MULTILINEAR CONVOLUTIONS DEFINED BY MEASURES ON SPHERES

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ABSTRACT. Let σ be Lebesgue measure on Σ_{n-1} and write $\sigma = (\sigma_1, \dots, \sigma_n)$ for an element of Σ_{n-1} . For functions $f_1, \dots, f_n$ on $\mathbf{R}$, define

$$T(f_1, \dots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - \sigma_1) \cdots f_n(x - \sigma_n) d\sigma, \quad x \in \mathbf{R}.$$

This paper partially answers the question: for which values of p and q is there an inequality

$$\|T(f_1, \dots, f_n)\|_q \leq C \|f_1\|_p \cdots \|f_n\|_p?$$

1. For a function f on $\mathbf{R}$ and $t \in \mathbf{R}$, define the translate f_t of f by $f_t(x) = f(x - t)$, $t \in \mathbf{R}$. An n -linear operator M taking n -tuples of functions on $\mathbf{R}$ to functions on $\mathbf{R}$ is called a multilinear convolution (see [2, 3, 4]) if

$$M((f_1)_t, \dots, (f_n)_t) = (M(f_1, \dots, f_n))_t, \quad t \in \mathbf{R}.$$

We are interested in certain multilinear convolutions M defined as follows. Let λ be a locally finite Borel measure on $\mathbf{R}^n$. If $f_1, \dots, f_n$ are continuous functions of compact support on $\mathbf{R}$, put

$$M(f_1, \dots, f_n)(x) = \int_{\mathbf{R}^n} f_1(x - x_1) \cdots f_n(x - x_n) d\lambda(x_1, \dots, x_n), \quad x \in \mathbf{R}.$$

Our question about such M is the question of L^p -boundedness: taking L^p norms with respect to Lebesgue measure, when do we have an inequality

$$\|M(f_1, \dots, f_n)\|_q \leq C \|f_1\|_p \cdots \|f_n\|_p?$$

If λ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^n$, then some results along these lines follow from [4]. This paper is concerned with the case that occurs when λ is σ , the Lebesgue measure on the unit sphere Σ_{n-1} in $\mathbf{R}^n$. Writing $\sigma = (\sigma_1, \dots, \sigma_n)$ for an element of Σ_{n-1} , we thus define

$$T_n(f_1, \dots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - \sigma_1) \cdots f_n(x - \sigma_n) d\sigma, \quad x \in \mathbf{R},$$

for, say, bounded Borel functions $f_1, \dots, f_n$ on $\mathbf{R}$. The problem is to determine the exponent pairs $(\frac{1}{p}, \frac{1}{q})$ such that there exists an inequality

$$(1) \quad \|T_n(f_1, \dots, f_n)\|_q \leq C(p, q) \|f_1\|_p \cdots \|f_n\|_p.$$

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FIGURE 1

To state our result, let $R = R(n)$ be the closed convex hull in $\mathbf{R}^2$ of the points $(0, 0)$, $(\frac{1}{n}, 1)$, $(\frac{n+1}{n+2}, 1)$, $(\frac{n+1}{n+3}, \frac{2}{n+3})$, $(\frac{n-1}{n+1}, 0)$. (See Figure 1.)

THEOREM. *If (1) holds, then $(\frac{1}{p}, \frac{1}{q})$ lies in the region R . Conversely, if $(\frac{1}{p}, \frac{1}{q})$ is in the region R and not on the two closed line segments forming the right-hand boundary D of R , then (1) holds. If the functions $f_1, \dots, f_n$ are restricted to be characteristic functions of subsets of $\mathbf{R}$, then (1) holds also when $(\frac{1}{p}, \frac{1}{q}) \in D$.*

The organization of this paper is as follows. §2 contains the proof of the necessity of the theorem's condition: that $(\frac{1}{p}, \frac{1}{q})$ lies in R if (1) holds. The rest of the paper contains the proof of the sufficiency. For technical reasons we will be concerned here with operators S more general than T_n : let $x \cdot y$ represent the scalar product of two vectors $x, y \in \mathbf{R}^n$, fix vectors $v_1, \dots, v_n$ in $\mathbf{R}^n$, and define $S = S(n; v_1, \dots, v_n)$ by

$$S(f_1, \dots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma) d\sigma.$$

We will prove by induction on the dimension n that if $v_1, \dots, v_n$ are linearly independent, then (A): the inequality

$$(2) \quad \|S(f_1, \dots, f_n)\|_q \leq C \|f_1\|_p \cdots \|f_n\|_p$$

holds if $(\frac{1}{p}, \frac{1}{q})$ is in R but not in D ; and (B): the inequality

$$(3) \quad \|S(I_1, \dots, I_n)\|_q \leq C \|I_1\|_p \cdots \|I_n\|_p$$

holds if $(\frac{1}{p}, \frac{1}{q})$ is in R and $I_1, \dots, I_n$ are characteristic functions of subsets of $\mathbf{R}$. (The constants in (2) and (3) depend on p, q , and the choice of $v_1, \dots, v_n$).

The induction starts with the case $n = 2$. This is handled in §4. We now outline the proof of (A) and (B) when $n > 2$.

To prove (B) it would be sufficient, by multilinear complex interpolation using the Lorentz spaces $L^{p,1}$ (see [1, §4.4 and 4.7]), to show that (3) holds whenever $(\frac{1}{p}, \frac{1}{q})$ is one of the five extreme points of R . The point $(0, 0)$ is trivial, while $(\frac{1}{n}, 1)$ is easy and is left to the reader. The point $(\frac{n+1}{n+2}, 1)$ is treated in §5, and $(\frac{n-1}{n+1}, 0)$ is considered in §3. The remaining point $(\frac{n+1}{n+3}, \frac{2}{n+3})$ is not dealt with directly. Rather, in §5 we split the positive operator S into a sum of two other positive operators: $S = S_1 + S_2$. It is shown in §5 that

$$(4) \quad \|S_1(I_1, \dots, I_n)\|_q \leq C \|I_1\|_p \cdots \|I_n\|_p$$

holds when $(\frac{1}{p}, \frac{1}{q}) = (1, 1)$. Since (4) holds also whenever (3) holds, it follows that (4) holds whenever $(\frac{1}{p}, \frac{1}{q}) \in R$. It is shown in §6 that

$$(5) \quad \|S_2(I_1, \dots, I_n)\|_q \leq C \|I_1\|_p \cdots \|I_n\|_p$$

holds when $(\frac{1}{p}, \frac{1}{q}) = (\frac{n}{n+2}, 0)$. Since (5) holds also whenever (3) holds, it follows that (5) holds whenever $(\frac{1}{p}, \frac{1}{q}) \in R$.

