STABLE EXTENSIONS OF HOMEOMORPHISMS ON THE PSEUDOARC

JUDY KENNEDY

ABSTRACT. We prove the following:

THEOREM. *If P' is a proper subcontinuum of the pseudoarc P , h' is a homeomorphism from P' onto itself, and Θ is an open set in P that contains P' , then there is a homeomorphism h from P onto itself such that $h|P' = h'$ and $h(x) = x$ for $x \notin \Theta$.*

A pseudoarc can be characterized as a chainable, hereditarily indecomposable continuum. Although the main result here extends a result of W. Lewis [LW] and answers in the affirmative a question he asked in that same paper, *Stable homeomorphisms of the pseudoarc*, this paper was motivated by a conversation the author had with Professor Marcy Barge, who works in the area of topological dynamics. Professor Barge is interested in knowing what sort of self-homeomorphisms the pseudoarc admits, and, in particular, what sort of homeomorphisms it admits with the property that some of the points of the space have periodic orbits. That pseudoarcs admit many such homeomorphisms will follow from a much earlier result of G. Lehnner [LG] and the result here. It will also follow, by combining the theorem proven here, G. Lehnner's theorem, and a theorem in [KJ], that any continuum with the property that all its proper nondegenerate subcontinua are pseudoarcs admits many such homeomorphisms. Examples of continua with this property include pseudocircles [B], pseudosolenoids [F], some examples due to T. Ingram [I] and H. Cook [C], as well as the pseudoarc itself.

We will prove the following theorem:

THEOREM. *If P' is a proper, nondegenerate subcontinuum of the pseudoarc P , $P' \subseteq o$, an open set in P , and h is a homeomorphism from P' onto itself, then h can be extended to a homeomorphism $\hat{h}$ of P onto itself such that $\hat{h}(x) = x$ for $x \notin o$.*

A continuum is *indecomposable* if no proper subcontinuum of it has interior, and it is *hereditarily indecomposable* if each subcontinuum is indecomposable. A *chain* is a collection of sets $C = \{C_0, C_1, \dots, C_n\}$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A continuum is *chainable* if it has open chain covers of arbitrarily small mesh. As mentioned previously, the pseudoarc P is a hereditarily indecomposable, chainable continuum [KB]. This continuum also has the fixed point property [H].

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The elements of a chain are called *links*. A chain $C = \{C_0, \dots, C_n\}$ is *taut* whenever $C_i \cap C_j \neq \emptyset$ if and only if $\bar{C}_i \cap \bar{C}_j \neq \emptyset$. A chain covers a set A *essentially* if there is a continuum Q contained in A such that each link contains a point of Q not in the closure of any other link. An open set o in a space X is *regular* if $\text{Int } \bar{o} = o$. In the discussion that follows we will assume that our open chain covers are regular, taut, and essential.

If B is a collection of sets, then B^* denotes the union of the sets in B . If C is a collection of sets, then C is an *amalgamation* of B if $B^* = C^*$ and each set in C is the union of some sets in B . If the closure of each set in B is a subset of a set in C , then B is said to *closure refine* C .

If X is a topological space, $H(X)$ will denote the collection of homeomorphisms from X onto X .

Suppose that d is a metric on the pseudoarc P that is compatible with its topology. Throughout the paper we will take distances with respect to this metric d , and for $x \in P$, $\varepsilon > 0$, $D_\varepsilon(x) = \{y \in P | d(x, y) < \varepsilon\}$. Also, if H and K are closed subsets of the pseudoarc, $d(H, K) = \min\{d(x, y) | x \in H, y \in K\}$. If $H = \{x\}$, we will write $d(x, K)$ for $d(\{x\}, K)$.

If $C = \{C_0, \dots, C_n\}$ is an open chain in a space X (which does not necessarily cover X), then for $c \in C$,

$$i(c, C) = \{y \in c | y \notin \bar{c}' \text{ for } c' \in C - \{c\}\},$$

and

$$j(c, C) = \{y \in \bar{c} | y \notin \bar{c}' \text{ for } c' \in C - \{c\}\}.$$

We will say that C has the *end link exit property* if $\partial C^* \subseteq j(c, C)$ where c is one of the end links of C . If $\partial C^* \subseteq j(c, C) \cup j(c', C)$ where c and c' are the end links of C , then C has the *weak end link exit property*. If C is an open chain with the end link exit property, $C = \{c_0, \dots, c_m\}$ and $\partial C^* \subseteq j(c_0, C)$, then $\mathbf{C}(C) = \{c_0 \cup (\bar{c}_0 \cap \partial C^*), c_1, \dots, c_m\}$. If $C = \{c_0, \dots, c_m\}$ is an open chain cover of $\bar{o}$, considered as space, where o is an open set in X , and $\partial_o \subseteq j(c_0, C)$, $\mathbf{I}(C) = \{\text{Int}(c_0), c_1, \dots, c_m\}$. Also, $C[k, l] = \{c_k, c_{k+1}, \dots, c_l\}$.

We will denote the nonnegative integers by $\mathbf{N}$. If $m, n \in \mathbf{N}$, $m < n$, let $[m, n] = \{m, m+1, \dots, n\}$. A function $f: [m', n'] \rightarrow [m, n]$ is called a (*light*) *pattern* provided $|f(i+1) - f(i)| \leq 1$ ($|f(i+1) - f(i)| = 1$, respectively) for $i = m', \dots, n'-1$. (The symbol $\rightarrow$ indicates that f is an onto function.) If $V = \{V_{m'}, V_{m'+1}, \dots, V_{n'}\}$ and $U = \{U_m, U_{m+1}, \dots, U_n\}$ are chain covers of the compactum X , and $f: [m', n'] \rightarrow [m, n]$ is a pattern, we will say that V *follows pattern f in U* provided $\bar{V}_i \subseteq U_{f(i)}$ for each $i \in [m', n']$. We call f a *pattern on U* . If $A \subseteq U$, we will say that f *prefers A* if whenever $V_i \in V$ and $\bar{V}_i \subseteq U_j \in A$, for some j , then $U_{f(i)} \in A$.

The chain C *properly covers* the chain D if D closure refines C and for every $c \in C$ there is some d in D such that $\bar{d} \subseteq c$. Suppose that $C = \{c(0), \dots, c(m)\}$, $D = \{d(0), \dots, d(m)\}$ and $E = \{e(0), \dots, e(n)\}$ are open chains covering the pseudoarc, X is a collection of links of C and $f: [0, n] \rightarrow [0, m]$ is a pattern that E follows in C . Then (E, f) is *compatible with D relative to (C, X)* provided that:

- (1) each of C and D is an amalgamation of E ;
- (2) $C(f(j)) \in X$ if $\bar{E}(j) \subseteq X^*$;
- (3) $C(i) = D(i)$ for every $C(i) \in X$;

(4) for each subchain $\tilde{E}$ of E with $\{C(f(k))|E(k) \in \tilde{E}\} \cap X \neq \emptyset$, $\tilde{E}$ is properly covered by $\{D(f(k))|E(k) \in \tilde{E}\}$.

