TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 310, Number 2, December 1988

ON JAMES' TYPE SPACES

ABDERRAZZAK SERSOURI

ABSTRACT. We study the spaces E which are isometric to their biduals E^{**}, and satisfy $\dim(E^{**}/E) < \infty$. We show that these spaces have several common points with the usual James' space.

Our study leads to a kind of classification of these spaces and we show that there are essentially four different basic structures for such spaces in the complex case, and five in the real case.

Introduction. In this paper we investigate some geometric properties of the Banach spaces E which are isometric to their biduals E^{**} and satisfy $\dim(E^{**}/E) = s < \infty$. This study was started in [**S**] where we considered the case $s = 1$.

In [**V**] Valdivia proved that if a Banach space E is such that $\dim(E^{**}/E) = s < \infty$, then $E = H \oplus G$, where H is a reflexive space, and G is a separable space (which satisfies $\dim(G^{**}/G) = s$).

Under the additional hypothesis that E is isometric to E^{**} we give an explicit construction of the spaces G and H appearing in Valdivia's decomposition. We also prove (Theorem 2.1) that the space G we construct is isomorphic to G^{**} and has a Schauder basis.

The basis structure of G is made precise in Theorem 5.1 where we prove that G has a shrinking basis $(e_n^{(j)})_{n\geq 0, 1\leq j\leq s}$ such that the sequence $(\sum_{p=0}^n e_p^{(j)})_{n\geq 0, 1\leq j\leq s}$ forms a boundedly complete basis of G, and such that all the basic sequences $(e_n^{(j)})_{n\geq 0}$, $1 \leq j \leq s$, are neighborly. Hence by a result of Bellenot [**B**] all the spaces $G^{(j)} = \overline{\text{sp}}[e_n^{(j)} : n \geq 0]$ are isomorphic to their biduals $G^{(j)**}$ and satisfy $\dim[G^{(j)**}/G^{(j)}] = 1$.

Unfortunately it is not clear whether or not we have $G = \bigoplus_{j=1}^s G^{(j)}$.

We investigate two other aspects of the geometry of the spaces E considered here. We first ask whether the space G is isometric to G^{**} and how such an isometry (if it exists) can be related to the given isometry A between E and E^{**}. More precisely we study how far $G^{\perp\perp}$ is from $A(G)$, or dually, how far $H = H^{\perp\perp}$ is from $A(H)$. The answers to these problems are given in Proposition 3.1.

With respect to these distortion properties, results of Theorem 5.1 lead to a classification of the spaces E considered here and assert that there are essentially four different basic structures in the complex case, and five in the real case. This will be illustrated by the examples we construct in §6.

The second geometrical aspect of the spaces E we consider is the behaviour of the onto isometries on E (§4). The main consequence of this study is the intrinsic character of all the (vector space) parameters we introduce. We prove that every onto isometry I on E respects both G and H, and in the particular case when

Received by the editors September 1, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B20.

$s = 1$, we have $I|_G = \varepsilon \operatorname{Id}_G$ for some unit scalar ε (a similar result holds in the general case, but it cannot be described easily). This implies in particular that the spaces E for which $E = G$ have only trivial isometries (this is the case for the usual James' space J).

A nice application of the results we obtain is the following remark due to Bellenot (personal communication):

There exists a separable Banach space (with a basis) E satisfying $\dim(E^{**}/E) = 1$, which is isomorphic to E^{**} but not isometric to E^{**} for any equivalent norm [**B**, ex. 4.6].

ACKNOWLEDGMENT. The author is glad to thank Professor Bernard Maurey for his precious help and his suggestions.

1. Notation and preliminaries. $\mathbf{K}$ will denote the scalar field $\mathbf{R}$ or $\mathbf{C}$.

Let us first recall briefly the definitions of the ω-dual of a Banach space, and of the ω-adjoint of an operator.

Let X be a Banach space and denote by $i_n^{[X]}$ the canonical injection of $X^{(n)}$, the nth dual of X, into $X^{(n+2)}$. Then the ω-dual $X^{(\omega)}$ of X is the injective limit of the sequence $((X^{(2n)}, i_{2n}^{[X]}))_{n\geq 0}$.

Let $T\colon X \to Y$ be an operator from X to Y, and denote by $T^{(n)}$ the nth-adjoint of T. Then the ω-adjoint $T^{(\omega)}$ of T is the operator from $X^{(\omega)}$ to $Y^{(\omega)}$ which is the injective limit of the following commutative diagram:

$$\begin{array}{ccccccccc}
X & \xrightarrow{i_0^{[X]}} & X & \xrightarrow{i_2^{[X]}} & X^{(4)} & \rightarrow & \cdots & & X^{(\omega)} \\
T\downarrow & & T^{**}\downarrow & & T^{(4)}\downarrow & & & & T^{(\omega)}\downarrow \\
Y & \xrightarrow[i_0^{[Y]}]{} & Y^{**} & \xrightarrow[i_2^{[Y]}]{} & Y^{(4)} & \rightarrow & \cdots & & Y^{(\omega)}
\end{array}$$

Let E be a Banach space which is isometric to E^{**} and which satisfies $\dim(E^{**}/E) = s < \infty$ and let $A\colon E \to E^{**}$ be an onto isometry between E and E^{**}.

It is easy to check that $A^{(\omega)}$ induces an onto isometry on $E^{(\omega)}$ and that $A^{-1*(\omega)}$ induces an onto isometry on $E^{*(\omega)}$. Moreover we have

$$(A^{(\omega)})^*|_{E^{*(\omega)}} = (A^{-1*(\omega)})^{-1} \quad \text{and} \quad (A^{-1*(\omega)})^*|_{E^{(\omega)}} = (A^{(\omega)})^{-1}.$$

(The spaces $E^{(\omega)}$ and $E^{*(\omega)}$ are both isometric to subspaces of $(E^{*(\omega)})^*$ and of $(E^{(\omega)})^*$ respectively.)

For convenience we will denote by $\mathscr{A}$ both of the isometries $A^{(\omega)}$ and $A^{-1*(\omega)}$, and the above two relations will be paraphrased by $\mathscr{A}^* = \mathscr{A}^{-1}$.

Now we are going to define a (nonexaustive) list of parameters which will be constantly used throughout this paper.

Let $E_0 = E$ and $E_1 = E^*$. For every $k \in \mathbf{Z}$ we define inductively the spaces E_k by $E_{k+2} = \mathscr{A}(E_k)$.

Notice that E_k is a subspace of E_{k+2} for every $k \in \mathbf{Z}$, and that the spaces E_{2k} and E_{2l+1} are in duality for every $k, l \in \mathbf{Z}$, since they are subspaces of $E^{(\omega)}$ and $E^{*(\omega)}$ respectively.

Using these dualities, we define on E_k two topologies: $\omega_k = \sigma(E_k, E_{k+1})$ and $\omega_k^* = \sigma(E_k, E_{k-1})$.

Let $(\xi_3^{(j)})_{1\leq j\leq s}$ be a basis of $E^\perp$ in $E^{(3)}$, and $(\xi_4^{(j)})_{1\leq j\leq s}$ be a basis of $E^{*\perp}$ in $E^{(4)}$. For every $k \in \mathbf{Z}$, and every j, $1 \leq j \leq s$, we define inductively the vectors

$\xi_k^{(j)} \in E_k$ by $\xi_{k+2}^{(j)} = \mathscr{A}(\xi_k^{(j)})$. We also introduce, for every $k \in \mathbf{Z}$, the 1-column vector $\xi_k = {}^t(\xi_k^{(1)}, \ldots, \xi_k^{(s)})$.

For every $k \in \mathbf{Z}$, we define two (s,s)-matrices M_k and N_k by

$$M_k = \langle \xi_{k+1}, {}^t\xi_k \rangle = (\langle \xi_{k+1}^{(i)}, \xi_k^{(j)} \rangle)_{1 \le i,j \le s}, \quad N_k = -M_{k+1}\, {}^tM_k^{-1}.$$

(All the matrices M_k are invertible; see Lemma 1.1).

For every $k \in \mathbf{Z}$ and for every $n \ge 0$, we define the 1-column vector $\tilde{\xi}_{k,n} = {}^t(\tilde{\xi}_{k,n}^{(1)}, \ldots, \tilde{\xi}_{k,n}^{(s)})$ by $\tilde{\xi}_{k,n} = \sum_{p=0}^n N_k^p \xi_{k-2p}$. Notice that for every $k \in \mathbf{Z}$, the vector $\tilde{\xi}_{k,n}^{(j)}$ is in E_k for every $n \ge 0$, $1 \le j \le s$.

For every $k \in \mathbf{Z}$, $n \ge 0$, we introduce the spaces:

$$G_k = \overline{\mathrm{sp}}[\xi_{k-2p}^{(j)} : p \ge 0, 1 \le j \le s],$$
$$H_k = \bigcap_{p \ge 0} \overline{E}_{k-2p}^{\omega_k^*}, \quad X_k = \mathrm{sp}[\xi_k^{(j)} : 1 \le j \le s],$$
$$\tilde{X}_{k,n} = \mathrm{sp}[\tilde{\xi}_{k,n}^{(j)} : 1 \le j \le s].$$

We will also consider the spaces $F = \bigcap_{k \in \mathbf{Z}} E_{2k}$ and $F_* = \bigcap_{k \in \mathbf{Z}} E_{2k+1}$.

The following lemma gathers some observations and elementary results which will be used frequently in this paper.

LEMMA 1.1. *For every $k \in \mathbf{Z}$, every $n \ge 0$, we have:*
(i) *E_k is the unique predual of E_{k+1},*
(ii) *$\mathscr{A} : E_k \to E_{k+2}$ is $\omega_k^* - \omega_{k+2}^*$ continuous,*
(iii) *$E_k = E_{k-2} \oplus X_k$, and more generally $E_k = E_{k-2n} \oplus (\bigoplus_{p=0}^{n-1} X_{k-2p})$,*
(iv) *$\xi_k^{(j)}|_{E_{k-3}} = 0$ for every $j \in [1,s]$,*
(v) *M_k is invertible,*
(vi) *$M_{k+2} = M_k$, and hence $N_{k+2} = N_k$,*
(vii) *$\mathscr{A}(\tilde{\xi}_{k,n}) = \tilde{\xi}_{k+2,n}$,*
(viii) *$\mathscr{A}(G_k) = G_{k+2}$, $\mathscr{A}(H_k) = H_{k+2}$, $\mathscr{A}(X_k) = X_{k+2}$, and $\mathscr{A}(\tilde{X}_{k,n}) = \tilde{X}_{k+2,n}$,*
(ix) *$\mathscr{A}(F) = F$, and $\mathscr{A}(F_*) = F_*$.*

PROOF. (i) and (ii) are consequences of a result of Godefroy [G].

