SUBORDINATION FAMILIES AND EXTREME POINTS

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ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a univalent function F in Δ the unit disk. Let B_0 denote the set of functions $\phi(z)$ analytic in Δ satisfying $|\phi(z)| < 1$ and $\phi(0) = 0$. We prove that if $f = F \circ \phi$ is an extreme point of $s(F)$, then ϕ is an extreme point of B_0 . Let $D = F(s)$ and $\lambda(w, \partial D)$ denote the distance between w and ∂D (boundary of D). We also prove that if ϕ is an extreme point of B_0 and $|\phi(e^{it})| < 1$ for almost all t , then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt = -\infty$ for almost all θ .

Introduction. Let $\Delta = \{z: |z| < 1\}$ and let $\mathbf{A}$ denote the set of functions analytic in Δ . Let B_0 denote the subset of $\mathbf{A}$ consisting of all functions ϕ that satisfy the conditions $|\phi(z)| < 1$, $\phi(0) = 0$. Let EB_0 denote the extreme points of B_0 . Let S denote the subset of $\mathbf{A}$ consisting of univalent functions f so that $f(z) = z + \dots$ in Δ .

Let F be in $\mathbf{A}$ and be univalent in Δ . Let $s(F)$ denote the subset of $\mathbf{A}$ consisting of functions f that are subordinate to F in Δ . This means that $f \in \mathbf{A}$, $f(0) = F(0)$, and $f(\Delta) \subset F(\Delta)$. These conditions are equivalent to the existence of $\phi \in B_0$ so that $f = F \circ \phi$. Note that $s(F) = \{F \circ \phi: \phi \in B_0\}$.

Let D denote $F(\Delta)$. It is known that $F \in H^p$ for all $p < 1/2$ [3, p. 50] and so if $f = F \circ \phi$ for $\phi \in B_0$, then $f \in H^p$ for $p < 1/2$ [3, pp. 10–11]. It follows that $\lim_{r \rightarrow 1} \int_1 f(re^{i\theta}) = f(e^{i\theta})$ exists almost everywhere.

We let $Es(F)$ denote the set of extreme points of $s(F)$. In [1] it was proved that if F' is in the Nevanlinna class and D is a Jordan domain subset to a half plane, then $Es(F) \subset \{F \circ \phi: \phi \in EB_0\}$. In that paper it was conjectured that this inclusion is valid for any univalent F . In [4] the inclusion was proved under the assumption F is univalent and D is a Jordan domain. In this paper we prove the inclusion for an arbitrary univalent function and so verify the conjecture made by the first author in [1].

In [1] it was also proved that if F' is in the Nevanlinna class and $\phi \in EB_0$, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$. It was conjectured that the integral was $-\infty$ for any univalent function F when $\phi \in EB_0$. (Note that this is trivially true if $|\phi(e^{it})| = 1$ on a set of positive measure since F is univalent.) A weaker conjecture is that if $F \circ \phi \in Es(F)$ and F is univalent, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty$.

In this paper we show that this weaker conjecture holds when F' is in the Nevanlinna class [3, p. 16]. We also take this opportunity to point out that in the proof

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of Theorem 8.25 in [5, pp. 143–144] there is a proof of this weaker conjecture when $F(\Delta)$ is a Jordan domain and no assumption on F' is made. Also it is known that $\int_0^{2\pi} \log \lambda(f(\phi(e^{it})), \partial D) dt = -\infty$ does not in general imply $F \circ \phi \in Es(F)$. This can be easily seen by considering the case $F(z) = ((1+z)/(1-z))^\alpha$ for $0 < \alpha < 1$ [5, pp. 131, 133]. We prove the interesting result that if F is univalent and $\phi \in EB_0$, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt = -\infty$ for almost all θ .

Functions subordinate to a univalent function.

THEOREM 1. *If F is a univalent function analytic in Δ and $f = F \circ \phi$ is an extreme point of $s(F)$, then ϕ is an extreme point of B_0 .*

PROOF. Assume $f = F \circ \phi \in Es(F)$ and $\phi \notin EB_0$. Then $\int_0^{2\pi} \log(1 - |\phi(e^{it})|) dt > -\infty$ [3, p. 125]. Clearly $|\phi(e^{it})| < 1$ almost everywhere and $\log(1 - |\phi(e^{it})|) \in L^1$. Let $P_z(t)$ for $|z| < 1$ denote $\operatorname{Re}((e^{it} + z)/(e^{it} - z))$. Since $\phi \in H^1$ we have $\phi(e^{it}) \in L^1$ and for $|z| < 1$, $\phi(z) = (1/2\pi) \int_0^{2\pi} P_z(t) \phi(e^{it}) dt$ [3, p. 34]. It follows that

$$(1) \quad 1 - |\phi(z)| \geq \frac{1}{2\pi} \int_0^{2\pi} P_z(t)(1 - |\phi(e^{it})|) dt.$$