We will make the following notational conventions: chains will be denoted with upper-case letters, and possibly additional symbols and links of chains with the associated lower-case letters, associated symbols, and link numbers. So, for example,

$$\begin{aligned} C_1 &\equiv \{c(1, 0), \dots, c(1, m)\} \equiv C_1[0, m]; \\ \tilde{D}_2 &\equiv \{\tilde{d}(2, k), \dots, \tilde{d}(2, l)\} \equiv \tilde{D}_2[k, l]; \\ F &\equiv \{f(1), \dots, f(n)\} \equiv \{f_1, \dots, f_n\} \equiv F[1, n]. \end{aligned}$$

The preceding is largely the terminology and notation of [OT] and [LW]. We will also make use of the following result from [LW].

THEOREM LW (THEOREM 6 OF [LW]). *Let C_0 , D_0 , and C_1 be chains covering the pseudoarc, with X a nonempty collection of links of C_0 and f_1 a pattern such that (C_1, f_1) is compatible with D_0 relative to (C_0, X) . Then for each $\varepsilon > 0$ there exists a chain E of mesh less than ε , a pattern f_2 , and a chain D_1 covering P such that D_1 follows the pattern f_1 in D_0 , and (E, f_2) is compatible with C_1 relative to $(D_1, \{d(1, j)|c(0, f_1(j)) \in X\})$.*

We will also need the following theorems due to G. R. Lehner [LG] and J. Krasinkiewicz and P. Minc [OT], respectively.

THEOREM LG. *Suppose $H_{1,1}, H_{1,2}, \dots, H_{1,n}$ are disjoint subcontinua of the pseudoarc P and that P is irreducible between each pair of them. Suppose T is a homeomorphism of $\bigcup_{i=1}^n H_{1,i}$ onto $\bigcup_{i=1}^n H_{2,i}$ where $H_{2,1}, H_{2,2}, \dots, H_{2,n}$ are subcontinua of P such that P is irreducible between each pair of them. Then T can be extended to a homeomorphism of P onto P .*

THEOREM KM. *Let X be a compactum. Then the following are equivalent:*

- (I) X is hereditarily indecomposable;
- (II) *for every pair of disjoint closed subsets A and B of X and for every pair of neighborhoods U of A and V of B , there exist closed subsets X_0, X_1 and X_2 of X such that*

- (1) $X = X_0 \cup X_1 \cup X_2$,
- (2) $A \subseteq X_0$ and $B \subseteq X_2$,
- (3) $X_0 \cap X_2 = \emptyset$,
- (4) $X_0 \cap X_1 \subseteq V$ and $X_1 \cap X_2 \subseteq U$.

A sequence $V = V_1, V_2, \dots$ of open chain covers of P is said to be a *defining sequence* for P if $\lim_i \text{mesh } V_i = 0$ and, for each i , V_i is an amalgamation of V_{i+1} . (Please recall that “open chain cover” really means “open taut essential regular chain cover.”) We will say that a set is V_j -open if it is a finite union of links of V_j for some j .

LEMMA 2. *Suppose that P' is a proper, nondegenerate subcontinuum of P , and $C = \{c_0, \dots, c_m\}$ is an open chain cover of P' , considered as space, of mesh less than ε . There is an open (in P) chain $D = \{d_0, \dots, d_m\}$ that covers P' , has mesh less than ε , has the end link exit property with $\partial D^* \supseteq j(d_0, D)$, and has the property that $d_i \cap P' = c_i$ for each $i \in [0, m]$.*

PROOF. Suppose P' is irreducible between $p \in i(c_0, C)$ and $g \in i(c_m, C)$. Suppose that $\varepsilon' = \text{mesh } C$ and $V_1, V_2, \dots$ is a defining sequence for P such that if

$$V'_i = \{v \cap P' | \bar{v} \cap P' \neq \emptyset, v \in V_i\},$$

then $V'_1, V'_2, \dots$ is a defining sequence for P' . Also, we may assume that $g \in i(v(i, n_i), V_i)$, where $v(i, n_i)$ is the last link of V_i . (It can easily be checked that such a sequence exists.) Choose $\sigma > 0$ so that $\sigma < \min\{d(\bar{c}_i, \bar{c}_j) | |i - j| > 1, i, j \in [0, m]\}$. There is an integer $p(1)$ so that $V'_{p(1)}$ closure refines C , and if $p \geq p(1)$, V_p has mesh less than $(\varepsilon - \varepsilon')/4$ and $\sigma/4$. Choose $\delta < \varepsilon - \varepsilon'$. We want for p to be in the first link of $V_{p(1)}$. Otherwise apply Theorem KM as follows: Suppose $v(p(1), a_{p(1)})$ is the first link of $V_{p(1)}$ that intersects P' . Let $B = \partial V_{p(1)}[a_{p(1)}, n_{p(1)}]^*$. Suppose o is an open set such that

$$B \subseteq o \subseteq \bar{o} \subseteq v(p(1), a_{p(1)}) \cup v(p(1), a_{p(1)} - 1) \\ - (\overline{\{v \in V_{p(1)} | v \neq v(p(1), a_{p(1)}) \text{ and } v \neq v(p(1), a_{p(1)} - 1)\}}^* \cup P').$$

Without loss of generality, assume $p \in i(v(p(1), b), V_{p(1)})$ and that u is open such that $p \in u \subseteq \bar{u} \subseteq i(v(p(1), b), V_{p(1)})$. There are closed sets X_0, X_1 , and X_2 such that

- (1) $X_0 \cup X_1 \cup X_2 = V_{p(1)}[a_{p(1)}, n_{p(1)}]^* \cup \bar{o}$;
- (2) $B \subseteq X_0$ and $p \in X_2$; and
- (3) $X_0 \cup X_1 \subseteq u$ and $X_1 \cap X_2 \subseteq o$.

Now form $\tilde{V}_{p(1)}$ as follows: Let $X = X_1 \cup X_2$. There is a chain T_0 such that $\{v \cap \text{Int}(X_0) | v \in V_{p(1)}[a_{p(1)}, n_{p(1)}], v \cap X_0 \neq \emptyset\}$ is an amalgamation of T_0 , and a chain T_1 such that $\{v \cap \text{Int}(X) | v \in V_{p(1)}[a_{p(1)}, n_{p(1)}], v \cap X \neq \emptyset\}$ is an amalgamation of T_1 , and T_0 and T_1 are open chains such that

- (4) $T_0^* \cup T_1^* = V_{p(1)}[a_{p(1)}, n_{p(1)}]^*$;
- (5) $\tilde{V}_{p(1)} = V_{p(1)}[0, a_{p(1)} - 1] \cup T_0 \cup T_1$ is an open chain cover of P that refines $V_{p(1)}$;
- (6) p is in the first link of $\tilde{V}_{p(1)}$ whose closure intersects P' and is not in the closure of any other link of $\tilde{V}_{p(1)}$;

(7) q is in the last link of $\tilde{V}_{p(1)}$ and not in the closure of any other link of $\tilde{V}_{p(1)}$; and

(8) if $\tilde{V}_{p(1)}^\# = \{v \in \tilde{V}_{p(1)} | \bar{v} \cap P' \neq \emptyset\}$, then $\partial \tilde{V}_{p(1)}^\# \subseteq j(\hat{v}, \tilde{V}_{p(1)}^\#)$, where $\hat{v}$ is the first link in $\tilde{V}_{p(1)}^\#$, and $p \in i(\hat{v}, \tilde{V}_{p(1)})$. Thus, we can always modify $V_{p(1)}$ so that p is in the first link of $V'_{p(1)}$ and q is in the last link of $V'_{p(1)}$, etc. In fact, we have actually shown that we could have chosen our sequence $V_1, V_2, \dots$ so that p is in the first link of V'_i and q is in the last link of V_i . Let us assume that our sequence has this property then, so that we will not have to worry about this again.

