THE BRAUER GROUP OF GRADED CONTINUOUS TRACE C^* -ALGEBRAS

ELLEN MAYCOCK PARKER

ABSTRACT. Let X be a locally compact Hausdorff space. The graded Morita equivalence classes of separable, $\mathbf{Z}_2$ -graded, continuous trace C^* -algebras which have spectrum X form a group, $\mathrm{GBr}^\infty(X)$, the infinite-dimensional graded Brauer group of X . Techniques from algebraic topology are used to prove that $\mathrm{GBr}^\infty(X)$ is isomorphic via an isomorphism w to the direct sum $\check{H}^1(X; \mathbf{Z}_2) \oplus \check{H}^3(X; \mathbf{Z})$. The group $\mathrm{GBr}^\infty(X)$ includes as a subgroup the ungraded continuous trace C^* -algebras, and the Dixmier-Douady invariant of such an ungraded C^* -algebra is its image in $\check{H}^3(X; \mathbf{Z})$ under w .

Introduction. The study of graded C^* -algebras has become particularly important since G. G. Kasparov's development of KK -theory for operator algebras [18]. In this paper, separable, $\mathbf{Z}_2$ -graded, continuous trace C^* -algebras are classified. The graded Morita equivalence classes of such algebras whose spectra are all the same locally compact Hausdorff space X form a group, called the infinite-dimensional graded Brauer group of X and denoted by $\mathrm{GBr}^\infty(X)$. Two invariants defined on $\mathrm{GBr}^\infty(X)$ provide useful insights into the structure of these C^* -algebras and relate the results presented here to previous work.

The constructions of J. Dixmier and A. Douady [3, 4, 5] form an important framework for the graded classification. Let X be a locally compact Hausdorff space, with countable base. Dixmier and Douady considered separable, stable, continuous trace C^* -algebras, with spectrum X . There is a canonical way to associate such an algebra A with a fiber bundle ξ_A over X with fiber $\mathcal{K}(\mathcal{H})$, the compact operators on an infinite-dimensional separable Hilbert space. Let $\mathcal{PU}(\mathcal{H})$ be the projective unitary group of $\mathcal{H}$, and let $\check{H}^*(X; \underline{G})$ denote the Čech cohomology of X with coefficients in the sheaf of germs of continuous functions from X to G , for G a group. Then the isomorphism class of ξ_A is an element of $\check{H}^1(X; \underline{\mathcal{PU}(\mathcal{H})})$, which can be shown to be isomorphic to $\check{H}^3(X; \mathbf{Z})$. They defined the Dixmier-Douady invariant $\delta(A) \in \check{H}^3(X; \mathbf{Z})$ of the algebra A , and proved that the invariant defines a one-to-one correspondence between isomorphism classes of such algebras and the elements of $\check{H}^3(X; \mathbf{Z})$.

Consider now the collection of graded, separable, continuous trace C^* -algebras, all with spectrum X . We will define $\mathrm{GBr}^\infty(X)$ as the set of equivalence classes of all such C^* -algebras under graded Morita equivalence, which is the graded version of strong Morita equivalence defined by M. Rieffel [22, 23]. It is important to note,

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however, that each equivalence class of $\mathrm{GBr}^\infty(X)$ can be uniquely represented, up to spectrum-preserving graded $*$ -isomorphism, by a C^* -algebra which is a separable, graded, stable, continuous trace C^* -algebra, with spectrum X . In the first sections of the paper, we choose such a representation, and delay a more thorough discussion of graded Morita equivalence until §5.

Let A be a separable, graded, stable, continuous trace C^* -algebra, with spectrum X . In this paper, a graded fiber bundle ξ_A is constructed from A using techniques parallel to those in the ungraded case. If $x \in X$ is an irreducible representation, then $A/\ker(x)$ is shown to be isomorphic, via a map which preserves the grading, to $\mathcal{K}_{\mathrm{gr}}(\mathcal{H})$, the graded compact operators on a separable, infinite-dimensional graded Hilbert space $\mathcal{H}$. The fiber of ξ_A over x is then $A/\ker(x)$. A topology on the total space $E(\xi_A)$ is given, and a structure group $\mathcal{PU}_{\mathrm{gr}}(\mathcal{H})$ for the bundle is defined. The original algebra A can be retrieved by considering the set of sections of ξ_A which vanish at ∞ . The correspondence between A and ξ_A lies at the heart of the main result: that $\mathrm{GBr}^\infty(X)$ is isomorphic to $\check{H}^1(X; \mathcal{PU}_{\mathrm{gr}}(\mathcal{H}))$, which in turn is isomorphic to the direct sum $\check{H}^1(X; \mathbb{Z}_2) \oplus \check{H}^3(X; \mathbb{Z})$. The isomorphism

$$w: \mathrm{GBr}^\infty(X) \rightarrow \check{H}^2(X; \mathbb{Z}_2) \oplus \check{H}^3(X; \mathbb{Z})$$

defines invariants $w_1^*(A) \in \check{H}^1(X; \mathbb{Z}_2)$ and $w_2^*(A) \in \check{H}^3(X; \mathbb{Z})$ for $A \in \mathrm{GBr}^\infty(X)$. When A is ungraded, it is shown that $w_2^*(A) = \delta(A)$ and $w_1^*(A) = 1$. The group structure is analyzed and an explicit inverse to an element of $\mathrm{GBr}^\infty(X)$ is constructed.

The correspondence between graded continuous trace C^* -algebras and graded fiber bundles allows the finite-dimensional cases considered by J.-P. Serre [12], P. Donovan and M. Karoubi [6], and R. Patterson [19, 20] to be included in $\mathrm{GBr}^\infty(X)$. The invariants of Donovan and Karoubi agree with those defined here. In addition, by applying the work of J. Phillips and I. Raeburn [21], it is shown that $w_1^*(A)$ is the obstruction to the grading automorphism of A being an inner automorphism. Using a construction of P. Green [11] for the correspondence between the isomorphism classes of continuous trace C^* -algebras and $\check{H}^3(X; \mathbb{Z})$, an alternate definition for the isomorphism w is given. This definition allows some modifications in the equivalence relation on $\mathrm{GBr}^\infty(X)$ to be made. Further applications of the infinite-dimensional graded Brauer group are anticipated in the context of Kasparov's KK -theory.

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1. Preliminaries. Let X be a locally compact Hausdorff space. The C^* -algebra of continuous maps from X to $\mathbb{C}$ which vanish at ∞ will be denoted by $C_0(X)$. We will assume that $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space, with inner product denoted by $(\ , \)_{\mathcal{H}}$. The group of unitary operators on $\mathcal{H}$, equipped with the strong operator topology, will be denoted by $\mathcal{U}(\mathcal{H})$, and the C^* -algebra of bounded operators on $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. If I is the identity operator, then S^1 is included into $\mathcal{U}(\mathcal{H})$ by mapping $s \in S^1$ to $sI \in \mathcal{U}(\mathcal{H})$. The

quotient $\mathcal{U}(\mathcal{H})/S^1$ is the projective unitary group of $\mathcal{H}$, denoted by $\mathcal{PU}(\mathcal{H})$, and is given the quotient topology. Let $\mathcal{K}(\mathcal{H})$ be the C^* -algebra of compact operators on $\mathcal{H}$. The automorphism of $\mathcal{K}(\mathcal{H})$, denoted by $\text{Aut}(\mathcal{K})$, will be given the topology of pointwise convergence.

Let A be a C^* -algebra. The spectrum of A , denoted by $\hat{A}$, is given the Jacobson topology [3, 3.1]. In this paper, attention will be restricted to separable, continuous trace C^* -algebras, whose spectra are all Hausdorff [3, 4.5.3], and have a countable base [3, 3.3.4]. A hermitian element $a \in A$ is called a positive element of A if there exists a $y \in A$ with $y \cdot y^* = a$. Let A^+ denote the set of positive elements of A . If t is a cardinal, then A is homogeneous of degree t if $\dim(H_\pi) = t$ for every nonzero irreducible representation π of A .

1.1. DEFINITION. Let A be a C^* -algebra with spectrum X . Then A is a continuous trace C^* -algebra if X is Hausdorff, and if, for every $x \in X$, there is an element $a \in A^+$ and a neighborhood V_x of x in X such that $v(a)$ is a rank one projection for every $v \in V_x$.

This definition is equivalent to the standard one of a continuous trace C^* -algebra [3, 4.5.3, 4.5.4]. The above characterization will be especially useful here.

1.2. DEFINITION. Suppose that X is a locally compact Hausdorff space. Let ξ be a family of C^* -algebras $\{\xi(x)\}_{x \in X}$ together with a set of maps from X to $\bigcup_{x \in X} \xi(x)$, called sections and denoted by $\Gamma(\xi)$, such that

- (i) the set of sections forms a $*$ -algebra under pointwise operations;
- (ii) the set $\{s(x) : s \in \Gamma(\xi), x \in X\}$ is dense in $\xi(x)$;
- (iii) the mapping $s \mapsto \|s(x)\|$ is continuous for every $s \in \Gamma(\xi)$;
- (iv) if $s : X \rightarrow \bigcup_{x \in X} \xi(x)$, then $s \in \Gamma(\xi)$ if, for every $x \in X$ and $\varepsilon > 0$, there is an $s' \in \Gamma(\xi)$ and a neighborhood V of x in X such that $\|s(y) - s'(y)\| < \varepsilon$ for all y in V .

Then ξ is called a continuous field of C^* -algebras over X [3, 10.1.2, 21, 1.3].