The proof of (A) is similar. Again, (2) is clear if $(\frac{1}{p}, \frac{1}{q})$ is $(0, 0)$ or $(\frac{1}{n}, 1)$. It is shown in §5 that (2) holds for $\frac{1}{q} = 1$ if $\frac{1}{n} \leq \frac{1}{p} < \frac{n+1}{n+2}$ and in §3 that (2) holds for $\frac{1}{q} = 0$ if $0 \leq \frac{1}{p} < \frac{n-1}{n+1}$. The proof is then completed by the inequalities

$$\|S_1(f_1, \dots, f_n)\|_1 \leq C \|f_1\|_1 \cdots \|f_n\|_1,$$

which is established in §5, and

$$\|S_2(f_1, \dots, f_n)\|_\infty \leq C \|f_1\|_p \cdots \|f_n\|_p,$$

which is established in §6 for $0 \leq \frac{1}{p} < \frac{n}{n+2}$.

2. Assume that (1) holds whenever the f_j are characteristic functions. We will observe that the inequalities

$$(6) \quad \frac{1}{q} \leq \frac{n}{p},$$

$$(7) \quad n - 1 + \frac{2}{q} \geq \frac{n + 1}{p},$$

and

$$(8) \quad n + \frac{1}{q} \geq \frac{n + 2}{p}$$

must hold. These inequalities imply that $(\frac{1}{p}, \frac{1}{q})$ lies in the region R . To see that (6) holds, set each f_j equal to $I_{[-L, L]}$, the characteristic function of $[-L, L]$. Then (1) implies

$$L^{1/q} \leq CL^{n/p}.$$

Letting $L \rightarrow \infty$ yields (6). Inequality (7) follows by setting $f_j = I_{[-\varepsilon, \varepsilon]}$ if $1 \leq j \leq n - 1$ and $f_n = I_{[1-2\varepsilon^2, 1+2\varepsilon^2]}$. Then (1) implies

$$\varepsilon^{n-1+2/q} < C\varepsilon^{(n+1)/p}.$$

As $\varepsilon \rightarrow 0$, (7) follows. To obtain (8), let f_1 be the characteristic function of $[2^{-1/2} - \varepsilon^2, 2^{-1/2} + \varepsilon^2]$, let f_2 be the characteristic function of $[-2^{-1/2} - \varepsilon^2, -2^{-1/2} + \varepsilon^2]$, and set $f_j = I_{[-\varepsilon, \varepsilon]}$ for $3 \leq j \leq n$. If B is the box in $\mathbf{R}^n$ whose characteristic function at $(x_1, \dots, x_n)$ is $\prod_{j=1}^n f_j(x_j)$, then the (surface area) measure of $(B + (t, t, \dots, t)) \cap \Sigma_{n-1}$ is of the order ε^n if $|t| \leq \varepsilon/2n$. Thus (1) implies $\varepsilon^{n+1/q} \leq C\varepsilon^{(n+2)/p}$, and this gives (8).

3. Suppose that $v_1, \dots, v_n$ are linearly independent vectors in $\mathbf{R}^n$. The purpose of this section is to establish the inequalities

$$(9) \quad \int_{\Sigma_{n-1}} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n |E_j|^{(n-1)/(n+1)},$$

and, for $0 \leq \frac{1}{p} < \frac{n-1}{n+1}$,

$$(10) \quad \int_{\Sigma_{n-1}} \prod_{j=1}^n f_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n \|f_j\|_p.$$

Here and through the paper I_j denotes the characteristic function of $E_j \subseteq \mathbf{R}$, $|E_j|$ denotes the Lebesgue measure of E_j , and the f_j are nonnegative functions on $\mathbf{R}$. The constants in (9) and (10) will depend on n and the choice of $v_1, \dots, v_n$, and the constant in (10) will also depend on p .

We begin by observing how (10) follows from (9). Inequality (9) is equivalent to

$$(11) \quad \int_{\Sigma_{n-1}} \prod_{j=1}^n f_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n \|f_j\|_{(n+1)/(n-1), 1}$$

where the norms are those of the Lorentz space $L^{(n+1)/(n-1), 1}(\mathbf{R})$. If $M = \max\{|v_j \cdot \sigma| : \sigma \in \Sigma_{n-1}, 1 \leq j \leq n\}$ and $\tilde{f}_j$ is the restriction of f_j to $[-M, M]$, then the left-hand side of (10) is unchanged if f_j is replaced by $\tilde{f}_j$. Thus (10) is a consequence of (11) and the inequalities

$$\|\tilde{f}_j\|_{(n+1)/(n-1), 1} \leq C\|\tilde{f}_j\|_p \leq C\|f_j\|_p$$

if $\frac{n+1}{n-1} < p \leq \infty$.

We begin the proof of (9) with a lemma.