For $j \in [0, m]$, let $d_j(1) = \{v \in V_{p(1)} | \emptyset \neq \bar{v} \cap P' \subseteq c_j\}^*$. Then $D(1) = \{d_0(1), \dots, d_m(1)\}$ is an open (in P) chain cover of P' of mesh less than $\varepsilon' + \delta/2$. Also, since $D(1)$ is an amalgamation of a subchain of $V_{p(1)}$, $D(1)$ has the end link exit property with $\partial D(1)^* \subseteq j(d_0(1), D(1))$, and $D'(1) = \{d \cap P' | d \in D(1)\}$ closure refines C .

Let $E = \{d_j(1) | j \in [1, m]\}^*$ and $\sigma' > 0$ such that

$$d(P', \partial D(1)^*) > \sigma', \quad d(E, \partial D(1)^*) > \sigma'.$$

Choose a sequence $p(2), p(3), \dots$ of increasing integers such that

- (1) $p(2) > p(1)$,
- (2) $V_{p(i)}$ has mesh less than $\sigma'/4$ for $i \geq 2$, and
- (3) $\{\overline{v \in V_{p(i)} | \bar{v} \cap P' \neq \emptyset}\}^* \subseteq \{v \in V_{p(i-1)} | \bar{v} \cap P' \neq \emptyset\}^*$.

For $j \in [0, m]$, $i \geq 2$, let $d_j(i) = \{v \in V_{p(i)} | \emptyset \neq \bar{v} \cap P' \subseteq c_j\}^*$. Then $D(i) = \{d_j(i) | 0 \leq j \leq m\}$ is an open (in P) chain cover of P' of mesh less than $\varepsilon' + \delta/2$, and $D(i)$ has the end link exit property. Further, $D'(i) = \{d \cap P' | d \in D(i)\}$ closure refines C . For $j \in [0, m]$ let $d_j = \bigcup_{i=1}^{\infty} d_j(i)$, and let $D = \{d_0, \dots, d_m\}$. It is clear that D is an open (in P) cover of P' and that $d_j \cap P' = c_j$. Suppose x and y are in $d_j \in D$. There are i and i' such that $x \in d_j(i)$ and $y \in d_j(i')$, and there are z and z' in c_j such that $d(z, x) < \delta/4$ and $d(z', y) < \delta/4$. Then

$$d(x, y) \leq d(x, z) + d(z, z') + d(z', y) < \delta/4 + \varepsilon' + \delta/4 = \varepsilon' + \delta/2 < \varepsilon' + \delta.$$

Thus, the mesh of D is less than $\varepsilon' + \delta = \varepsilon$.

Suppose that $\bar{d}_i \cap \bar{d}_j \neq \emptyset$. There is $w \in \bar{d}_i \cap \bar{d}_j$ and we can find $w' \in d_i(l)$ for some l , and $\hat{w} \in d_j(l')$ for some l' , such that $d(w', w) < \sigma/4$ and $d(w, \hat{w}) < \sigma/4$. There are p' and $\hat{p}$ in c_i and c_j , respectively, such that $d(w', p') < \sigma/4$ and $d(\hat{w}, \hat{p}) < \sigma/4$. Then $d(p', \hat{p}) \leq d(p', w') + d(w', w) + d(w, \hat{w}) + d(\hat{w}, \hat{p}) < \sigma$. By the choice of σ , this means that $|i - j| \leq 1$ and that D is a (taut) chain.

Consider finally ∂D^* . $D^* = D(1)^*$, since

$$D(1)^* = \{v \in V_{p(1)} | \bar{v} \cap P' \neq \emptyset\}^* \supseteq D(i)^* \quad \text{for } i > 1$$

(by the choice of σ'). If $x \in \partial D^*$, $x \in \partial D(1)^*$. Suppose $x \in \bar{d}_j$ for some j . There is $x' \in D(1)^*$ such that $d(x', x) < \sigma'/4$, $x' \in d_j(l)$ for some l . If $l > 1$, there is $\hat{x} \in c_j$ so that $d(x', \hat{x}) < \sigma'/4$. Then $d(x, \hat{x}) \geq d(x, x') + d(x', \hat{x}) < \sigma'/2$, which cannot be because of the choice of σ' . Then $l = 1$ and $x' \in d_j(1)$. But $x' \in d_j(1)$ only if $j = 0$, so we are done. $\square$

We will refer to the process of applying Lemma 2 as fattening an open chain in $P'(C)$ to obtain an open chain in $P(D)$.

LEMMA 3. *Suppose that P' is a nondegenerate, proper subcontinuum of P and $C = \{c_0, \dots, c_m\}$ is an open (in P) chain cover of P' , with $\partial C^* \subseteq j(c_m, C)$, and P' is covered essentially by C . There is an open chain $D = \{d_0, \dots, d_{2m}\}$ such that*

- (1) D refines C ,
- (2) $\partial D^* \subseteq j(d_0, D) \subseteq i(c_m, C)$,
- (3) $P' \cap d_{m+i} = P' \cap c_i$ for $i \in [0, m]$, and
- (4) $P' \cap \bar{d}_i = \emptyset$ for $i \in [0, m-1]$.

PROOF. There is a point $p \in i(c_0, C) \cap P'$. Find disjoint open sets u and v such that $p \in u \subseteq \bar{u} \subseteq i(c_0, C)$ and $\partial C^* \subseteq v$, $\bar{v} \cap P' = \emptyset$, $\bar{v} \cap \bar{c}_i = \emptyset$ for $i \in [0, m-1]$. If $X = \bar{C}^*$, we can apply Theorem 2 of [OT] (which is actually a theorem of Krasinkiewicz and Minc, but is stated in the form we need in [OT]) to find closed sets X_0, X_1 , and X_2 of X such that

- (1) $p \in X_2$ and $\partial C^* \subseteq X_0$,
- (2) $X = X_0 \cup X_1 \cup X_2$,
- (3) $X_0 \cap X_2 = \emptyset$,
- (4) $X_0 \cap X_1 \subseteq u$ and $X_1 \cap X_2 \subseteq v$.