Let $E(\xi) = \bigcup_{x \in X} \xi(x)$ be the total space of ξ . If $p: E(\xi) \rightarrow X$ by $p(y) = x$ for $y \in \xi(x)$, then $E(\xi)$ can be equipped with the tube topology [5, 1.2]. The set of sections of ξ which vanish at ∞ , denoted by $\Gamma_0(\xi)$, forms a C^* -algebra A with the norm defined by $\|s\| = \sup_{x \in X} \|s(x)\|$ for $s \in \Gamma_0(\xi)$. Its spectrum $\hat{A}$ is the space X [3, 10.4.1], and we can then consider an element of X to be an irreducible representation [3, 10.4.4]. Let ξ and ξ' be two continuous fields of C^* -algebras over X . A function $\varphi: \xi \rightarrow \xi'$ is an isomorphism if φ is the union $\bigcup_{x \in X} \varphi_x$ of isomorphisms $\varphi_x: \xi(x) \rightarrow \xi'(x)$ such that $\varphi(\Gamma(\xi)) = \Gamma(\xi')$. A continuous field of Hilbert spaces may be defined in a manner similar to the definition of a continuous field of C^* -algebras, where each $\xi(x)$ is a now separable Hilbert space [3, 10.1.2].

We recall some elementary sheaf theory. The references [27 and 2] provide more detail. Let X be a paracompact Hausdorff space. If G is an abelian group, then $\check{H}^*(X; \underline{G})$ is the Čech cohomology of X with coefficients in $\underline{G}$, the sheaf of germs of continuous functions from X into G . If G is nonabelian, then the cohomology set $\check{H}^1(X; \underline{G})$ can be defined [13, p. 38]. Let $0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$ be a short exact sequence of groups such that G_1 is contained in the center of G_2 ; we can then construct the following exact sequence [10]:

$$\cdots \rightarrow \check{H}^1(X; \underline{G}_1) \rightarrow \check{H}^1(X; \underline{G}_2) \rightarrow \check{H}^1(X; \underline{G}_3) \rightarrow \check{H}^2(X; \underline{G}_1).$$

1.3. DEFINITION. Let A be a C^* -algebra. Then A is a $(\mathbf{Z}_2\text{-})$ graded C^* -algebra if A can be expressed as the direct sum $A^{(0)} \oplus A^{(1)}$, where $A^{(i)}$, $i = 0, 1$, are selfadjoint, closed linear subspaces of A , closed under $*$, and such that if $a_i \in A^{(i)}$, $a_j \in A^{(j)}$, then $a_i a_j \in A^{(i+j)}$, where addition is modulo 2. If $a \in A^{(i)}$, then a is said to have degree i .

Alternatively, a grading on A may be induced from an automorphism α of order 2 on A in the following way. Let $A^{(i)} = \{a \in A: \alpha(a) = (-1)^i a\}$, $i = 0, 1$. Then $A^{(0)} \oplus A^{(1)} = A$ is a grading for A . If a grading for A is given, the automorphism α can be defined by $\alpha(a_0 + a_1) = a_0 + (-a_1)$. An element a of a graded C^* -algebra A is called homogeneous if $a \in A^{(i)}$. A C^* -algebra A is trivially graded if $A^{(0)} = A$ and $A^{(1)} = 0$. If A and B are two graded C^* -algebras, a $*$ -homomorphism $\psi: A \rightarrow B$ is graded if $\psi(A^{(i)}) \subset B^{(i)}$.

The grading of $\mathcal{H}(\mathcal{H})$ will now be constructed; the resulting graded C^* -algebra will be denoted by $\mathcal{H}_{\text{gr}}(\mathcal{H})$. First, we will say that $\mathcal{H}$ is a graded Hilbert space, if it is graded in the following way. Suppose that $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ are two copies of $\mathcal{H}$. Since there is an isomorphism $\mathcal{H} \approx \mathcal{H} \oplus \mathcal{H}$, we may write $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$. An alternate grading on $\mathcal{H}$ uses a unitary operator J of $\mathcal{H}$ with $J^2 = 1$: define $\mathcal{H}^{(i)} = \{h \in \mathcal{H}: J(h) = (-1)^i h\}$, where we consider only those operators J for which $\mathcal{H}^{(i)}$ is infinite-dimensional. A direct computation verifies that, if J and J' are two unitary operators of order 2 of a graded Hilbert space $\mathcal{H}$ which determine the same grading of $\mathcal{H}$, then $J = J'$.

An operator T on $\mathcal{H}$ is said to be of degree i , $i = 0, 1$, if $T(\mathcal{H}^{(j)}) \subset \mathcal{H}^{(i+j)}$, for $j = 0, 1$. Define a grading for $\mathcal{L}(\mathcal{H})$ by letting $\mathcal{L}^{(i)}(\mathcal{H})$ be the set of bounded operators of degree i . For convenience, a matrix is often used to describe a graded operator. A degree 0 operator can be represented by a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, where $A: \mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(0)}$ and $D: \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(1)}$. Similarly, a degree 1 operator can be written as $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ with $B: \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(0)}$ and $C: \mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(1)}$.

The compact operators on a graded Hilbert space $\mathcal{H}$ can be graded by defining $\mathcal{K}_{\text{gr}}^{(i)}(\mathcal{H})$ to be the compact operators of $\mathcal{H}$ of degree i . A unitary J of order 2 on $\mathcal{H}$ may also be used to define the grading on $\mathcal{K}(\mathcal{H})$ (respectively $\mathcal{L}(\mathcal{H})$); let $T \in \mathcal{K}_{\text{gr}}^{(i)}(\mathcal{H})$ (respectively $\mathcal{L}^{(i)}(\mathcal{H})$) if $JTJ^{-1} = (-1)^i T$, for $i = 0, 1$. We can easily check that if $J, J' \in \mathcal{U}(\mathcal{H})$ are of order 2 and induce the same grading on $\mathcal{K}_{\text{gr}}(\mathcal{H})$, then $J = \pm J'$. A graded elementary C^* -algebra is a graded C^* -algebra which is isomorphic to $\mathcal{K}_{\text{gr}}(\mathcal{H})$, for $\mathcal{H}$ a graded Hilbert space. The spectrum of a graded C^* -algebra A is the usual spectrum of A regarded as an ungraded algebra.

We now define the graded tensor product of A and B [24, p. 61; 18, 2.6]. Let A be a graded C^* -algebra. A graded state on A is a positive linear functional s defined on A such that $\|s\| = 1$ and $s = 0$ on $A^{(1)}$. If A and B are separable, graded continuous trace C^* -algebras, let $A \hat{\otimes} B$ denote the algebraic graded tensor product of A and B , where the elements of $A \hat{\otimes} B$ are graded by

$$\deg(a \hat{\otimes} b) = \deg(a) + \deg(b).$$

The product and involution are defined by

$$\begin{aligned} (a \hat{\otimes} b)(a' \hat{\otimes} b') &= (-1)^{\deg(b) \deg(a')} (aa' \hat{\otimes} bb'), \\ (a \hat{\otimes} b)^* &= (-1)^{\deg(a) \deg(b)} (a^* \hat{\otimes} b^*). \end{aligned}$$

If s and t are graded states on A and B , respectively, let

$$s\hat{\otimes}t(x^*x) = \sum_{i,j=1}^n s(a_i^*a_j)t(b_i^*b_j)$$

for $x = \sum_1^n a_j\hat{\otimes}b_j \in A\hat{\otimes}B$. Then a C^* -norm may be defined on $A\hat{\otimes}B$ by

$$\|x\|_*^2 = \sup_{s,t,y} \frac{s\hat{\otimes}t(y^*x^*xy)}{s\hat{\otimes}t(y^*y)}$$

where the supremum is taken over all graded states s on A , t on B , and over all $y \in A\hat{\otimes}B$ with $s\hat{\otimes}t(y^*y) \neq 0$. Let $A\hat{\otimes}B$ denote the completion of $A\hat{\otimes}B$ with respect to the norm $\|\cdot\|_*$.

Note that $A\hat{\otimes}B$ defined above is the graded analogue of the minimal tensor product of A and B . In the case considered here, A and B are continuous trace, so $A\hat{\otimes}B$ agrees with the graded version of the maximal tensor product [1, 16.4]. Thus there is no ambiguity when we refer to the graded tensor product $A\hat{\otimes}B$.

We say that a graded C^* -algebra A is stable if $A \approx A\hat{\otimes}\mathcal{K}_{\text{gr}}(\mathcal{H})$, via a graded $*$ -isomorphism. Let X be a locally compact Hausdorff space, with countable base. Then we define $\mathcal{S}(X)$ to be the category whose objects are separable, graded, stable, C^* -algebras with continuous trace, with spectrum X . We note that the grading of A must be nontrivial; in addition, we require that the grading automorphism α of A fix X . It is useful to observe that every element of $\mathcal{S}(X)$ is homogeneous of degree $\aleph_0$ [21, 1.12]. A morphism of $\mathcal{S}(X)$ is a graded $*$ -homomorphism. Let $\text{GBr}^\infty(X)$ denote the set of graded isomorphism classes of elements of $\mathcal{S}(X)$.

Let ξ be a fiber bundle over X with fiber F a C^* -algebra, and group G . Then ξ is a graded fiber bundle if $F = F^{(0)} \oplus F^{(1)}$ is a graded C^* -algebra and if the group G is contained in the subgroup of $\text{Aut}(F)$ whose elements preserve the grading of F . We note that the local trivializations $h_i: \mathcal{U}_i \times F \rightarrow \xi|_{\mathcal{U}_i}$, for $\{\mathcal{U}_i\}_{i \in I}$ an open cover of X , must preserve the grading on the fiber. In addition, ξ may be written as the Whitney sum $\xi = \xi^{(0)} \oplus \xi^{(1)}$. One example of a graded fiber bundle is a Clifford algebra bundle. If ξ is a real vector bundle over X with a Riemannian metric, then the complexified Clifford algebra bundle of ξ , denoted by $C(\xi)$, is a bundle of graded C^* -algebras such that $C(\xi)_x = C(F_x) \otimes_{\mathbf{R}} \mathbf{C}$, where $C(F_x)$ is the Clifford algebra associated to the fiber over x . Let ξ be an ungraded fiber bundle with fiber F a C^* -algebra. Then ξ may be given a trivial grading corresponding to the trivial grading of the fiber F . In this case, $\xi^{(0)} = \xi$, and $\xi^{(1)} = 0$. If ξ is a graded fiber bundle over x , then $\Gamma_0(\xi)$, the algebra of sections of ξ which vanish at ∞ , is graded as follows: for $s \in \Gamma_0(\xi)$, $\deg(s) = i$ if $s(x) \in F_x^{(i)}$ for every $x \in X$. If ξ_1 and ξ_2 are graded fiber bundles, then $\varphi: \xi_1 \rightarrow \xi_2$ is a graded homomorphism of graded fiber bundles if φ is a homomorphism of fiber bundles which preserves the grading on each fiber.