LEMMA 3.1. Suppose E is a Lebesgue measurable subset of $\{x = (x_1, \dots, x_n) \in \mathbf{R}^n : x_j \geq 0, 1 \leq j \leq n\}$. Then if $|E|$ is the Lebesgue measure of E , we have

$$|E|^{n+1} \leq \frac{(n+1)^n}{n!} \prod_{j=1}^n \int_E x_j dx.$$

PROOF. It is enough to prove this in the case of a bounded E with $|E| > 0$. In this case, define

$$\bar{x}_j = \frac{1}{|E|} \int_E x_j dx.$$

Put

$$l(x) = \sum_{j=1}^n \frac{x_j}{\bar{x}_j}, \quad x \in \mathbf{R}^n,$$

and

$$T(r) = \{x: x_j \geq 0, 1 \leq j \leq n; l(x) \leq r\}.$$

Define $E^+ = E \cap T(n)$ and $E^- = E \sim E^+$. Since $\int_E (n - l(x)) dx = 0$, we have

$$\int_{E^-} (l(x) - n) dx = \int_{E^+} (n - l(x)) dx \leq \int_{T(n)} (n - l(x)) dx.$$

Of all sets

$$F \subseteq \{x: x_j \geq 0, 1 \leq j \leq n; l(x) > n\}$$

such that

$$\int_F (l(x) - n) dx \leq \int_{T(n)} (n - l(x)) dx,$$

the one with largest $|F|$ will be the set

$$F_0 = \{x: x_j \geq 0, 1 \leq j \leq n; n < l(x) \leq s\} = T(s) \sim T(n),$$

where $s > n$ is chosen to have

$$\int_{F_0} (l(x) - n) dx = \int_{T(n)} (n - l(x)) dx.$$

Since

$$|T(r)| = \frac{r^n}{n!} \prod_{j=1}^n \bar{x}_j, \quad \int_{T(r)} l(x) dx = \frac{nr^{n+1}}{(n+1)!} \prod_{j=1}^n \bar{x}_j,$$

it follows that $s = n + 1$. Then

$$|E| = |E^+| + |E^-| \leq |T(n)| + |F_0| = |T(s)| = \frac{(n+1)^n}{n!} \prod_{j=1}^n \bar{x}_j.$$

COROLLARY 3.2. (a) *If E is a Lebesgue measurable subset of $\mathbf{R}^n$, then*

$$|E|^{n+1} \leq \frac{2^n(n+1)^n}{n!} \prod_{j=1}^n \int_E |x_j| dx.$$

(b) *If $w_1, \dots, w_n$ are linearly independent in $\mathbf{R}^n$, there is a constant C (depending on n and $w_1, \dots, w_n$) such that*

$$|E|^{n+1} \leq C \prod_{j=1}^n \int_E |w_j \cdot x| dx$$

for measurable $E \subseteq \mathbf{R}^n$.

(c) *If $w_1, \dots, w_n$ are linearly independent in $\mathbf{R}^n$, there is a constant C (depending on n and $w_1, \dots, w_n$) such that*

$$(\sigma(E))^{n+1} \leq C \prod_{j=1}^n \int_E |w_j \cdot \sigma| d\sigma$$

for measurable $E \subseteq \Sigma_{n-1}$.

PROOF. Statement (a) is an immediate consequence of Lemma 3.1. The proof of (b) is an easy change of variables argument: With e_j denoting the j th unit vector

in $\mathbf{R}^n$, define a linear mapping T on $\mathbf{R}^n$ by $e_j \cdot Tx = w_j \cdot x$, $1 \leq j \leq n$. Then, writing $|T|$ for $|\det(T)|$,

$$\int_E |w_j \cdot x| dx = \frac{1}{|T|} \int_E |e_j \cdot Tx| |T| dx = \frac{1}{|T|} \int_{T(E)} |e_j \cdot y| dy.$$

Thus

$$\begin{aligned} |T| \cdot |E|^{n+1} &= |T|^{-n} |T(E)|^{n+1} \leq |T|^{-n} \frac{2^n(n+1)^n}{n!} \prod_{j=1}^n \int_{T(E)} |e_j \cdot y| dy \\ &= \frac{2^n(n+1)^n}{n!} \prod_{j=1}^n \int_E |w_j \cdot x| dx, \end{aligned}$$

where the inequality follows from (a). To obtain (c), fix $E \subseteq \Sigma_{n-1}$ and define $\tilde{E} = \{r\sigma : \sigma \in E, 0 \leq r \leq 1\}$. Then

$$|\tilde{E}| = \frac{1}{n} \sigma(E) \quad \text{and} \quad \int_{\tilde{E}} |w_j \cdot x| dx = \frac{1}{n+1} \int_E |w_j \cdot \sigma| d\sigma.$$

Thus (c) follows from (b).

LEMMA 3.3. *Fix linearly independent $v_1, \dots, v_{n-1}$ in $\mathbf{R}^n$. Define $F: \Sigma_{n-1} \rightarrow \mathbf{R}^{n-1}$ by $F(\sigma) = (v_1 \cdot \sigma, \dots, v_{n-1} \cdot \sigma)$ and define $D(\sigma)$ to be the determinant of the matrix whose rows are the vectors $\sigma, v_1, v_2, \dots, v_{n-1}$. Then, with $|A|$ denoting the Lebesgue measure of a subset A of $\mathbf{R}^{n-1}$, we have*

$$\int_E |D(\sigma)| d\sigma \leq 2|F(E)|$$

for all $E \subseteq \Sigma_{n-1}$.

PROOF. Let B_{n-1} be the closed unit ball in $\mathbf{R}^{n-1}$ and parametrize the “top half” Σ_{n-1}^+ of Σ_n by the map

$$G: x = (x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, \sqrt{1 - |x|^2}) = (\sigma_1, \dots, \sigma_n) = \sigma, \\ x \in B_{n-1}.$$

The map G carries the measure dx/σ_n on B_{n-1} to the measure $d\sigma$ on Σ_{n-1}^+ . Write $v_j = (v_j^1, \dots, v_j^n)$ for $1 \leq j \leq n-1$. Then the Jacobian J_H of the map $H \doteq F \circ G$ is the determinant of the matrix $[v_j^i - v_j^n x_i/\sigma_n]_{1 \leq i, j \leq n-1}$. A little linear algebra shows that $|J_H| = |D(\sigma)|/\sigma_n$. Thus, if $y = H(x) = F(\sigma)$, then

$$dy = |J_H| dx = \frac{|D(\sigma)|}{\sigma_n} dx = |D(\sigma)| d\sigma,$$

at least on any subset of Σ_{n-1}^+ where F is one-to-one. A similar argument applies to Σ_{n-1}^- . Since F is one-to-one on the two components of the complement of Σ_{n-1} with respect to the hyperplane spanned by $v_1, \dots, v_{n-1}$, the lemma follows.