All the other properties, except (v), are elementary and can be proved by inductive arguments.

Let us prove (v). Using (i) and (iv), it is clear that the vectors $(\xi_{k+1}^{(j)})_{1 \le j \le s}$ form a basis of $E_{k-2}^{\perp}$, where the polar is taken in E_{k+1}.

If $\det(M_k) = 0$, then there exist scalars $(\alpha_j)_{1 \le j \le s}$, not all zero, such that $\sum_{j=1}^s \alpha_j \langle \xi_{k+1}^{(j)}, \xi_k^{(i)} \rangle = 0$ for every $i \in [1,s]$ (since the rows of the matrix M_k are not linearly independent). From (iii), we deduce that $\sum_{j=1}^s \alpha_j \langle \xi_{k+1}^{(j)}, x \rangle = 0$ for every $x \in E_k$, which is equivalent to $\sum_{j=1}^s \alpha_j \xi_{k+1}^{(j)} = 0$ by (i). This means that the vectors $(\xi_{k+1}^{(j)})_{1 \le j \le s}$ are not linearly independent, which is a contradiction. Then $\det(M_k) \ne 0$ and hence M_k is invertible. □

Another result which will be used very often in this paper is the following well-known theorem.

THE BIPOLAR THEOREM. *Let X be a Banach space, and Y be a subspace of X. Then, if $Y^{\perp\perp}$ denotes the bipolar of Y in X^{**} we have:* $B(Y^{\perp\perp}) = \overline{B(Y)}^{\sigma(X^{**},X^{*})}$, *where $B(Y)$ and $B(Y^{\perp\perp})$ are the unit balls of Y and $Y^{\perp\perp}$ respectively.*

2. The decomposition theorem. This part is essentially devoted to the proof of the following theorem, which is the first main result of this paper:

THEOREM 2.1. *With our notations, we have for every $k \in \mathbf{Z}$:*
(i) $E_k = G_k \oplus H_k$.
(ii) *H_k is reflexive.*
(iii) *G_k is isomorphic to G_k^{**}, and* $\dim(G_k^{**}/G_k) = s$.
(iv) *The vectors $(\xi_{k-2n}^{(j)})_{n\geq 0, 1\leq j\leq s}$ form a shrinking basis of G_k.*
(v) *The vectors $(\tilde{\xi}_{k,n}^{(j)})_{n\geq 0,1\leq j\leq s}$ form a boundedly complete basis of G_k.*
(vi) *The Schauder decomposition $H_k \oplus (\bigoplus_{n=0}^{\infty} X_{k-2n})$ of E_k is monotone.*
In particular H_k is 1-complemented with respect to the decomposition $E_k = H_k \oplus G_k$.
(vii) *The Schauder decomposition $\bigoplus_{n=0}^{\infty} X_{k-2n}$ of G_k is bimonotone.*

The proof of this theorem will be decomposed into several lemmas. Its assertions will be proved in Propositions 2.3, 2.10, 2.11, 2.12, and 2.13.

LEMMA 2.2. *For every $k \in \mathbf{Z}$, there exists a norm-one projection $\pi_k \colon E_k \to E_{k-2}$ such that $\pi_k(E_k) = E_{k-2}$, and* $\ker \pi_k = X_k$.

PROOF. It is easy to see that if π_k is constructed, we can take $\pi_{k-2} = \mathscr{A}^{-1} \circ \pi_k \circ \mathscr{A}$ (resp. $\pi_{k+2} = \mathscr{A} \circ \pi_k \circ \mathscr{A}^{-1}$).

The lemma is then proved since we can take for $\pi_3 \colon E^{(3)} \to E^*$ (resp. $\pi_4 \colon E^{(4)} \to E^{**}$) the canonical projection, whose kernel is $E^{\perp}$ (resp. $E^{*\perp}$). □

REMARK. Throughout this paper, the notation π_k will always mean the projections described in Lemma 2.2.

An immediate consequence of this lemma is

PROPOSITION 2.3. *For every $k \in \mathbf{Z}$, the sequence $(X_{k-2n})_{n\geq 0}$ is a reverse monotone Schauder decomposition of G_k.*

The first key lemma of this paper is

LEMMA 2.4. *For every $k \in \mathbf{Z}$, every $n \geq 1$, we have*
(i) $E_{k-2n}^{\perp} = \bigoplus_{p=0}^{n-1} X_{k+1-2p}$,
(ii) $(\bigoplus_{p=0}^{n-1} X_{k-2p})^{\perp} = E_{k-2n-1} \oplus \tilde{X}_{k+1,n}$,
where the polars are taken in E_{k+1}.

PROOF. This lemma is an easy consequence of a dimensional argument and the following:

Claim. $\langle \tilde{\xi}_{k+1,p}, {}^t\xi_{k-2q} \rangle = \delta_{pq} N_{k+1}^{p} M_k$, for every $k \in \mathbf{Z}$, every $p, q \geq 0$.

Let us now prove the lemma by assuming the claim.

(i) By Lemma 1.1(iv) it is clear that $(\bigoplus_{p=0}^{n-1} X_{k+1-2p}) \subset E_{k-2n}^{\perp}$, and the equality holds since $\dim(\bigoplus_{p=0}^{n-1} X_{k+1-2p}) = ns$, $E_{k-2n}^{\perp} = (E_k/E_{k-2n})^*$ (Lemma 1.1(i)) and hence $\dim(E_{k-2n}^{\perp}) = \dim(E_k/E_{k-2n}) = ns$ (Lemma 1.1(iii)).

(ii) Again by 1.1(iv) we have $E_{k-2n-1} \subset (\bigoplus_{p=0}^{n-1} X_{k-2p})^{\perp}$, and using the claim we see that $\tilde{X}_{k+1,n} \subset (\bigoplus_{p=0}^{n-1} X_{k-2p})^{\perp}$. The equality (ii) of Lemma 2.4 holds since:

(1) E_{k-1-2n} and $\tilde{X}_{k+1,n}$ are in direct sum (Lemma 1.1(iii)) and hence

$$\operatorname{codim}_{E_{k+1}}[E_{k-1-2n} \oplus \tilde{X}_{k+1,n}] = \dim\left[\left(\bigoplus_{p=0}^{n} X_{k+1-2p}\right)/\tilde{X}_{k+1,n}\right] = ns.$$

(2) $(\bigoplus_{p=0}^{n-1} X_{k-2p})^{*} = E_{k+1}/(\bigoplus_{p=0}^{n-1} X_{k-2p})^{\perp}$ and hence

$$\operatorname{codim}_{E_{k+1}}\left(\bigoplus_{p=0}^{n-1} X_{k-2p}\right)^{\perp} = ns.$$

PROOF OF THE CLAIM. It is clear by 1.1(iv) that $\langle\tilde{\xi}_{k+1,p}, {}^{t}\xi_{k-2q}\rangle = 0$ if $q > p$. On the other hand for $0 \le q < p$, we have

$$\begin{aligned}\langle\tilde{\xi}_{k+1,p}, {}^{t}\xi_{k-2q}\rangle &= \langle N_{k+1}^{q}\xi_{k+1-2q} + N_{k+1}^{q+1}\xi_{k-1-2q}, {}^{t}\xi_{k-2q}\rangle \\ &= N_{k+1}^{q}(M_{k-2q} + N_{k+1}{}^{t}M_{k-1-2q}) = 0.\end{aligned}$$

A similar computation will show that $\langle\tilde{\xi}_{k+1,p}, \xi_{k-2p}\rangle = N_{k+1}^{p}M_{k}$. This concludes the proof of the claim and of Lemma 2.4. □

As an immediate corollary we have the following useful result:

COROLLARY 2.5. *For every $k \in \mathbf{Z}$, every $n \ge 0$, we have $E_{k-2n}^{\perp\perp} = E_{k-2n} \oplus \tilde{X}_{k+2,n}$, where the bipolar is taken in E_{k+2}.*

REMARK. This corollary implies that $H_k = \bigcap_{n\ge 0}(E_{k-2n-2} \oplus \tilde{X}_{k,n})$ for every $k \in \mathbf{Z}$. This equality is very useful and will be used frequently.

We are now going to prove the second key lemma of this paper.

LEMMA 2.6. *For every $k \in \mathbf{Z}$, every $m, n \ge 0$, every 1-row matrices α_{k+2p}, $-m \le p \le n$, and every vector $x \in E_{k-2n-2}$ we have the following inequalities:*

(i) $$\left\|\sum_{p=1}^{m} \alpha_{k+2p}\xi_{k+2p} + \alpha_k\tilde{\xi}_{k,n} + x\right\| \le \left\|\sum_{p=-n}^{m} \alpha_{k+2p}\xi_{k+2p} + x\right\|.$$

(ii) $$\left\|\sum_{p=1}^{m} \alpha_{k+2p}\xi_{k+2p} + \alpha_{k-2n}N_k^{-n}\tilde{\xi}_{k,n} + x\right\| \le \left\|\sum_{p=-n}^{m} \alpha_{k+2p}\xi_{k+2p} + x\right\|.$$

REMARK. Roughly speaking, the lemma asserts that, modulo the matrices N_k, the norm of the vector $\sum_k \alpha_k\xi_k + x$ decreases if we "propagate" (to the left or to the right) any coefficient of its expansion. In particular the decomposition $\bigoplus_{n=0}^{\infty} X_{k-2n}$ satisfies a kind of neighborly property.

PROOF. Observe first that it suffices to prove the lemma for $n = 1$ (the other parameters have to be arbitrary). For the other values of n, the lemma can be proved by an easy inductive argument.

PROOF OF PART (i). For $n = 1$, the assertion will be proved by an inductive argument on the value of m.

Step 1. *The case where* $m = 0$:

Let α_k, α_{k-2} be two 1-row matrices, and let $x \in E_{k-4}$. Put $e = \alpha_k \xi_k + \alpha_{k-2}\xi_{k-2} + x$, and assume that $\|e\| = 1$.

By Corollary 2.5 (and the bipolar theorem), choose an ultrafilter $\mathscr{F}$, matrices $\lambda_{k-2,\beta}$ ($\beta \in \mathscr{F}$), and vectors $x_\beta \in E_{k-4}$ ($\beta \in \mathscr{F}$) such that:

$$\|\lambda_{k-2,\beta}\xi_{k-2} + x_\beta\| \leq 1 \quad \text{for every } \beta \in \mathscr{F},$$
$$e = \omega_k^*\text{-}\lim_{\beta\in\mathscr{F}}(\lambda_{k-2,\beta}\xi_{k-2} + x_\beta).$$

Let $\lambda_{k-2} = \lim_{\beta\in\mathscr{F}} \lambda_{k-2,\beta}$ and $\overline{x} = \omega_k^*\text{-}\lim_{\beta\in\mathscr{F}} x_\beta$, and let us prove that $\lambda_{k-2} = \alpha_{k-2} - \alpha_k N_k$. (The nets $(\lambda_{k-2,\beta})_{\beta\in\mathscr{F}}$, and $(x_\beta)_{\beta\in\mathscr{F}}$ are bounded by Lemma 2.2, and the basis properties of the vectors $(\xi_{k-2}^{(j)})_{1\leq j\leq s}$.)