Applying Jensen's inequality we obtain from (1)

$$(2) \quad \log(1 - |\phi(z)|) \geq \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|) dt.$$

Hence we deduce from (2) that

$$(3) \quad \exp \left(\frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|)^2 dt \right) \leq (1 - |\phi(z)|)^2.$$

Now define

$$g(z) = \frac{z}{16} \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log(1 - |\phi(e^{it})|)^2 dt.$$

Note that $g(0) = 0$, $g \not\equiv 0$, and g is analytic in Δ . We also have

$$|g(z)| < \frac{1}{16} \exp \left(\frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|)^2 dt \right)$$

and so this fact and (3) imply

$$(4) \quad |g(z)| < \frac{1}{16}(1 - |\phi(z)|)^2.$$

Since F is univalent, it follows from [7, p. 22] that

$$(5) \quad \frac{1}{4}(1 - |z|^2)|F'(z)| \leq \lambda(F(z), \partial D) \leq (1 - |z|^2)|F'(z)|$$

for all z in Δ . We may assume, without loss of generality, that $F \in S$ and so it is known that

$$(6) \quad \frac{1 - |z|}{(1 + |z|)^3} \leq |F'(z)|$$

for all z in Δ [7, p. 21]. It follows from (5) and (6) that

$$(7) \quad \frac{1}{16}(1 - |z|)^2 \leq \lambda(F(z), \partial D).$$

Therefore $\frac{1}{16}(1 - |\phi(z)|)^2 \leq \lambda(F(\phi(z)), \partial D) = \lambda(f(z), \partial D)$. Hence, by (4) and (7) we have $|g(z)| < \lambda(f(z), \partial D)$ for all z in Δ . We conclude that $f(z) \pm g(z) \in F(\Delta)$ for all z in Δ . Since F is univalent, $f(0) = F(0)$, and $g(0) = 0$, we have $f \pm g \in s(F)$ and $g \equiv 0$. This contradicts the assumption that $f \in Es(F)$.

Hence $\phi \in EB_0$ and the proof is complete.

REMARK. The conjecture made by the first author in [1] is now proved.

THEOREM 2. *If F is a univalent function analytic in Δ , $f = F \circ \phi \in Es(F)$, and F' is in the Nevanlinna class, then*

$$(8) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D) dt = -\infty.$$

PROOF. Theorem 1 implies $F \circ \phi \in EB_0$ and so we have $\int_0^{2\pi} \log(1 - |\phi(e^{it})|) dt = -\infty$ [3, p. 125]. It follows from this fact and the assumption that F' is in the Nevanlinna class that (8) holds [1, p. 440].

REMARK. This result was previously known under the assumptions that $F(\Delta)$ is a Jordan domain and $\phi \in EB_0$ [5, pp. 143–144]. We conjecture that (8) holds in general. The following theorem is indirect evidence for this conjecture.

THEOREM 3. *If F is a univalent function analytic in Δ and $f = F \circ \phi \in Es(F)$, then*

$$(9) \quad \inf_r \int_0^{2\pi} \log \lambda(F(re^{it}), \partial D) dt = -\infty.$$

PROOF. We first prove that if $\phi \in B_0$, then

$$(10) \quad \inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt = \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt.$$

It is easy to verify by a computation that $-\log(1 - |z|^2)$ is a C^∞ subharmonic function in Δ . It follows easily by another computation that $-\log(1 - |\phi(z)|^2)$ is subharmonic in Δ and so $-\int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt$ is an increasing function of r [3, p. 9].

First consider the case $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt = -\infty$. Then by Fatou's lemma we have

$$(11) \quad +\infty = - \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt \leq \lim_{r \rightarrow 1} - \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt.$$

Since $-\int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt$ is an increasing function of r , we have

$$\lim_{r \rightarrow 1} - \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt = \sup_r - \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt = +\infty.$$

It follows that $\inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt = -\infty$ and (10) holds in this case.