2. Construction of the fiber bundle associated to a graded C^* -algebra.

The aim of this section is to identify each element of $\text{GBr}^\infty(X)$ with one of a Čech cohomology group. Then the powerful techniques of cohomology theory can be used to analyze $\text{GBr}^\infty(X)$. The key step in this identification is the construction of a continuous field of graded C^* -algebras from an element of $\mathcal{S}(X)$. This continuous

field is then shown to be a fiber bundle. Before proceeding to the actual construction, it is necessary to make some remarks concerning graded representations of a graded C^* -algebra.

Let $A \in \mathcal{G}(X)$, and suppose that $\pi: A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ is a representation of A . Then π is a graded representation if $\mathcal{H}_\pi$ is a separable, graded, infinite-dimensional Hilbert space, and π is a graded $*$ -homomorphism. As in the ungraded case, a subspace K of a graded Hilbert space $\mathcal{H}$ is said to be invariant under a graded representation $\pi: A \rightarrow \mathcal{L}(\mathcal{H})$ if $\pi(A)K \subset K$. An irreducible graded representation π of $A \in \mathcal{G}(X)$ is a graded representation such that if K is an invariant subspace of π , then $K = 0$ or $\mathcal{H}$. The quotient $A/\ker(\pi)$ is graded in the following way. Let $a \in A$ be a homogeneous element of A . Let $[a]$ be the equivalence class of a in $A/\ker(\pi)$. Define $\deg([a]) = \deg(a)$. Since π is graded, this definition is well defined. It then follows that the quotient map $q: A \rightarrow A/\ker(\pi)$ is graded, and that the homomorphism $\varphi: A/\ker(\pi) \rightarrow \mathcal{K}_{\text{gr}}(\mathcal{H}_\pi)$ is a graded isomorphism.

2.1. LEMMA. *Let $A \in \mathcal{G}(X)$. Every element $x \in X$ can be represented by a nontrivial irreducible graded representation.*

PROOF. Let $x \in X$ and let $\pi: A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ be a representative of the equivalence class x . Suppose that α is the grading automorphism of A ; then α preserves the kernel of π . Since $A/\ker(\pi) \approx \mathcal{K}_{\text{gr}}(\mathcal{H}_\pi)$, α induces the standard grading on $\mathcal{K}(\mathcal{H}_\pi)$. There exists a $J \in \mathcal{U}(\mathcal{H}_\pi)$ which induces this grading on $\mathcal{K}(\mathcal{H}_\pi)$. Use J to define a grading on $\mathcal{H}_\pi$ as in §1. Then $\pi': A \rightarrow \mathcal{L}(\mathcal{H}_\pi^{(0)} \oplus \mathcal{H}_\pi^{(1)})$ by $\pi'(a) = \pi(a)$ is a graded representation of A . Since π is irreducible, π' is also irreducible. And $\ker(\pi) = \ker(\pi')$ implies that π and π' determine the same equivalence class of x . $\square$

It is now possible to construct the continuous field of graded elementary C^* -algebras associated to an element of $\mathcal{G}(X)$. Let $A \in \mathcal{G}(X)$. Every $x \in X$ may be identified with an irreducible representation of A on a graded Hilbert space $\mathcal{H}$, and this representation is graded by the above lemma. Then the continuous field ξ_A is the family of C^* -algebras $\{\xi(x)\}_{x \in X}$, where $\xi(x) = A/\ker(x)$, together with the set of sections $\Gamma(\xi_A)$ defined as follows. For every $a \in A$, let $s_a: x \mapsto a_x$, where a_x denotes the image of a in $A/\ker(x)$. Let $\mathcal{S} = \{s_a: a \in A\}$. Then $\Gamma(\xi_A)$ is the set of maps $s': X \rightarrow \bigcup_{x \in X} \xi(x)$ with the property: for every $\varepsilon > 0$ and every $x \in X$, there exists a neighborhood V of x in X and a map $s \in \mathcal{S}$ such that $\|s(y) - s'(y)\| < \varepsilon$ for every $y \in V$. Note that since A is homogeneous of degree $\aleph_0$, then for every $x \in X$, $A/\ker(x)$ is isomorphic to $\mathcal{K}(\mathcal{H})$. Since A is graded, each $\xi(x)$ is graded and the isomorphism between $A/\ker(x)$ and $\mathcal{K}_{\text{gr}}(\mathcal{H})$ preserves the grading of $\xi(x)$ induced from A . This construction is the graded analogue of the Dixmier-Douady construction [3, 10.5].

We can proceed now to show that the continuous field ξ_A is a graded fiber bundle, with base X and fiber $\mathcal{K}_{\text{gr}}(\mathcal{H})$. First, it is necessary to identify the group of the proposed fiber bundle. Let $\text{Aut}^0(\mathcal{K})$ be the subgroup of $\text{Aut}(\mathcal{K})$ whose elements preserve the grading of $\mathcal{K}_{\text{gr}}(\mathcal{H})$. Then $\text{Aut}^0(\mathcal{K})$ inherits the topology of pointwise convergence from $\text{Aut}(\mathcal{K})$. Let

$$\mathcal{U}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathcal{U}(\mathcal{K}) \right\}, \quad \mathcal{U}_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathcal{U}(\mathcal{K}) \right\}.$$

Let $\mathcal{U}_{\text{gr}}(\mathcal{H}) = \mathcal{U}_0 \cup \mathcal{U}_1$; $\mathcal{U}_{\text{gr}}(\mathcal{H})$ is a closed subgroup of $\mathcal{U}(\mathcal{H})$ which inherits the strong operator topology from $\mathcal{U}(\mathcal{H})$. Define $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ to be the quotient $\mathcal{U}_{\text{gr}}(\mathcal{H})/S^1$.

2.2. PROPOSITION. $\text{Aut}^0(\mathcal{H})$ is homeomorphic to $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$.

PROOF. Define a function $\varphi: \mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow \text{Aut}^0(\mathcal{H})$ by $\varphi(U)(T) = UTU^*$, for $U \in \mathcal{U}_{\text{gr}}(\mathcal{H})$ and $T \in \mathcal{H}$. It is clear that the kernel of φ is S^1 , and that $\varphi(\mathcal{U}) \in \text{Aut}^0(\mathcal{H})$ for every $\mathcal{U} \in \mathcal{U}_{\text{gr}}(\mathcal{H})$. We next show that φ is surjective. Suppose that $\Phi \in \text{Aut}^0(\mathcal{H})$. There exists a unitary U such that $\Phi(T) = UTU^*$ for every $T \in \mathcal{H}(\mathcal{H})$, and in particular, for every $T \in \mathcal{H}_{\text{gr}}(\mathcal{H})$. Since $\Phi \in \text{Aut}^0(\mathcal{H})$, then $\deg(T) = i$ implies that $\deg(\Phi(T)) = i$, $i = 0, 1$. It can be shown that U is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $a, b, c, d \in \mathcal{U}(\mathcal{H})$, by choosing an orthonormal basis for $\mathcal{H}$, making some appropriate choices for T , and then computing UTU^* for these cases.

Using the definition of the strong operator topology, it is easy to show that the map $\varphi: \mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow \text{Aut}^0(\mathcal{H})$ is continuous. Therefore, the quotient map $\bar{\varphi}: \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow \text{Aut}^0(\mathcal{H})$ is bijective and continuous. To complete the proof that $\bar{\varphi}$ is a homeomorphism, it can be shown, following the argument of [7, 5.40], that $\bar{\varphi}^{-1}$ is continuous. $\square$

2.3. THEOREM. Let $A \in \mathcal{G}(X)$. Then ξ_A is a graded fiber bundle with base space X , fiber $\mathcal{H}_{\text{gr}}(\mathcal{H})$, and group $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$.

PROOF. The construction above of ξ_A gives the base space X , the fiber $\mathcal{H}_{\text{gr}}(\mathcal{H})$, and the total space $E(\xi_A) = \bigcup_{x \in X} A/\ker(x)$, which is equipped with the tube topology. Let $p: E(\xi_A) \rightarrow X$ by $p(y) = x$ when $y \in A_x$. It is straightforward to check that $\text{Aut}^0(\mathcal{H}) \approx \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ is an effective topological transformation group for ξ_A . The rest of the defining conditions for a fiber bundle are satisfied by the following proposition.

2.4. PROPOSITION. There exist coordinate neighborhoods $\{\mathcal{U}_i\}_{i \in I}$ of X and graded homeomorphisms $h_i: \mathcal{U}_i \times \mathcal{H}_{\text{gr}}(\mathcal{H}) \rightarrow p^{-1}(\mathcal{U}_i)$ which satisfy

- (i) $ph_i(x, T) = x$, for every $x \in \mathcal{U}_i$, $T \in \mathcal{H}_{\text{gr}}(\mathcal{H})$;
- (ii) if $h_{i,x}: \mathcal{H}_{\text{gr}}(\mathcal{H}) \rightarrow p^{-1}(x)$ is defined by setting $h_{i,x}(T) = h_i(x, T)$, then, for each pair $i, j \in I$, and each $x \in \mathcal{U}_i \cap \mathcal{U}_j$, the homeomorphism $h_{i,x}^{-1} \circ h_{j,x}: \mathcal{H}_{\text{gr}}(\mathcal{H}) \rightarrow \mathcal{H}_{\text{gr}}(\mathcal{H})$ coincides with an element of $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$;
- (iii) for each $i, j \in I$, the map $g_{i,j}: \mathcal{U}_i \cap \mathcal{U}_j \rightarrow \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ defined by $g_{i,j}(x) = h_{i,x}^{-1} \circ h_{j,x}$ is continuous.