PROOF OF (9). For $E \subseteq \Sigma_{n-1}$, let $v_j \cdot E = \{v_j \cdot \sigma : \sigma \in E\} \subseteq \mathbf{R}$. We will show the existence of C such that

$$(12) \quad \sigma(E) \leq C \prod_{j=1}^n |v_j \cdot E|^{(n-1)/(n+1)}, \quad E \subseteq \Sigma_{n-1}.$$

Then (9) follows if $E = \{\sigma: v_j \cdot \sigma \in E_j, 1 \leq j \leq n\}$. For fixed l , define $F_l(\sigma)$ to be $(v_1 \cdot \sigma, \dots, v_{l-1} \cdot \sigma, v_{l+1} \cdot \sigma, \dots, v_n \cdot \sigma)$. Then, using $|\cdot|$ to indicate Lebesgue measure in both $\mathbf{R}^{n-1}$ and $\mathbf{R}$,

$$|F_l(E)| \leq \prod_{\substack{j=1 \\ j \neq l}}^n |v_j \cdot E|,$$

so

$$(13) \quad \prod_{i=1}^n |F_l(E)| \leq \prod_{j=1}^n |v_j \cdot E|^{n-1}.$$

By Lemma 3.3,

$$(14) \quad \int_E |D_l(\sigma)| d\sigma \leq 2|F_l(E)|,$$

where $D_l(\sigma)$ is the determinant of the matrix whose rows are the vectors $\sigma, v_1, v_2, \dots, v_{l-1}, v_{l+1}, \dots, v_n$. But $D_l(\sigma) = w_l \cdot \sigma$, where w_l is some vector orthogonal to the linear span of $\{v_j\}_{j \neq l}$. Since $v_1, \dots, v_n$ are linearly independent, it follows that $w_1, \dots, w_n$ are linearly independent also. Now (12) is a consequence of (13), (14), and (c) of Corollary 3.2.

4. Fix linearly independent v_1 and v_2 in $\mathbf{R}^2$ and write $e(\theta) = (\cos \theta, \sin \theta)$. Define

$$S(f_1, f_2)(x) = \int_0^{2\pi} f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) d\theta.$$

Consider the inequalities

$$(2') \quad \|S(f_1, f_2)\|_q \leq C \|f_1\|_p \|f_2\|_p,$$

$$(3') \quad \|S(I_1, I_2)\|_q \leq C \|I_1\|_p \|I_2\|_p.$$

The purpose of this section is to show that

$$(A') \quad (2') \text{ holds if } \left(\frac{1}{p}, \frac{1}{q}\right) \text{ is in the region } R \text{ but not on either of the lines } 1 + \frac{2}{q} = \frac{3}{p}, 2 + \frac{1}{q} = \frac{4}{p}, \text{ and}$$

$$(B') \quad (3') \text{ holds if } \left(\frac{1}{p}, \frac{1}{q}\right) \text{ is in } R.$$

Bearing in mind the procedure for proving (A) and (B) which was laid out in §1, we begin by noting again that (2') holds when $(\frac{1}{p}, \frac{1}{q}) = (0, 0)$ (trivially) and when $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, 1)$ (a consequence of Fubini's theorem and Hölder's inequality). Additionally, we recall from §3 that (2') holds for $\frac{1}{q} = 0$ when $\frac{1}{p} < \frac{1}{3}$ and that (3') holds when $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{3}, 0)$. The next step is

LEMMA 4.1. *Inequality (2') holds when $(\frac{1}{p}, \frac{1}{q}) = (\frac{3}{4}, 1)$.*

PROOF. Suppose $f_1, f_2 \geq 0$. Then

$$\|S(f_1, f_2)\|_1 = \int_{-\infty}^{\infty} f_1(x) \int_0^{2\pi} f_2(x + v_1 \cdot e(\theta) - v_2 \cdot e(\theta)) d\theta dx.$$

Write

$$Tf(x) = \int_0^{2\pi} f(x + v_1 \cdot e(\theta) - v_2 \cdot e(\theta)) d\theta.$$

It is enough to check that

$$(15) \quad \|Tf\|_4 \leq C\|f\|_{4/3}.$$

Now

$$\begin{aligned} Tf(x) &= \int_0^{2\pi} f(x + |v_1 - v_2| \cos \theta) d\theta \\ &= 2 \int_{-|v_1 - v_2|}^{|v_1 - v_2|} f(x - u) \frac{du}{\sqrt{|v_1 - v_2|^2 - u^2}}. \end{aligned}$$

Since the function $h(u) = (|v_1 - v_2|^2 - u^2)^{-1/2}$ is in the Lorentz space $L^{2,\infty}$ on $[-|v_1 - v_2|, |v_1 - v_2|]$, inequality (15) follows from Comment 1.4 on p. 121 of [5].

