0) = 0, \qquad \lim\limits_{\eta\to\infty} \eta^{p/(q+1-p)}f(\eta) = 0. & (1.12)\end{cases}$$

We shall show that such a solution indeed exists for appropriate values of p and q.

THEOREM. *Suppose that $N \geq 1$, $p > 2$ and*

$$p - 1 < q < p - 1 + p/N. \tag{1.13}$$

Then there exists a nontrivial solution f of Problem (I); *f has compact support.*

The conditions on p and q in this theorem ensure that the singularity of $W(x,t)$ at $(0,0)$ is stronger than that of $E(x,t)$. For instance

$$E(0,t) = \psi(0)t^{-1/\mu} \quad \text{and} \quad W(0,t) = f(0)t^{-1/(q-1)},$$

and, by (1.8) and (1.13), $q - 1 < p - 2 + (p/N) = \mu$.

The solution $f(\eta)$ we obtain is of the form

$$f(\eta)\begin{cases} >0 & \text{for } 0\le\eta<\eta_0,\\ =0 & \text{for } \eta_0\le\eta<\infty,\end{cases}$$

where η_0 is some positive number. At the point η_0, we find that

$$\lim_{\eta\uparrow\eta_0}\frac{|f'(\eta)|^{p-2}f'(\eta)}{f(\eta)}=-\frac{\eta_0}{\beta}. \tag{1.14}$$

Observe that the lower bound for q in (1.13) ensures that $\beta>0$; the upper bound will be needed to ensure the behavior of $f(\eta)$ as $\eta\to\infty$, required by (1.12).

It is interesting to note that for the corresponding elliptic equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+u^q=0,\qquad u\ge 0, \tag{1.15}$$

there exists a very singular solution if

$$p-1<q<N\frac{p-1}{N-p}\quad\text{if } 1<p<N$$

or

$$p-1<q\quad\text{if } p=N.$$

It is given explicitly by

$$u(x)=\gamma_{N,p,q}|x|^{-p/(q+1-p)},$$

where

$$\gamma_{N,p,q}=\left[\left(\frac{p}{q+1-p}\right)^{p-1}\left(\frac{pq}{q+1-p}-N\right)\right]^{1/(q+1-p)}.$$

For further details we refer to Friedman and Veron who give in [**3** and **4**] a complete classification of the isolated singularities of equation (1.15).

In this paper we have imposed the condition $p>2$, leaving the range $1<p<2$ unexplored. We intend to return to this range in a forthcoming paper.

The organization of this paper is the following. In §2 we place Problem (I) in the context of a class of problems, and derive a few properties of its solutions. In particular we shall prove that a solution is a decreasing function whenever it is positive. In §3, we formulate equation (1.10) as a system of three first order differential equations by introducing appropriate variables, dependent as well as independent. In this setting the desired solution corresponds to an orbit connecting two given curves in the three-dimensional solution space. In §4 we establish the existence of such an orbit by means of a shooting argument, not unlike that used in [**8**].

2. Preliminaries. We shall consider the more general boundary value problem

$$\text{(II)}\begin{cases}(|u'|^{p-2}u')'+\dfrac{N-1}{x}|u'|^{p-2}u'+\dfrac{x}{\beta}u'+f(u)=0, & x>0, & (2.1)\\ u(x)\ge 0\quad(\not\equiv 0), & x\ge 0, & (2.2)\\ u'(0)=0,\qquad \lim\limits_{x\to\infty}u(x)=0 & & (2.3)\end{cases}$$

in which $p > 2$, $\beta > 0$, $N \geq 1$ and

$$f(u) = (\alpha/\beta)u - u^q, \tag{2.4}$$

where $\alpha > 0$ and $q > 1$. Clearly we obtain Problem (I) if we set

$$\alpha = \frac{p}{q-p+1}, \qquad \beta = \frac{p(q-1)}{q-p+1} \tag{2.5}$$

and require *in addition* that

$$u(x) = o(x^{-\alpha}) \quad \text{as } x \to \infty. \tag{2.6}$$

By a solution of Problem (II) we shall mean a function $u \in C^1([0,\infty))$ such that $|u'|^{p-2}u' \in C^1(0,\infty)$ which satisfies (2.1)–(2.3). In fact, $|u'|^{p-2}u' \in C^1([0,\infty))$ and it is readily shown that

$$\lim_{x\downarrow 0} \frac{|u'(x)|^{p-2}u'(x)}{x} = (|u'|^{p-2}u')'(0) = -\frac{1}{N}f(u(0)). \tag{2.7}$$

The main result of this paper is the following existence theorem.