Then $P' \subseteq \text{Int } X_2$. Define $d_0 = (c_m \cap X_0) - \bar{v}$. For $i \in [1, m-1]$, define $d_i = c_{m-i} \cap X_0$. Define $d_m = c_0$, and for $i \in [m+1, 2m]$, $d_i = c_{i-m} \cap (X_1 \cup X_2)$. Finally, $D = \{d_0, \dots, d_{2m}\}$. We have the chain we wanted. $\square$

Given an open, in P' , chain cover $C = \{c_0, \dots, c_m\}$, we will refer to the process of obtaining the chains (open in P) that Lemmas 1-3 guarantee us as fattening and doubling C .

LEMMA 4. Suppose that $C_0 = C_0[0, m_0]$, $D_0 = D_0[0, m_0]$, and $\hat{C}_1 = \hat{C}_1[0, \hat{m}_1]$ are chains covering the pseudoarc P , such that $\hat{C}_1$ closure refines C_0 and D_0 , $X \subseteq C_0$, $\hat{f}_1$ is a pattern, and $(\hat{C}_1, \hat{f}_1)$ is compatible with D_0 relative to (C_0, X) . Further, suppose C'_1 is an open chain cover of P so that C'_1 closure refines $\hat{C}_1$, $\hat{C}_1$ is an amalgamation of C'_1 , and $X' = \{c \in X | c \cap (C_0 - X)^* = \emptyset\}$. If $C_1 = \{\tilde{C}^* | \tilde{C} \text{ is a maximal subchain of } C'_1 \text{ such that for some } \hat{c} \in \hat{C}_1, \tilde{C}^* \subseteq \hat{c} \text{ and } i(\hat{c}, \hat{C}_1) \cap \tilde{C}^* \neq \emptyset\}$, then C_1 is an open chain cover of P , $C_1 = C_1[0, m_1]$, and $f_1: [0, m_1] \rightarrow [0, m_0]$ is a pattern defined by $f_1(i) = \hat{f}_1(j)$, where $c(1, i) \subseteq \hat{c}(1, j)$ and $c(1, i) \cap i(\hat{c}(1, j), \hat{C}_1) \neq \emptyset$, unless $c(0, \hat{f}_1(j)) \notin X'$, $\overline{c(1, i)} \subset X'^*$, in which case $f_1(i)$ is defined so that f_1 will prefer X' , then (C_1, f_1) is compatible with D_0 relative to (C_0, X') .

PROOF. This proof is straightforward and will be omitted. $\square$

LEMMA 5. Suppose that P' is a proper nondegenerate subcontinuum of P , and that $p' \in P'$. Suppose further that (1) $C_0 = C_0[0, m_0]$, $D_0 = D_0[0, m_0]$, and $C_1 = C_1[0, m_1]$ are chains covering P such that

- (1) C_1 closure refines C_0 and D_0 ;
- (2) $p' \in i(c(0, m_0), C_0) \cap i(d(0, m_0), D_0) \cap i(c(1, m_1), C_1)$;
- (3) $\text{mesh } C_1 < \frac{1}{6}\delta$ where $\delta = \min\{d(p', e) | p' \notin e, e \in C_0 \cup D_0\}$;
- (4) $X \subseteq C_0$ such that $P' \cap \bar{X}^* = \emptyset$; and
- (5) f_1 is a pattern with (C_1, f_1) compatible with D_0 relative to (C_0, X) .

Then for each $\varepsilon > 0$ there exists a chain D_2 of mesh less than ε , a pattern f_2 , and a chain D_1 covering P so that D_1 follows the pattern f_1 in D_0 , $p' \in i(d(1, m_1), D_1) \cap i(d(2, m_2), D_2)$ and (D_2, f_2) is compatible with C_1 relative to (D_1, X'_1) where $X'_1 = \{c \in X_1 | c \cap (C_1 - X_1)^* = \emptyset\}$ and $X_1 = \{c(1, j) | c(0, f_1(j)) \in X\}$.

PROOF. Suppose $\varepsilon > 0$. There are open sets u, v, w such that (1) $\{u, v, w\}$ is an open chain with $c(1, m_1) = u \cup v \cup w$; (2) $p' \in w - \overline{u \cup v}$; (3) $\partial\{u, v, w\}^* = \partial(c(1, m_1)) \subseteq \partial u$; and (4) $(\bar{w} \cup \bar{v}) \cap \overline{c(1, m_1 - 1)} = \emptyset$. Modify C_0 and D_0 next: $\hat{C}_0 = \hat{C}_0[0, m_0 + 1]$ is defined by

$$\hat{c}(0, i) = \begin{cases} c(0, i) & \text{for } i < m_0, \\ (c(0, m_0) - w) \cup v & \text{for } i = m_0, \\ u \cup v \cup w & \text{for } i = m_0 + 1, \end{cases}$$

and $\hat{D}_0 = \hat{D}_0[0, m_0 + 1]$ is defined by

$$\hat{d}(0, i) = \begin{cases} d(0, i) & \text{for } i < m_0, \\ (d(0, m_0) - w) \cup v & \text{for } i = m_0, \\ u \cup v \cup w & \text{for } i = m_0 + 1. \end{cases}$$

Further, define $\hat{C}_1 = \hat{C}_1[0, m_1 + 2]$ by

$$\hat{c}(1, i) = \begin{cases} c(1, i) & \text{for } i < m_1, \\ u & \text{for } i = m_1, \\ v & \text{for } i = m_1 + 1, \\ w & \text{for } i = m_1 + 2. \end{cases}$$

Then $\hat{C}_1$ follows $\hat{f}_1$ in $\hat{C}_0$ where $\hat{f}_1(i) = f_1(i)$ for $i < m_1$, $\hat{f}_1(m_1) = m_0$, $\hat{f}_1(m_1 + 1) = \hat{f}_1(m_1 + 2) = m_0 + 1$, and $(\hat{C}_1, \hat{f}_1)$ is compatible with $\hat{D}_0$ relative to $(\hat{C}_0, X)$. Theorem LW allows us to find a chain $\hat{D}_2 = \hat{D}_2[0, \hat{m}_2]$ of mesh less than ε , a pattern $\hat{f}_2$, and a chain $\hat{D}_1 = \hat{D}_1[0, m_1 + 2]$ covering P so that $\hat{D}_1$ follows $\hat{f}_1$ in $\hat{D}_0$ and $(\hat{D}_2, \hat{f}_2)$ is compatible with $\hat{C}_1$ relative to $(\hat{D}_1, \{\hat{d}(1, j) | \hat{c}(0, \hat{f}_1(j)) \in X\})$. Note that p' must be in either $\hat{d}(1, m_1 + 1)$ or $\hat{d}(1, m_1 + 2)$, for $\hat{f}_1(m_1 + 1) = \hat{f}_1(1, m_1 + 2) = m_0 + 1$, $\hat{f}_1(i) < m_0 + 1$ for $i < m_1 + 1$, and $p' \in i(\hat{d}(0, m_0 + 1), \hat{D}_0)$. Let $D_1 = D_1[0, m_1]$, where

$$d(1, i) = \begin{cases} \hat{d}(1, i) & \text{for } i < m_1, \\ \bigcup_{j=0}^2 \hat{d}(1, j + m_1) & \text{for } i = m_1. \end{cases}$$

Then D_1 follows f_1 in D_0 . Define $f'_2: [0, \hat{m}_2] \rightarrow [0, m_1]$ by $f'_2(i) = \hat{f}_2(i)$ for i such that $\hat{f}_2(i) < m_1$, $f'_2(i) = m_1$ for $\hat{f}_2(i) \geq m_1$, so that $\hat{D}_2$ follows f'_2 in D_1 . Further, $(\hat{D}_2, f'_2)$ is compatible with C_1 relative to $(D_1, \{d(1, j) | c(0, f_1(j)) \in X\})$. Note that $p' \in i(d(1, m_1), D_1)$. Let $\{d(1, j) | c(0, f_1(j)) \in X\} = X_1$ and let $X'_1 = \{d \in X_1 | d \cap (D_1 - X_1)^* = \emptyset\}$.