Indeed

$$\langle e, {}^t\xi_{k-1}\rangle = \langle \alpha_k\xi_k + \alpha_{k-2}\xi_{k-2}, {}^t\xi_{k-1}\rangle = \alpha_k M_{k+1} + \alpha_{k-2}{}^tM_k,$$
$$\langle e, {}^t\xi_{k-1}\rangle = \lim_{\beta\in\mathscr{F}}\langle \lambda_{k-2,\beta}\xi_{k-2}, {}^t\xi_{k-1}\rangle = \lambda_{k-2}{}^tM_k.$$

We then deduce that $\lambda_{k-2} = \alpha_{k-2} - \alpha_k N_k$, and hence $\overline{x} = \alpha_k\xi_k + \alpha_k N_k \xi_{k-2} + x$. Moreover

$$\|\overline{x}\| \leq \lim_{\beta\in\mathscr{F}} \|x_\beta\| \leq \lim_{\beta\in\mathscr{F}} \|\lambda_{k-2,\beta}\xi_{k-2} + x_\beta\| \leq 1,$$

which is the desired result. (The second inequality uses Lemma 2.2.)

Step 2. *The induction.* Suppose that Lemma 2.6(i) was proved for the value $m-1$. Let us prove it for the value m (recall that $n = 1$).

Let $e = \sum_{p=-1}^m \alpha_{k+2p}\xi_{k+2p} + x$, where $(\alpha_{k+2p})_{-1\leq p\leq m}$ are matrices, and $x \in E_{k-4}$, and suppose that $\|e\| = 1$.

Choose an ultrafilter $\mathscr{F}$, matrices $\lambda_{k+2p,\beta}$ $(-1 \leq p \leq m-1, \beta \in \mathscr{F})$ and vectors $x_\beta \in E_{k-4}$ such that:

$$\left\|\sum_{p=-1}^{m-1} \lambda_{k+2p,\beta}\xi_{k+2p} + x_\beta\right\| \leq 1 \quad \text{for every } \beta \in \mathscr{F},$$
$$e = \omega_{k+2m}^*\text{-}\lim_{\beta\in\mathscr{F}}\left(\sum_{p=-1}^{m-1} \lambda_{k+2p,\beta}\xi_{k+2p} + x_\beta\right).$$

Let $\lambda_{k+2p} = \lim_{\beta\in\mathscr{F}} \lambda_{k+2p,\beta}$, for $-1 \leq p \leq m-1$, and $\overline{x} = \omega_{k+2m}^*\text{-}\lim_{\beta\in\mathscr{F}} x_\beta$, and let us compute these limits.

For every p, $-1 \leq p \leq m-1$, we have (putting $\lambda_{k+2m} = 0$):

$$\alpha_{k+2p+2}M_{k+1} + \alpha_{k+2p}{}^tM_k = \langle e, {}^t\xi_{k+1+2p}\rangle = \lambda_{k+2p+2}M_{k+1} + \lambda_{k+2p}{}^tM_k.$$

It is easily seen that these equalities imply

$$\lambda_{k+2p} = \alpha_{k+2p} - \alpha_{k+2m}N_k^{m-p}, \quad \text{for every } p,\ -1 \leq p \leq m-1,$$

and hence $\overline{x} = x + \sum_{p=-1}^m \alpha_{k+2m}N_k^{m-p}\xi_{k+2p}$.

Now using the inductive hypothesis we obtain

$$\left\|\sum_{p=1}^{m}\alpha_{k+2p}\xi_{k+2p}+\alpha_k\xi_k+\alpha_kN_k\xi_{k-2}+x\right\|$$
$$=\left\|\sum_{p=1}^{m-1}\lambda_{k+2p}\xi_{k+2p}+\lambda_k\xi_k+\lambda_kN_k\xi_{k-2}+\overline{x}\right\|$$
$$\le\lim_{\beta\in\mathcal{F}}\left\|\sum_{p=1}^{m-1}\lambda_{k+2p,\beta}\xi_{k+2p}+\lambda_{k,\beta}\xi_k+\lambda_{k,\beta}N_k\xi_{k-2}+x_\beta\right\|$$
$$\le\lim_{\beta\in\mathcal{F}}\left\|\sum_{p=-1}^{m-1}\lambda_{k+2p,\beta}\xi_{k+2p}+x_\beta\right\|\le 1.$$

This concludes the proof of the inductive argument, and hence the proof of part (i).

PROOF OF PART (ii). Let $e=\sum_{p=-1}^{m}\alpha_{k+2p}\xi_{k+2p}+x$, where α_{k+2p} are matrices, and $x\in E_{k-4}$, and choose a *norm one* vector $f\in E_{k+2m+1}$ which normalizes the vector

$$e'=\sum_{p=1}^{m}\alpha_{k+2p}\xi_{k+2p}+\alpha_{k-2}N_k^{-1}\xi_k+\alpha_{k-2}\xi_{k-2}+x\quad(\text{i.e. }\langle f,e'\rangle=\|e'\|).$$

The vector f has a decomposition $f=\sum_{q=-1}^{m}\alpha_{k+1+2q}\xi_{k+1+2q}+y$, where α_{k+1+2q}, $-1\le q\le m$, are matrices, and $y\in E_{k-3}$. Using the orthogonality relations (Lemma 2.4(ii)), and the result of part (i), we obtain

$$\|e'\|=\langle e',f\rangle$$
$$=\left\langle\sum_{p=1}^{m}\alpha_{k+2p}\xi_{k+2p}+\alpha_{k-2}N_k^{-1}(\xi_k+N_k\xi_{k-2})+x;\ \sum_{q=-1}^{m}\alpha_{k+1+2q}\xi_{k+1+2q}+y\right\rangle$$
$$=\left\langle\sum_{p=1}^{m}\alpha_{k+2p}\xi_{k+2p}+\alpha_{k-2}N_k^{-1}(\xi_k+N_k\xi_{k-2})+x;\ \sum_{q=1}^{m}\alpha_{k+1+2q}\xi_{k+1+2q}+\alpha_{k+1}(\xi_{k+1}+N_{k+1}\xi_{k-1})+y\right\rangle$$
$$=\left\langle\sum_{p=-1}^{m}\alpha_{k+2p}\xi_{k+2p}+x;\ \sum_{q=1}^{m}\alpha_{k+1+2q}\xi_{k+1+2q}+\alpha_k(\xi_{k+1}+N_{k+1}\xi_{k-1})+y\right\rangle$$
$$\le\|e\|\cdot\left\|\sum_{q=1}^{m}\alpha_{k+1+2q}\xi_{k+1+2q}+\alpha_k(\xi_{k+1}+N_{k+1}\xi_{k-1})+y\right\|$$
$$\le\|e\|.\quad\square$$

The first important consequence of the above lemma is

COROLLARY 2.7. *For every* $k \in \mathbf{Z}$, $n \geq 0$, *we have* $\|\alpha\xi_k\| = \|\alpha\tilde{\xi}_{k,n}\|$ *for every 1-row matrix* α. *In particular* $\|\xi_k^{(j)}\| = \|\tilde{\xi}_{k,n}^{(j)}\|$ *for every* $j \in [1, s]$.

PROOF. $\|\alpha\xi_k\| \geq \|\alpha\tilde{\xi}_{k,n}\|$ by 2.6(i) ["shift" the α-coefficient of ξ_k], and $\|\alpha\tilde{\xi}_{k,n}\| \geq \|\alpha\xi_k\|$ by 2.6(ii) ["shift" the 0-coefficient of ξ_{k-2n-2}].

The particular case is obtained using the equalities $\tilde{\xi}_{k,n}^{(j)} = \alpha_j\tilde{\xi}_{k,n}$, where the α_j's are the 1-row matrices $(\delta_{ij})_{1\leq i\leq s}$. □

DEFINITION 2.8. Let $\mathscr{U}$ be a nontrivial ultrafilter on $\mathbf{N}$. For every $k \in \mathbf{Z}$, every $j \in [1, s]$ we define vectors $w_k^{(j)}$, and (s, s)-matrices σ_k and the spaces W_k in the following way:

$$w_k^{(j)} = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \tilde{\xi}_{k,n}^{(j)},$$

$$(\sigma_k\xi_{k+2})^{(j)} + w_k^{(j)} = \omega_{k+2}^*\text{-}\lim_{n\in\mathscr{U}} \tilde{\xi}_{k,n}^{(j)},$$

$$W_k = \mathrm{sp}[w_k^{(j)} : 1 \leq j \leq s].$$

REMARKS. (i) These definitions make sense in view of Corollary 2.7.

(ii) We will see later that $(\xi_{k-2n}^{(j)})_{n\geq 0, 1\leq j\leq s}$ is a shrinking basis of G_k. Then the vectors $w_k^{(j)}$ and the matrices σ_k do not depend on the choice of $\mathscr{U}$.

(iii) The spaces W_k and the matrices σ_k will play an important role in the study of the distortion properties of the space H_k and G_k with respect to $\mathscr{A}$ (Proposition 3.1) and in the classification of the space E considered in this paper (§5).

In the following lemma we will summarize some properties of the vectors $w_k^{(j)}$ and the matrices σ_k. For convenience we will introduce the 1-column vector $w_k = {}^t(w_k^{(1)}, \ldots, w_k^{(s)})$.

LEMMA 2.9. *For every* $k \in \mathbf{Z}$, *we have the following*:

(i) $w_k^{(j)} \in H_k$ *for every* $j \in [1, s]$,

(ii) $w_{k+2} = \mathscr{A}(w_k)$ *and* $\sigma_{k+2} = \sigma_k$,

(iii) $\sigma_{k+1}M_k{}^t\sigma_k = M_k$.

In particular the matrix σ_k *is invertible, and* $N_k\sigma_k = \sigma_k N_k$.

(iv) $w_k = (\mathbf{1}_s + N_k\sigma_k)\tilde{\xi}_{k,n} + N_k^n w_{k-2n-2}$ *for every* $n \geq 0$. ($\mathbf{1}_s$ *denotes the identity* (s, s)*-matrix.*)

In particular $\langle w_k, {}^t\xi_{k+1}\rangle = {}^tM_k - \sigma_k M_{k+1}$.