The proof of Proposition 2.4 will be delayed until §6. This will conclude the proof that ξ_A is a graded fiber bundle. $\square$

3. $\text{GBr}^\infty(X) \approx \check{H}^1(X; \underline{\mathbb{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbb{Z}})$. We now prove that $\text{GBr}^\infty(X)$ is isomorphic to $\check{H}^1(X; \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}))$, which in turn is isomorphic to $\check{H}^1(X; \underline{\mathbb{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbb{Z}})$, and discuss the group structure of each. It is first shown that $\check{H}^1(X; \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}))$ and $\text{GBr}^\infty(X)$ are isomorphic, as sets. Let $\mathcal{B}(X)$ be the category whose objects are graded fiber bundles over X , with fiber $\mathcal{H}_{\text{gr}}(\mathcal{H})$ and group $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$. A morphism between objects of $\mathcal{B}(X)$ is a graded homomorphism of graded fiber bundles.

Then $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ can be regarded as the set formed from the graded isomorphism classes of elements of $\mathcal{B}(X)$. Let $\xi \in \mathcal{B}(X)$. Define the functions τ and τ' as follows:

$$\begin{aligned} \tau: \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) &\rightarrow \text{GBr}^\infty(X) & \text{by } \tau([\xi]) &= [\Gamma_0(\xi)], \\ \tau': \text{GBr}^\infty(X) &\rightarrow \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \text{by } \tau'([A]) &= [\xi_A]. \end{aligned}$$

It will be shown that τ and τ' are well-defined natural functions, such that τ' is inverse to τ .

The following proposition verifies that τ and τ' are well defined.

3.1. PROPOSITION. (i) *If ξ and $\xi' \in \mathcal{B}(X)$ such that $[\xi] = [\xi']$ in $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$, then $[\Gamma_0(\xi)] = [\Gamma_0(\xi')]$ in $\text{GBr}^\infty(X)$.*
(ii) *If A and $B \in \mathcal{G}(X)$ such that $[A] = [B]$ in $\text{GBr}^\infty(X)$, then $[\xi_A] = [\xi_B]$ in $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$.*

PROOF. (i) If $f: E(\xi) \rightarrow B(\xi')$ is a graded, fiber-preserving isomorphism, it is easy to verify that $\Gamma_0(f)$ is a graded isomorphism from $\Gamma_0(\xi)$ to $\Gamma_0(\xi')$.

(ii) Suppose that $\varphi: A \rightarrow B$ is a graded $*$ -isomorphism. Let $x \in X$ correspond to $\ker(\pi)$, where π is an irreducible graded representation of A . Let $\pi' = \pi\varphi^{-1}: B \rightarrow \mathcal{L}(\mathcal{H})$. Consider the following diagram, which defines $\bar{\varphi}_x$.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ q_A \downarrow & & \downarrow q_B \\ A/\ker(\pi) & \xrightarrow{\bar{\varphi}_x} & B/\ker(\pi') \end{array}$$

Note that $\bar{\varphi}_x$ is a graded isomorphism for each $x \in X$. Hence φ_x is a graded isomorphism from each fiber of ξ_B . Let $\Phi = \bigcup_{x \in X} \bar{\varphi}_x$. Then Φ is a graded isomorphism from ξ_A to ξ_B . $\square$

The next proposition verifies that τ' is inverse to τ .

3.2. PROPOSITION. *Let $A \in \mathcal{G}(X)$ and $\xi \in \mathcal{B}(X)$. Then*

- (i) *A and $\Gamma_0(\xi_A)$ are isomorphic as graded C^* -algebras;*
- (ii) *ξ and $\xi_{\Gamma_0(\xi)}$ are isomorphic as graded fiber bundles.*

PROOF. (i) By [3, 10.5.4], there is an isomorphism which maps an element $a \in A$ to the section s_a of ξ_A defined by $s_a(x) = a_x$, for $x \in X$, where a_x is the image of a in A/x . Since the projection $a: A \rightarrow A/x$ preserves the grading, the isomorphism $a \mapsto s_a$ preserves the grading.

(ii) Let $y_x \in \mathcal{H}_{\text{gr}}(\mathcal{H}) = \xi_x$. There is a section $s: X \rightarrow E(\xi)$ by $s(x) = y_x$ for every $x \in X$. Let $q_x: \Gamma_0(\xi) \rightarrow \Gamma_0(\xi)/x$ be the quotient map, and let s_x denote the image of s under q_x . The canonical isomorphism between ξ_x and $\Gamma_0(\xi)/x$ is then defined by $y_x \mapsto s_x$ [3, 10.5.2]. This isomorphism is graded on each fiber since q_x preserves the grading. Hence ξ and $\xi_{\Gamma_0(\xi)}$ are isomorphic as graded fiber bundles. $\square$

Therefore, there is a one-to-one correspondence between $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ and $\text{GBr}^\infty(X)$. Before proceeding to the discussion of their operations, it will be shown that τ is natural. Suppose $f: X \rightarrow Y$ is a map, where X and Y are locally compact

Hausdorff spaces, each with countable base. Let $\xi \in \mathcal{B}(Y)$ and $B \in \mathcal{G}(Y)$. Then f induces the functions

$$f^*: \check{H}^1(Y; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \rightarrow \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \quad \text{by } [\xi] \mapsto [f^*(\xi)]$$

and

$$\bar{f}: \text{GBr}^\infty(Y) \rightarrow \text{GBr}^\infty(X) \quad \text{by } [\Gamma_0(\xi_B)] \mapsto [\Gamma_0(f^*\xi_B)].$$

Then the following diagram commutes:

$$\begin{array}{ccc} \check{H}^1(Y; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \xrightarrow{f^*} & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \\ \tau \downarrow & & \downarrow \tau \\ \text{GBr}^\infty(Y) & \xrightarrow{\bar{f}} & \text{GBr}^\infty(X). \end{array}$$

Next, the operations for $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ and $\text{GBr}^\infty(X)$ are discussed. In addition, it is shown that τ and τ' respect these operations. The fiberwise graded tensor product of graded fiber bundles is the operation of $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$. Specifically, if $\xi, \xi' \in \mathcal{B}(X)$, let $[\xi] \hat{\otimes}_X [\xi'] = [\xi \hat{\otimes}_X \xi']$. This fiberwise tensor product on infinite-dimensional bundles must be carefully defined; see [9, p. 78] for a more complete discussion of the ungraded case. Let ξ_0 denote the trivial bundle over X with fiber $\mathcal{H}_{\text{gr}}(\mathcal{H})$. Then the identity element of $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ is $[\xi_0]$.

Let $A, B \in \mathcal{G}(X)$. Then A and B are $C_0(X)$ -modules, and we define $[A] \hat{\otimes}_X [B] = [A \hat{\otimes}_{C_0(X)} B]$. Note that the operation $\hat{\otimes}_{C_0(X)}$ is not the usual algebraic tensor product, but a graded version of a C^* -algebraic construction due to Rieffel and Green [11]. By Propositions 3.1 and 3.2, $[A] \hat{\otimes}_X [B] = [\Gamma_0(\xi_A \hat{\otimes}_X \xi_B)]$. It is clear that the identity element of $\text{GBr}^\infty(X)$ is the equivalence class of the C^* -algebra of maps from X to $\mathcal{H}_{\text{gr}}(\mathcal{H})$ which vanish at ∞ . It is immediate that $\tau([\xi_0]) = 1_{\text{GBr}^\infty(X)}$. We have, for $\xi, \xi' \in \mathcal{B}(X)$, that

$$\tau([\xi] \hat{\otimes}_X [\xi']) = \tau([\xi]) \hat{\otimes}_X \tau([\xi']).$$

We now can proceed to the definition of the function w . Let $w_1: \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow \mathbf{Z}_2$ be defined by $w_1([a]) = (-1)^{\deg(a)}$. It is easy to check that w_1 is well defined. Recall that the Bockstein homomorphism $\delta_j^*: \check{H}^j(X; \underline{S}^1) \rightarrow \check{H}^{j+1}(X; \underline{\mathbf{Z}})$ associated to the exact sequence $1 \rightarrow \mathbf{Z} \rightarrow \mathbf{R} \rightarrow S^1 \rightarrow 1$ is an isomorphism. The short exact sequence $1 \rightarrow S^1 \rightarrow \mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow 1$ induces the following exact sequence

$$(I) \quad \cdots \rightarrow \check{H}^1(X; \underline{S}^1) \rightarrow \check{H}^1(X; \underline{\mathcal{U}}_{\text{gr}}(\mathcal{H})) \rightarrow \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X; \underline{S}^1).$$

Let $w_2^* = \delta_2^* \tilde{\delta}_1^*$. Define

$$w: \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \rightarrow \check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^3(X; \underline{\mathbf{Z}}) \quad \text{by } w(x) = (w_1^*(x), w_2^*(x)).$$

Using the exactness of (I) and the definition of w_1 , it is straightforward to verify the following lemma.

3.3. LEMMA. *Let $x \in \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$. Then $w(x) = (1, 0)$ implies that x is the identity element in $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$.*

3.4. PROPOSITION. Let $\xi, \xi' \in \mathcal{B}(X)$, and let β be the Bockstein homomorphism associated to the sequence $1 \rightarrow \mathbf{Z} \xrightarrow{(\times 2)} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \rightarrow 1$, where $r(n) = (-1)^n$. Then

$$w([\xi \hat{\otimes}_X \xi']) = (w_1^*([\xi]) \cdot w_1^*([\xi']), w_2^*([\xi]) + w_2^*([\xi']) + \beta(w_1^*([\xi']) \cup w_1^*([\xi])).$$

The proof parallels that of [6, Lemma 10] and will be omitted.

