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ODD PRIMARY PERIODIC PHENOMENA IN THE CLASSICAL ADAMS SPECTRAL SEQUENCE

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ABSTRACT. We study certain periodic phenomena in the cohomology of the mod p Steenrod algebra which are related to the polynomial generators $v_n \in \pi_*BP$. A chromatic resolution of the E_2 term of the classical Adams spectral sequence is constructed.

One of the major goals of homotopy theory is the understanding of $\pi_*(S^0)$, the stable homotopy groups of spheres. A technique for studying these groups is by the construction of certain "systematic families" of classes, first due to M. G. Barratt [**3**]. One way to express this idea is as follows. Let X be a finite complex. (All "spaces" and "complexes" are objects in the stable category localized at a prime p.) A self-map of degree i, $v\colon \Sigma^i X \to X$, is nonnilpotent if the k-fold composition $v^k = (v \circ v \circ \cdots \circ v)\colon \Sigma^{ki}X \to X$ is essential for all $k > 0$.

DEFINITION (1). *For a given nonnilpotent map v, a class $\alpha \in \pi_j(S^0)$ is v-periodic if α can be decomposed as $S^t \hookrightarrow X/X^{(t-1)} \xrightarrow{\overline{\alpha}} S^{t-j}$, where $X^{(k)}$ denotes the k-skeleton of X, and the composite $\Sigma^{ki}X \xrightarrow{v^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\overline{\alpha}} S^{t-j}$ is essential for all $k > 0$* [**4**].

A v-periodic class $\alpha \in \pi_*(S^0)$ determines an infinite "systematic family" in the following manner. For each $k > 0$ there exists an integer ε with $0 \le \varepsilon \le \dim X$, such that the composite

$$S^{ki+\varepsilon} \hookrightarrow \Sigma^{ki}[X/X^{(\varepsilon-1)}] \xrightarrow{v^k} X \xrightarrow{p} X/X^{(t-1)} \xrightarrow{\overline{\alpha}} S^{t-j}$$

is essential (since the composite above is essential for all k), so that each $k > 0$ determines a class (or classes) in $\pi_{(ki+\varepsilon-t+i)}(S^0)$. Here are several well-known examples of this sort of phenomenon.

EXAMPLE (2). Let M_p denote the mod p Moore space ($p \ge 3$). Then Adams [**1**] has constructed a nonnilpotent map $A\colon \Sigma^q M_p \to M_p$, where $q = 2(p-1)$. This map determines a family of nontrivial classes $\{\alpha_t\}$, $t \ge 1$. with $\alpha_t \in \pi_{qt-1}(S^0)$ given by the following diagram:

$$\begin{tikzcd}
\Sigma^{qt}M_p \arrow[r, "A^t"] & M_p \arrow[d, "p"] \\
S^{qt} \arrow[u] \arrow[r, "\alpha_t"] & S^1
\end{tikzcd}$$

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EXAMPLE (3). Let $V(1)$ denote the cofiber of the map A above. Then for $p \geq 5$, there is a nonnilpotent map $B\colon \Sigma^{2(p^2-1)}V(1) \longrightarrow V(1)$ which determines a family of nontrivial classes $\{\beta_t\}$, $t \geq 1$, with $\beta_t \in \pi_{[2(p^2-1)t-q-1]}(S^0)$, by including $S^{2(p^2-1)t}$ into the bottom cell of $\Sigma^{2(p^2-1)t}V(1)$ and pinching out onto the top cell of $V(1)$ [**17**].

EXAMPLE (4). Let $V(2)$ denote the cofiber of B. Then for $p \geq 7$ there is a map $C\colon \Sigma^{2(p^3-1)}V(2) \to V(2)$ which determines a family of nontrivial classes $\{\gamma_t\}$ in $\pi_*(S^0)$ in a similar manner [**13**].

Nonnilpotent self-maps of finite complexes have been classified by Devinatz, Hopkins and J. Smith [**6**] as part of the affirmative solution of the Nilpotence Conjecture. Part of this result can be stated as follows.