PROOF. (i) It is clear that $\tilde{\xi}_{k,n}^{(j)} \in E_{k-2m-2} \oplus \tilde{X}_{k,m}$ for every $m \leq n$. Hence by definition $w_k^{(j)} \in \bigcap_{m\geq 1} \overline{E_{k-2m-2} \oplus \tilde{X}_{k,m}}^{\omega_k^*} = H_k$ since all the spaces $E_{k-2m-2} \oplus \tilde{X}_{k,m}$ are ω_k^*-closed (see Corollary 2.5).

(ii) This is an easy consequence of Lemma 1.1(ii)–(vii).

(iii) We will compute the scalar product matrix $\langle w_{k+1}, {}^t w_k\rangle$ in two different ways:

$$\begin{aligned}\langle w_{k+1}, {}^t w_k\rangle &= \lim_{n\in\mathscr{U}} \langle \tilde{\xi}_{k+1,n}, {}^t w_k\rangle \\ &= \lim_{n\in\mathscr{U}} \lim_{m\in\mathscr{U}} \langle \tilde{\xi}_{k+1,n}, {}^t(\tilde{\xi}_{k,m} - \sigma_k \xi_{k+2})\rangle \\ &= -{}^t M_{k+1} {}^t\sigma_k + \lim_{n\in\mathscr{U}} \langle \tilde{\xi}_{k+1,n}, {}^t\tilde{\xi}_{k,n}\rangle \\ &= -{}^t M_{k+1} {}^t\sigma_k + M_k,\end{aligned}$$

$$\begin{aligned}\langle w_{k+1}, {}^t w_k\rangle &= \lim_{n\in\mathscr{U}} \langle w_{k+1}, {}^t(\tilde{\xi}_{k,n} - \sigma_k\xi_{k+2})\rangle \\ &= \lim_{n\in\mathscr{U}} \lim_{m\in\mathscr{U}} \langle \tilde{\xi}_{k+1,m} - \sigma_{k+1}\xi_{k+3}, {}^t(\tilde{\xi}_{k,n} - \sigma_k \xi_{k+2})\rangle \\ &= \sigma_{k+1} M_k {}^t\sigma_k - {}^t M_{k+1} {}^t\sigma_k.\end{aligned}$$

By comparing these two results we deduce that $\sigma_{k+1} M_k {}^t\sigma_k = M_k$. Since the matrix M_k is invertible, this implies $(\det\sigma_{k+1})(\det\sigma_k) = 1$, and in particular that σ_k is invertible.

We also have

$$\begin{aligned}-\sigma_k N_k &= \sigma_{k+2} M_{k+1} {}^t M_k^{-1} = M_{k+1} {}^t\sigma_{k+1}^{-1} {}^t M_k^{-1} \\ &= M_{k+1} {}^t(\sigma_{k+1} M_k)^{-1} = M_{k+1} {}^t(M_k {}^t\sigma_k^{-1})^{-1} \\ &= -N_k\sigma_k.\end{aligned}$$

(iv) It suffices to prove that $w_k = (\mathbf{1}_s + N_k\sigma_k)\xi_k + N_k w_{k-2}$.

$$\begin{aligned}w_k &= \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \tilde{\xi}_{k,n} = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} (\xi_k + N_k\tilde{\xi}_{k-2,n-1}) \\ &= (\mathbf{1}_s + N_k\sigma_k)\xi_k + N_k w_{k-2}.\end{aligned}$$

In particular we have

$$\langle w_k, {}^t\xi_{k+1}\rangle = (\mathbf{1}_s + \sigma_k N_k)\langle \xi_k, {}^t\xi_{k+1}\rangle = {}^t M_k - \sigma_k M_{k+1}.$$

Lemma 2.9 is then proved. □

We are now able to start the proof of Theorem 2.1. We first introduce the following notation:

NOTATION. By Lemma 1.1(iii) and Lemma 2.2, there exists for every $x \in E_k$ (and every $k \in \mathbf{Z}$) a sequence $(\mu_{k,n}(x))_{n\geq 0}$ of 1-row matrices such that

$$x = \sum_{p=0}^{n} \mu_{k,p}(x)\xi_{k-2p} + \pi_{k-2n}\circ\cdots\circ\pi_{k-2}\circ\pi_k(x) \quad \text{for every } n \geq 0.$$

(Using the orthogonality relations we see that $\mu_{k,n}(x) = \langle x, {}^t\tilde{\xi}_{k+1,n}\rangle {}^t N_{k+1}^{-n} {}^t M_k^{-1}$.)

By Lemma 2.6(ii) we have

$$\|\mu_{k,n}(x) N_k^{-n}\tilde{\xi}_{k,n} + \pi_{k-2n}\circ\cdots\circ\pi_{k-2}\circ\pi_k(x)\| \leq \|x\|$$

for every $x \in E_k$ and every $n \geq 0$.

So, we can define on E_k a linear operator P_k by

$$P_k(x) = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \left[x + \mu_{k,n}(x) N_k^{-n}\tilde{\xi}_{k,n} - \sum_{p=0}^{n} \mu_{k,p}(x)\xi_{k-2p}\right].$$

PROPOSITION 2.10. *The operator P_k is a norm one projection on H_k and satisfies* $\ker P_k = G_k$.

PROOF. It is clear that $\|P_k\| \leq 1$, and by an argument similar to the one we used in proving that $w_k \in H_k$, it can be seen that $P_k(E_k) \subset H_k$.

It is easy to check that for every $h \in H_k$ and every $n \geq 0$, we have $h = \mu_{k,0}(h)\tilde{\xi}_{k,n} + \pi_{k-2n} \circ \cdots \circ \pi_{k-2} \circ \pi_k(k)$. This implies that $\mu_{k,n}(h) = \mu_{k,0}(h)N_k^n$ for every $n \geq 0$, and hence $P_k(h) = h$.

Altogether the above observations imply that P_k is a norm one projection, and that $P_k(E_k) = H_k$.

It remains to prove that $\ker P_k = G_k$.

By Proposition 2.3, every $g \in G_k$ satisfies $g = \sum_{p=0}^{\infty} \mu_{k,p}(g)\xi_{k-2p}$ with norm convergence for the series, and by Lemma 2.6(ii) we have

$$\left\| \mu_{k,n}(g)N_k^{-n}\tilde{\xi}_{k,n} + \sum_{p=n+1}^{\infty} \mu_{k,p}(g)\xi_{k-2p} \right\| \leq \left\| \sum_{p=n}^{\infty} \mu_{k,p}(g)\xi_{k-2p} \right\|.$$

This implies that $P_k(g) = 0$, and hence $G_k \subset \ker P_k$.

To prove the converse inclusion we will need the following claim:

Claim. For every $x \in E_k$, there exists a 1-row matrix $\overline{\mu}_k(x)$ such that $\overline{\mu}_k(x) = \lim_{n\in\mathscr{U}} \mu_{k,n}(x)N_k^{-n}$ (the limit holds in $\mathbf{K}^s$).

Assuming the claim, let us continue the proof of the proposition. Let $x \in E_k$ be such that $P_k(x) = 0$, which is equivalent to

$$x = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \left[-\mu_{k,n}(x)N_k^{-n}\tilde{\xi}_{k,n} + \sum_{p=0}^{n} \mu_{k,p}(x)\xi_{k-2p} \right].$$

This implies, using the claim, that for every matrix λ we have

$$x + \lambda w_k = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \left[(\lambda - \overline{\mu}_k(x))\tilde{\xi}_{k,n} + \sum_{p=0}^{n} \mu_{k,p}(x)\xi_{k-2p} \right].$$

Now since $\langle \xi_k - w_k, {}^t\xi_{k+1} \rangle = \sigma_k M_{k+1}$ is invertible there exists a matrix $\overline{\lambda}(x)$ such that

$$\langle x + \overline{\lambda}(x)w_k, {}^t\xi_{k+1} \rangle = (\mu_{k,0}(x) - \overline{\mu}_k(x) + \overline{\lambda}(x))\langle \xi_k, {}^t\xi_{k+1} \rangle$$

and then

$$x + \overline{\lambda}(x)w_k = \omega_k\text{-}\lim_{n\in\mathscr{U}} \left[(\overline{\lambda}(x) - \overline{\mu}(x))\tilde{\xi}_{k,n} + \sum_{p=1}^{n} \mu_{k,p}(x)\xi_{k-2p} \right].$$

(This choice of the matrix λ allows us to take weak convergence instead of weak* convergence!!) and in particular $x + \overline{\lambda}(x)w_k \in G_k$ (since the vectors which appear inside the limit are all in G_k).

Finally $0 = P_k(x + \overline{\lambda}(x)w_k) = \overline{\lambda}(x)w_k$, and hence $x \in G_k$.

PROOF OF THE CLAIM. It suffices to prove that the sequence $(\mu_{k,n}(x)N_k^{-n})_{n\geq 0}$ is bounded in $\mathbf{K}^s$. Indeed

$$\begin{aligned}
\|\mu_{k,n}(x)N_k^{-n}\xi_k\| &= \|\mu_{k,n}(x)N_k^{-n}\tilde{\xi}_{k,n}\| && \text{(by Corollary 2.7)}\\
&\leq \left\|\sum_{p=0}^{n}\mu_{k,p}(x)\xi_{k-2p}\right\| && \text{(by Lemma 2.6(ii))}\\
&\leq 2\|x\| && \text{(by Lemma 2.2).}
\end{aligned}$$

The boundedness of the sequence $(\mu_{k,n}(x)N_k^{-n})_{n\geq 0}$ now follows from the basis properties of the vectors $(\xi_k^{(j)})_{1\leq j\leq s}$.

This finishes the proof of the claim and of Proposition 2.10. □

PROPOSITION 2.11. *The Schauder decomposition* $H_k\oplus(\bigoplus_{n=0}^{\infty}X_{k-2n})$ *of* E_k *is monotone (for every* $k\in\mathbf{Z}$*).*

PROOF. $H_k\oplus(\bigoplus_{n=0}^{\infty}X_{k-2n})$ is a Schauder decomposition of E_k since $\bigoplus_{n=0}^{\infty}X_{k-2n}$ is a Schauder decomposition of G_k (Proposition 2.3) and since $E_k = H_k\oplus G_k$.

To prove that this decomposition is monotone we will use the fact that $G_{k-1}\subset H_k^{\perp}$, where the polar is taken in E_{k+1} (since Lemma 2.4(ii) implies $G_k^{\perp} = H_{k+1}$, and hence $H_k^{\perp} = G_{k-1}^{\perp\perp}$).