Finally, there is an open chain cover D'_2 of P such that $D'_2 = D'_2[0, m'_2]$ closure refines $\hat{D}_2$, $\hat{D}_2$ is an amalgamation of D'_2 and $p' \in i(d'(2, m'_2), D'_2)$ with $d'(2, m'_2) \subseteq i(d(1, m_1), D_1)$. Let $D_2 = D_2[0, m_2] = \{\tilde{D}^* | \tilde{D} \text{ is a maximal subchain of } D'_2 \text{ such that } \tilde{D}^* \subseteq d \in \hat{D}_2 \text{ and } \tilde{D}^* \cap i(d, \hat{D}_2) \neq \emptyset \text{ for some } d \in \hat{D}_2\}$. Without loss of generality, assume $p' \in i(d(2, m_2), D_2)$. Then D_2 is an open chain cover of P and D_2 follows the pattern f_2 in D_1 , where $f_2(i) = f'_2(j)$ when $d'(2, j)$ is that unique link of D'_2 such that $d(2, i) \subseteq d'(2, j)$ and $d(2, i) \cap i(d'(2, j), D'_2) \neq \emptyset$, unless $d(1, f'_2) \notin X'_1$, $\overline{d(2, i)} \subseteq X'_1$, in which case $f_2(i)$ is defined so that f_2 prefers X'_1 . (We are applying Lemma 4 here.) We are done. $\square$

For $g, k \in H(P')$ define $\rho'(g, k) = \text{lub}\{d(g(x), k(x)) | x \in P'\}$ and for $g, k \in H(P)$, define $\rho(g, k) = \text{lub}\{d(g(x), k(x)) | x \in P\}$. Our strategy in the proof that follows will be the following. There is a sequence $u_1, u_2, \dots$, of open sets of P such that $\bar{u}_1 \subseteq o$, $\bar{u}_{i+1} \subseteq u_i$ for $i \geq 1$, and $\bigcap u_i = P'$. We will construct inductively a sequence $g_1, g_2, \dots$ of homeomorphisms of P so that

- (1) $g_1(x) = x$ for $x \notin o$ and $g_{i+1}(x) = x$ for $x \notin g_i \cdot g_{i-1} \cdot \dots \cdot g_1(u_1)$;
- (2) $g_i(P') = P'$ for each i ;
- (3) $\rho'(g_i|P', h_i) < 2^{-(i+1)}$ and $\rho'(g_i^{-1}|P', h_i^{-1}) < 2^{-(i+1)}$ for $i > 1$, where $h_i = h_{i-1} \cdot g_{i-1}^{-1}|P'$; and

- (4) $\rho(g_i, 1_{P'}) < 2^{-i}$ and $\rho(g_i^{-1}, 1_{P'}) < 2^{-i}$ for $i > 1$, where $1_{P'}$ denotes the identity on P' .

We will then define $g(x) = \lim_i g_i \cdot \dots \cdot g_1(x)$ and g will be the homeomorphism we desired (as we later prove).

PROOF OF THE THEOREM. Since the pseudoarc has the fixed point property, there is some $p' \in P'$ such that $h(p') = p'$. Further, we may assume that there is some $q' \in P'$ such that P' is irreducible from p' to q' and $h(q') = q'$. (If this is not the case, then we can extend P' and h using Theorem LG, so that it is the case in the new continuum with the extended homeomorphism, and then prove the theorem for that homeomorphism and continuum.) Let us also assume, without loss of generality, that $q \in P - \bar{o}$ such that q and p' are in different composants of P .

Suppose that $V_1, V_2, \dots$ is a defining sequence for P with the following properties:

- (1) $\text{mesh } V_i < 2^{-i}$ for each i ;
- (2) for $i \in \mathbf{N}$, $V_i = \{v(i, 0), \dots, v(i, n_i)\}$ with $q \in i(v(i, 0), V_i)$ and $p' \in i(v(i, n_i), V_i)$;
- (3) if $\overline{v(1, i)} \cap (P - o) \neq \emptyset$ and $\overline{v(1, j)} \cap P' \neq \emptyset$, then $|j - i| > 5$;
- (4) for $i \geq 1$, $\text{mesh } V_{i+1} \leq \frac{1}{5} \min\{|d(v, v')|v, v' \text{ are nonadjacent links of } V_i\} \cup \{d(x, v)|(x = p' \text{ and } v \neq v(i, n_i)) \text{ or } (x = q \text{ and } v \neq v(i, 0)), v \in V_i\}$; and
- (5) for $i \geq 1$, $q' \in i(v(i, b'_i), V_i)$, where $v(i, b'_i)$ is the first link of V_i whose closure intersects P' . (To see that this last property can be satisfied, take a defining sequence with the other four properties, modify the chains involved using Lemma 3 (fattening and doubling in the subchains containing P' in their unions), and then doing any additional modifications to ensure that the result is a defining sequence with all five properties satisfied.)

Let $\varepsilon_1 < \frac{1}{4}$, $\varepsilon_1 > 0$, such that if $\rho'(k, 1) < \varepsilon_1$, then $\rho'(k^{-1}, 1) < \frac{1}{4}$ for $k \in H(P')$. There is $\delta_1 > 0$, $\delta_1 < \varepsilon_1/2$, such that if $d(x, y) < \delta_1$, $x, y \in P'$, then $d(h(x), h(y)) < \varepsilon_1/2$ and $d(h^{-1}(x), h^{-1}(y)) < \varepsilon_1/2$. Then there is $\tau_1 > 0$, $\tau_1 < \delta_1/2$, such that if $d(x, y) < \tau_1$, $x, y \in P'$, then $d(h(x), h(y)) < \delta_1/2$ and $d(h^{-2}(x), h^{-2}(y)) < \delta_1/2$. Also choose τ_1 so that if $\rho'(h, g) < \tau_1$, $\rho'(h^{-1}, g^{-1}) < \delta_1$ for $g \in H(P')$, and choose $\tau'_1 > 0$, $\tau'_1 < \tau_1/2$ so that if $d(x, y) < \tau'_1$, $x, y \in P'$, then $d(h(x), h(y)) < \tau_1/2$.