THEOREM 1. *Suppose $N \geq 1$, $p > 2$, $\beta > 0$ and*

$$\alpha > N, \quad q > 1.$$

Then Problem (II) *has a solution u with the property* $\lim_{x\to\infty} x^\alpha u(x) = 0$.

We begin, in this section, by deriving a few properties of solutions of Problem (II).

LEMMA 1. *Suppose $u(x)$ is a solution of Problem* (II) *in which $\alpha \geq N$. Then*
(i) *$u(x) \leq A$ for all $x \geq 0$; $A = (\alpha/\beta)^{1/(q-1)}$;*
(ii) *$u(x)$ is nonincreasing on $[0,\infty)$;*
(iii) *$u'(x) < 0$ at points $x > 0$ where $u(x) > 0$.*

PROOF. (i) Suppose to the contrary that

$$u(\bar{x}) = \sup\{u(x) : x > 0\} > A. \tag{2.8}$$

Then at $x = \bar{x}$: $u > A$, $u' = 0$, $(|u'|^{p-2}u')' \leq 0$. If $\bar{x} > 0$, we deduce, however, from the differential equation that

$$(|u'|^{p-2}u')'(\bar{x}) = -f(u(\bar{x})) > 0,$$

i.e. a contradiction. If, on the other hand, $\bar{x} = 0$, we obtain a contradiction to (2.8) from (2.7). Therefore (2.8) must be false, and $u \leq A$ on $[0,\infty)$.

(ii) Suppose that for some $x_0 > 0$, $u'(x_0) > 0$. Then in view of (2.7) there exists a point $\bar{x} \in (0, x_0)$ such that at $\bar{x}$

$$0 \leq u < A, \qquad u' = 0, \qquad (|u|^{p-2}u')' \geq 0. \tag{2.9}$$

We shall show in Lemma 2 that $u(\bar{x}) > 0$. This means, according to equation (2.1) that $(|u'|^{p-2}u')' < 0$ at $x = \bar{x}$, contradicting (2.9). Thus $u'(x) \leq 0$ for all $x \geq 0$.

(iii) It follows from the proof of part (ii) that $u'(x) < 0$ at every $x > 0$, where $0 < u(x) < A$. Thus it remains to exclude the case that for some $a > 0$,

$$u(x) \begin{cases} = A, & 0 \leq x \leq a, \\ < A, & a < x < \infty. \end{cases}$$

We assert that there exists a $\delta > 0$ such that $u'(x) < 0$ for $a < x < a+\delta$. For, if no such δ can be found, there exists a sequence $\{x_n\}$, $x_n \downarrow a$, such that $u'(x_n) = 0$ and $0 < u(x_n) < A$. By equation (2.1) the points $\{x_n\}$ can only be strict maxima. Since by the continuity of u', there will be minima between the points $\{x_n\}$, where $u' = 0$, equation (2.1) yields a contradiction.

If we multiply equation (2.1) by x^{N-1}, we can write it as

$$(x^{N-1}|u'|^{p-2}u')' + \frac{1}{\beta}x^N u' + x^{N-1}f(u) = 0. \tag{2.10}$$

Integration over (a, x), where $a < x < a + \delta$, now yields, after some elementary manipulations:

$$\begin{aligned} x^{N-1}|u'|^{p-1} &+ \frac{x^N}{\beta}[A - u(x)] \\ &= \frac{N}{\beta}\int_a^x s^{N-1}[A - u(s)]\,ds + \int_a^x s^{N-1}f(u(s))\,ds. \end{aligned} \tag{2.11}$$

Because u is decreasing on $(a, a + \delta)$

$$\int_a^x s^{N-1}[A - u(s)]\,ds \le \frac{1}{N}(x^N - a^N)[A - u(x)] \tag{2.12}$$

and, remembering the definition (2.4) of f:

$$\begin{aligned} \int_a^x s^{N-1}f(u(s))\,ds &\le qA^{q-1}\int_a^x s^{N-1}[A - u(s)]\,ds \\ &\le \frac{qA^{q-1}}{N}(x^N - a^N)[A - u(x)]. \end{aligned} \tag{2.13}$$

Hence, if we substitute (2.12) and (2.13) into (2.11), and divide by $x^N[A - u(x)]$, we obtain

$$\frac{1}{\beta} \le \left(\frac{1}{\beta} + \frac{q}{N}A^{q-1}\right)\left\{1 - \left(\frac{a}{x}\right)^N\right\}.$$

If we now let x tend to a, we arrive at the contradiction $1/\beta \le 0$.

Thus, we have excluded the possibility that $u = A$ on some interval $[0, a]$, $a > 0$. This completes the proof of the lemma.