Let $e = h+\sum_{p=0}^{n}\alpha_{k-2p}\xi_{k-2p}$ ($e\in E_k$, and $h\in H_k$). Choose a norm one vector $f\in E_{k+1}$ which normalizes e, and let $f=\sum_{q=0}^{n+2}\alpha_{k+1-2q}\xi_{k+1-2q}+y$, for some $y\in E_{k-2n-5}$, be a decomposition of f. Then using the orthogonality relations we obtain

$$\begin{aligned}
\|e\| &= \langle e,f\rangle\\
&= \left\langle e,\sum_{q=0}^{n}\alpha_{k+1-2q}\xi_{k+1-2q}+\alpha_{k-2n-1}(\xi_{k-2n-1}+N_{k+1}\xi_{k-2n-3})+y\right\rangle\\
&= \Bigg\langle h+\sum_{p=0}^{n+1}\alpha_{k-2p}\xi_{k-2p};\\
&\qquad \sum_{q=0}^{n}\alpha_{k+1-2q}\xi_{k+1-2q}+\alpha_{k-2n-1}(\xi_{k-2n-1}+N_{k+1}\xi_{k-2n-3})+y\Bigg\rangle\\
&\leq \left\|h+\sum_{p=0}^{n+1}\alpha_{k-2p}\xi_{k-2p}\right\| \quad \text{(by Lemma 2.6(i)).} \quad \square
\end{aligned}$$

The statements of Propositions 2.3, 2.10 and 2.11 are nothing but the assertions (i), (vi), (vii) of Theorem 2.1. We now turn to the proof of 2.1(ii)–(iii).

PROPOSITION 2.12. *For every* $k\in\mathbf{Z}$*, the space* G_k *is isomorphic to* G_k^{**} *and satisfies* $\dim(G_k^{**}/G_k)=s$*. This implies in particular that the space* H_k *is reflexive.*

PROOF. Since the matrix σ_k is invertible and the vectors $(\xi_{k+2}^{(j)})_{1\leq j\leq s}$ are linearly independent, the vectors $((\sigma_k\xi_{k+2})^{(j)})_{1\leq j\leq s}$ are linearly independent, and hence the vectors $(w_k^{(j)} + (\sigma_k \xi_{k+2})^{(j)})_{1\le j\le s}$ are also linearly independent and belong to $\overline{G}_k^{\omega^*_{k+2}} \setminus G_k$.

Then $s \le \dim(G_k^{**}/G_k) \le \dim(E_k^{**}/E_k) = s$.

On the other hand, Lemma 2.4(ii) implies that $G_k^\perp = H_{k+1}$, hence $G_k^* = E_{k+1}/G_k^\perp \approx G_{k+1}$ (Proposition 2.10), and then $G_k^{**} \approx G_{k+2} \equiv G_k$.

Now since $E_k = H_k \oplus G_k$ we have $\dim(E_k^{**}/E_k) = \dim(H_k^{**}/H_k) + \dim(G_k^{**}/G_k)$, which implies clearly that H_k is reflexive. □

We are now going to prove the assertions (iv) and (v) of Theorem 2.1.

PROPOSITION 2.13. *For every $k \in \mathbf{Z}$, the sequence $(\xi_{k-2n}^{(j)})_{n\ge 0, 1\le j\le s}$ is a shrinking (Schauder) basis of G_k, and the sequence $(\tilde{\xi}_{k,n}^{(j)})_{n\ge 0,1\le j\le s}$ is a boundedly complete basis of G_k.*

PROOF. (i) Since all the bases $(\xi_{k-2n}^{(j)})_{1\le j\le s}$ of X_{k-2n} (k is fixed, and $n \ge 0$ is free) are isometrically equivalent it is sufficient to prove that $\bigoplus_{n=0}^\infty X_{k-2n}$ is a shrinking Schauder decomposition of G_k.

We have seen before that $G_k^* \approx G_{k+1}$, and it is easily seen that the restriction map gives such an isomorphism (i.e. $T_k\colon G_{k+1} \to G_k^*\colon T_k(g) = g|_{G_k}$).

Let $\Pi_{k,n}\colon G_k \to G_k$ be the natural projections on G_k with ranks X_{k-2n}, and let us prove that the projections $(\Pi_{k,n}^*)_{n\ge 0}$ induce a Schauder decomposition of G_k^*, or equivalently that the projections $(T_k^{-1}\Pi_{k,n}^* T_k)_{n\ge 0}$ induce a Schauder decomposition of G_{k+1}.

Since $G_{k+1} = \overline{\mathrm{sp}}[\tilde{X}_{k+1,n}\colon n \ge 0]$, the above-mentioned result will be proved if we show that $T_k^{-1}\Pi_{k,n}^* T_k(G_{k+1}) = \tilde{X}_{k+1,n}$. Let us prove this.

$$\begin{aligned}
\langle \Pi_{k,n}^* T_k(\tilde{\xi}_{k+1,p}), {}^t\xi_{k-2q}\rangle &= \langle T_k(\tilde{\xi}_{k+1,p}), \Pi_{k,n}({}^t\xi_{k-2q})\rangle \\
&= \langle \tilde{\xi}_{k+1,p}, \delta_{nq}\, {}^t\xi_{k-2n}\rangle \\
&= \delta_{nq}\delta_{np} N_{k+1}^n M_k \\
&= \langle \delta_{np}\tilde{\xi}_{k+1,n}, {}^t\xi_{k-2q}\rangle.
\end{aligned}$$

This implies that $T_k^{-1}\Pi_{k,n}^* T_k(\tilde{\xi}_{k+1,p}) = \delta_{np}\tilde{\xi}_{k+1,n}$, which is the desired result. Now by [**LT**] this means that the projections $(\Pi_{k,n})_{n\ge 0}$ induce a shrinking Schauder decompositoin of G_k, and hence the basis $(\xi_{k-2n}^{(j)})_{n\ge 0,1\le j\le s}$ of G_k is shrinking.

(ii) By the preceding computations the sequence $(\tilde{X}_{k,n})_{n\ge 0}$ is a boundedly complete decomposition of G_k [**LT**]. Since all the bases $(\tilde{\xi}_{k,n}^{(j)})_{1\le j\le s}$ of $\tilde{X}_{k,n}$ (k is fixed, and $n \ge 0$ is free) are isometrically equivalent (Corollary 2.7), we deduce that the sequence $(\tilde{\xi}_{k,n}^{(j)})_{n\ge 0,1\le j\le s}$ forms a boundedly complete basis of G_k.

This proves Proposition 2.13, and concludes the proof of Theorem 2.1.

3. Regularity properties. We have seen in Proposition 2.12 that the spaces G_k are isomorphic to G_k^{**}, and it is quite natural to ask whether G_k is isometric to G_k^{**}, and how such an isometry is related to $\mathscr{A}$.

Since $G_k^{\perp\perp}$ (the bipolar is taken in E_{k+2}) is a natural isometric copy of G_k^{**}, we will ask whether we have $G_k^{\perp\perp} = \mathscr{A}(G_k)$. Similarly we will ask whether $H_k = \mathscr{A}(H_k)$.

The answer to these problems (which are dual to each other) is given by the following proposition. We will use the spaces F_k defined by $F_{2k} = F$, and $F_{2k+1} = F_*$.

PROPOSITION 3.1. *For every $k \in \mathbf{Z}$, we have:*

(i) $\dim(W_k) + \dim[(G_{k+2} \cap G_k^{\perp\perp})/G_k] = s$. *In particular $G_k^{\perp\perp} = \mathscr{A}(G_k)$ if and only if $W_k = \{0\}$.*

(ii) $\dim(W_{k+1}) = \dim(H_k/F_k)$. *In particular $H_k = \mathscr{A}(H_k)$ if and only if $W_{k+1} = \{0\}$.*

PROOF. (i) Since $G_k^{\perp\perp} \cap G_{k+2} = G_k \oplus \{g \in G_k^{\perp\perp} : g = \alpha\xi_{k+2}\}$,

$$\dim[(G_{k+2} \cap G_k^{\perp\perp})/G_k] = \dim[\{\alpha \in \mathbf{K}^s : \alpha\xi_{k+2} \in G_k^{\perp\perp}\}].$$

But $\alpha\xi_{k+2} \in G_k^{\perp\perp}$ if and only if $\alpha\sigma_k^{-1}w_k = 0$. Indeed, by definition $\alpha(\xi_{k+2} + \sigma_k^{-1}w_k) \in G_k^{\perp\perp}$ for every matrix α, so $\alpha\xi_{k+1} \in G_k^{\perp\perp}$ implies $\alpha\sigma_k^{-1}w_k \in G_k$, and since $\alpha\sigma_k^{-1}w_k \in H_k$ we deduce that $\alpha\sigma_k^{-1}w_k = 0$. The converse implication is trivial. Hence

$$\begin{aligned}\dim[(G_{k+2} \cap G_k^{\perp\perp})/G_k] &= \dim[\{\alpha \in \mathbf{K}^s : \alpha\sigma_k^{-1}w_k = 0\}] \\ &= \dim[\{\beta \in \mathbf{K}^s : \beta w_k = 0\}] \\ &= s - \dim W_k.\end{aligned}$$

If $W_k = \{0\}$, the above equality implies that $G_k^{\perp\perp} = G_{k+2} = \mathscr{A}(G_k)$, since $\dim(G_{k+2}/G_k) = \dim(G_k^{\perp\perp}/G_k) = s$. The converse implication is trivial.

(ii) To prove the second assertion of Proposition 3.1, we need the following lemma:

LEMMA 3.2. *For every $k \in \mathbf{Z}$ we have*

(i) $F_k = H_k \cap [\bigcap_{j=1}^s \ker(\xi_{k+1}^{(j)})]$,

(ii) $(w_{k+1}^{(j)} - \xi_{k+1}^{(j)}) \in H_k^\perp$ *for every $j \in [1, s]$.*

Let us continue the proof of the proposition assuming the lemma.

$$\begin{aligned}\dim(H_k/F_k) &= \dim(\mathrm{sp}[\xi_{k+1}^{(j)}|_{H_k} : 1 \le j \le s]) \\ &= \dim(\mathrm{sp}[w_{k+1}^{(j)}|_{H_k} : 1 \le j \le s]) \\ &= \dim(W_{k+1})\end{aligned}$$

since $w_{k+1}^{(j)} \in H_{k+1} = G_k^\perp$, and $E_k = G_k \oplus H_k$.

Notice now that $\mathscr{A}(H_k) = H_k$ if and only if $H_k = F_k$. Indeed $\mathscr{A}(H_k) = H_k$ implies $H_k \subset E_{k-2n}$ for every $n \ge 0$, and hence $H_k = F_k$ (since $F_k \subset H_k$). The converse implication is trivial. Then by the above equality $\mathscr{A}(H_k) = H_k$ if and only if $W_{k+1} = \{0\}$.

PROOF OF LEMMA 3.2. (i) We will use the notation $\mu_{k,n}(x)$ introduced in the proof of Proposition 2.10.