Let I be a subset of $[0, 2\pi]$ to be determined later and define

$$\begin{aligned} J = [0, 2\pi] \sim I, \quad S_1(f_1, f_2)(x) &= \int_I f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) d\theta, \\ S_2(f_1, f_2)(x) &= \int_J f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) d\theta. \end{aligned}$$

We will show that I can be chosen to have

$$(16) \quad \|S_1(f_1, f_2)\|_1 \leq C\|f_1\|_1\|f_2\|_1,$$

$$(17) \quad \|S_2(f_1, f_2)\|_\infty \leq C(p)\|f_1\|_p\|f_2\|_p, \quad \frac{1}{p} < \frac{1}{2},$$

and

$$(18) \quad \|S_2(I_1, I_2)\|_\infty \leq C\|I_1\|_2\|I_2\|_2.$$

Since $S = S_1 + S_2$ and since S_1 and S_2 are $L^p - L^q$ bounded whenever S is, it will follow by interpolation from (16)–(18), from Lemma 4.1, and from the remarks preceding Lemma 4.1 that (A') and (B') hold. Since (17) follows from (18) as (10) follows from (9) in §3, we will be concerned with only (16) and (18). Write $v_i = r_i e(\phi_i)$ for $i = 1, 2$ and $v_1 - v_2 = re(\phi)$, where $0 \leq \phi, \phi_i < 2\pi$. Note that ϕ, ϕ_1 , and ϕ_2 are all distinct since v_1 and v_2 are linearly independent. Then, if $f_1, f_2 \geq 0$,

$$\|S_1(f_1, f_2)\|_1 = \int_{-\infty}^{\infty} f_1(x) \int_I f_2(x + r \cos(\theta - \phi)) d\theta dx.$$

Thus (16) is true if

$$\int_I f_2(x + r \cos(\theta - \phi)) d\theta \leq C\|f_2\|_1.$$

The change of variables $t = \cos(\theta - \phi)$ then shows that (16) holds if $|\sin(\theta - \phi)|$ is bounded away from zero on I . It is therefore enough to show that (18) holds whenever $|\sin(\theta - \phi_1)\sin(\theta - \phi_2)|$ is bounded away from zero on J . We will show below that if $|\sin(\theta - \phi_1)\sin(\theta - \phi_2)| \geq \delta > 0$ on J , then

$$(19) \quad (r_1 r_2)^{1/2} \delta |E| \leq 9|v_1 \cdot e(E)|^{1/2} |v_2 \cdot e(E)|^{1/2}, \quad E \subseteq J.$$

Here $v_i \cdot e(E) = \{v_i \cdot e(\theta) : \theta \in E\}$. From this (18) will follow just as (9) followed from (12) in §3. Now J can be partitioned into 9 subsets J_i , $1 \leq i \leq 9$, on each of which $v_j \cdot e(\theta) = r_j \cos(\theta - \phi_i)$ is monotone for both $j = 1$ and $j = 2$. If $E_i = E \cap J_i$, then

$$\prod_{j=1}^2 |v_j \cdot e(E_i)|^{1/2} = \prod_{j=1}^2 \left(\int_{E_i} r_j |\sin(\theta - \phi_j)| d\theta \right)^{1/2} \geq (r_1 r_2)^{1/2} \delta |E_i|.$$

This gives (19).

5. Suppose that $v_1, \dots, v_n$ are linearly independent in $\mathbf{R}^n$, $n \geq 3$. The purpose of this section is to establish the inequalities, for $f_1, \dots, f_n \geq 0$,

$$(20) \quad \int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} \prod_{j=1}^n I_j(x - v_j \cdot \sigma) d\sigma dx \leq C \prod_{j=1}^n |E_j|^{(n+1)/(n+2)}$$

and, for $\frac{1}{n} \leq \frac{1}{p} < \frac{n+1}{n+2}$,

$$(21) \quad \int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} \prod_{j=1}^n f_j(x - v_j \cdot \sigma) d\sigma dx \leq C \prod_{j=1}^n \|f_j\|_p.$$

We will also introduce operators S_1 and S_2 such that $S = S_1 + S_2$ and show that

$$(22) \quad \|S_1(f_1, \dots, f_n)\|_1 \leq C \prod_{j=1}^n \|f_j\|_1.$$

As before, the constants in (20)–(22) will depend on n and the choice of $v_1, \dots, v_n$. The constant in (21) will also depend on p .

Recall from §3 that the linear map $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by $e_j \cdot Tx = v_j \cdot x$, $x \in \mathbf{R}^n$. Fix a unit vector u_n with $Tu_n = (c, c, \dots, c)$ for some $c \in \mathbf{R}$, and let $\{u_1, \dots, u_n\}$ be an orthonormal basis for $\mathbf{R}^n$. Let η be Lebesgue measure on Σ_{n-2} . Parametrize (almost all of) Σ_{n-1} by

$$\sigma = \sum_{j=1}^{n-1} r \eta_j u_j + \operatorname{sgn}(r) \sqrt{1 - r^2} u_n \quad \text{where } \eta = (\eta_1, \dots, \eta_{n-1}) \in \Sigma_{n-2}, \quad -1 \leq r \leq 1.$$

Then

$$d\sigma = |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} d\eta$$

and, for a function f on $\mathbf{R}$,

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} f((x, \dots, x) - T\sigma) d\sigma dx \\ &= \int_{-\infty}^{\infty} \int_{-1}^1 \int_{\Sigma_{n-2}} f\left((x, \dots, x) - \operatorname{sgn}(r) \sqrt{1-r^2} (c, \dots, c) \right. \\ & \quad \left. - T\left(\sum_{k=1}^{n-1} r \eta_k u_k\right)\right) d\eta |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} dx \\ &= \int_{-\infty}^{\infty} \int_{-1}^1 \int_{\Sigma_{n-2}} f\left((x, \dots, x) - r T\left(\sum_{k=1}^{n-1} \eta_k u_k\right)\right) d\eta |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} dx. \end{aligned}$$

In particular, the left-hand side of (21) is equal to

$$\int_{-1}^1 F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}},$$

where

$$F(r) = \int_{-\infty}^{\infty} \int_{\Sigma_{n-2}} \prod_{j=1}^n f_j \left(x - r e_j \cdot T \left(\sum_{k=1}^{n-1} \eta_k u_k \right) \right) d\eta dx.$$

At the end of this section we will prove that

$$(23) \quad \int_{-1}^1 F(r) |r|^{n-2} dr \leq C \prod_{j=1}^n \|f_j\|_1$$

and, if $f_j = I_j$,

$$(24) \quad F(r) \leq C r^{1-n^2/(n+2)} \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