THEOREM (5). *(Nilpotence theorem) Let X be a finite complex. A self-map $v\colon \Sigma^i X \to X$ is nonnilpotent if and only if the induced homomorphism BP_*v is nonnilpotent in $BP_*(X)$.*

Here BP is the mod p Brown-Peterson spectrum, where

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots],$$

with $|v_i| = 2(p^i - 1)$. The three examples above all represent multiplication by a generator in BP-homology. Here BP_*A is the map $\cdot v_1$ in BP_*M_p, $BP_*B = \cdot v_2$, and $BP_*C = \cdot v_3$. Two other interesting maps representing v_i's have been studied at the prime 2. These are $v_1^4\colon \Sigma^8 M_2 \longrightarrow M_2$ and $v_2^8\colon \Sigma^{48}Y \to Y$, where Y is a certain four cell complex. Adams and Barratt have used the first map and Davis and Mahowald have used the second map to produce families in $\pi_*(S^0)$ at the prime 2 in [**1** and **4**].

Since these systematic families in π_*S^0 are associated with v_n-self-maps, one obvious way to investigate this sort of thing is by way of the Adams-Novikov spectral sequence. Here the E_2 term is $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, with the spectral sequence converging to π_*X, completed at p. For the sake of convenience, we denote $\mathrm{Ext}_{BP_*BP}(BP_*, M)$ by $\mathrm{Ext}(M)$, for a BP_*BP-comodule M. Let I_n denote the prime ideal $(p, v_1, v_2, \dots, v_{n-1})$ in BP_*. Then the connecting homomorphisms in Ext associated to the short exact sequences

$$0 \longrightarrow BP_*/I_{n-1} \xrightarrow{v_{n-1}} BP_*/I_{n-1} \longrightarrow BP_*/I_n \longrightarrow 0$$

yield

$$\mathrm{Ext}^0(BP_*/I_n) \xrightarrow{\delta} \mathrm{Ext}^1(BP_*/I_{n-1}) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathrm{Ext}^n(BP_*).$$

Clearly there is a class $v_n^t \in \mathrm{Ext}^0(BP_*/I_n)$. Denote the class $(\delta\delta\cdots\delta)(v_n^t) \in \mathrm{Ext}^n(BP_*)$ by $gr_t^{(n)}$, where $gr^{(n)}$ is meant to represent the "nth Greek letter". It is shown in [**13**] that for $n = 1$, 2 and 3, these classes in Ext survive the Adams-Novikov spectral sequence to represent the classes α_t, β_t and γ_t, respectively. The following conjecture generalizes these results.

CONJECTURE (6). *For p a sufficiently large prime, depending on n, $gr_t^{(n)}$ is a nontrivial class in $\mathrm{Ext}^n(BP_*)$ which survives the Adams-Novikov spectral sequence to represent a nontrivial homotopy class in π_*S^0.*

The process of investigating $\mathrm{Ext}(BP_*)$ by means of the n-fold connecting homomorphism shown above can be set up formally as the Chromatic spectral sequence of [**13**], which filters the Adams-Novikov spectral sequence E_2 term into v_n-periodic subquotients, known as the "chromatic filtration". This can be geometrically realized by spectra [**15**].

A natural question to ask is: how does this machinery of v_n-self-maps of finite complexes and their associated systematic families in $\pi_* S^0$ appear in the classical Adams spectral sequence? At the prime 2, this question was answered in [**9**] and [**16**]. There, a fair amount of technical machinery was necessary to start the analysis. For odd primes, the question may be answered in a much simpler fashion.

Recall that the classical Adams spectral sequence (abbreviated by "clASS") at a prime p has $E_2 \cong \mathrm{Ext}_A^{s,t}(\mathbf{Z}/p, \mathbf{Z}/p) \Rightarrow \pi_{t-s}(S^0)\hat{}$, where A denotes the mod p Steenrod algebra and $\hat{}$ denotes completion at p. Let A_n denote the Hopf subalgebra generated by $\{\beta, \mathcal{P}^1, \dots, \mathcal{P}^{p^{n-1}}\}$ if p is odd, $\{\mathrm{Sq}^1, \dots, \mathrm{Sq}^{2^n}\}$ if $p = 2$. Then $A = \varinjlim_n A_n$, so that

$$\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p) \cong \varprojlim_n \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p).$$

We can use information about the cohomology of the finite Hopf algebra A_n, then, to infer results about the clASS E_2 term.