An explicit inverse to an arbitrary element of $\mathrm{GBr}^\infty(X)$ will now be given. Let $A \in \mathcal{G}(X)$ and let ξ_A be the graded fiber bundle associated to A . Let $\bar{\xi}_A$ be the fiber bundle which is topologically identical to ξ_A , and where the elements in each fiber have the same grading as the corresponding ones of ξ_A . The fiber of $\bar{\xi}_A$ is $\mathcal{K}_{\mathrm{gr}}(\mathcal{H})$; let $\bar{\xi}_A$ have the following fiberwise operations, for every $x, y \in \mathcal{K}_{\mathrm{gr}}(\mathcal{H})$, $c \in \mathcal{C}$:

addition:	$(x, y) \mapsto x + y$
scalar multiplication:	$(c, x) \mapsto \bar{c}x$
multiplication:	$(x, y) \mapsto (-1)^{\deg(x) \deg(y)} xy$
involution:	$x \mapsto x^*$
norm:	$x \mapsto \ x\ $

Denote the new multiplication by $x \times y$.

3.5. PROPOSITION. Let $A \in \mathcal{G}(X)$. Then $[\bar{\xi}_A]$ is inverse to $[\xi_A]$ in

$$\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\mathrm{gr}}(\mathcal{H})).$$

PROOF. By Lemma 3.4, it is sufficient to show that $w([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = (1, 0)$. Let d_{ij} be the transition functions for ξ_A , $i, j \in I$. Then the transition functions for $\bar{\xi}_A$ are also d_{ij} . Hence $w_1^*([\xi_A]) = w_1^*([\bar{\xi}_A])$, so $w_1^*([\xi_A]) \cdot w_1^*([\bar{\xi}_A]) = 1$.

To calculate $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A])$, we need to do the following computation. Let g_{ij} (respectively g'_{ij}) be the element of $\mathcal{U}_{\mathrm{gr}}(\mathcal{H})$ which implements the transition function d_{ij} for ξ_A (d'_{ij} for $\bar{\xi}_A$). Let $g_{ij}g_{jk} = u_{ijk}g_{ik}$ and $g'_{ij}g'_{jk} = u'_{ijk}g'_{ik}$. Then

$$\begin{aligned} (g_{ij} \hat{\otimes} g'_{ij})(g_{jk} \hat{\otimes} g'_{jk}) &= (-1)^{\deg(g'_{ij}) \deg(g_{jk})} (g_{ij}g_{jk}) \hat{\otimes} (g'_{ij} \times g'_{jk}) \\ &= u_{ijk}u'_{ijk}(g_{ik} \hat{\otimes} g'_{ik}), \end{aligned}$$

since $\deg(g_{jk}) + \deg(g'_{jk}) = 0$. Hence $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = w_2^*([\xi_A]) + w_2^*([\bar{\xi}_A])$. But $u'_{ijk} = \bar{u}_{ijk}$, the complex conjugate of u_{ijk} . Therefore $w_2^*([\xi_A]) = -w_2^*([\bar{\xi}_A])$, or $w_2^*([\xi_A \hat{\otimes}_X \bar{\xi}_A]) = 0$. $\square$

If $A \in \mathcal{G}(X)$, the inverse element to $[A] \in \mathrm{GBr}^\infty(X)$ is the element $[\Gamma_0(\bar{\xi}_A)]$. This completes the verification that $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\mathrm{gr}}(\mathcal{H}))$ and $\mathrm{GBr}^\infty(X)$ are groups, and that the function $\tau: \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\mathrm{gr}}(\mathcal{H})) \rightarrow \mathrm{GBr}^\infty(X)$ is a group homomorphism.

It is shown below that w is an isomorphism. Let $T = \mathcal{U}_0/S^1$. Let $\eta: \mathcal{U}_{\mathrm{gr}}(\mathcal{H}) \rightarrow \mathbf{Z}_2$ be defined by $\eta(a) = (-1)^{\deg(a)}$. Then we have the following diagram of short

exact sequences of groups, where γ and $\tilde{\gamma}$ are inclusions cf. [6]:

$$(II) \quad \begin{array}{ccccccc} & & & \downarrow & & \downarrow & \\ & & & 1 & & 1 & \\ 1 & \rightarrow & S^1 & \xrightarrow{\tilde{\gamma}} & \mathcal{U}_0 & \rightarrow & T \rightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \rightarrow & S^1 & \xrightarrow{\gamma} & \mathcal{U}_{\text{gr}}(\mathcal{H}) & \xrightarrow{\nu} & \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow 1 \\ & & & & \downarrow \eta & & \downarrow w_1 \\ & & & & \mathbf{Z}_2 & = & \mathbf{Z}_2 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

Diagram (II) induces

$$(III) \quad \begin{array}{ccccccc} \check{H}^1(X; \underline{S}^1) & \xrightarrow{\tilde{\gamma}^*} & \check{H}^1(X; \underline{\mathcal{U}}_0) & & & & \\ \downarrow = & & \downarrow & & & & \\ \cdots \rightarrow \check{H}^1(X; \underline{S}^1) & \xrightarrow{\gamma^*} & \check{H}^1(X; \underline{\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \xrightarrow{\nu^*} & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \xrightarrow{\tilde{\delta}_1^*} & \check{H}^2(X; \underline{S}^1) \\ & & \downarrow \eta^* & \nearrow \tilde{\nu}^* & & & \\ & & \check{H}^1(X; \underline{\mathbf{Z}}_2) & & & & \end{array}$$

It is easy to verify that $\check{H}^1(X; \underline{\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ is a group; hence diagram (III) is a commutative diagram of groups. The set $\mathcal{U}_0$ is contractible [18], so $\check{H}^1(X; \underline{\mathcal{U}}_0) = 0$ by [14], and therefore $\gamma^* = 0$. In addition, η^* is injective. Let $\varsigma: \mathbf{Z}_2 \rightarrow \mathcal{U}_{\text{gr}}(\mathcal{H})$ be defined by $\varsigma(+1) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $\varsigma(-1) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then $\eta\varsigma = 1_{\mathbf{Z}_2}$, so η^* is surjective. Let $\tilde{\nu}^* = \nu^*(\eta^*)^{-1}$. Then diagram (III) reduces to the following exact sequence:

$$0 \rightarrow \check{H}^1(X; \underline{\mathbf{Z}}_2) \xrightarrow{\tilde{\nu}^*} \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X; \underline{S}^1).$$

One result of the theorem below is the fact that $\tilde{\delta}_1^*$ is surjective; hence

$$(IV) \quad 0 \rightarrow \check{H}^1(X; \underline{\mathbf{Z}}_2) \xrightarrow{\tilde{\nu}^*} \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) \xrightarrow{\tilde{\delta}_1^*} \check{H}^2(X; \underline{S}^1) \rightarrow 0$$

is exact. It is also shown that the sequence (IV) splits.

3.6. THEOREM. *w is an isomorphism.*

PROOF. It is necessary to show that $\tilde{\delta}_1^*$ is surjective. Let $\theta: \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{U}_{\text{gr}}(\mathcal{H})$ by $\theta(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, and $\bar{\theta}: \mathcal{P}\mathcal{U}(\mathcal{H}) \rightarrow \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ by $\bar{\theta}([a]) = [\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}]$. It is easy to check that $\bar{\theta}$ is well defined. Let ξ_2 be the trivial bundle over X with fiber $M = M_2(\mathbf{C})$. The grading of M is defined by

$$M^{(0)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbf{C} \right\} \quad \text{and} \quad M^{(1)} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : b, c \in \mathbf{C} \right\}.$$

Note that $\bar{\theta}^*([\xi]) = [\xi \hat{\otimes}_X \xi_2]$. We have the following commutative diagram, where the sequences are exact:

$$(V) \quad \begin{array}{ccccccc} 1 & \rightarrow & S^1 & \rightarrow & \mathcal{U}_{\text{gr}}(\mathcal{H}) & \rightarrow & \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow 1 \\ & & \downarrow = & & \downarrow \theta & & \downarrow \bar{\theta} \\ 1 & \rightarrow & S^1 & \rightarrow & \mathcal{U}(\mathcal{H}) & \rightarrow & \mathcal{P}\mathcal{U}(\mathcal{H}) \rightarrow 1 \end{array}$$

This induces the commutative diagram:

$$(VI) \quad \begin{array}{ccccccc} \cdots \rightarrow & \check{H}^1(X; \underline{S}^1) & \rightarrow & \check{H}^1(X; \underline{\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \rightarrow & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \xrightarrow{\delta_1^*} \check{H}^2(X; \underline{S}^1) \\ & \downarrow = & & \downarrow \theta^* & & \downarrow \bar{\theta}^* & \downarrow = \\ \cdots \rightarrow & \check{H}^1(X; \underline{S}^1) & \rightarrow & \check{H}^1(X; \underline{\mathcal{U}}(\mathcal{H})) & \rightarrow & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}(\mathcal{H})) & \xrightarrow{\delta_1^*} \check{Y}^2(X; \underline{S}^1) \end{array}$$

Since δ_1^* is an isomorphism, $\tilde{\delta}_1^*$ is surjective. Note that $w_1 \tilde{\nu} = 1_{\mathbf{Z}_2}$, so w_1^* is surjective. Hence w is surjective, since both w_1^* and $\delta_2^* \tilde{\delta}_1^*$ are. Lemma 3.3 implies that w is injective. $\square$

4. Interpretations of the invariants w_1^* and w_2^* . Let A be a separable, stable, continuous trace C^* -algebra, with spectrum X . Then the Dixmier-Douady invariant of A , $\delta(A)$, is the image of the fiber bundle constructed from A under the composite

$$\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}(\mathcal{H})) \xrightarrow{\delta_1^*} \check{H}^2(X; \underline{S}^1) \xrightarrow{\delta_2^*} \check{H}^3(X; \underline{\mathbf{Z}}).$$