Let us see how (20) follows from (23) and (24): Write Π for $\prod_{j=1}^n |E_j|$. If $\Pi \geq 1$, then, by (24),

$$\begin{aligned} \int_{-1}^1 F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leq C \Pi^{n/(n+2)} \int_{-1}^1 |r|^{(n-2)/(n+2)} \frac{dr}{\sqrt{1-r^2}} \\ &\leq C \Pi^{(n+1)/(n+2)}. \end{aligned}$$

If, on the other hand, $0 < \Pi < 1$, let $\delta = \Pi^{2/(n+2)}$ and split the integral. First,

$$\begin{aligned} \int_{\delta-1}^{1-\delta} F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leq \delta^{-1/2} C \int_{\delta-1}^{1-\delta} F(r) |r|^{n-2} dr \\ &\leq \delta^{-1/2} C \Pi = C \Pi^{(n+1)/(n+2)}, \end{aligned}$$

where the second inequality follows from (23). Then, from (24),

$$\begin{aligned} \int_{\{1-\delta \leq |r| \leq 1\}} F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} &\leq C \Pi^{n/(n+2)} \int_{1-\delta}^1 r^{(n-2)/(n+2)} \frac{dr}{\sqrt{1-r^2}} \\ &\leq C \Pi^{n/(n+2)} \delta^{1/2} = C \Pi^{(n+1)/(n+2)}. \end{aligned}$$

To deduce (21) we require the estimate

$$(25) \quad F(r) \leq C r^{1-n/s} \prod_{j=1}^n \|f_j\|_s,$$

valid if $\frac{1}{s} = \frac{n}{n+2} - \varepsilon$ and $\varepsilon > 0$ is small. This analogue of (24) will also be established at the end of this section. Fix a small $\varepsilon > 0$ and put $\frac{1}{s} = \frac{n}{n+2} - \varepsilon$. Also fix δ with $0 < \delta < 1$. Then (25) gives

$$(26) \quad \int_{-1}^1 F(r) |r|^{n-2} \frac{dr}{(1-r^2)^{1-\delta}} \leq C \prod_{j=1}^n \|f_j\|_s.$$

We want to interpolate (26) with (23) to obtain (21) in the form

$$(27) \quad \int_{-1}^1 F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j=1}^n \|f_j\|_p,$$

for any $\frac{1}{p}$ slightly smaller than $\frac{n+1}{n+2}$. (Since (21) holds with $\frac{1}{p} = \frac{1}{n}$, one more interpolation will then establish (21) for $\frac{1}{n} \leq \frac{1}{p} < \frac{n+1}{n+2}$.) The method of interpolating between (23) and (26) is to apply complex multilinear interpolation to the operator $(f_1, \dots, f_n) \mapsto F(r)$ where, for the operator's range, we interpolate with change of measure between spaces $L^1([-1, 1], |r|^{n-2} dr / (1-r^2)^\alpha)$, $0 \leq \alpha \leq 1 - \delta$. (The theorems justifying this interpolation can be found in §§4.4 and 5.4 of [1].) The result is that (27) holds when

$$\frac{1}{p} = 1 - \frac{1}{2} \left(\frac{1}{1-\delta} \right) \left(\frac{2}{n+2} + \varepsilon \right).$$

We now define

$$S_1(f_1, \dots, f_n)(x) = \int_{\gamma-1}^{1-\gamma} \int_{\Sigma_{n-2}} \prod_{j=1}^n f_j \cdot \left(x - \operatorname{sgn}(r) c \sqrt{1-r^2} - e_j \cdot T \left(\sum_{k=1}^{n-1} r \eta_k u_k \right) \right) d\eta |r|^{n-2} \frac{dr}{\sqrt{1-r^2}},$$

and $S_2(f_1, \dots, f_n)(x)$ similarly, but with $\int_{\gamma-1}^{1-\gamma}$ replaced by $\int_{\{1-\gamma \leq |r| \leq 1\}}$. Here $\gamma \in (0, 1)$ will be fixed subject to a further restriction in §6. Then $S = S_1 + S_2$. Since

$$\|S_1(f_1, \dots, f_n)\|_1 = \int_{\gamma-1}^{1-\gamma} F(r) |r|^{n-2} \frac{dr}{\sqrt{1-r^2}}$$

if $f_1, \dots, f_n \geq 0$, (22) is a consequence of (23).

Here is the promised proof of (23): If f is a nonnegative function on $\mathbf{R}^n$ and $|T|$ is the absolute value of the determinant of T , then

$$\begin{aligned} & |T|^{-1} \int_{\mathbf{R}^n} f(x) dx \\ &= \int_{\mathbf{R}^n} f \left(T \left(\sum_{k=1}^n x_k u_k \right) \right) dx_1 \cdots dx_n \\ &= \int_{-\infty}^{\infty} \int_{\mathbf{R}^{n-1}} f \left(c(x, \dots, x) + T \left(\sum_{k=1}^{n-1} x_k u_k \right) \right) dx_1 \cdots dx_{n-1} dx \\ &\geq |c|^{-1} \int_{-\infty}^{\infty} \int_0^1 \int_{\Sigma_{n-2}} f \left((x, \dots, x) + T \left(\sum_{k=1}^{n-1} r \eta_k u_k \right) \right) d\eta r^{n-2} dr dx. \end{aligned}$$

Substituting $\prod_{j=1}^n f_j(x_j)$ for $f(x_1, \dots, x_n)$ gives (23).

The proofs of (24) and (25) depend on the induction hypotheses: that (A) and (B) of §1 are true in dimension $n-1$. We begin by stating a fact from linear algebra which can be proved by comparing determinants:

If $v_1, \dots, v_{n-1}, (1, 1, \dots, 1)$ are linearly independent in $\mathbf{R}^n$, then the vectors $((e_j - e_1) \cdot v_1, (e_j - e_1) \cdot v_2, \dots, (e_j - e_1) \cdot v_{n-1})$, $2 \leq j \leq n$, are linearly independent in $\mathbf{R}^{n-1}$.