In the following lemma we discuss the behavior of a nonnegative solution of (2.1) near a zero.

LEMMA 2. *Let u be a nonnegative solution of equation* (2.1) *such that at some point $x_0 \ge 0$, $u(x_0) = u'(x_0) = 0$. Suppose $\alpha \ge N$. Then*

(i) *$u(x) \equiv 0$ for all $x \ge x_0$;*

(ii) *if $x_0 > 0$ and $u > 0$ in a left neighborhood of x_0, then*

$$\lim_{x \uparrow x_0} u^{-1}|u'|^{p-1} = x_0/\beta.$$

PROOF. (i) If not, we may assume that $u(x) > 0$ and $u'(x) > 0$ for $x \in (x_0, x_0 + \varepsilon)$, where ε is some sufficiently small positive number.

We multiply the equation by x^{N-1} and integrate over (x_0, x), where $x_0 < x < x_0 + \varepsilon$. This yields

$$x^{N-1}|u'|^{p-1} + \frac{x^N}{\beta}u + \frac{\alpha - N}{\beta}\int_{x_0}^x s^{N-1}u(s)\,ds = \int_{x_0}^x s^{N-1}u^q(s)\,ds.$$

Hence, because $\alpha \geq N$,

$$\frac{x^N}{\beta}u(x) \leq \int_{x_0}^{x} s^{N-1}u^q(s)\,ds \leq \frac{1}{N}(x^N - x_0^N)u^q(x)$$

or

$$\frac{1}{\beta} \leq \frac{1}{N}u^{q-1}(x)\left\{1 - \left(\frac{x_0}{x}\right)^N\right\}.$$

If we now let x tend to x_0, we arrive at a contradiction.

(ii) Choose $\varepsilon > 0$ so small that $u > 0$ and $u' < 0$ in $(x_0 - \varepsilon, x_0)$. Then, as in part (i) we obtain

$$x^{N-1}|u'|^{p-1} + \frac{\alpha - N}{\beta}\int_x^{x_0} s^{N-1}u(s)\,ds = \frac{x^N}{\beta}u + \int_x^{x_0} s^{N-1}u^q(s)\,ds$$

when $x \in (x_0 - \varepsilon, x_0)$. Hence, because $\alpha \geq N$,

$$\frac{|u'(x)|^{p-1}}{u(x)} \leq \frac{x}{\beta} + \frac{\int_x^{x_0} s^{N-1}u^q(s)\,ds}{x^{N-1}u(x)}$$

and

$$\frac{|u'(x)|^{p-1}}{u(x)} > \frac{x}{\beta} - \frac{\alpha - N}{\beta}\frac{\int_x^{x_0} s^{N-1}u(s)\,ds}{x^{N-1}u(x)}.$$

If we let $x \uparrow x_0$ in these inequalities, we obtain the required limit.

REMARK. The limit in part (ii) of Lemma 2 is equivalent to

$$\lim_{x\uparrow x_0}\left|\frac{p-1}{p-2}v'\right|^{p-1} = \frac{x_0}{\beta}, \tag{2.14}$$

where $v = u^{(p-2)/(p-1)}$.

3. The system. The limit (2.14) suggests we introduce the new dependent variables

$$v = u^{(p-2)/(p-1)} \quad \text{and} \quad w = -\frac{p-1}{p-2}v'.$$

Equation (2.1) can then be written as the first order system

$$\text{(III)}\begin{cases} v' = -\dfrac{p-2}{p-1}w, & (3.1)\\ v(w^{p-1})' = w^p - \dfrac{N-1}{x}vw^{p-1} - \dfrac{x}{\beta}w + \dfrac{\alpha}{\beta}w - v^{1+\mu} & (3.2)\end{cases}$$

where $\mu = (q-1)(p-1)/(p-2)$, because $w \geq 0$ by Lemma 1.

To simplify the left side of (3.2), we also introduce a new independent variable t through the symbolic equation

$$\frac{d}{dt} = (p-1)vw^{p-2}\frac{d}{dx}.$$

This yields the system

$$\text{(IV)}\begin{cases} v' = -(p-2)vw^{p-1},\\ w' = w^p - \dfrac{N-1}{x}vw^{p-1} - \dfrac{x}{\beta}w + g(v),\\ x' = (p-1)vw^{p-2},\end{cases}$$

where primes now denote differentiation with respect to t and

$$g(v) = (\alpha/\beta)v - v^{1+\mu}. \tag{3.3}$$

By Lemmas 1 and 2 it suffices to find a solution $(v(t), w(t), x(t))$ defined for $t > 0$ with the properties