We have seen that every $h \in H_k$ satisfies $h = \mu_{k,0}(h)\tilde{\xi}_{k,n} + \pi_{k-2n} \circ \cdots \circ \pi_{k-2} \circ \pi_k(h)$ for every $n \ge 0$, and that $\mu_{k,0}(h) = \langle h, {}^t\xi_{k+1}\rangle^t M_k^{-1}$.

Then such an $h \in H_k$ is in F_k if and only if $\mu_{k,0}(h) = 0$ (since the vectors $\tilde{\xi}_{k,n}$ are linearly independent) which is equivalent to $\langle h, {}^t\xi_{k+1}\rangle = 0$, i.e.: $h \in \bigcap_{j=1}^s \ker(\xi_{k+1}^{(j)})$.

This proves (i) since $F_k \subset H_k$.

(ii) We have seen before that $G_{k-1} \subset H_k^\perp$. The conclusion of the lemma holds since $H_k^\perp$ is ω_{k+1}^*-closed, and since

$$w_{k+1}^{(j)} - \xi_{k+1}^{(j)} = \omega_{k+1}^*\text{-}\lim_{n\in\mathscr{U}}(N_{k-1}\tilde{\xi}_{k-1,n})^{(j)}.$$

This concludes the proof of Lemma 3.2, and of Proposition 3.1. □

We conclude this section with the following result, which is in the same spirit of ideas as Lemma 3.2:

LEMMA 3.3. *For every $k \in \mathbf{Z}$ we have $W_{k+1} = (G_k \oplus F_k)^\perp$, where the polar is taken in E_{k+1}.*

PROOF. Suppose that we have proved that $W_{k+1} \subset (F_k \oplus G_k)^\perp$, and let us prove that the equality holds.

Indeed since $F_k \subset H_k$ and $(G_k \oplus F_k)^\perp = (E_k/G_k \oplus F_k)^*$ we have

$$\dim(F_k \oplus G_k)^\perp = \dim(G_k \oplus H_k/G_k \oplus F_k) = \dim(H_k/F_k) = \dim(W_{k+1}).$$

To prove that $W_{k+1} \subset (F_k \oplus G_k)^\perp$ it suffices to show that $w_{k+1}^{(j)} \in (F_k \oplus G_k)^\perp$ for every $j \in [1, s]$.

We have seen before that $w_{k+1}^{(j)} \in H_{k+1} = G_k^\perp$. On the other hand $w_{k+1}^{(j)} \in F_k^\perp$ since $\xi_{k+1}^{(j)} \in F_k^\perp$ (Lemma 3.2(i)) and $(w_{k+1}^{(j)} - \xi_{k+1}^{(j)}) \in H_k^\perp \subset F_k^\perp$.

This concludes the proof of Lemma 3.3. □

4. Structure of the onto isometries. Theorem 4.1 will describe the structure of the onto isometries on E. Its main application is Theorem 4.5 which asserts that the "space parameters" corresponding to E are intrinsic.

Let $I\colon E \to E$ be an onto isometry of E, and let $I^{(\omega)}$ and $I^{-1*(\omega)}$ be the onto isometries induced by I and I^{-1*} on $E^{(\omega)}$ and $E^{*(\omega)}$ respectively.

We will denote by $\mathscr{I}$ both of these two isometries. Observe that we have $\mathscr{I}^* = \mathscr{I}^{-1}$ (in the sense that $(I^{(\omega)})^*|_{E^{*(\omega)}} = (I^{-1*(\omega)})^{-1}$ and $(\overline{I}^{1(\omega)*})^*|_{E^{(\omega)}} = (I^{(\omega)})^{-1}$) and $(\mathscr{I}^{-1})^* = \mathscr{I}$. (Notice that $\mathscr{I}^{-1}$ coincides with the "isometry" which corresponds to I^{-1}.)

THEOREM 4.1. *With our notations, for every $k \in \mathbf{Z}$, and every $n \geq 0$, we have:*

(i) *$\mathscr{I}$ respects the spaces $E_k, G_k, H_k, X_k, \tilde{X}_{k,n}, W_k, F$ and F_*, where the expression "$\mathscr{I}$ respects Z "means" $\mathscr{I}(Z) = Z$".*

(ii) *If Λ_k is the (s,s)-matrix such that $\mathscr{I}(\xi_k) = \Lambda_k \xi_k$ then*

(a) $\Lambda_{k+1} M_k {}^t\Lambda_k = M_k$,

(b) $\Lambda_{k+2} = N_k \Lambda_k N_k^{-1}$ *and* $\Lambda_{k+2} = \sigma_k^{-1} \Lambda_k \sigma_k$,

(c) $\mathscr{I}(\tilde{\xi}_{k,n}) = \Lambda_k \tilde{\xi}_{k,n}$ *and* $\mathscr{I}(w_k) = \Lambda_k w_k$.

REMARK. The relation (ii)(a) implies $(\det \Lambda_{k+1}) \cdot (\det \Lambda_k) = 1$, hence all the matrices Λ_k are invertible. This is not surprising since the Λ_k's are the matrices associated to the isometries $\mathscr{I}|_{X_k}$.

The key lemma in the proof of Theorem 3.1 is the following result [G].

LEMMA 4.2. *Let X be a Banach space such that $X^* \not\supset l^1$. Then every onto isometry of X^* is ω^*-ω^* continuous.*

Applying the above result twice, we see that for every onto isometry I of E, there exists an onto isometry J of E such that $J^{**} = AIA^{-1}$. If we denote by $\mathscr{J}$ the "isometry" corresponding to J, the above equation implies $\mathscr{J} = \mathscr{A}\mathscr{I}\mathscr{A}^{-1}$ and $\mathscr{I} = \mathscr{A}^{-1}\mathscr{J}\mathscr{A}$ (in the sense that $J^{(\omega)} = A^{(\omega)}I^{(\omega)}(A^{(\omega)})^{-1}$, etc.).

We will also need the following elementary result:

LEMMA 4.3. *If $T\colon X \to Y$ is an onto isomorphism, and if Z is a subspace of Y, then $T^*(Z^\perp) = [T^{-1}(Z)]^\perp$.*

PROOF OF THEOREM 4.1.

Proof of $\mathscr{I}(E_k) = E_k$. We will prove this by induction on k.

It is clear that $\mathscr{I}(E_k) = E_k$ for every $k \geq 0$. ($E_k = E^{(k)}$: the usual dual of order k.)

Suppose now that we have proved that for some k, $\mathscr{I}(E_k) = E_k$ for every onto isometry I of E.

Then $\mathscr{I}(E_{k-2}) = \mathscr{A}^{-1}\mathscr{J}\mathscr{A}(E_{k-2}) = \mathscr{A}^{-1}\mathscr{J}(E_k) = \mathscr{A}^{-1}(E_k) = E_{k-2}$.

Proof of $\mathscr{I}(F) = F$ and $\mathscr{I}(F_*) = F_*$.

$$\mathscr{I}(F) = \mathscr{I}\left(\bigcap_{k\in\mathbf{Z}} E_{2k}\right) = \bigcap_{k\in\mathbf{Z}} \mathscr{I}(E_{2k}) = F.$$

The argument is similar for F^*.

Proof of $\mathscr{I}(X_k) = X_k$ and $\mathscr{I}(G_k) = G_k$. By Lemma 2.4 we have $X_k = E_{k-3}^{\perp_{E_k}}$, then by applying Lemma 4.3 to the isometry $\mathscr{I}^{-1}\colon E_{k-1} \to E_{k-1}$, we obtain

$$\mathscr{I}(X_k) = (\mathscr{I}^{-1})^*(E_{k-3}^{\perp_{E_k}}) = [\mathscr{I}(E_{k-3})]^{\perp_{E_k}} = X_k.$$

Now since $G_k = \overline{\mathrm{sp}}(X_{k-2n})$ we deduce that $\mathscr{I}(G_k) = G_k$.

Proof of $\Lambda_{k+1}M_k{}^t\Lambda_k = M_k$ and $\Lambda_{k+2} = N_k\Lambda_kN_k^{-1}$.

$$\begin{aligned} M_k &= \langle \xi_{k+1}, {}^t\xi_k\rangle = \langle \mathscr{I}^*\mathscr{I}(\xi_{k+1}), {}^t\xi_k\rangle \\ &= \langle \mathscr{I}(\xi_{k+1}), \mathscr{I}({}^t\xi_k)\rangle = \langle \Lambda_{k+1}\xi_{k+1}, {}^t(\Lambda_k\xi_k)\rangle \\ &= \Lambda_{k+1}M_k{}^t\Lambda_k. \end{aligned}$$

Applying the above relation twice we obtain

$$\Lambda_{k+2} = M_{k+1}{}^t\Lambda_{k+1}^{-1}M_{k+1}^{-1} = M_{k+1}{}^t(M_k{}^t\Lambda_k^{-1}M_k^{-1})^{-1}M_{k+1}^{-1} = N_k\Lambda_kN_k^{-1}.$$

Proof of $\mathscr{I}(\tilde{X}_{k,n}) = \tilde{X}_{k,n}$ and $\mathscr{I}(\tilde{\xi}_{k,n}) = \Lambda_k\tilde{\xi}_{k,n}$. An easy inductive argument shows that $\Lambda_{k-2n} = N_k^{-n}\Lambda_kN_k^n$ for every $n \geq 0$.

Then

$$\begin{aligned} \mathscr{I}(\tilde{\xi}_{k,n}) &= \mathscr{I}\left(\sum_{p=0}^{n} N_k^p\xi_{k-2p}\right) = \sum_{p=0}^{n} N_k^p\mathscr{I}(\xi_{k-2p}) \\ &= \sum_{p=0}^{n} N_k^p\Lambda_{k-2p}\xi_{k-2p} = \Lambda_k\tilde{\xi}_{k,n}. \end{aligned}$$

This relation implies in particular that $\mathscr{I}(\tilde{X}_{k,n}) = \tilde{X}_{k,n}$.

Proof of $\mathscr{I}(H_k) = H_k$.

$$\mathscr{I}(H_k) = \mathscr{I}\left[\bigcap_{n\geq 0}(E_{k-2n-2}\oplus \tilde{X}_{k,n})\right] = \bigcap_{n\geq 0}\mathscr{I}(E_{k-2n-2}\oplus \tilde{X}_{k,n})$$
$$= \bigcap_{n\geq 0}[\mathscr{I}(E_{k-2n-2})\oplus \mathscr{I}(\tilde{X}_{k,n})] = H_k.$$

Proof of $\mathscr{I}(W_k) = W_k$ *and* $\mathscr{I}(w_k) = \Lambda_k w_k$. We will use the fact that the operator $\mathscr{I}: E_k \to E_k$ is ω_k^*-ω_k^* continuous (Lemma 4.2).