Now suppose $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt > -\infty$. We have $\log(1 - |\phi(e^{it})|^2) \in L^1$, and it follows by the type of arguments used in the proof of Theorem 1 that

$$(12) \quad -\log(1 - |\phi(re^{i\theta})|^2) \leq \frac{-1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|^2) dt.$$

Hence by integrating both sides of (12), using Fubini's theorem and the fact that $(1/2\pi) \int_0^{2\pi} P_z(t) d\theta = 1$ we have

$$(13) \quad - \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt \leq - \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt.$$

It follows that

$$(14) \quad \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt \leq \inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt.$$

A direct application of Fatou's lemma gives

$$(15) \quad - \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt \leq \varliminf_{r \rightarrow 1} - \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt.$$

As we have seen in the preceding argument, the expression on the right-hand side of (15) is equal to $-\inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt$ and so we have

$$(16) \quad \inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt \leq \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt.$$

It follows from (14) and (16) that (10) holds in this case.

To complete the proof of the theorem we may assume without loss of generality that $F \in S$. Now suppose $f = F \circ \phi \in Es(F)$. Then by Theorem 1, $\phi \in EB_0$. Since $F'(z) \neq 0$ in Δ , we have $\log |F'(z)|$ and $\log |F'(\phi(z))|$ harmonic in Δ . It follows that

$$\int_0^{2\pi} \log |F'(\phi(re^{it}))| dt = 2\pi \log |F'(\phi(0))| = 2\pi \log |F'(0)| = 0$$

since $F'(0) = 1$. It follows from this fact and (5) that

$$(17) \quad \int_0^{2\pi} \log \lambda(F(\phi(re^{it})), \partial D) dt \leq \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2) dt.$$

It follows from (10), (17) and the fact that $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt = -\infty$ when $\phi \in EB_0$ that (9) holds and the proof is complete.

The next two theorems are technical results needed for the proof of Theorem 6.

THEOREM 4. *If F is a bounded univalent function analytic in Δ , $\phi \in B_0$, $|\phi(e^{it})| < 1$ for almost all t , then*

$$(18) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| dt < +\infty$$

for almost all θ .

PROOF. It is easy to see that the integrand in (18) is a measurable function of (t, θ) on $[0, 2\pi] \times [0, 2\pi]$ since it is almost everywhere the limit of continuous functions on $[0, 2\pi] \times [0, 2\pi]$. Since F is univalent and bounded, it follows easily from Theorem 5.2 in [7, p. 129] that there exists a constant B such that

$$(19) \quad \int_0^{2\pi} |F'(re^{i\theta})| d\theta \leq \frac{B}{(1-r)^{1/2-1/320}}$$

for all $r < 1$. Since $|\phi(e^{it})| < 1$ for almost all t , (19) implies

$$(20) \quad \int_0^{2\pi} |F'(\phi(e^{it})e^{i\theta})| d\theta \leq \frac{B}{(1 - |\phi(e^{it})|)^{1/2 - 1/320}}$$

for almost all t . It follows from (20) that

$$(21) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| d\theta \leq B$$

for almost all t . Hence (21) implies that

$$(22) \quad 0 \leq \int_0^{2\pi} \left(\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| d\theta \right) dt < +\infty.$$

The Tonelli-Hobson theorem and (22) imply that

$$(23) \quad 0 \leq \int_0^{2\pi} \left(\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| d\theta \right) dt < +\infty.$$

It follows from (23) that (18) holds and the proof is complete.

REMARK. The set of θ of measure 2π on which (18) holds depends on ϕ . Also the exponent $1/2$ in (18) is not the smallest possible. Clearly $1/2 - 1/320$ will also suffice.

THEOREM 5. *If F is a univalent function analytic in Δ , $\phi \in B_0$, and $|\phi(e^{it})| < 1$ for almost all t , then*

$$(24) \quad \int_0^{2\pi} \log \left[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| \right] dt < +\infty$$

for almost all θ .

PROOF. We first show that it is sufficient to consider the case F is univalent and bounded. If F is not bounded, then $F(\Delta)$ cannot be the entire plane, so there is a point b in the complement of $F(\Delta)$. By a simple argument [8, pp. 302–303] there is a number c such that $g(z) = [(F(z) - b)^{1/2} + c]^{-1}$ is univalent and bounded in Δ . So $F(z) = b + (1/g(z) - c)^2$ for $|z| < 1$ and g is a bounded univalent function. It follows that we have

$$(25) \quad \begin{aligned} & \int_0^{2\pi} \log \left[(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| \right] dt \\ &= 2\pi \log 2 + \int_0^{2\pi} \log |1 - cg(\phi(e^{it})e^{i\theta})| dt - 3 \int_0^{2\pi} \log |g(\phi(e^{it})e^{i\theta})| dt \\ & \quad + \int_0^{2\pi} \log (1 - |\phi(e^{it})|^2)^{1/2} |g'(\phi(e^{it})e^{i\theta})| dt. \end{aligned}$$

Since $1 - cg(z)$ and $g(z)$ are in H^p for all $p < 1/2$ [3, p. 50], the first and second integrals on the right-hand side of (25) are finite [3, p. 17]. Since we will prove that the third integral on the right is finite, the proof will be complete.