Find $i_1 \in \mathbf{N}$ such that $\text{mesh } V_{i_1} < \tau'_1$. Let $V_{i_1} = C_1$, and let

$$C_1 = \{c(1, 0), \dots, c(1, m_1)\}$$

where $p' \in i(c(1, m_1), C_1)$, $q \in i(c(1, 0), C_1)$, and $q' \in i(c(1, b_1), C_1)$. Then $C_1[b_1, m_1]$ essentially covers P' , and $P - o \subseteq C_1[0, b_1 - 4]$.

For $i \in [b_1, m_1]$, let $c^\#(1, i) = c(1, i) \cap P'$, and let $C_1^\# = \{c^\#(1, i)|i \in [b_1, m_1]\}$. Use Lemma 2 to obtain the open chains E_1 and F_1 as follows:

(1) Fatten $C_1^\#$ in $C_1[b_1, m_1]$, obtaining the open chain $\hat{E}_1 = \hat{E}_1[b_1, m_1]$ so that $\bar{\hat{E}}_1^* \cap \overline{C_1[0, b_1 - 1]}^* = \emptyset$ and $\text{mesh } \hat{E}_1 < \tau'_1$.

(2) Fatten $h(C_1^\#) = \{h(c^\#)|c^\# \in C_1^\#\}$ in $\hat{E}_1^*$ obtaining the open chain $F_1 = F_1[b_1, m_1]$.

(3) Let $E_1 = E_1[b_1, m_1]$ be defined by $e(1, i) = \hat{e}(1, i) \cap F_1^*$ for $i \in [b_1, m_1]$.

(4) Further, do these fattenings so that $\partial E_1^* \subseteq j(e(1, m_1), E_1) \subseteq i(c(1, m_1), C_1)$ and $\partial F_1^* \subseteq j(f(1, m_1), F_1)$. (Note $E_1^* = F_1^*$.)

We are almost ready to construct D_1 . Let us make one more assumption: The links in F_1 may not be V_j -open for any j , although we can construct E_1 so that those links are V_j -open for some j . However, with an arbitrarily small enlargement of each link, we can make the links of F_1 V_k -open for some k . Of course, it will no longer necessarily be the case that $h(c^\#(1, i)) = f(1, i) \cap P'$ for $i \in [b_1, m_1]$. We

assume then that the links of F_1 and of E_1 are all V_k -open for some k . Let

$$\begin{aligned} D_1 &= \{c(1, 0), c(1, 1), \dots, c(1, b_1 - 1), \\ &\quad (c(1, b_1) - F_1^*) \cup f(1, b_1), \dots, (c(1, m_1) - F_1^*) \cup f(1, m_1)\} \\ &= D_1[0, m_1] = \{d(1, 0), \dots, d(1, m_1)\}, \end{aligned}$$

and let $X_1 = C_1[0, b_1 - 1] = D_1[0, b_1 - 1]$. Choose $i_2 \in \mathbb{N}$ so that $i_2 > i_1$, the links of both D_1 and C_1 are V_{i_2} -open, and $\text{mesh } V_{i_2} < \tau_2$ where (a) $\varepsilon_2 > 0$ such that $\varepsilon_2 < 1/8$, $\varepsilon_2 < \tau_1'$, (b) $\delta_2 > 0$, $\delta_2 < \varepsilon_2/2$, such that if $x, y \in P'$ and $d(x, y) < \delta_2$, then $d(h(x), h(y)) < \varepsilon_2/2$ and $d(h^{-1}(x), h^{-1}(y)) < \varepsilon_2/2$, (c) $\tau_2 > 0$, $\tau_2 < \delta_2/2$ such that if $x, y \in P'$ and $d(x, y) < \tau_2$, then $d(h(x), h(y)) < \delta_2/2$ and (d) V_{i_2} closure refines C_1 and D_1 . Also, choose i_2 so that $\text{mesh } V_{i_2} < \frac{1}{4} \min\{d(v, v')\}$ for $i \in [0, m_1 - 1]$, $v = c(1, i) - E_1^*$, $v' = E_1^*$ or $v = d(1, i) - F_1^*$, $v' = F_1^*$, $\bar{v}(i_2, n_{i_2}) \subseteq i(d(1, m_1), D_1)$, and $\text{mesh } V_{i_2} < \frac{1}{6} \min\{d(p', e) | p' \notin e, e \in C_1 \cup D_1\}$.

Let $C_2 = \{c(2, 0), \dots, c(2, m_2)\} = V_{i_2}$ where $q \in c(2, 0)$, $p' \in c(2, m_2)$, $q' \in c(2, b_2)$. Now C_2 follows some pattern $f_1: [0, m_2] \rightarrow [0, m_1]$ in C_1 where f_1 prefers X_1 . Then (C_2, f_1) is compatible with D_1 relative to (C_1, X_1) , and we can apply Lemma 5. Choose $\sigma_2 < \frac{1}{16}$, $\sigma_2 > 0$, $\sigma_2 < \tau_2$, $\sigma_2 < \text{Lebesgue number for } \{h(c \cap P') | c \in C_2[b_2, m_2]\}$. Then there exist chains $\hat{D}_2$ and $\hat{D}_3$ covering P so that $\hat{D}_2$ follows f_1 in D_1 , $\text{mesh } \hat{D}_3 < \sigma_2$, $\hat{D}_3$ follows a pattern $\hat{f}_2$ in $\hat{D}_2$, $p' \in i(\hat{d}(2, m_2), \hat{D}_2) \cap i(\hat{d}(3, m_3), \hat{D}_3)$, and $(\hat{D}_3, \hat{f}_2)$ is compatible with C_2 relative to $(\hat{D}_2, X'_2)$ (where $\hat{D}_2 = \hat{D}_2[0, m_2]$, $X_2 = \{c(2, j) | c(1, f_1(j)) \in X'_1\}$ and $X'_2 = \{c \in X_2 | c \cap (C_2 - X_2)^* = \emptyset\}$).

Without loss of generality assume that $\hat{D}_2$ and $\hat{D}_3$ are V_l -open for some l . We need for q' to be in $i(\hat{d}(2, b_2), \hat{D}_2)$, but this may not be the case. We can, however, modify $\hat{D}_2$ and $\hat{D}_3$ so that this will be the case, without losing the other properties we want these chains to have.

Let $D_2^\# = \{h(c \cap P') | c \in C_2, c \cap P' \neq \emptyset\}$. Fatten $D_2^\#$ (using Lemma 2) and then enlarge the links slightly to obtain the chain F_2 with the end link exit property so that

- (1) $F_2 = \{f(2, i) | h(c(2, i) \cap P') \subseteq f(2, i) \cap P', i \in [b_2, m_2]\}$;
- (2) $\partial F_2^* \subseteq i(\hat{d}(2, m_2), \hat{D}_2) \cap i(c(2, m_2), C_2) \cap j(f(2, m_2), F_2)$;
- (3) F_2^* is V_j -open for some j ;
- (4) $F_2^* \subseteq C_2[b_2, m_2]^* \cap \tilde{D}_3^*$, where $\tilde{D}_3$ is that unique subchain of $\hat{D}_3$ such that $\tilde{D}_3^*$ contains P' and each link of $\tilde{D}_3$ intersects P' and
- (5) $\overline{f(2, i)} \subseteq f(1, f_1(i))$ for $i \in [b_2, m_2]$.