$$0 < v(0) \le B, \quad w(0) = 0, \quad x(0) = 0$$

and

$$\lim_{t\to\infty} v(t) = 0, \qquad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \beta w^{p-1}(t) < \infty, \tag{3.4}$$

where

$$B = A^{(p-2)/(p-1)} = (\alpha/\beta)^{1/\mu}. \tag{3.5}$$

The set of critical points of (IV) in the half space $x > 0$ consists of the two half lines

$$L_1 = \{(v, w, x)\colon v = 0,\ w = 0,\ x > 0\},$$
$$L_2 = \{(v, w, x)\colon v = B,\ w = 0,\ x > 0\}$$

and the curve

$$C = \{(v, w, x)\colon v = 0,\ w^{p-1} = x/\beta,\ x > 0\}.$$

(See Figure 1.)

On this curve C we define for every $\gamma > 0$ the point

$$P_\gamma = (0, (\gamma/\beta)^{1/(p-1)}, \gamma).$$

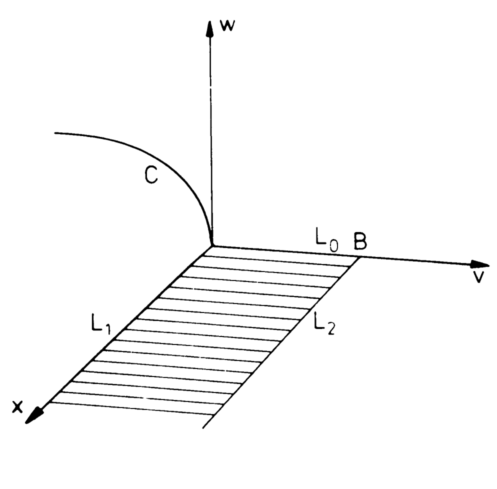

FIGURE 1. The sets L_1, L_2 and C

Near P_γ, the flow of (IV) is described by the linear system obtained by linearizing (IV) at P_γ. The coefficient matrix M of this system is given by

$$M = \begin{pmatrix} -(p-2)\frac{\gamma}{\beta} & 0 & 0 \\ \frac{\alpha-N+1}{\beta} & (p-1)\frac{\gamma}{\beta} & -\frac{1}{\beta}(\frac{\gamma}{\beta})^{1/(p-1)} \\ (p-1)(\frac{\gamma}{\beta})^{(p-2)/(p-1)} & 0 & 0 \end{pmatrix}.$$

Its eigenvalues are

$$\lambda_1 = -(p-2)\frac{\gamma}{\beta}, \quad \lambda_2 = (p-1)\frac{\gamma}{\beta}, \quad \lambda_3 = 0$$

and their corresponding eigenvectors:

$$\begin{aligned} e_1 &= \left((p-2)\frac{\gamma}{\beta}, -\frac{(\alpha-N)(p-2)+2p-3}{(2p-3)\beta}, -(p-1)\left(\frac{\gamma}{\beta}\right)^{(p-2)/(p-1)}\right), \\ e_2 &= (0,1,0), \\ e_3 &= (0,(\gamma/\beta)^{1/(p-1)},(p-1)\gamma). \end{aligned}$$

In view of (3.4) we look for an orbit which enters a point P_γ on C. For such orbits we have the following result [**5**, p. 127].

LEMMA 3. *For each $\gamma > 0$ there exists a unique (up to translation) nonconstant solution*

$$U(t,\gamma) = (v(t,\gamma), w(t,\gamma), x(t,\gamma))$$

of (IV) *such that* $\lim_{t\to\infty} U(t,\gamma) = P_\gamma$. *In addition, as $t \to \infty$, $U(t,\gamma)$ enters P_γ along the eigenvector e_1 at P_γ.*

Because λ_1 and e_1 depend continuously on γ, we may choose the translation so that $U(0,\gamma)$ is a continuous function. Thus, from classical results on the continuous dependence of solutions of ordinary differential equations on the initial value and a parameter in the equation, we may deduce the following lemma.

LEMMA 4. *Let $\gamma^* > 0$. For given constants $\varepsilon, T > 0$ there exists a constant $\delta > 0$ such that if $|\gamma - \gamma^*| < \delta$, then*

$$|U(t,\gamma) - U(t,\gamma^*)| \le \varepsilon \quad \text{for all } t \ge T.$$

Here $|\cdot|$ denotes the usual norm in $\mathbf{R}^3$.