Consider $E(n) = E(Q_0, Q_1, \dots, Q_n)$, the $\mathbf{F}_p$ exterior algebra on the first $n+1$ Milnor generators. Then $E(n)$ is a Hopf subalgebra of A_n, where we denote the inclusion by $i\colon E(n) \hookrightarrow A_n$. Recall also that for $n \geq 0$ there is a spectrum $BP\langle n\rangle$, known as the Baas-Sullivan spectrum [**2**] (or as the Johnson-Wilson spectrum in [**14**]), such that $\pi_*(BP\langle n\rangle) \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n]$, where $|v_n| = 2p^n - 2$. Its cohomology is given by $H^*(BP\langle n\rangle) \cong A \otimes_{E(n)} \mathbf{Z}/p$, (where, as in the sequel, all cohomology groups are assumed to have $\mathbf{Z}/p$ coefficients, unless otherwise specified). Then the clASS converging to $\pi_*(BP\langle n\rangle)$ has

$$\begin{aligned} E_2(BP\langle n\rangle) &= \mathrm{Ext}_A(H^*(BP\langle n\rangle), \mathbf{Z}/p) \\ &= \mathrm{Ext}_A(A \otimes_{E(n)} \mathbf{Z}/p, \mathbf{Z}/p) \\ &\cong \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p) \text{ by change of rings} \\ &\cong \mathbf{Z}/p[q_0, v_1, \dots, v_n], \end{aligned}$$

converging to $\pi_*(BP\langle n\rangle) \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n]$, where the class "$v_i$" $= \{Q_i\}$ in $\mathrm{Ext}_{E(n)}^{1,2p^n-1}(\mathbf{Z}/p, \mathbf{Z}/p)$ represents the homotopy class v_i and multiplication by q_0 corresponds to multiplication by p in $\pi_* BP\langle n\rangle$. Here the E_2 term is concentrated in even dimensions, so that the clASS collapses from that stage. The inclusion map $i\colon E(n) \hookrightarrow A_n$ given above induces the restriction map in cohomology

$$i^*\colon \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p) \to \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p) = \mathbf{Z}/p[q_0, v_1, \dots, v_n].$$

DEFINITION (7). *A class $x \in \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ is said to represent v_i^k if the restriction $i^*(x)$ is $v_i^k \in \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)$.*

With these conventions, we can state our first main result.

THEOREM A. *For all $n \geq 1$, p an odd prime, there exist classes $u_1, u_2, \ldots, u_n$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ such that*

(i) $\mathbf{Z}/p[q_0, u_1, u_2, \ldots, u_n] \subset \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$,

(ii) $i^*(u_n) = v_n^p \in \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)$,

(iii) $i^*(u_i) = v_i^{p^{n-i+1}} \in \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)$ *for* $1 \leq i \leq n$,

(iv) u_n *is a non-zero-divisor in* $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$.

Thus $u_i \in \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ represents $v_i^{p^{n-i+1}}$. We hereafter abuse notation and write $\mathbf{Z}/p[q_0, v_1^{p^n}, v_2^{p^{n-1}}, \ldots, v_n^p] \subset \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$. At the prime 2, the best that one can show is that $\mathbf{Z}/2[h_0, v_1^{N_1}, v_2^{N_2}, \ldots, v_n^{2^{n+1}}] \subset \mathrm{Ext}_{A_n}(\mathbf{Z}/2, \mathbf{Z}/2)$, where N_i is some (possibly very large) integer [**9**]. The proof in the mod 2 case requires the use of Koszul-type resolutions [**5**, **9**], together with a theorem of Lin and Wilkerson, rather than the simpler machinery used below. It should be noted that there are possibly many classes in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ representing $v_i^{p^k}$, one of which we will explicitly produce in the proof of the theorem. For notational ease, we will let $w_i \subset \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ denote the coset of classes which represent $v_i^{p^{n-i+1}}$. An easy inspection of the May spectral sequence converging to $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ shows that there is only one class in the same bigrading as the class u_n of the theorem, so that v_n^p is uniquely represented.