In particular, the vectors $((e_j - e_1) \cdot Tu_1, \dots, (e_j - e_1) \cdot Tu_{n-1})$, $2 \leq j \leq n$, are linearly independent in $\mathbf{R}^{n-1}$. Let f_j^r denote the dilate of f_j defined by $f_j^r(x) = f_j(rx)$ and define I_j^r similarly. We have

$$\begin{aligned} F(r) &= r \int_{-\infty}^{\infty} \int_{\Sigma_{n-2}} \prod_{j=1}^n f_j^r \left(x - e_j \cdot T \left(\sum_{k=1}^{n-1} \eta_k u_k \right) \right) d\eta dx \\ &= r \int_{-\infty}^{\infty} f_1^r(x) \int_{\Sigma_{n-2}} \prod_{j=2}^n f_j^r \left(x - (e_j - e_1) \cdot T \left(\sum_{k=1}^{n-1} \eta_k u_k \right) \right) d\eta dx \\ &= r \int_{-\infty}^{\infty} f_1^r(x) \int_{\Sigma_{n-2}} \prod_{j=2}^n f_j^r(x - w_j \cdot \eta) d\eta dx, \end{aligned}$$

where $w_j = ((e_j - e_1) \cdot Tu_1, \dots, (e_j - e_1) \cdot Tu_{n-1})$, $2 \leq j \leq n$. Define

$$S_{n-1}(f_2, \dots, f_n)(x) = \int_{\Sigma_{n-2}} \prod_{j=2}^n f_j(x - w_j \cdot \eta) d\eta.$$

Since $\|I_j^r\|_p = |r|^{-1/p} \|I_j\|_p$, (24) follows from

$$\|S_{n-1}(I_2^r, \dots, I_n^r)\|_{(n+2)/2} \leq C \prod_{j=2}^n \|I_j^r\|_{(n+2)/n},$$

which is the case $(\frac{1}{p}, \frac{1}{q}) = (\frac{n}{n+2}, \frac{2}{n+2})$ of (3) in dimension $n-1$. Finally, (25) follows similarly from

$$\|S_{n-1}(f_2^r, \dots, f_n^r)\|_{s'} \leq C \prod_{j=2}^n \|f_j^r\|_s, \quad \frac{1}{s} + \frac{1}{s'} = 1,$$

(that is the case $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{s}, \frac{1}{s'})$ of (2) in dimension $n-1$) which is true because $(\frac{1}{s}, \frac{1}{s'}) \in R(n-1)$ if $\frac{1}{s} = \frac{n}{n+2} - \varepsilon$ and $\varepsilon > 0$ is small.

6. With S_2 as defined in §5, the purpose of this section is to prove the inequalities

$$(28) \quad \|S_2(I_1, \dots, I_n)\|_{\infty} \leq C \prod_{j=1}^n |E_j|^{n/(n+2)},$$

and, if $0 \leq \frac{1}{p} < \frac{n}{n+2}$,

$$(29) \quad \|S_2(f_1, \dots, f_n)\|_{\infty} \leq C \prod_{j=1}^n \|f_j\|_p.$$

As before, the constants will depend on $n, v_1, \dots, v_n$, and (in (29)) on p . Now (29) follows from (28) as (10) followed from (9) in §3. For $0 < \delta < 1$, let $M(\delta) = \{\sigma \in \Sigma_{n-1} : |\sigma \cdot u_n| \leq \delta\}$. Since

$$S_2(I_1, \dots, I_n)(x) = \int_{M(\sqrt{2\gamma-\gamma^2})} \prod_{j=1}^n I_j(x - v_j \cdot \sigma) d\sigma,$$

(28) will follow when we show that for some fixed $\delta = \delta(v_1, \dots, v_n) \in (0, 1)$ we have

$$\int_{M(\delta)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

For $j = 1, \dots, n$ define $\tilde{v}_j = v_j/|v_j|$. For $\delta > 0$ define $N(\delta)$ to be the set

$$\{\sigma \in \Sigma_{n-1} : |\sigma - \tilde{v}_j|, |\sigma + \tilde{v}_j| > \delta \text{ for } 1 \leq j \leq n\}.$$

Since $v_j \cdot u_n = e_j \cdot Tu_n = c \neq 0$ for $1 \leq j \leq n$, there is some positive δ such that $M(\delta) \subseteq N(\delta)$. We will show, then, that

$$(30) \quad \int_{N(\delta)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

Let S_j be the linear span of $\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$. Since $\bigcap_{j \neq j_0} S_j$ is the set of scalar multiples of v_{j_0} , there is an $\varepsilon > 0$ such that, for any j_0 ,

$$\text{dist}(\sigma, S_j) < \varepsilon \text{ for all } j \neq j_0 \text{ implies } \min\{|\sigma + \tilde{v}_{j_0}|, |\sigma - \tilde{v}_{j_0}|\} < \delta \quad (\sigma \in \Sigma_{n-1}).$$

In particular,

$$N(\delta) \subseteq \bigcup_{j=1}^n \{\sigma \in \Sigma_{n-1} : \text{dist}(\sigma, S_j) \geq \varepsilon\}.$$

Thus, for fixed $E_1, \dots, E_n$, there is some j_0 ($1 \leq j_0 \leq n$) such that if

$$N(\delta, j_0) = N(\delta) \cap \{\sigma : \text{dist}(\sigma, S_{j_0}) \geq \varepsilon\},$$

then

$$\int_{N(\delta)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \leq n \int_{N(\delta, j_0)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma.$$

So fix j_0 . Inequality (30) will follow from

$$(31) \quad \int_{N(\delta, j_0)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \leq C \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

Let $\{w_1, \dots, w_{n-1}, \tilde{v}_{j_0}\}$ be an orthonormal basis for $\mathbf{R}^n$. Parametrize (almost all of) Σ_{n-1} by

$$\sigma = \sum_{k=1}^{n-1} r\eta_k w_k + \text{sgn}(r)\sqrt{1-r^2}\tilde{v}_{j_0} \quad \text{where } \eta = (\eta_1, \dots, \eta_{n-1}) \in \Sigma_{n-2}, \quad -1 \leq r \leq 1.$$