To prove the existence of a solution u of Problem (II) which converges to zero at infinity sufficiently fast, we use a shooting argument, shooting backward from P_γ for different values of $\gamma > 0$. The idea of the proof will then be to show that there exists a $\gamma_0 > 0$ and a time $T_0 \in (-\infty, \infty)$ such that

$$\lim_{t \downarrow T_0} U(t,\gamma_0) \in L_0 \stackrel{\text{def}}{=} \{(v,w,x) \colon v > 0,\ w = 0,\ x = 0\}.$$

The function $U(t,\gamma_0)$ is then readily found to correspond to a solution of Problem (II).

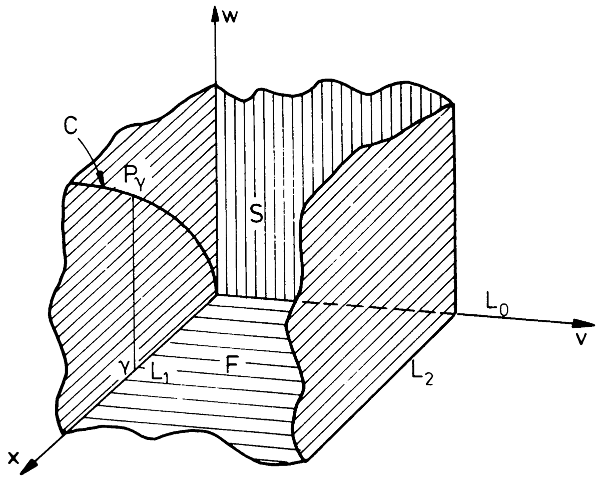

FIGURE 2. The sets S and F

4. The existence proof. We shall need the sets

$$S = \{(v, w, x) \colon 0 < v < B,\ w > 0,\ x > 0\},$$

$$F = \{(v, w, x) \colon 0 < v < B,\ w = 0,\ x > 0\},$$

where B was defined in (3.5). (See Figure 2.)

Because the vector e_1 points from P_γ into S, the orbit $U(t, \gamma)$, when it emerges from P_γ as time runs backward, will enter S. We shall denote the time when it leaves S again—time still running backward—by T_γ:

$$T_\gamma = \sup\{t \colon U(t, \gamma) \notin S\}.$$

If it does not leave S, i.e. $U(t, \gamma) \in S$ for all $t \in (T_{\min}, \infty)$, where $(T_{\min}, \infty)$ is its maximal interval of existence, we set $T_\gamma = T_{\min}$.

We shall denote the set of values of $\gamma \in \mathbf{R}^+$ for which $U(t, \gamma)$ leaves S through the bottom by G:

$$G = \{\gamma > 0 \colon T_\gamma > -\infty \text{ and } U(T_\gamma, \gamma) \in F\}.$$

LEMMA 5. *Suppose $\alpha > N$. Then there exists a number $\gamma_1 > 0$ such that if $\gamma \in (0, \gamma_1)$, then $\gamma \in G$.*

PROOF. Choose

$$\gamma_1 = \min\left\{\left(\frac{\alpha - N}{\beta}\right)^{1/\mu},\ \beta\left(\frac{p-1}{p-2}\right)^{p-1}\right\}$$

and define the set

$$S_1 = \{(v, w, x) \colon 0 < v < \gamma_1,\ 0 < x < \gamma_1 - v,\ 0 < w < (x/\beta)^{1/(p-1)}\}.$$

(See Figure 3.)

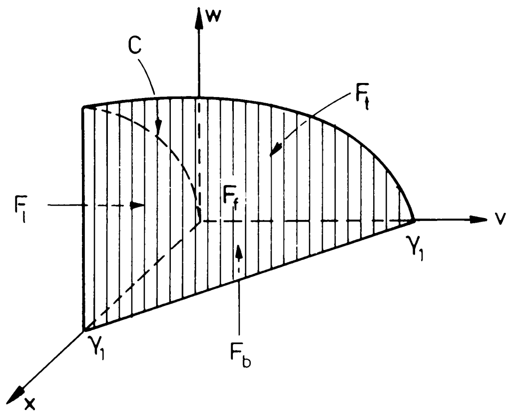

FIGURE 3. The set S_1

We compute where on the boundary ∂S_1 of S_1, $U(t,\gamma)$ may enter or leave S_1.

On the front F_f of S_1,

$$F_f = \{(v,w,x)\colon 0 < v < \gamma_1,\ x = \gamma_1 - v,\ 0 < w < (x/\beta)^{1/(p-1)}\}$$

the outward pointing normal vector is given by $n_f = (1,0,1)$ (not normalized). Let X denote the vector field determined by (IV). Then

$$\begin{aligned} n_f \cdot X &= vw^{p-2}\{(p-1) - (p-2)w\} \\ &> vw^{p-2}\{(p-1) - (p-2)(\gamma_1/\beta)^{1/(p-1)}\} > 0 \end{aligned}$$

in view of the definition of γ_1. Thus, F_f is part of the exit set of S_1.