Let $D_2 = \{(\hat{d}(2, i) - F_2^*) \cup f(2, i) | i \in [0, m_2]\}$, where we define $f(2, i) = \emptyset$ for $i \in [0, b_2 - 1]$, so that D_2 is an open chain cover of P so that D_2 follows f_1 in D_1 , $q' \in i(d(2, b_2), D_2)$, $p' \in i(d(2, m_2), D_2)$, and $q \in i(d(2, 0), D_2)$.

Find i_3 so that V_{i_3} closure refines $\hat{D}_3$, D_2 , $\hat{D}_2$, and C_2 ; each of $\hat{D}_3$, D_2 , $\hat{D}_2$ and C_2 is an amalgamation of V_{i_3} ; and no link of V_{i_3} intersects both $\hat{d}(2, i) - F_2^*$ and F_2^* for some $i < m_2$. If $D_3 = D_{31} \cup D_{32}$ where $D_{31} = \{\tilde{D}^* | \tilde{D} \text{ is a maximal subchain of } V_{i_3} \text{ such that (1) } \tilde{D}^* \subseteq d \text{ for some } d \in \hat{D}_3, (2) \text{ no closure of a link of } \tilde{D} \text{ is in } F_2^*, \text{ and (3) } \tilde{D}^* \cap i(d, \hat{D}_3) \neq \emptyset\}$, and $D_{32} = \{d \in V_{i_3} | d \subseteq F_2^*\}$, then $D_3 = D_3[0, m_3]$ is an open chain cover of P and without loss of generality we can assume that $p' \in i(d(3, m_3), D_3)$. It will follow that $q' \in i(d(3, b_3), D_3)$ where $D_3[b_3, m_3]$ essentially

covers P' , no proper subchain of $D_3[b_3, m_3]$ covers P' , and $\overline{D_3[0, b_3 - 1]}^* \cap P' = \emptyset$. Then D_3 follows the pattern f_2 in D_2 where we define $f_2(i) = \hat{f}_2(j)$ where j is that unique integer such that $\hat{d}(3, j) \supseteq d(3, i)$ and $d(3, i) \cap i(\hat{d}(3, j), \hat{D}_3) \neq \emptyset$, (1) unless $\overline{d(3, i)} \subseteq X'_2$, but $d(2, \hat{f}_2(j)) \notin X'_2$, in which case we define f_2 so that f_2 prefers X'_2 , or, (2) $d(3, i) \subseteq F_2^*$, in which case $f_2(i) = k$ where $f(2, k)$ is some link of F_2 such that $f(2, k) \supseteq \hat{d}(3, i)$. It will follow that (D_3, f_2) is compatible with C_2 relative to (D_2, X'_2) .

Continue this process. Choose $\sigma_3 > 0$, $\sigma_3 < \sigma_2$, $\sigma_3 < 1/32$, $\sigma_2 < \text{Lebesgue number for } \{h^{-1}(d \cap P') | d \in D_3[b_3, m_3]\}$. There exist chains $C_3 = C_3[0, m_3]$ and $C_4 = C_4[0, m_4]$ so that (1) $\text{mesh } C_4 < \sigma_3$; (2) C_4 follows a pattern f_3 in C_3 ; (3) C_3 follows f_2 in C_2 ; (4) f_3 prefers $X'_3 = \{c \in X_3 | c \cap (C_3 - X_3)^* = \emptyset\}$, where $X_3 = \{c(3, j) | c(2, f_2(j)) \in X'_2\}$; (5) (C_4, f_3) is compatible with D_3 relative to (C_3, X'_3) ; and (6)(a) $q \in i(c(3, 0), C_3) \cap i(c(4, 0), C_4)$, (b) $q' \in i(c(3, b_3), C_3) \cap i(c(4, b_4), C_4)$ (b_4 being that unique integer such that $C_4[b_4, m_4]$ essentially covers P' and $P' \cap \overline{C_4[0, b_4 - 1]}^* = \emptyset$), and (c) $p' \in i(c(3, m_3), C_3) \cap i(c(4, m_4), C_4)$. Having all this, we continue, choosing D_4, D_5 accordingly, etc., obtaining sequences $C_1, C_2, \dots$, and $D_1, D_2, \dots$, of open chain covers of P , and a sequence $f_1, f_2, \dots$, of patterns such that

(1) $C_i = \{c(i, 0), \dots, c(i, m_i)\}$, and $D_i = \{d(i, 0), \dots, d(i, m_i)\}$ such that $q \in c(i, 0) = d(i, 0)$, $q' \in i(c(i, b_i), C_i) \cap i(d(i, b_i), D_i)$, and

$$p' \in i(d(i, m_i), D_i) \cap i(c(i, m_i), C_i);$$

(2) C_{i+1} follows f_i in C_i and D_{i+1} follows f_i in D_i ;

(3) f_i prefers $X'_i \subseteq C_i \cap D_i$ and $P - o \subseteq X'_i$; and

(4) $\lim_i \text{mesh } C_i = 0 = \lim_i \text{mesh } D_i$.

Thus, by Theorem 6 of [KR], we have induced a homeomorphism $g_1 \in H(P)$ defined by $g_1(x) = \bigcap_{i=1}^{\infty} d(i, j(x, i))$ where for each i , $x \in c(i, j(x, i))$ and $f_{i-1}(j(x, i)) = j(x, i-1)$ ($i > 1$). By construction $g_1(x) = x$ for $x \notin o$ and $g_1(p') = p'$, $g_1(q') = q'$. Also, since P' is irreducible from p' to q' , $g_1(P') = P'$. Thus, if $x \in P'$, there is an infinite sequence $j(x, 1), j(x, 2), \dots$, of integers such that for $i \geq 2$, $x \in c(i, j(x, i))$ and $f_{i-1}(j(x, i)) = j(x, i-1)$, and, in particular, $x \in c(1, j(x, 1))$ and $g_1(x) \in d(1, j(x, 1)) = (c(1, j(x, 1)) - F_1^*) \cup f(1, j(x, 1))$. Hence, $g_1(x) \in f(1, j(x, 1))$, $x \in e(1, j(x, 1))$, and $h(x) \in f(1, j(x, 1))$. Then $d(g_1(x), h(x)) < \tau_1$ ($\text{mesh } F_1 < \tau_1$ by construction). Likewise $d(g_1^{-1}(x), h^{-1}(x)) < \delta_1$, and $\rho'(1_{P'}, hg_1^{-1}|P') < \varepsilon_1$, $\rho'(1_{P'}, g_1 h^{-1}) < \frac{1}{4}$. (Let $1_{P'}$ denote the identity of P' .) Now we just continue this process. Let $h_2 = hg_1^{-1}|P' \in H(P')$, and choose $\rho_2 > 0$ such that $\rho_2 > \rho'(1_{P'}, h_2)$ and $\rho_2 > \rho'(1_{P'}, h_2^{-1})$, $\rho_2 < \frac{1}{4}$ (unless $h_2 = g_1|P'$, in which case we are done). There are $\tau'_{21}, \tau_{21}, \delta_{21}, \varepsilon_{21} > 0$ such that