PROOF. Let A_n^* denote the dual of the Hopf algebra A_n. Then there is an extension of Hopf algebras:

$$\mathbf{F}_p \to P_n \to A_n^* \to E_n \to \mathbf{F}_p,$$

where P_n is the truncated polynomial algebra $\mathbf{Z}/p[\xi_1, \xi_2, \ldots, \xi_n]/(\xi_{n-i}^{p^{i+1}})$ and E_n denotes the $\mathbf{F}_p$ exterior algebra $E(\tau_0, \tau_1, \ldots, \tau_n)$. Here $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. Associated to this short exact sequence, we have a Cartan-Eilenberg spectral sequence (CESS) converging to $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$, with E_2 term given by

$$\mathrm{Ext}_{P_n}(\mathbf{Z}/p, \mathrm{Ext}_{E_n}(\mathbf{Z}/p, \mathbf{Z}/p))$$

[**14**]. To analyze this spectral sequence, we first note that $\mathrm{Ext}_{E_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ is a polynomial algebra on $n+1$ generators, which we denote by $\mathbf{Z}/p[a_0, a_1, \ldots, a_n]$, where a_i has bigrading $(1, 2p^i - 1)$. The spectral sequence collapses from E_2 for odd primes [**12**], as one can see by filtering the dual of the Steenrod algebra by the number of τ's in a term. This filtration leads to an E_2 term filtration in terms of the a_i's, which is preserved by the differentials in the CESS, for $p > 2$, so that there can be no nontrivial differentials. Hence the E_2 term gives a filtered version of $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$. The P_n-coaction on $H^*E_n = \mathbf{Z}/p[a_0, a_1, \ldots, a_n]$ is given by $\psi(a_k) = \Sigma \xi_{k-i}^{p^i} \otimes a_i$. Thus, the P_n-coaction on the class a_n^p is $\psi(a_n^p) = \Sigma \xi_{n-i}^{p^{i+1}} \otimes a_i^p = 1 \otimes a_n^p \in P_n \otimes H^*E_n$. Since a_n^p is primitive in H^*E_n, it yields a nontrivial cohomology class in

$$E_2 = \mathrm{Ext}_{P_n}(\mathbf{Z}/p, H^*E_n).$$

Further, the map

$$(\cdot a_n^p)\colon \mathbf{Z}/p[a_0, a_1, \ldots, a_n] \to \mathbf{Z}/p[a_0, a_1, \ldots, a_n]$$

is the inclusion of a direct summand as a map of of P_n-comodules, since $\psi(x)$ can have a term containing a_n^p if and only if a_n^p divides x. If we let u_n denote the class in $\mathrm{Ext}_{P_n}(\mathbf{Z}/p, H^*E_n) = E_0\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ given by the map $(\cdot a_n^p)$, then u_n is a nontrivial class in bidegree $(p, 2p^{n+1}-1)$. Further, the map $(\cdot a_n^p)$ induces

$$(\cdot a_n^p)\colon \mathrm{Ext}_{P_n}(\mathbf{Z}/p, H^*E_n) \to \mathrm{Ext}_{P_n}(\mathbf{Z}/p, H^*E_n)$$

which is also the inclusion of a direct summand. Thus the class $u_n \in \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ representing $(\cdot a_n^p)$ is a non-zero-divisor.

To produce the classes u_i for $i < n$ of the theorem, one notes that $\psi(a_i^{p^{n-i+1}}) = 1 \otimes a_i^{p^{n-i+1}}$, so that $(a_i^{p^{n-i+1}})$ is a primitive in H^*E_n. Let u_i denote the class in $E_0\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ representing $(\cdot a_i^{p^{n-i+1}})$. The u_i's are not necessarily non-zero-divisors, however, since the map

$$(\cdot a_i^{p^{n-i+1}})\colon \mathbf{Z}/p[a_0, a_1, \dots, a_n] \longrightarrow \mathbf{Z}/p[a_0, a_1, \dots, a_n]$$

is not the inclusion of a direct summand of P_n-comodules (a class $x \in H^*E_n$ might have $a_i^{p^{n-i+1}}$ as a factor of $\psi(x)$ even if x is not divisible by $a_i^{p^{n-i+1}}$).