On the top F_t of S_1,

$$F_t = \{(v,w,x)\colon 0 < v < \gamma_1,\ 0 < x < \gamma_1 - v,\ x = \beta w^{p-1}\}$$

the outward pointing normal n_t is given by

$$n_t = (0, (p-1)\beta w^{p-2}, -1)$$

and

$$\begin{aligned} n_t \cdot X &= (p-1)\beta w^{p-2}\left\{g(v) - \frac{N-1}{x} v w^{p-1}\right\} - (p-1)vw^{p-2} \\ &= (p-1)vw^{p-2}\{(\alpha - N) - \beta v^{\mu}\} > 0, \end{aligned}$$

where we have used the definition of γ_1 again. Thus F_t is also part of the exit set of S_1.

On the left-hand side F_l of S_1,

$$F_l = \{(v,w,x)\colon v = 0,\ 0 < x < \gamma_1,\ 0 < w < (x/\beta)^{1/(p-1)}\},$$

we have $v' = 0$, $x' = 0$ and

$$w' = w\{w^{p-1} - (x/\beta)\} < 0.$$

Therefore, this side is invariant, and orbits in F_l converge to L_1.

Finally, on the bottom F_b of S_1:

$$F_b = \{(v,w,x)\colon 0 < v < \gamma_1,\ w = 0,\ 0 < x < \gamma_1 - v\}$$

we have $w' = g(v) > 0$ because $\gamma_1 < B$. Hence F_b is part of the entrance set of S_1.

Thus, we have shown that the entrance set of S_1 consists of only F_b, and therefore that if a trajectory leaves S_1 when time runs backward, it must do so through F_b.

Because $F_b \subset F$, it remains to prove that (i) the trajectory $U(t,\gamma)$ enters S_1 from $P_\gamma \in C$ when t decreases from $+\infty$, and (ii) that it must leave S_1 again at some $t_0 > -\infty$. Now, according to Lemma 3, $U(t,\gamma)$ enters P_γ along the vector $-e_1$ at P_γ. Thus as regards (i) it suffices to prove that the vector

$$e_1 = \left((p-2)\frac{\gamma}{\beta}, -\frac{(\alpha-N)(p-2)+2p-3}{(2p-3)\beta}, -(p-1)\left(\frac{\gamma}{\beta}\right)^{(p-2)/(p-1)}\right)$$

points from P_γ into S_1.

The point P_γ lies on the curve where the left-hand side F_l and the top F_t of S_1 intersect. The outward pointing normals on F_l and F_t are

$$n_l = (-1,0,0), \quad n_t = (0,(p-1)\beta(\gamma/\beta)^{(p-2)/(p-1)}, -1).$$

Therefore

$$e_1 \cdot n_l = -(p-2)\gamma/\beta < 0$$

and

$$e_1 \cdot n_t = -\frac{(p-1)(p-2)}{2p-3}(\alpha - N)\left(\frac{\gamma}{\beta}\right)^{(p-2)/(p-1)} < 0$$

because $\alpha > N$, whence indeed, e_1 points into S_1.

To prove (ii), suppose to the contrary that $U(t,\gamma) \in S_1$ for all $t \in \mathbf{R}$. Then, by (IV),

$$v(t) \nearrow \bar{v}, \quad x(t) \searrow \bar{x} \qquad \text{as } t \to -\infty, \tag{4.1}$$

where $0 < \bar{v} \leq \gamma_1$ and $0 \leq \bar{x} < \gamma_1$. In addition

$$\liminf_{t\to-\infty} w(t) = 0, \tag{4.2}$$

since otherwise $x(t) \to -\infty$ as $t \to -\infty$, and $U(t,\gamma)$ would leave S_1 as t decreases. Also, because $w^{p-1}(t) < x(t)/\beta$ when $U(t,\gamma) \in S_1$,

$$w' \geq w^p - \frac{x}{\beta}w + \frac{\alpha - N + 1}{\beta}v - v^{1+\mu} > w^p - \frac{x}{\beta}w + \frac{1}{\beta}v$$

because $\alpha > N$ and $v < \gamma_1 \leq \{(\alpha-N)/\beta\}^{1/\mu}$. Thus, in view of (4.1),

$$w' > w^p - \frac{\gamma_1}{\beta}w + \frac{1}{2\beta}\bar{v} \tag{4.3}$$

for $-t$ large enough.