Let $\bar{\theta}: \mathcal{P}\mathcal{U}(\mathcal{H}) \rightarrow \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ be the map defined in the proof of Theorem 3.6. The composite $\mathcal{P}\mathcal{U}(\mathcal{H}) \xrightarrow{\bar{\theta}} \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \xrightarrow{w_1} \mathbf{Z}_2$ maps every element of $\mathcal{P}\mathcal{U}(\mathcal{H})$ to $+1$, so $w_1^* \bar{\theta}^*$ is the zero map. Therefore, it is straightforward to compute the following:

4.1. PROPOSITION. $w(\bar{\theta}^*[\xi_A]) = \delta(A)$.

There is an alternate way to view w_2^* . Since $\mathcal{U}_{\text{gr}}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$, we can consider the commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \rightarrow & S^1 & \rightarrow & \mathcal{U}(\mathcal{H}) & \rightarrow & \mathcal{P}\mathcal{U}(\mathcal{H}) \rightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \rightarrow & S^1 & \rightarrow & \mathcal{U}_{\text{gr}}(\mathcal{H}) & \rightarrow & \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H}) \rightarrow 1 \end{array}$$

This induces the following diagram:

$$\begin{array}{ccccccc} \cdots \rightarrow & \check{H}^1(X; \underline{\mathcal{U}}(\mathcal{H})) & \rightarrow & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}(\mathcal{H})) & \xrightarrow{\delta_1^*} & \check{H}^2(X; \underline{S}^1) & \xrightarrow{\delta_2^*} \check{H}^3(X; \underline{\mathbf{Z}}) \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \cdots \rightarrow & \check{H}^1(X; \underline{\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \rightarrow & \check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H})) & \xrightarrow{\delta_1^*} & \check{H}^2(X; \underline{S}^1) & \xrightarrow{\delta_2^*} \check{H}^3(X; \underline{\mathbf{Z}}) \end{array}$$

The homomorphism from $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}_{\text{gr}}(\mathcal{H}))$ to $\check{H}^1(X; \underline{\mathcal{P}\mathcal{U}}(\mathcal{H}))$, which is induced from the inclusion, maps $[\xi]$ to $[\xi^*]$, where ξ^* is the ungraded $\mathcal{P}\mathcal{U}(\mathcal{H})$ -bundle underlying ξ . We now have the following proposition.

4.2. PROPOSITION. *Let $A \in \mathcal{G}(X)$. Let ξ_A be the $\mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ -module constructed from A . Let A^* be A considered as an ungraded C^* -algebra. Then $\xi_{A^*} = (\xi_A)^*$ is the ungraded $\mathcal{P}\mathcal{U}(\mathcal{H})$ -bundle underlying ξ_A , and $w_2^*[\xi_A] = \delta(A^*)$.*

The invariant w_1^* measures the grading of the given graded C^* -algebra. We have the following characterization.

4.3. PROPOSITION. *Let $A \in \mathcal{G}(X)$. Then $w_1^*[\xi_A] = 1$ if and only if $A \approx A' \hat{\otimes} M_2(\mathbf{C})$, where A' is a separable, stable, continuous trace C^* -algebra, with spectrum X , such that $(A')^{(0)} = A'$ and $(A')^{(1)} = 0$.*

PROOF. We have $[A] = [A' \hat{\otimes} M_2(\mathbf{C})]$ if and only if $[\xi_A] = [\xi_{A'} \hat{\otimes}_X \xi_2]$ if and only if $[\xi_A]$ is in the image of θ^* if and only if $w_1^*[\xi_A] = 1$. $\square$

We can also apply the work of J. Phillips and I. Raeburn [21] to interpret w_1^* . Recall that associated to a graded C^* -algebra is a grading automorphism of order 2. Suppose that A is a separable, stable, continuous trace C^* -algebra, with spectrum X . Let $\text{Inn}(A)$ denote the automorphisms of A which are implemented by unitaries in the multiplier algebra, and let $\text{Aut}_{C_0(X)}(A)$ denote the automorphisms of A which fix $C_0(X)$. There is a map $\varphi: \text{Aut}_{C_0(X)}(A) \rightarrow \check{H}^1(X; \underline{S}^1)$ which fits into the following short exact sequence [21, 2.1]:

$$(VII) \quad 0 \rightarrow \text{Inn}(A) \rightarrow \text{Aut}_{C_0(X)}(A) \xrightarrow{\varphi} \check{H}^1(X; \underline{S}^1) \approx \check{H}^2(X; \underline{\mathbf{Z}}) \rightarrow 0.$$

Let $\mathcal{H}$ be a Hilbert space and suppose that J is a unitary of degree 2 on $\mathcal{H}$, which is used to define a grading on $\mathcal{H}$. It is straightforward to check that, for $m \in \mathcal{U}_{\text{gr}}(\mathcal{H})$, $w_1^*[m] = (mJ)(Jm)^{-1}$ is a well defined method of computing w_1^* . Note that J defines an automorphism of order 2 which gives the grading on each fiber of ξ_A . Let $i: \underline{\mathbf{Z}}_2 \rightarrow S^1$ be the inclusion, and suppose that α is the automorphism of A which determines the grading of A . Using this definition of w_1^* , we can calculate that $\varphi(\alpha) = i^*w_1^*([\xi_A])$.

A grading operator of A is a selfadjoint unitary g contained in the multiplier algebra of A , such that $A^{(i)} = \{a \in A: gag^* = (-1)^i a\}$ for $i = 0, 1$. The short exact sequence (VII) then implies that $w_1^*[\xi_A] = 1$ when the grading of A is determined by a grading operator.

Donovan and Karoubi [6] consider the case where ξ is a fiber bundle over a finite complex X , with fiber F a simple central graded $\mathbf{C}$ -algebra [30]. The isomorphism classes of such bundles form a group, $\text{GBr}U(X)$. They prove that [6, Theorem 11]

$$\text{GBr}U(X) \approx \check{H}^0(X; \underline{\mathbf{Z}}_2) \oplus \check{H}^1(X; \underline{\mathbf{Z}}_2) \oplus \text{Tors}(\check{H}^3(X; \underline{\mathbf{Z}})).$$

This isomorphism defines invariants $u_1[\xi] \in \check{H}^1(X; \underline{\mathbf{Z}}_2)$ and $u_2[\xi] \in \text{Tors}(\check{H}^3(X; \underline{\mathbf{Z}}))$ for the element $[\xi] \in \text{GBr}U(X)$. Let ξ_0 be the trivial bundle over X with fiber $\mathcal{H}_{\text{gr}}(\mathcal{H})$. Given $\xi \in \text{GBr}U(X)$, we can include ξ into $\text{GBr}^\infty(X)$ by mapping $[\xi] \rightarrow [\xi_0 \hat{\otimes}_X \xi]$. It can then be verified that $w_j^*[\xi] = u_j[\xi]$, $j = 1, 2$. The case where the fiber is a simple central $\mathbf{R}$ -algebra [6, 19, 20, 30] can be considered by first complexifying the given bundle and then mapping it into $\text{GBr}^\infty(X)$ as above.

Let V be a real n -dimensional vector bundle over X with fiber F . Suppose that V is equipped with a Riemannian metric. Let $C(V)$ denote the Clifford algebra bundle of V , and let $C(V) \otimes_{\mathbf{R}} \mathbf{C}$ denote the complexification of $C(V)$. Let $w_i(V) \in \check{H}^i(X; \underline{\mathbf{Z}}_2)$, $i = 1, 2$, denote the usual Stiefel-Whitney classes of V . Let $\beta: \check{H}^2(X; \underline{\mathbf{Z}}_2) \rightarrow \check{H}^3(X; \underline{\mathbf{Z}})$ be the Bockstein homomorphism associated to the short

exact sequence $1 \rightarrow \mathbf{Z} \xrightarrow{(\times 2)} \mathbf{Z} \xrightarrow{\tau} \mathbf{Z}_2 \rightarrow 1$. Then, using [6, p. 165], we obtain the result that

$$w_1(V) = w_1^*([C(V) \otimes_{\mathbf{R}} \mathbf{C}]) \quad \text{and} \quad \beta w_2(V) = w_2^*([C(V) \otimes_{\mathbf{R}} \mathbf{C}]).$$

5. Graded Morita equivalence. Let A be a separable graded continuous trace C^* -algebra with spectrum X . Then $A \hat{\otimes} \mathcal{K}_{\text{gr}}(\mathcal{H})$ is an element of $\mathcal{G}(X)$. If B is another graded continuous trace C^* -algebra, we would like to define an equivalence between A and B which would imply that $[A \hat{\otimes} \mathcal{K}_{\text{gr}}(\mathcal{H})] = [B \hat{\otimes} \mathcal{K}_{\text{gr}}(\mathcal{H})]$ in $\text{GBr}^\infty(X)$. The work in this section determines that the appropriate equivalence is graded Morita equivalence, which is based on the standard definition of strong Morita equivalence. In [22], M. Rieffel presented the theory for ungraded C^* -algebras.

In an unpublished note [11], P. Green gives a variant on the construction of the Dixmier-Douady invariant for ungraded continuous trace C^* -algebras. We now consider a graded version of Green's development. Let $A \in \mathcal{G}(X)$. By Lemma 6.2 below, there exists a locally finite open cover $\{\mathcal{U}_i\}_{i \in I}$ of X such that, for every $i \in I$, there exists $a_i \in A^{(0)}$ with $x(a_i)$ a degree 0 rank one projection for all $x \in \mathcal{U}_i$. Let $p_i(x) = x(a_i)$. Suppose $i, j \in I$ and $x \in \mathcal{U}_i \cap \mathcal{U}_j$. Let $\text{Im}(p_i(x)) = \mathbf{C}e_{i,x}$ and $\text{Im}(p_j(x)) = \mathbf{C}e_{j,x}$, for $e_{i,x}$ and $e_{j,x}$ some chosen unit vectors of $\mathcal{H}$. There exists a partial isometry $b \in \mathcal{L}(\mathcal{H})$ whose initial space is $\mathbf{C}e_{j,x}$ and whose range is $\mathbf{C}e_{i,x}$. Let $c \in A$ such that $x(c) = b$. Then $x(a_i c a_j) = x(c) \neq 0$, and in some neighborhood of x , $v(a_i c a_j)$ is a rank one operator.