That $i^*(u_i) = v_i^{p^{n-i+1}}$ follows from the fact that the edge homomorphism of the CESS of an extension is the restriction map. Equivalently, the result is clear from the following commutative diagram of Hopf algebras and the naturality of the CESS:

$$\begin{tikzcd}
\mathbf{F}_p \arrow[r] & P_n \arrow[r] \arrow[d] & A_n^* \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & \mathbf{F}_p \arrow[d] \\
 & \mathbf{F}_p \arrow[r] & E(n)^* \arrow[r] & E(n)^* \arrow[r] & \mathbf{F}_p
\end{tikzcd}$$

This completes the proof of the theorem.

We now use these classes representing v_i^k in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ to define what it means for elements in $\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ to be v_i-periodic or v_i-torsion.

DEFINITION (8). *Let $S \subset \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$ be the multiplicative set consisting of the elements which represent $v_i^{p^k}$ for some k. Define $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)(v_i^{-1})$ to be the ring $S^{-1}\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$.*

Note that this definition is independent of the power of $v_i^{p^{n-i+1}}$ chosen. Let

$$p_n\colon \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p) \to \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$$

denote the restriction map in cohomology. Then these localizations fit together into the following tower:

$$\begin{array}{ccc} \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p) & & \\ \downarrow & & \\ \vdots & & \\ \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p,\mathbf{Z}/p) & \longrightarrow & \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p,\mathbf{Z}/p)(v_i^{-1}) \\ \downarrow & & \\ \mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p) & \longrightarrow & \mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)(v_i^{-1}) \\ \vdots & & \\ \downarrow & & \\ \mathrm{Ext}_{A_i}(\mathbf{Z}/p,\mathbf{Z}/p) & \longrightarrow & \mathrm{Ext}_{A_i}(\mathbf{Z}/p,\mathbf{Z}/p)(v_i^{-1}). \end{array} \tag{9}$$

Taking the inverse limit, we obtain a map

$$f_i\colon \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p) \longrightarrow \varprojlim_n \{\mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)(v_i^{-1})\}. \tag{10}$$

This allows us to make the following definition.

DEFINITION (11). *A class $a \in \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p)$ is v_i-periodic if $f_i(a) \neq 0$ and is v_i-torsion if $f_i(a) = 0$.*

This definition is equivalent to the following. Let

$$q_n\colon \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p) \to \mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)$$

denote the restriction map. A class $a \in \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p)$ is v_i-periodic if and only if for each n such that $\hat{a} = q_n(a) \neq 0$, we have $\hat{a}(v_i^{p^{n-i+1}})^s \neq 0$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)$ for all $s \geq 0$, where we use the informal notation for any representative for a power of v_i. A class $a \in \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p)$ is v_i-torsion if and only if for each n such that $\hat{a} = q_n(a) \neq 0$, there is some $s > 0$ such that $\hat{a}(v_i^{p^{n-i+1}})^s = 0$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)$. Note that for some N sufficiently large, $\hat{a} = q_n(a) \neq 0$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p,\mathbf{Z}/p)$, for all $n \geq N$.

THEOREM B. *If a class $a \in \mathrm{Ext}_A(\mathbf{Z}/p,\mathbf{Z}/p)$ is v_n-periodic, then a is also v_{n+k}-periodic for all $k > 0$. Equivalently, if a is v_n-torsion, then a is also v_i-torsion for all $i < n$.*

This result is known in the setting of BP_*BP-comodules by a result of Johnson and Yosimura [**7**]. At the prime 2, this appears as Theorem C in [**9**]. The proof for odd primes is similar to that for the prime 2.