It follows from (4.3) that there exists a number $w_1 > 0$ such that if for some time $t_1 \in \mathbf{R}$, $w(t_1) < w_1$, then there exists a time $t_2 < t_1$ such that $w(t_2) = 0$ and $w'(t_2) > 0$. Thus, in that case, $U(t,\gamma)$ would leave S_1 at $t = t_2$ in backward time. Since, by assumption, this cannot happen, we must conclude that $w(t) \geq w_1$ for all $-t$ sufficiently large. This contradicts (4.2), and completes the proof of (ii).

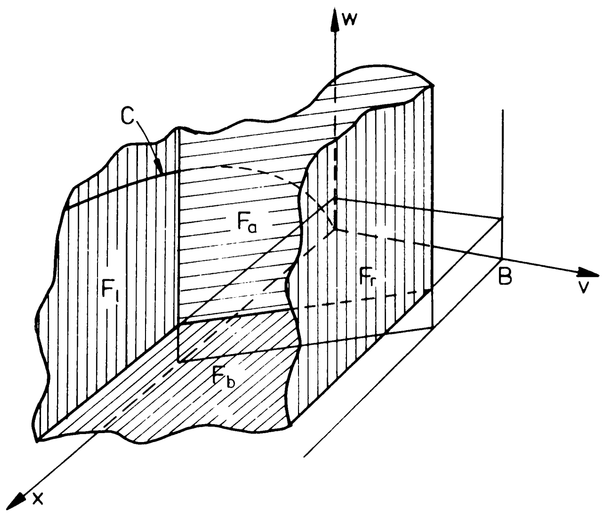

FIGURE 4. The set S_2

LEMMA 6. *There exists a number* $\gamma_2 > 0$ *such that if* $\gamma > \gamma_2$, *then* $\gamma \notin G$.

PROOF. Set

$$M = \max\{g(v)\colon 0 \le v \le B\}, \quad \gamma_2 = \beta(1+M) + \frac{p-1}{p-2}B$$

and define the set S_2 (see Figure 4).

$$S_2 = \left\{(v,w,x)\colon 0 < v < B,\ w > 1,\ x > \gamma_2 - \frac{p-1}{p-2}v\right\}.$$

As in the proof of Lemma 5, we determine the entrance set and the exit set of S_2.

On the bottom F_b of S_2:

$$F_b = \left\{(v,w,x)\colon 0 < v < B,\ w = 1,\ x > \gamma_2 - \frac{p-1}{p-2}v\right\}$$

w' is given by

$$w' = 1 - \frac{N-1}{x}v - \frac{x}{\beta} + g(v).$$

Hence, because $g(v) < M$,

$$w' < 1 + M - \frac{1}{\beta}\left(\gamma_2 - \frac{p-1}{p-2}B\right) = 0$$

in view of the definition of γ_2. Therefore, F_b is part of the exit set of S_2.

On the back F_a of S_2:

$$F_a = \left\{(v,w,x)\colon 0 < v < B,\ w > 1,\ x = \gamma_2 - \frac{p-1}{p-2}v\right\}$$

an outward pointing normal vector n_a is given by

$$n_a = \left(-\frac{p-1}{p-2}, 0, -1\right).$$

Therefore, on F_a,

$$n_a \cdot X = (p-1)vw^{p-2}(w-1) > 0$$

whence F_a is also part of the exit set of S_2.

On the right-hand side F_r of S_2:

$$F_r = \left\{(v,w,x)\colon v = B,\ w \geq 1,\ x \geq \gamma_2 - \frac{p-1}{p-2}B\right\}$$

we have

$$v' = -(p-2)Bw^{p-1} < 0,$$

so that F_r is part of the entrance set of S_2.

Finally, as we saw in the proof of Lemma 5, the left-hand side F_l of S_2 is invariant.

Thus, we have found that the entrance set of S_2 is F_r. Therefore, as time is reversed, orbits can only leave S_2 through F_r.

In view of the definition of γ_2, $P_\gamma \in F_l$ if $\gamma > \gamma_2$. Since e_1 points into S_2, it follows that $U(t,\gamma) \in S_2$ if $\gamma > \gamma_2$ and t is large. As we have seen $U(t,\gamma)$ can only leave S through F_r. Because $F_r \cap F = \varnothing$ this means that $\gamma \notin G$ if $\gamma > \gamma_2$.

We may conclude from Lemmas 5 and 6 that as t decreases from $+\infty$, $U(t,\gamma_0)$ enters S and *must* leave it again through F when γ is small, and *cannot* leave it through F when γ is large. Thus, if $\gamma_0 = \sup\{\gamma > 0\colon \gamma \in G\}$ then $\gamma_0 < \infty$. We shall show that $U(t,\gamma_0)$ is a solution of (IV) which passes through or tends to the half-line

$$L_0 = \{(v,w,x)\colon v > 0,\ w = 0,\ x = 0\}$$

as t decreases.