Then

$$\mathscr{I}(w_k) = \mathscr{I}[\omega_k^*\text{-}\lim_{n\in\mathscr{U}}\tilde{\xi}_{k,n}] = \omega_k^*\text{-}\lim_{n\in\mathscr{U}}\mathscr{I}(\tilde{\xi}_{k,n}) = \Lambda_k w_k.$$

In particular we have $\mathscr{I}(W_k) = W_k$.

Proof of $\Lambda_{k+2} = \sigma_k^{-1}\Lambda_k\sigma_k$. By an argument similar to above we obtain

$$\sigma_k\Lambda_{k+2}\xi_{k+2} + \Lambda_k w_k = \mathscr{I}(\sigma_k\xi_{k+2} + w_k) = \Lambda_k(\sigma_k\xi_{k+2} + w_k)$$

and hence $\sigma_k\Lambda_{k+2}\xi_{k+2} = \Lambda_k\sigma_k\xi_{k+2}$.

Since the vectors $(\xi_{k+2}^{(j)})_{1\leq j\leq s}$ are linearly independent we deduce that $\sigma_k\Lambda_{k+2} = \Lambda_k\sigma_k$.

This concludes the proof of Theorem 4.1. □

An immediate consequence of Theorem 4.1 is

COROLLARY 4.4. *Let* E *be such that* $\dim E^{**}/E = 1$, *and* $E = G_0$. *Then every onto isometry of* E *is trivial* (*i.e.* $I = \varepsilon\,\mathrm{Id}_E$, $|\varepsilon| = 1$).

PROOF. All the matrices Λ_k are reduced to scalars, which then must satisfy $|\Lambda_k| = 1$. On the other hand by Theorem 4.1 we have $\Lambda_{2k} = \Lambda_0$ for every $k \in \mathbf{Z}$, and since $E = G_0$, this implies that $I = \Lambda_0 \cdot \mathrm{Id}_E$. □

REMARK. We will see later that the James' space J satisfies the assumptions of the corollary. Hence it has only trivial onto isometries. This is also the case for its dual space J^*.

REMARK. We will see later (Theorem 5.1) that we can assume that $N_0 = N_1 = \mathbf{1}_s$. Hence, if the space E is such that $E = G_0$, every onto isometry I on E satisfies $I(\sum_0^\infty \alpha_n\xi_{-2n}) = \sum_0^\infty \alpha_n\Lambda\xi_{-2n}$ for some square matrix Λ.

As an application of Theorem 4.1 we are going to show that all the spaces defined before are intrinsic, in the sense that they not depend neither on the choice of the isometry A, nor on the choice of the bases $(\xi_3^{(j)})_{1\leq j\leq s}$ and $(\xi_4^{(j)})_{1\leq j\leq s}$ of $E^\perp$ and $E^{*\perp}$ respectively.

THEOREM 4.5. *The spaces* $E_k, G_k, H_k, X_k, \tilde{X}_{k,n}, W_k, F$ *and* F_* *are all intrinsic.*

PROOF. Assume that we have proved that the spaces E_k are intrinsic. Then all the others are also intrinsic. Indeed:

The H_k's, F, F_* are intrinsic by their definitions.

The X_k's are intrinsic since $X_k = E_{k-3}^{\perp_{E_k}}$; hence this is also the case for the G_k's.

The W_k's are intrinsic since $W_k = (G_{k-1}\oplus F_{k-1})^{\perp_{E_k}}$.

The $\tilde{X}_{k,n}$'s are intrinsic since $\tilde{X}_{k,n} = (\mathrm{Id}_{E_k} - \pi_{k-2n}\circ\cdots\circ\pi_{k-2}\circ\pi_k)\ (E_{k-2n-2}^{\perp\perp_{E_k}})$ and since the projections π_k are intrinsic.

Let us prove now that the spaces E_k are intrinsic.

It is clear (by their definitions) that the spaces E_k depend at most on the isometry A.

Let A and B be two isometries, and let us denote by $E_k^{(A)}$ and $E_k^{(B)}$ the spaces corresponding to A and B respectively.

It is clear that $E_k^{(A)} = E_k^{(B)}$ for every $k \geq 0$.

Assume now that we have proved that $E_k^{(A)} = E_k^{(B)}$ for some k. Then by applying Theorem 4.1 to the isometry $I = B^{-1}A$ we obtain $E_{k-2}^{(B)} = \mathscr{B}^{-1}(E_k^{(B)}) = \mathscr{B}^{-1}(E_k^{(A)}) = \mathscr{B}^{-1}\mathscr{A}(E_{k-2}^{(A)}) = E_{k-2}^{(A)}$, since it is easy to check that $\mathscr{I} = \mathscr{B}^{-1}\mathscr{A}$.

This concludes the proof of Theorem 4.5.

5. Reduction of the parameters and classification. The following theorem is our second main result. It will allow us to "classify" the spaces E considered in this paper.

THEOREM 5.1. *It is possible to choose the isometry $A: E \to E^{**}$ and the bases $(\xi_3^{(j)})_{1\leq j\leq s}$, and $(\xi_4^{(j)})_{1\leq j \leq s}$ in such a way that the corresponding parameters satisfy:*

(i) $M_k = (-1)^k \mathbf{1}_s$, *and hence* $N_k = \mathbf{1}_s$, *for every* $k \in \mathbf{Z}$.

(ii) $\sigma_0 = \overline{\sigma} \oplus \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)^{\otimes c}$ *in the complex case, and*

$$\sigma_0 = \overline{\sigma} \oplus \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}^{\otimes c} \oplus \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\varphi_d & -\sin\varphi_d \\ \sin\varphi_d & \cos\varphi_d \end{pmatrix}$$

in the real case, where $\overline{\sigma}$ is a diagonal unitary matrix, and $\varphi_i \not\equiv 0$ $[\pi]$ for $1 \leq i \leq d$. Moreover $\sigma_1 = {}^t\sigma_0^{-1}$.

(iii) $\dim W_k = \operatorname{Card}\{j \in [1,s] : w_k^{(j)} \neq 0\}$, *for every* $k \in \mathbf{Z}$.

Let us first make precise the meaning of the notations used in the theorem.

For every natural number n, $\mathbf{1}_n$ denotes the (n,n)-identity matrix.

If R and T are two square matrices, $R \oplus T$ denotes the square matrix $\left(\begin{smallmatrix} R & 0 \\ 0 & T \end{smallmatrix}\right)$, and $R^{\otimes n}$ denotes the matrix $R \oplus \cdots \oplus R$ n-times.

REMARK. In the case when $N_0 = N_1 = \mathbf{1}_s$, it is quite natural to consider the spaces $G_k^{(j)} = \overline{\text{sp}}[\xi_{k-2n}^{(j)} : n \geq 0]$. Unfortunately we were not able to decide whether or not we have $G_k = \bigoplus_{j=1}^s G_k^{(j)}$. The interest of the spaces $G_k^{(j)}$ comes from the fact that they satisfy $G_k^{(j)**} \approx G_k^{(j)}$, and $\dim(G_k^{(j)**}/G_k^{(j)}) = 1$. Indeed, by Theorem 2.1(iv)–(v) and Lemma 2.6, the sequence $(\xi_{k-2n}^{(j)})_{n\geq 0}$ is a bimonotone, neighborly, and shrinking basis of $G_k^{(j)}$ and its summing sequence $(\sum_{p=0}^n \xi_{k-2p}^{(j)})_{n\geq 0}$ is a boundedly complete basis of $G_k^{(j)}$; hence by [**B**] we have

$$G_k^{(j)**} = G_k^{(j)} \oplus \mathbf{K} \cdot (w_k + \sigma_k \xi_{k+2})^{(j)} \approx G_k^{(j)} \oplus \mathbf{K} \cdot \xi_{k+2}^{(j)} \equiv G_k^{(j)}.$$

Proofs of (ii) and (iii) are purely algebraic. They will make use of the following *norm-one* operators $\hat{\pi}_k : E_k \to E_k$ defined by

$$\hat{\pi}_k(x) = \omega_k^*\text{-}\lim_{n\in\mathscr{U}} \pi_{k-2n} \circ \cdots \circ \pi_{k-2} \circ \pi_k(x).$$

We will use the following properties of the operators $\hat{\pi}_k$:

LEMMA 5.2. *For every* $k \in \mathbf{Z}$, *we have*
(i) $\hat{\pi}_k(w_k) = -\sigma_k N_k w_k$,
(ii) $\hat{\pi}_k(x) = P_k(x) - \langle x, {}^t w_{k+1}\rangle {}^t M_k^{-1} w_k$ *for every* $x \in E_k$.

PROOF. (i) By Lemma 2.9(iv) we have

$$\hat{\pi}_k(w_k) = \omega_k^*\text{-}\lim_{n\in\mathscr{U}}[w_k - (\mathbf{1}_\beta + \sigma_k N_k)\tilde{\xi}_{k,n}] = -\sigma_k N_k w_k.$$

(ii) We have seen before that for every $h \in H_k$ we have $h = \langle h, {}^t\xi_{k+1}\rangle {}^t M_k^{-1}\tilde{\xi}_{k,n} + \pi_{k-2n}\circ\cdots\circ\pi_{k-2}\circ\pi_k(h)$. Then by taking ω_k^*-limits we obtain $h = \langle h, {}^t\xi_{k+1}\rangle {}^t M_k^{-1} w_k + \hat{\pi}_k(h)$.

Since $\hat{\pi}_k(G_k) = \{0\}$, and by 3.2(ii) and 3.3, the above equality is equivalent to $\hat{\pi}_k(x) = P_k(x) - \langle x, {}^t w_{k+1}\rangle {}^t M_k^{-1} w_k$ for every $x \in E_k$. □

Another ingredient in the proof of Theorem 5.1 is the change of bases formulas (with respect to a fixed isometry $A\colon E \to E^{**}$).

Let C and C_* be two invertible (s,s)-matrices, and consider the new bases ξ_3' and ξ_4' defined by $\xi_3' = C_*\xi_3$ and $\xi_4' = C\xi_4$.

It is easy to check that the parameters corresponding to ξ_3' and ξ_4' satisfy

$$(*)\qquad \begin{cases} \text{(i) } \xi_k' = C_k\xi_k, & \text{(ii) } M_k' = C_{k+1}M_k{}^tC_k, \\ \text{(iii) } N_k' = C_kN_kC_k^{-1}, & \text{(iv) } \tilde{\xi}_{k,n}' = C_k\tilde{\xi}_{k,n}, \\ \text{(v) } w_k' = C_kw_k, & \text{(vi) } \sigma_k' = C_k\sigma_kC_k^{-1}, \end{cases}$$

where we have put $C_{2k} = C$ and $C_{2k+1} = C_*$ for every $k \in \mathbf{Z}$.