Now replace $\{\mathcal{U}_i\}_{i \in I}$ with a locally finite refinement such that for all $i, j \in I$, there exists $c_{ij} \in A$ with $x(a_i c_{ij} a_j) = x(c_{ij}) \neq 0$ for all $x \in \mathcal{U}_i \cap \mathcal{U}_j$. Let $b_{ij}(x) = x(c_{ij})$. Note that the fact that $x(a_i)$ and $x(a_j)$ are degree 0 projections implies that $b_{ij}(x)$ is a homogeneous operator for every $x \in \mathcal{U}_i \cap \mathcal{U}_j$. Hence c_{ij} is a homogeneous element of A . Since $b_{kj}(x)b_{ji}(x)$ and $b_{ki}(x)$ are, for $x \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, two partial isometries with the same one-dimensional initial space and range, there exists an element $\gamma_{ijk}(x) \in S^1$ such that

$$b_{kj}(x)b_{ji}(x)b_{ki}(x)^* = \gamma_{ijk}(x) \cdot I.$$

The $\{\gamma_{ijk}\}$ form a 2-cocycle in $C^2(X; \underline{S}^1)$. It can be verified that the cohomology class $[\{\gamma_{ijk}\}] \in \check{H}^2(X; \underline{S}^1)$ is independent of the choices made. Let $A \in \mathcal{G}(X)$. Then define $w'(A) = (w'_1(A), w'_2(A))$ where $w'_1(A) = \{(-1)^{\deg(c_{ji})}\}$ and $w'_2(A) = \delta_2^*[\{\gamma_{ijk}\}]$. It can be shown that $w'(A) = w[\xi_A]$.

It is appropriate now to turn to a definition of graded Morita equivalence. Let A and B be graded C^* -algebras and M a graded left A -module and right B -module. Then, for $i, j = 0, 1$, one has $A^{(i)}M^{(j)} \subset M^{(i+j)}$ and $M^{(i)}B^{(j)} \subset M^{(i+j)}$. If A is a graded C^* -algebra and M a graded A -module, an A -valued inner product on M is a function $\langle \cdot, \cdot \rangle_A: M \times M \rightarrow A$ where $\langle M^{(i)}, M^{(j)} \rangle_A \subset A^{(i+j)}$.

5.1 DEFINITION. Two graded C^* -algebras A and B are graded Morita equivalent if there exists a graded left- A -right- B -bimodule M equipped with A - and B -valued inner products $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$ satisfying:

(a) the requirements for strong Morita equivalence:

- (1) $\langle x, x \rangle_A \geq 0$; $\langle x, x \rangle_B \geq 0$;
- (2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$; $\langle x, y \rangle_B^* = \langle y, x \rangle_B$;
- (3) $\langle ax, y \rangle_A = a \langle x, y \rangle_A$; $\langle x, yb \rangle_B = \langle x, y \rangle_B b$;

- (4) $\langle xb, y \rangle_A = \langle x, yb^* \rangle_A$; $\langle ax, y \rangle_B = \langle x, a^*y \rangle_B$;
- (5) $\langle x, y \rangle_A z = x \langle y, z \rangle_B$;
- (6) $\langle ax, ax \rangle_B \leq \|a\|^2 \langle x, x \rangle_B$; $\langle xb, xb \rangle_A \leq \|b\|^2 \langle x, x \rangle_A$;

for $x, y, z \in M$, $a \in A$, $b \in B$;

(b) the graded requirements:

- (1) the span of $\langle M^{(i)}, M^{(j)} \rangle_A$ is dense in $A^{(i+j)}$;
- (2) the span of $\langle M^{(i)}, M^{(j)} \rangle_B$ is dense in $B^{(i+j)}$.

M is called a graded A - B -equivalence bimodule.

Note that if A and B are graded Morita equivalent, they are strong Morita equivalent. The definition of graded Morita equivalence is justified by the following proposition.

5.2. PROPOSITION. *Let A and $B \in \mathcal{G}(X)$. If A and B are graded Morita equivalent, then A and B are isomorphic as graded C^* -algebras.*

PROOF. Suppose that M is an A - B -equivalence bimodule. It will be shown that $w[\xi_A] = w[\xi_B]$. Let $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$ be a locally finite open cover of X with elements $a_i \in A^{(0)}$ chosen for each i , such that $x(a_i)$ is a degree 0 rank one projection for every $x \in \mathcal{U}_i$, and such that for each i , there exists $m_i \in A^{(0)}$ with $\langle m_i, m_i \rangle_A = a_i$. Property (b) of Definition 5.1 guarantees the existence of m_i .

Let $i, j \in I$. Suppose $x \in \mathcal{U}_i \cap \mathcal{U}_j$. Let $c_{ij} \in A$ be chosen as before. When A and B are strong Morita equivalent, there is a homeomorphism between $\hat{A}$ and $\hat{B}$ [22, 6.2.7]. Let $\hat{x}$ be an irreducible representation of B associated to x under this homeomorphism. Then $\hat{x}(\langle m_i, m_i \rangle_B)$ is a rank one projection for every i [11]. Define $\hat{c}_{ij} = \langle m_i, c_{ij} m_j \rangle_B$. It is easy to check that $\hat{x}(\langle m_i, m_i \rangle_B \hat{c}_{ij} \langle m_j, m_j \rangle_B) = \hat{x}(\hat{c}_{ij}) \neq 0$. So $\hat{x}(\hat{c}_{ij})$ is a rank one operator with initial space equal to $\text{Im}(\hat{x} \langle m_j, m_j \rangle_B)$ and range equal to $\text{Im}(\hat{x} \langle m_i, m_i \rangle_B)$. Using the properties of Definition 5.1, one can compute that the c_{ij} and the $\hat{c}_{ij}$ define the same cocycle in $C^2(\mathcal{U}; \underline{S}^1)$. Therefore, $w'_2(A) = w'_2(B)$ so $w_2^*[\xi_A] = w_2^*[\xi_B]$.

Since the m_i and m_j are chosen to be of degree 0, we can see that $\deg(\hat{c}_{ji}) = \deg(c_{ji})$. And $[A] = [B]$ in $\text{GBr}^\infty(X)$ implies that A and B are isomorphic as graded C^* -algebras. $\square$

5.3. COROLLARY. *Let A and B be separable, graded continuous trace C^* -algebras with spectrum X . Suppose that A and B are graded Morita equivalent. Then $A \hat{\otimes} \mathcal{K}_{\text{gr}}(\mathcal{H})$ and $B \hat{\otimes} \mathcal{K}_{\text{gr}}(\mathcal{H})$ are isomorphic as graded C^* -algebras.*

6. The proof of Proposition 2.4. In the ungraded case, the fact that the continuous field ξ_A constructed from a separable, stable, continuous trace C^* -algebra A is a fiber bundle is based on [3, 10.7.11]. Proposition 2.4 is a graded version of this lemma. The proof of this proposition requires that we verify that the constructions in [3, Chapter 10, §§6–7] can be done in the graded setting.

Let F_1 be the category whose objects are pairs $(\mathcal{H}, e_0)$, where $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$ is a graded Hilbert space and $e_0 \in \mathcal{H}^{(0)}$ is a unit vector. A morphism between $(\mathcal{H}, e_0)$ and $(\mathcal{H}', e'_0)$ is a graded isomorphism $u: \mathcal{H} \rightarrow \mathcal{H}'$ such that $u(e_0) = e'_0$. Let F_2 be the category whose object are pairs (A, p) , where A is a graded elementary C^* -algebra of infinite dimension and p is a degree 0 projection of rank one. A morphism between (A, p) and (A', p') in F_2 is a graded isomorphism $g: A \rightarrow A'$ such that $g(p) = p'$. Note that, given a degree 0 projection of rank one on the

graded Hilbert space $\mathcal{H}$, we may assume that it is a degree 0 projection whose image is in $\mathcal{H}^{(0)}$. Let the functor $\alpha: F_1 \rightarrow F_2$ be defined by $\alpha(\mathcal{H}, e_0) = (A, p)$, where $A = \mathcal{K}_{\text{gr}}(\mathcal{H})$ and $p: \mathcal{H} \rightarrow \mathbf{C}e_0$ is the projection. If $u: (\mathcal{H}, e_0) \rightarrow (\mathcal{H}', e'_0)$ is a morphism in F_1 , then define $\alpha(u): \mathcal{K}_{\text{gr}}(\mathcal{H}) \rightarrow \mathcal{K}_{\text{gr}}(\mathcal{H}')$ by $\alpha(u)(T)(y) = u(T(u^{-1}(y)))$, for $T \in \mathcal{K}_{\text{gr}}(\mathcal{H})$ and $y \in \mathcal{H}'$. It is easy to check that $\alpha(u)$ is a morphism in F_2 .

Let $(\mathcal{H}, e_0) \in F_1$, and let $(A, p) = \alpha(\mathcal{H}, e_0)$. Then Ap can be given an inner product by $(a, b)_{Ap} = (ae_0, be_0)_{\mathcal{H}}$. A grading on Ap is defined as follows. Suppose $p_0: \mathcal{H}^{(0)} \rightarrow \mathbf{C}e_0$ is the projection. Let $\varsigma: \mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(0)}$ and $\gamma: \mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(1)}$ be maps. Then a typical element of $(Ap)^{(0)}$ has the form $\begin{pmatrix} \varsigma p_0 & 0 \\ 0 & 0 \end{pmatrix}$ and a typical element of $(Ap)^{(1)}$ has the form $\begin{pmatrix} 0 & 0 \\ \varsigma p_0 & 0 \end{pmatrix}$. An easy computation verifies that $(Ap)^{(i)}(Ap)^{(j)} \subset (Ap)^{(i+j)}$, for $i, j = 0, 1$. Suppose that $(\mathcal{H}, e_0) \in F_1$ and $\alpha(\mathcal{H}, e_0) = (A, p)$. If we define $\varphi: Ap \rightarrow \mathcal{H}$ by $\varphi(a) = ae_0$, for $a \in Ap$, then we can check that φ is a graded isometric isomorphism.