Let

$$T_0 = \inf\{T \in [-\infty,\infty)\colon U(t,\gamma_0) \in S \text{ for all } T < t < \infty\}.$$

We shall consider the cases $T_0 > -\infty$ and $T_0 = -\infty$ in succession. (i) $T_0 > -\infty$. Since $U(t,\gamma)$ depends continuously on γ for t bounded (Lemma 4), $U(T_0,\gamma_0) \in \partial F$. In addition, $x' > 0$ when $U \in S$, and therefore $U(T_0,\gamma_0) \in \partial F \cap \{x < \gamma_0\}$. However, the half-lines

$$L_1 = \{(v,w,x)\colon v = 0,\ w = 0,\ x > 0\}$$

and

$$L_2 = \{(v,w,x)\colon v = B,\ w = 0,\ x > 0\}$$

are invariant. Therefore, we are left with the conclusion that $U(T_0,\gamma_0) \in L_0 \cap \{v \leq B\}$.

(ii) $T_0 = -\infty$. In this case $x'(t) > 0$ and $x(t) > 0$ for all $t \in \mathbf{R}$ and hence $x(t) \to x_0$ as $t \to -\infty$, where $x_0 \geq 0$. Similarly, $v'(t) < 0$ and $v(t) < B$ for all $t \in \mathbf{R}$ whence $v(t) \to v_0$ as $t \to -\infty$, where $0 < v_0 \leq B$. It follows because x' has one sign that $\liminf_{t\to-\infty} x'(t) = 0$. This means, since $v_0 > 0$, that $\liminf_{t\to-\infty} w(t) = 0$.

We assert that

(a) $x_0 = 0$;

(b) $\lim_{t\to-\infty} w(t) = 0$.

To prove (a), suppose that $x_0 > 0$. Then there exists a sequence $\{t_k\}$, $t_k \to -\infty$ as $k \to \infty$ such that $U(t_k,\gamma_0) \to F \cup L_2$ as $k \to \infty$. However, because orbits leave S through F when time runs backward and L_2 is invariant, this possibility must be ruled out, and (a) is proved.

Suppose now that (b) does not hold, i.e., there exists a number $\varepsilon > 0$ and a sequence $\{\tau_k\}$, $\tau_k \to -\infty$ as $k \to \infty$ such that $w(\tau_k) \geq \varepsilon$ and $w'(\tau_k) = 0$ for all $k \geq 1$. Writing $v_k = v(\tau_k)$, $w_k = w(\tau_k)$ and $x_k = x(\tau_k)$, the equation for w' becomes

$$\frac{N-1}{x_k} v_k w_k^{p-1} = w_k^p - \frac{x_k}{\beta} w_k + g(v_k).$$

If we now let $k \to \infty$, the left-hand side becomes unbounded, whilst the right-hand side remains bounded. Hence, we obtain a contradiction, and (b) is proved as well.

We conclude from (a) and (b) that $U(t, \gamma_0) \to L_0 \cap \{v \leq B\}$ as $t \to -\infty$.

Finally, we conclude that both if $T_0 > -\infty$ and if $T_0 = -\infty$, the orbit $\{U(t, \gamma_0): T_0 \leq t < \infty\}$, where equality applies if $T_0 > -\infty$, connects the half-line L_0 to the curve C.

This completes the proof of Theorem 1.

REFERENCES

1. H. Brezis, L. A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. **95** (1986), 185–209.
2. M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Analysis (to appear).
3. A. Friedman and L. Veron, *Solutions singulieres d'équations quasilinéaires elliptiques*, C.R. Acad. Sci. Paris **302** (1986), 147–150.
4. ——, *Singular solutions of some quasilinear elliptic equations*, Arch. Rational Mech. Anal. **96** (1986), 259–287.
5. J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer, New York, 1983.
6. S. Kamin and L. A. Peletier, *Singular solutions of the heat equation with absorption*, Proc. Amer. Math. Soc. **95** (1985), 205–210.
7. L. Oswald, *Isolated, positive singularities for a non-linear heat equation*, Houston J. Math. (to appear).
8. L. A. Peletier and D. Terman, *A very singular solution of the porous media equation with absorption*, J. Differential Equations **65** (1986), 396–410.

MATHEMATICAL INSTITUTE, UNIVERSITY OF LEIDEN, LEIDEN, THE NETHERLANDS

DEPARTMENT OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN, JILIN, CHINA