Now suppose g is a bounded univalent function analytic in Δ . Theorem 4 implies that

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |g'(\phi(e^{it})e^{i\theta})| dt < +\infty$$

for almost all θ . It is easy to deduce (24) from this fact and this completes the proof.

REMARK. The set of θ of measure 2π on which (24) holds depends on ϕ .

THEOREM 6. *If F is a univalent function analytic in Δ , $\phi \in EB_0$, and $|\phi(e^{it})| < 1$ for almost all t , then*

$$(26) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt = -\infty$$

for almost all θ .

PROOF. Since F is univalent we have from (5) that

$$(27) \quad \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) \leq (1 - |\phi(e^{it})|^2) |F'(\phi(e^{it})e^{i\theta})|$$

for almost all t and all θ . It follows from (27) that

$$(28) \quad \begin{aligned} & \int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt \\ & \leq \frac{1}{2} \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt + \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})e^{i\theta})| dt. \end{aligned}$$

Since $\phi \in EB$, we have $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2) dt = -\infty$ [3, p. 125] and so (26) follows from this fact, (24), and (28).

REMARK. The set of θ of measure 2π on which (26) holds depends on ϕ .

COROLLARY. *If F is a univalent function analytic in Δ , $\phi \in B_0$, and $F \circ \phi \in Es(F)$, then*

$$(29) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D) dt = -\infty$$

for almost all θ .

PROOF. This follows directly from Theorem 1 and Theorem 6.

Our next result generalizes the previous theorem.

THEOREM 7. *If F is a univalent function analytic in Δ , $\phi \in EB_0$, $|\phi(e^{it})| < 1$ for almost all t , and ψ is an inner function such that $|\psi(0)| \neq 1$, then*

$$(30) \quad \int_0^{2\pi} \log \lambda(F(\phi(e^{it})\psi(e^{i\theta})), \partial D) dt = -\infty$$

for almost all θ .

PROOF. We just prove that under the assumption that F is a bounded univalent function in Δ then

$$(31) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})\psi(e^{i\theta}))| dt < +\infty$$

for almost all θ . The proof of (30) follows from (31) in the way that Theorem 6 was deduced from Theorems 4 and 5. We note that Fatou's lemma gives

$$(32) \quad \begin{aligned} & \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})\psi(e^{i\theta}))| d\theta \\ & \leq \lim_{r \rightarrow 1} \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})\psi(re^{i\theta}))| d\theta. \end{aligned}$$

Littlewood's inequality [3, pp. 10–11; 6, p. 243] can easily be generalized so that we have

$$(33) \quad \int_0^{2\pi} |F'(\phi(e^{it})\psi(re^{i\theta}))| d\theta \leq \frac{r + |\psi(0)|}{r - |\psi(0)|} \int_0^{2\pi} |F'(\phi(e^{it})re^{i\theta})| d\theta$$

whenever $|\psi(0)| < r$. Since F is a bounded univalent function, (19) and (13) imply that when $|\psi(0)| < r$,

$$(34) \quad \int_0^{2\pi} |F'(\phi(e^{it})\psi(re^{i\theta}))| d\theta \leq \frac{r + |\psi(0)|}{r - |\psi(0)|} \frac{B}{(1 - |\phi(e^{it})|r)^{1/2-1/320}}.$$

It follows from (32) and (34) that we have

$$(35) \quad \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})\psi(e^{i\theta}))| d\theta \leq \frac{1 + |\psi(0)|}{1 - |\psi(0)|} B.$$

We infer from (35) that

$$(36) \quad \int_0^{2\pi} \left(\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2} |F'(\phi(e^{it})\psi(e^{i\theta}))| d\theta \right) dt < +\infty.$$

Hence (31) follows from (36) and the Tonelli-Hobson theorem.

REMARK. The set of θ of measure 2π on which (30) holds depends on ϕ and ψ .

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