(a) $\varepsilon_{21} > \frac{1}{2}(\frac{1}{4} - \rho_2)$, $\rho_2, \frac{1}{8}$ and if $\rho'(l, h_2) < \varepsilon_{21}$ for $l \in H(P')$, then $\rho(l^{-1}, h_2^{-1}) < \frac{1}{8}$;

(b) $\delta_{21} < \varepsilon_{21}/2$ and if $x, y \in P'$, $d(x, y) < \delta_{21}$, then

$$d(h_2(x), h_2(y)), d(h_2^{-1}(x), h_2^{-1}(y)), d(h_2^{-2}(x), h_2^{-2}(y)) < \varepsilon_{21}/2;$$

(c) $\tau_{21} < \delta_{21}/2$ and if $x, y \in P'$, $d(x, y) < \tau_{21}$, then

$$d(h_2(x), h_2(y)), d(h_2^{-1}(x), h_2^{-1}(y)), d(h_2^{-2}(x), h_2^{-2}(y)) < \delta_{21}/2;$$

(d) if $l \in H(P')$ and $\rho'(h_2, l) < \tau_{21}$, $\rho(h_2^{-1}, l^{-1}) < \delta_{21}$; and

(e) $\tau'_{21} < \tau_{21}/2$ and if $x, y \in P'$, $d(x, y) < \tau'_{21}$, then

$$d(h_2(x), h_2(y)), d(h_2^{-1}(x), h_2^{-1}(y)), d(h_2^{-2}(x), h_2^{-2}(y)) < \tau_{21}/2.$$

There is a sequence $u_1, u_2, \dots$, of open sets of P such that $\bar{X}_1^* \cap \bar{u}_1 = \emptyset$, $P' \subseteq \dots \subseteq \bar{u}_3 \subseteq u_2 \subseteq \bar{u}_2 \subseteq u_1 \subseteq \bar{u}_1$, $P' = \bigcap_i u_i$. Find $i_{21} \in \mathbb{N}$ such that $\text{mesh } V_{i_{21}} < \tau'_{21}$, $\text{mesh } V_{i_{21}} < \mu_1$, where $\overline{D_{\mu_1}(P')} \subseteq E_1^*$, and if $\tilde{C}_1$ denotes the maximal subchain of $V_{i_{21}}$ with the property that the closure of each link of $\tilde{C}_1$ intersects P' , then $\overline{\tilde{C}_1^*} \cap (g_1(P - u_1)) = \emptyset$ and there are at least five links of $V_{i_{21}}$ between any link in $\tilde{C}_1$ and any link intersecting $g_1(P - u_1)$. Let $C_{21} = V_{i_{21}}$ and from here, with $X_{21} = \{c \in C_{21} | c \notin \tilde{C}_1\}$, construct a stable homeomorphism $g_2 \in H(P)$ such that $g_2(x) = x$ for $x \in g_1(P - u_1)$, $g_2(p') = p'$ and $g_2(q') = q'$. Further construct g_2 so that $\rho'(1_{P'}, h_2 g_2^{-1} | P') < \frac{1}{8}$, $\rho'(1_{P'}, g_2 h_2^{-1}) < \frac{1}{8}$, and $\rho(1_P, g_2) < \frac{1}{4}$, $\rho(1_P, g_2^{-1}) < \frac{1}{4}$. Note that if $x \in P - u_1$, $g_2 g_1(x) = g_1(x)$. Continue, obtaining a sequence $g_1, g_2, \dots$, of stable homeomorphisms such that for each i

- (1) $g_i(x) = x$ for $x \in g_{i-1} \cdot g_{i-2} \cdots g_1(P - u_{i-1})$;
- (2) $g_i(p') = p'$, $g_i(q') = q'$;
- (3) $\rho(1, g_i) < 2^{-i}$ and $\rho(1, g_i^{-1}) < 2^{-i}$ (unless $i = 1$);
- (4) $\rho'(1_{P'}, h_i g_i^{-1} | P') < 2^{-(i+1)}$, $\rho'(1_{P'}, g_i h_i^{-1}) < 2^{-(i+1)}$, for $h_i = h_{i-1} g_{i-1}^{-1} | P'$ ($i > 2$).

Define $g: P \rightarrow P$ by $g = \lim_i g_i \cdots g_1$. By construction, g is a continuous function from P onto P such that $g(P') = P'$. Further, if $x \notin P'$, $x \in P - U_i$ for some i , and $g(x) = g_i \cdots g_1(x) \notin P'$. Thus, g is one-to-one on $P - P'$. If $x \in P'$,

$$\begin{aligned} d(x, g_1 h^{-1}(x)) &< \frac{1}{4}, \quad d(x, g_2 h_2^{-1}(x)) = d(x, g_2 g_1 h^{-1}(x)) < \frac{1}{8}, \\ d(x, g_3 h_3^{-1}(x)) &= d(x, g_3 g_2 g_1 h^{-1}(x)) < \frac{1}{16}, \quad \text{etc.} \end{aligned}$$

Then $g_1 h^{-1}(x)$, $g_2 g_1 h^{-1}(x)$, $\dots$, converges to x for $x \in P'$, and, thus, gh^{-1} is a homeomorphism on P' , and, in fact, $1_{P'}$. So, $g|P' = h$ and g is one-to-one on all of P , and g is an extension of h . We have the homeomorphism we desired. $\square$

COROLLARY 1. *If P' is a proper subcontinuum of X , a continuum with the property that each of its nondegenerate proper subcontinua is a pseudoarc, h' is a homeomorphism from P' onto itself, and o is an open set in P such that $P' \subseteq o$, then there is a homeomorphism $h \in H(P)$ such that $h|P' = h'$ and $h(x) = x$ for $x \notin o$.*

PROOF. This follows from Theorem 8 of [KJ] and our main theorem.

COROLLARY 2. *Suppose that (1) $n_1, n_2, \dots$, is a sequence of positive integers, (2) $o_1, o_2, \dots$, is a sequence of disjoint open sets in the continuum X with the property that each of its nondegenerate proper subcontinua is a pseudoarc, and (3) for $i \in \mathbb{N}$, P_i is a nondegenerate subcontinuum contained in o_i with $x_i \in P_i$. Then there is a homeomorphism $h \in H(X)$ such that for each $i \in \mathbb{N}$, $h(P_i) = P_i$, x_i has period n_i under h , and $h(x) = x$ for $x \notin \bigcup o_i$.*

PROOF. Use Theorem LG to find $h_i \in H(P_i)$ such that x_i has period n_i , and then extend h_i to $\hat{h}_i \in H(P)$ such that $\hat{h}_i(x) = x$ for $x \notin o_i$ and $\hat{h}_i|P_i = h_i$. Then define

$$h(x) = \begin{cases} h_i(x) & \text{if } x \in o_i \text{ for some } i, \\ x & \text{otherwise.} \end{cases} \quad \square$$

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DEPARTMENT OF FOUNDATIONS, ANALYSIS, AND TOPOLOGY, DIVISION OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19716