Then the inequalities $|\sigma \pm \tilde{v}_{j_0}| > \delta$ imply $|r| > \alpha$ for some $\alpha = \alpha(\varepsilon) > 0$. Thus, if $V = \{\sigma : \text{dist}(\sigma, S_{j_0}) \geq \varepsilon\}$, then

$$\begin{aligned} & \int_{N(\delta, j_0)} \prod_{j=1}^n I_j(v_j \cdot \sigma) d\sigma \\ & \leq \int_{\{\alpha \leq |r| \leq 1\}} \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j \left(v_j \cdot \left[\sum_{k=1}^{n-1} r\eta_k w_k + \text{sgn}(r)\sqrt{1-r^2}\tilde{v}_{j_0} \right] \right) I_V(\eta, r) d\eta \\ & \quad \cdot I_{j_0}(|v_{j_0}|\text{sgn}(r)\sqrt{1-r^2})|r|^{n-2} \frac{dr}{\sqrt{1-r^2}}. \end{aligned}$$

Then (31) will follow if the right-hand side of the inequality above can be bounded by

$$C \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

To simplify notation we work only with $\int_{\alpha}^1$ —the integral $\int_{-1}^{-\alpha}$ is treated similarly. Write

$$F(r) = \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j \left(v_j \cdot \left[\sum_{k=1}^{n-1} r \eta_k w_k + \sqrt{1-r^2} \tilde{v}_{j_0} \right] \right) I_V(\eta, r) d\eta.$$

Then the desired inequality is

$$\int_{\alpha}^1 F(r) I_{j_0} \left(|v_{j_0}| \sqrt{1-r^2} \right) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j=1}^n |E_j|^{n/(n+2)}.$$

Since

$$\int_{\alpha}^1 I_{j_0} \left(|v_{j_0}| \sqrt{1-r^2} \right) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C |E_{j_0}|,$$

the duality between the Lorentz spaces $L^{(n+2)/n,1}$ and $L^{(n+2)/2,\infty}$ (on the interval $[\alpha, 1]$ with the measure $r^{n-2} dr / \sqrt{1-r^2}$) shows that it is enough to prove the following for $\lambda > 0$:

$$(32) \quad \text{if } E = \{r \in [\alpha, 1] : F(r) > \lambda\}, \text{ then } \lambda \left(\int_{\alpha}^1 I_E(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \right)^{2/(n+2)} \\ \leq C \prod_{j \neq j_0} |E_j|^{n/(n+2)}.$$

To prove (32), let us assume for the moment the inequalities

$$(33) \quad F(r) \leq C \prod_{j \neq j_0} |E_j|^{(n-2)/n}, \quad \alpha \leq r \leq 1,$$

$$(34) \quad \int_{\alpha}^1 F(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j \neq j_0} |E_j|.$$

Write I for the integral

$$\int_{\alpha}^1 I_E(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}}.$$

If $I \neq 0$, then

$$\lambda \leq C \prod_{j \neq j_0} |E_j|^{(n-2)/n}$$

follows from (33). Also, the inequality

$$\lambda I \leq \int_{\alpha}^1 F(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j \neq j_0} |E_j|$$

follows from (34). These last two inequalities then yield (32) in the form

$$\lambda^{1+2/n} I^{2/n} \leq C \prod_{j \neq j_0} |E_j|.$$

To prove (33) we begin by defining vectors $u_j \in \mathbf{R}^{n-1}$, $j \neq j_0$, by $u_j = (v_j \cdot w_1, \dots, v_j \cdot w_{n-1})$. We will need to know that these $n-1$ vectors are linearly independent in $\mathbf{R}^{n-1}$. This is true because the absolute value of the determinant of the matrix whose rows are the vectors u_j is the same as the absolute value of the determinant of the matrix $\tilde{V} \cdot \tilde{W}$. Here $\tilde{V}$ is the (nonsingular) matrix whose rows are the vectors $v_1, \dots, v_{j_0-1}, v_{j_0+1}, \dots, v_n, \tilde{v}_{j_0}$ and $\tilde{W}$ is the matrix whose columns are the (orthonormal) vectors $w_1, \dots, w_{n-1}, \tilde{v}_{j_0}$.

Now for $j \neq j_0$

$$v_j \cdot \left[\sum_{k=1}^{n-1} r \eta_k w_k + \sqrt{1-r^2} \tilde{v}_{j_0} \right] = r u_j \cdot \eta + \sqrt{1-r^2} v_j \cdot \tilde{v}_{j_0},$$

so

$$F(r) \leq \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j(r u_j \cdot \eta + \sqrt{1-r^2} v_j \cdot \tilde{v}_{j_0}) d\eta.$$

By (9) in dimension $n-1$, the right-hand side of the inequality above is bounded by

$$C r^{-(n-1)(n-2)/n} \prod_{j \neq j_0} |E_j|^{(n-2)/2}.$$

Since $\alpha \leq r \leq 1$, this yields (33). To obtain (34), we begin by noting that

$$(35) \quad \int_{\alpha}^1 F(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq \int_V \prod_{j \neq j_0} I_j(v_j \cdot \sigma) d\sigma.$$

Let $D(\sigma)$ be the determinant of the matrix whose rows are the vector σ and the vectors v_j , $j \neq j_0$. It follows from Lemma 3.3 that

$$(36) \quad \int_V \prod_{j \neq j_0} I_j(v_j \cdot \sigma) |D(\sigma)| d\sigma \leq 2 \prod_{j \neq j_0} |E_j|.$$

Since

$$V = \{\sigma : \text{dist}(\sigma, S_{j_0}) \geq \varepsilon\}$$

and since $D(\sigma) = 0$ if and only if $\sigma \in S_{j_0}$, it follows that

$$|D\sigma| \geq \beta = \beta(\varepsilon) > 0 \quad \text{if } \sigma \in V.$$

Together with (35) and (36), this gives (34).

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