Let $(A, p) \in F_2$, and construct the graded Hilbert space Ap . Note that p is a unit vector of Ap . Then define a functor $\beta: F_2 \rightarrow F_1$ by $\beta(A, p) = (Ap, p)$. If $g: (A, p) \rightarrow (A', p')$ is a morphism of F_2 , then $\beta(g): (Ap, p) \rightarrow (A'p', p')$ is defined by $\beta(g)(ap) = g(a)p'$, for $a \in A$. Suppose the pair (A, p) is an object of F_2 . One has $\alpha\beta(A, p) = (\mathcal{K}(Ap), p)$. The homomorphism $\psi: A \rightarrow \mathcal{K}(Ap)$ defined by $\psi(a)(x) = ax$ for each $a \in A$ and $x \in Ap$, is a graded isomorphism.

The functors α and β will now be extended to the case of continuous fields. Let $\xi(\mathcal{H}_x)$ be a continuous field of graded Hilbert spaces over X . Suppose that $s \in \Gamma(\xi(\mathcal{H}_x))$ such that $\|s(x)\| = 1$ for every $x \in X$, and that $s(x) \in \mathcal{H}_x^{(0)}$ for $x \in X$. Then s is called a degree 0 unit section for $\xi(\mathcal{H}_x)$. Let ξ be a continuous field of graded elementary C^* -algebras over X . An element $r \in \Gamma(\xi)$ is called a degree 0 rank one section if $r(x)$ is a degree 0 rank one projection for every $x \in X$. Let $\mathcal{F}_1$ be the category whose objects are pairs $(\xi(\mathcal{H}_x), s)$ where $\xi(\mathcal{H}_x)$ is a continuous field of graded Hilbert spaces over X and s is a degree 0 unit section of $\xi(\mathcal{H}_x)$. A morphism $\varsigma: (\xi(\mathcal{H}_x), s) \rightarrow (\xi(\mathcal{H}'_x), s')$ is defined by $\varsigma = \bigcup_{x \in X} \varsigma_x$, where $\varsigma_x: \mathcal{H}_x \rightarrow \mathcal{H}'_x$ is a graded isomorphism for every $x \in X$, and $\varsigma(s) = s'$. Let $\mathcal{F}_2$ be the category whose objects are pairs (ξ, p) where ξ is a continuous field of graded elementary C^* -algebras and where p is a degree 0 rank one section for ξ . A morphism $\eta: (\xi, p) \rightarrow (\xi', p')$ is defined by $\eta = \bigcup_{x \in X} \eta_x$, where $\eta_x: \xi(x) \rightarrow \xi'(x)$ is a graded isomorphism for every $x \in X$ and $\eta(p) = p'$.

Suppose that $(\xi(\mathcal{H}_x), s) \in \mathcal{F}_1$. Then a degree 0 rank one section for the continuous field $\xi(\mathcal{K}_{\text{gr}}(\mathcal{H}_x))$ can be constructed as follows. Let $r_s: X \rightarrow E(\xi(\mathcal{K}_{\text{gr}}(\mathcal{H}_x)))$ by $r_s(x)(h) = (h, s(x))_{\mathcal{H}_x} s(x)$ where $h \in \mathcal{H}_x$. Then $r_s(x)$ is a degree 0 rank one projection for every $x \in X$. There is a functor $\alpha: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ defined by $\alpha(\xi(\mathcal{H}_x), s) = (\xi(\mathcal{K}_{\text{gr}}(\mathcal{H}_x)), r_s)$. If $\varsigma: (\xi(\mathcal{H}_x), s) \rightarrow (\xi(\mathcal{H}'_x), s')$ is a morphism of $\mathcal{F}_1$, then let $\alpha(\varsigma) = \bigcup_{x \in X} \alpha(\varsigma_x)$. The next result follows immediately.

6.1. LEMMA. *If $\varsigma: \xi(\mathcal{H}_x) \rightarrow \xi(\mathcal{H}'_x)$ is a graded isomorphism, then the induced map $\alpha(\varsigma): \xi(\mathcal{K}_{\text{gr}}(\mathcal{H}_x)) \rightarrow \xi(\mathcal{K}_{\text{gr}}(\mathcal{H}'_x))$ is a graded isomorphism.*

Let $(\xi, p) \in \mathcal{F}_2$ where $\xi(x) = A_x$. Define a functor $\beta: \mathcal{F}_2 \rightarrow \mathcal{F}_1$ by $\beta(\xi, p) = (\xi(A_x p(x)), p)$, where $p(x)$ is the unit vector of $A_x p(x)$ for every $x \in X$. If $\eta: (\xi, p) \rightarrow (\xi', p')$ is a morphism of $\mathcal{F}_2$, then let $\beta(\eta)$ be defined as $\beta(\eta) = \bigcup_{x \in X} \beta(\eta_x)$. The following lemma is a graded version of Definition 1.1.

6.2. LEMMA. *Let $A \in \mathcal{G}(X)$. For each $x \in X$, there exists an element $a \in A^{(0)}$ and a neighborhood V_x of x in X such that, for every $v \in V_x$, $v(a)$ is a rank one projection of degree 0.*

PROOF. By Lemma 2.1, we may assume that the elements of X are graded representations. Let $x \in X$ such that $x: A \rightarrow \mathcal{L}(\mathcal{H}_x)$, where $\mathcal{H}_x$ is a separable, graded, infinite-dimensional Hilbert space. Let $e_0 \in \mathcal{H}_x^{(0)}$ be a unit vector. Let P_x be the degree 0 projection $\begin{pmatrix} p_0 & 0 \\ 0 & 0 \end{pmatrix}$, where $p_0: \mathcal{H}_x^{(0)} \rightarrow \mathbb{C}e_0$. Since $\text{Im}(x) = \mathcal{K}(\mathcal{H}_x)$, there exists $a_1 \in A$ with $x(a_1) = P_x$. We may assume that $\deg(a_1) = 0$ since x is a graded homomorphism. Applying the proof of [3, 4.4.2] to a_1 , we can construct an a such that $x(a) = p_x$ and with the property that there exists a neighborhood V_x of x in X such that $v(a)$ is a rank one projection for every $v \in V_x$. Then $\deg(a) = 0$ so $\deg v(a) = 0$ for every $v \in V_x$. $\square$

6.3. LEMMA. *Let ξ_A be the continuous field constructed from $A \in \mathcal{G}(X)$ as defined in §2. Then there exists an open cover $\{\mathcal{U}_i\}_{i \in I}$ of X such that for every $i \in I$, there is a fiber-preserving, graded isomorphism*

$$h_i: \mathcal{U}_i \times \mathcal{K}_{\text{gr}}(\mathcal{H}) \rightarrow \xi|_{\mathcal{U}_i}$$

where $\mathcal{H}$ is a graded Hilbert space.

PROOF. The continuous field ξ_A has the following property: for each $x \in X$, there exists a neighborhood V of x and a map $p: V \rightarrow E(\xi_A)$ such that $p(y)$ is a degree 0 rank one projection for every $y \in V$ [3, 10.5.8]. Let $\{\mathcal{U}_i\}_{i \in I}$ be a locally finite open cover X of such neighborhoods, with associated degree 0 rank one sections p_i . Let $\xi_A(x) = A_x$. The α and β constructions for continuous fields imply that

$$\alpha\beta(\xi_A|_{\mathcal{U}_i}, p_i) = (\xi(\mathcal{K}_{\text{gr}}(A_x p_i(x))), p_i).$$

Let $\psi_x: \xi_A|_{\{x\}} \rightarrow \mathcal{K}_{\text{gr}}(A_x p_i(x))$ be the graded isomorphism constructed earlier for each $x \in X$. Let $\psi_i = \bigcup_{x \in \mathcal{U}_i} \psi_x$. By [3, 10.7.6(ii)], ψ_i is an isomorphism. Then $k_i = \psi_i^{-1}$ is a fiber-preserving, graded isomorphism.

The algebra A is stable, so ξ_A is locally trivial of rank $\aleph_0$ [21, 1.12]. Then there is a graded isomorphism $\varphi_x: A_x p_i(x) \rightarrow \mathcal{H}$, where $\mathcal{H}$ is a separable, graded, infinite-dimensional Hilbert space. Let $\varphi_i = \bigcup_{x \in \mathcal{U}_i} \varphi_x$; by [3, 10.7.6(i)] and [3, 10.8.7], φ_i is a graded isomorphism between trivial continuous fields of Hilbert spaces. Let $\zeta_i = \alpha(\varphi_i^{-1})$; ζ_i is graded by Lemma 6.1 and is clearly fiber-preserving. The coordinate function h_i for ξ_A can then be defined as:

$$h_i: \mathcal{U}_i \times \mathcal{K}_{\text{gr}}(\mathcal{H}) \xrightarrow{\zeta_i} \xi(\mathcal{K}_{\text{gr}}(A_x p_i(x))) \xrightarrow{k_i} \xi_A|_{\mathcal{U}_i}. \quad \square$$

Every $h_{i,x}$ is a homeomorphism since it is a $*$ -isomorphism. An easy argument using the product topology for $\mathcal{U}_i \times \mathcal{K}_{\text{gr}}(\mathcal{H})$ verifies that h_i is a homeomorphism. Since each h_i is graded, the composite $h_{i,x}^{-1} \circ h_{j,x}$ coincides with an element of $\text{Aut}^0(\mathcal{H}) \approx \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$, for every i and j . It is straightforward to verify that $g_{ij}: \mathcal{U}_i \cap \mathcal{U}_j \rightarrow \mathcal{P}\mathcal{U}_{\text{gr}}(\mathcal{H})$ is continuous.

This completes the proof of Proposition 2.4. $\square$

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DEPARTMENT OF MATHEMATICS, WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS 02181

Current address: Department of Mathematics and Computer Science, DePauw University, Greencastle, Indiana 46135