PROOF. The proof uses a map of algebras which is essentially the total reduced power operation. Let t be an indeterminate of degree $2(p-1)$, and let

$$P_t = \sum_{n \geq 0} P^n t^n$$

be the total reduced power operation. Let

$$r\colon A^* \to A^*[t]$$

denote the action of P_t on the left. Then r is a map of right A-algebras given by

$$(12)\qquad \begin{aligned} r(\tau_{n+1}) &= \tau_{n+1} + \tau_n t^{p^n} \quad \text{if } n \geq -1,\\ r(\xi_{n+1}) &= \xi_{n+1} + \xi_n t^{p^n} \quad \text{if } n \geq 0. \end{aligned}$$

Recall that $(A//A_n)^*$ is isomorphic to

$$\mathbf{Z}/p[\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p, \xi_{n+1}, \dots] \otimes E(\tau_{n+1}, \tau_{n+2}, \dots),$$

both as algebras and as right A-modules. Similarly,

$$(A//E(n))^* \cong \mathbf{Z}/p[\xi_1, \xi_2, \dots] \otimes E(\tau_{n+1}, \tau_{n+2}, \dots).$$

The following lemma follows easily from (12).

LEMMA 13. *There are inclusions*

$$r(A//A_{n+1})^* \subset (A//A_n)^*[t],$$
$$r(E//E(n+1))^* \subset (E//E(n))^*[t].$$

By Lemma 13, we have maps (after suitable change of rings)

$$r^*\colon \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p) \to \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)[t],$$
$$r^*\colon \mathrm{Ext}_{E(n+1)}(\mathbf{Z}/p, \mathbf{Z}/p) \to \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)[t],$$

which are ring homomorphisms, since r is given by a map of algebras. The image of $\mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p)$ is contained in the ideal generated by t^{p^n}. Note that if a class $a \in \mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ has nontrivial restriction $\hat{a} \in \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p)$ and $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$, then $r(\hat{a}) = \hat{a}$. (Here, as in the rest of the paper, we use $\hat{a}$ to denote any nontrivial restrictions of $a \in \mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ in $\mathrm{Ext}_{A_m}(\mathbf{Z}/p, \mathbf{Z}/p)$, for all $m > 0$.)

LEMMA 14. *The induced map*

$$r^*\colon \mathrm{Ext}_{E(n+1)}(\mathbf{Z}/p, \mathbf{Z}/p) \to \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)[t]$$

has the values

$$r^*(v_{i+1}) = v_{i+1} + v_i t^{p^i} \in \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)[t]$$

whenever $1 \leq i \leq n$.

PROOF. We compute in the bar construction. Let $\pi\colon A^* \to E(n)^*$ be the natural restriction and ε denote the augmentation. Then the change of rings isomorphism

$$\mathrm{Ext}_A(\mathbf{Z}/p, (A//E(n))^*) \cong \mathrm{Ext}_{E(n)}(\mathbf{Z}/p, \mathbf{Z}/p)$$

is given in terms of the bar construction by

$$\sum [a_i' | a_i''] \mapsto \sum \varepsilon(a_i')[\pi a_i''] \in E(n)^*,$$

where

$$\sum [a_i' | a_i''] \in (A//E(n))^* \otimes A^*.$$

Recall that $H^*E(n) \cong \mathbf{Z}/p[q_0, v_1, \ldots, v_n]$, where $v_i = \{Q_i\}$, corresponding to $\partial\tau_i$ in the bar resolution for $E(n)^*$. Consider the element $\sum[\xi_{i-j}^{p^j}|\tau_j] = \partial[\tau_i]$. It is a cycle since $\partial^2 = 0$ and further, $\sum \varepsilon(\xi_{i-j}^{p^j})[\pi\tau_j] = \partial[\tau_i]$, since $\varepsilon(\xi_{i-j}^{p^j}) = 0$ unless $j = i$. So it follows that $\sum[\xi_{i-j}^{p^j}|\tau_j]$ is a representative for v_i. The element $r^*(v_{i+1})$ is therefore represented by

$$\sum[r(\xi_{i+1-j}^{p^j})|\tau_j] = \sum[(\xi_{i-j}t^{p^{i-j}})^{p^j}|\tau_j] = \sum[\xi_{i-j}^{p^j}|\tau_j]t^{p^i}$$

which represents $v_i t^{p^i}$.

COROLLARY 15. *If $x \in \mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p)$ represents $v_i^{p^k}$ then*

$$r^*(x) = x + yt^{p^{k+i}} \in \mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)[t],$$

where y represents $v_i^{p^k}$.

PROOF. This follows from naturality (Lemma 13 and Lemma 14).

PROOF OF THEOREM B. Let $a \in \mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ be v_{i+1}-torsion. It suffices to show that a is v_i-torsion. Let n be sufficiently large so that the restriction $q_n(a) \neq 0$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$. Let $\hat{a}$ denote both $q_{n+1}(a)$ and $q_n(a)$, as above. Since a is v_{i+1}-torsion, there is some integer s such that $x\hat{a} = 0$ in $\mathrm{Ext}_{A_{n+1}}(\mathbf{Z}/p, \mathbf{Z}/p)$, where x represents $v_{i+1}^{p^s}$. Then

$$0 = r^*(x\hat{a}) = (x + yt^{p^{s+i}})r^*(\hat{a}) = x\hat{a} + y\hat{a}t^{p^{s+i}} = 0 + y\hat{a}t^{p^{s+i}},$$

where y represents $v_i^{p^s}$ by the above corollary. Thus $\hat{a}$ is v_i-torsion in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$, implying our result.

It should be remarked that the total reduced power operation r can be factored through the Davis-Mahowald splitting, which decomposes $A \otimes_{A_n} \mathbf{Z}/p[x, x^{-1}]$ as a sum of $A \otimes_{A_{n-1}} \mathbf{Z}/p$'s [**8**].

As an easy consequence of Theorem B, we have the following corollary.

COROLLARY C. *There is a filtration, which we call the chromatic filtration,*

$$Ext_A(\mathbf{Z}/p, \mathbf{Z}/p) = F_{-1} \supset F_0 \supset F_1 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$$

such that F_n/F_{n+1} is the subquotient of classes that are v_k-torsion for all $k \leq n$ and v_j-periodic for all $j \geq n+1$.

PROOF. Let $F_n = \ker(f_n)$, where the map f_n is given in Definition (11). The result follows immediately from Theorem B.

One should think of this chromatic filtration in the following manner:

$$\begin{array}{ccc}
\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p) & \longrightarrow & (q_0\text{-periodic quotient}) \\
\cup & & \\
(q_0\text{-torsion subgroup}) & \longrightarrow & (v_1\text{-periodic subquotient}) \\
\cup & & \\
(v_1\text{-torsion subgroup}) & \longrightarrow & (v_2\text{-periodic subquotient}) \\
\cup & & \\
(v_2\text{-torsion subgroup}) & \longrightarrow & (v_3\text{-periodic subquotient}) \\
\cup & & \\
\vdots & & \\
\cup & & \\
(v_n\text{-torsion subgroup}) & \longrightarrow & (v_{n+1}\text{-periodic subquotient}) \\
\cup & & \\
\vdots & &
\end{array}$$

PROPOSITION (16). *The chromatic filtration of* $\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ *is complete.*

PROOF (MAHOWALD). Recall that v_n^p is a non-zero-divisor in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$. For each class $a \in \mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$, there is some integer n such that $\hat{a} = q_n(a) \neq 0$ in $\mathrm{Ext}_{A_n}(\mathbf{Z}/p, \mathbf{Z}/p)$. So for each such n, the class $\hat{a}$ is v_n-periodic. Hence

$$\bigcap_{n\geq 0} (v_n\text{-torsion subgroup}) = 0,$$

completing the proof.

Haynes Miller has constructed a chromatic spectral sequence converging to $\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ using the collapsing of the CESS for p odd [**11**]. This allows one to define v_n-periodicity in $\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ in another manner. It is not hard to show that if a class $a \in \mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ is v_n-torsion in Miller's definition, then it is also v_n-torsion in the sense given above. The converse seems to be quite difficult to prove, because of the intractability of the chromatic SS differentials. It is conjectured that the two definitions of v_n-periodicity in $\mathrm{Ext}_A(\mathbf{Z}/p, \mathbf{Z}/p)$ agree.

The chromatic filtration given above is intimately tied in with the idea of "root invariants" in stable homotopy. See [**10** or **16**] for a partial explanation of this relationship.

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