PROOF OF THEOREM 5.1.

Proof of (i). The proof will be divided in two steps. In the first one we will prove the existence of an isometry $A\colon E \to E^{**}$ for which $N_0 = N_1 = \mathbf{1}_s$. In the second step we will show that by a suitable change of bases we may assume that the above-mentioned isometry satisfies $M_0 = -M_1 = \mathbf{1}_s$.

Step 1. Assume that we have proved that there exists an isometry $A\colon E \to E^{**}$ for which $N_0 = N_1 = \mathbf{1}_s$.

If we take $C = \mathbf{1}_s$ and $C_* = M_0^{-1}$, the change of bases formulas $(*)$ imply that $M_0' = -M_1' = \mathbf{1}_s$.

Step 2. Let $A\colon E \to E^{**}$ be a given onto isometry between E and E^{**}. We are going to construct a new isometry $B\colon E \to E^{**}$ for which $N_0^{(B)} = N_1^{(B)} = \mathbf{1}_s$.

For every $x \in E$, define

$$I(x) = \omega_0^*\text{-}\lim_{n\in\mathscr{U}}\left[\sum_{p=0}^{n}\mu_{0,p}(x)N_0^{-1}\xi_{-2p} + \mathscr{A}^{-1}\left(x - \sum_{p=0}^{n-1}\mu_{0,p}(x)\xi_{-2p}\right)\right],$$

$$J(x) = \omega_0^*\text{-}\lim_{n\in\mathscr{U}}\left[\sum_{p=0}^{n-1}\mu_{0,p}(x)N_0\xi_{-2p} - \mathscr{A}\left(x - \sum_{p=0}^{n}\mu_{0,p}(x)\xi_{-2p}\right)\right],$$

where the vectors ξ_k denote those corresponding to the isometry A.

Let us show first that these definitions make sense. Indeed, using Lemma 2.6 we obtain

$$\begin{aligned}\|x\| &= \left\|\sum_{p=0}^{n} \mu_{0,p}(x)\xi_{-2p} + \pi_{-2n}\circ\cdots\circ\pi_{-2}\circ\pi_0(x)\right\| \\ &\geq \left\|\sum_{p=0}^{n} \mu_{0,p}(x)N_0^{-1}\xi_{-2p+2} + \mu_{0,n}(x)\xi_{-2n} + \pi_{-2n}\circ\cdots\circ\pi_{-2}\circ\pi_0(x)\right\| \\ &= \left\|\sum_{p=0}^{n} \mu_{0,p}(x)N_0^{-1}\xi_{-2p+2} + \left(x - \sum_{p=0}^{n-1}\mu_{0,p}(x)\xi_{-2p}\right)\right\| \\ &= \left\|\sum_{p=0}^{n} \mu_{0,p}(x)N_0^{-1}\xi_{-2p} + \mathscr{A}^{-1}\left(x - \sum_{p=0}^{n-1}\mu_{0,p}\xi_{-2p}\right)\right\|.\end{aligned}$$

This shows that I is well defined, and that $\|I\| \leq 1$.

Similarly, using Lemma 2.6 again and Lemma 2.2 (for the last inequality) we obtain

$$\begin{aligned}\|x\| &= \left\|\sum_{p=0}^{n} \mu_{0,p}(x)\xi_{-2p} + \pi_{-2n}\circ\cdots\circ\pi_{-2}\circ\pi_0(x)\right\| \\ &\geq \left\|\mu_{0,0}(x)\xi_0 + \sum_{p=0}^{n-1} \mu_{0,p}(x)N_0\xi_{-2p-2} + \pi_{-2n}\circ\cdots\circ\pi_{-2}\circ\pi_0(x)\right\| \\ &= \left\|\mu_{0,0}(x)\xi_2 + \sum_{p=0}^{n-1} \mu_{0,p}(x)N_0\xi_{-2p} + \mathscr{A}\left(x - \sum_{p=0}^{n}\mu_{0,p}(x)\xi_{-2p}\right)\right\| \\ &\geq \left\|\sum_{p=0}^{n-1} \mu_{0,p}(x)N_0\xi_{-2p} + \mathscr{A}\left(x - \sum_{p=0}^{n}\mu_{0,p}(x)\xi_{-2p}\right)\right\|.\end{aligned}$$

Hence J is well defined and satisfies $\|J\| \leq 1$.

Claim. $JI = IJ = \mathrm{Id}_E$, hence both I and J are isometries of E, and are inverse to each other.

Assume that we have proved the claim, and prove that the isometry $B = AI\colon E \to E^{**}$ satisfies $N_0^{(B)} = N_1^{(B)} = \mathbf{1}_s$.

Recall that the vectors $\xi_k^{(B)}$ are constructed starting from fixed bases ξ_3 and ξ_4, i.e.: $\xi_3^{(B)} = \xi_3$, and $\xi_4^{(B)} = \xi_4$. Now applying Theorem 4.1 to the isometry J we obtain

$$\begin{aligned}\xi_2^{(B)} &= \mathscr{B}^{-1}(\xi_4) = \mathscr{J}\mathscr{A}^{-1}(\xi_4) = \Lambda_2^{(J)}\xi_2 \\ &= N_0\Lambda_0^{(J)}N_0^{-1}\xi_2 = N_0\xi_2\end{aligned}$$

since it is clear that N_0 is equal to the matrix of $\mathscr{J}|_{X_0}$.

Then

$$\begin{aligned}&M_0^{(B)} = \langle\xi_3^{(B)}, {}^t\xi_2^{(B)}\rangle = M_0{}^tN_0, \quad M_1^{(B)} = \langle\xi_4^{(B)}, {}^t\xi_3^{(B)}\rangle = M_1, \\ &N_0^{(B)} = -M_1^{(B)t}(M_0^{(B)})^{-1} = \mathbf{1}_s, \quad N_1^{(B)} = -M_0^{(B)t}(M_1^{(B)})^{-1} = \mathbf{1}_s.\end{aligned}$$

PROOF OF THE CLAIM. It is clear that for every $g \in G_0$ we have: $I(g) = \sum_{n=0}^{\infty} \mu_{0,n}(g) N_0^{-1} \xi_{-2n}$ and $J(g) = \sum_{n=0}^{\infty} \mu_{0,n}(g) N_0 \xi_{-2n}$ with norm convergence for the series. Then both I and J respect G_0 and satisfy $JI|_{G_0} = IJ_{|G_0} = \mathrm{Id}_{G_0}$.

Let us prove now that I and J also respect H_0.

We have seen before that every $h \in H_0$ satisfies $h = \mu_{0,0}(h)\tilde{\xi}_{0,n} + \pi_{-2n} \circ \cdots \circ \pi_{-2} \circ \pi_0(h)$ for every $n \geq 0$. Then $I(h)$ is given by

$$I(h) = \omega_0^*\text{-} \lim_{n \in \mathscr{U}} [\mu_{0,0}(h) N_0^{-1} \tilde{\xi}_{0,n} + \mathscr{A}^{-1}(h - \mu_{0,0}(h)\tilde{\xi}_{0,n-1})],$$

and a similar argument as in 2.9(i) gives $I(h) \in H_0$. We prove similarly that $J(H_0) \subset H_0$.

To prove the claim it is now enough to prove that $JI|_{H_0} = IJ_{|H_0} = \mathrm{Id}_{H_0}$, since $E = G_0 \oplus H_0$. To do this we need to compute, for every $h \in H_0$, the expressions of $I(h)$ and $J(h)$.

$$\begin{aligned} (1) \qquad I(h) &= \omega_0^*\text{-} \lim_{n \in \mathscr{U}} [\mu_{0,0}(h) N_0^{-1} \tilde{\xi}_{0,n} + \mathscr{A}^{-1}(h - \mu_{0,0}(h)\tilde{\xi}_{0,n-1})] \\ &= \omega_0^*\text{-} \lim_{n \in \mathscr{U}} [\mu_{0,0}(h) N_0^{-1} \xi_0 + \mathscr{A}^{-1}(h)] \\ &= \mu_{0,0}(h) N_0^{-1} \xi_0 + \mathscr{A}^{-1}(h). \end{aligned}$$

$$\begin{aligned} (2) \qquad J(h) &= \omega_0^*\text{-} \lim_{n \in \mathscr{U}} [\mu_{0,0}(h) N_0 \tilde{\xi}_{0,n-1} + \mathscr{A}(h - \mu_{0,0}(h)\tilde{\xi}_{0,n})] \\ &= \omega_0^*\text{-} \lim_{n \in \mathscr{U}} [-\mu_{0,0}(h)\xi_2 + \mathscr{A}(h)] \\ &= \pi_2[\mathscr{A}(h) - \mu_{0,0}(h)\xi_2] = \pi_2 \circ \mathscr{A}(h). \end{aligned}$$

Hence $JI(h) = \pi_2 \circ \mathscr{A}[\mu_{0,0}(h) N_0^{-1} \xi_0 + \mathscr{A}^{-1}(h)] = h$, and one can also check that $IJ(h) = h$.

This concludes the proof of the claim and of part (i) of Theorem 5.1.

From now on, we will always suppose that the isometry $A\colon E \to E^{**}$, and the bases ξ_3 and ξ_4 are chosen in such a way that $M_k = (-1)^k \mathbf{1}_s$ for every $k \in \mathbf{Z}$.

To preserve this property, all the forthcoming change of bases will be done only with matrices C and C_* which satisfy $C_* {}^tC = \mathbf{1}_s$.

PROOF OF 5.1(ii).

1*st case*: *the complex case.* Without loss of generality (by doing a suitable change of bases if necessary) we can assume that σ_0 is a Jordan matrix, i.e.: $\sigma_0 = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r$ where each ρ_j, $1 \leq j \leq r$, is a square matrix of the form

$$\rho_j = \lambda_j \begin{pmatrix} 1 & & & \multicolumn{2}{c}{0} \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 1 \end{pmatrix}.$$

[We use the fact σ_0 is invertible, i.e. $\lambda_j \neq 0$, to obtain this special form for the matrices ρ_j.] Hence $\sigma_1 = {}^t\sigma_0^{-1}$ (Lemma 2.9(iii) satisfies $\sigma_1 = {}^t\rho_1^{-1} \oplus {}^t\rho_2^{-1} \oplus \cdots \oplus {}^t\rho_r^{-1}$).

We are now going to prove that for every j, $1 \leq j \leq r$, we have $|\lambda_j| = 1$ and $\mathrm{rank}(\rho_j) \leq 2$. This will be proved only by using the fact that the sequences

$(\hat{\pi}_k^n(w_k^{(j)}))_{n\geq 0}$ are bounded for every $k \in \mathbf{Z}$, $1 \leq j \leq s$, and the formulas $\hat{\pi}_k(w_k) = -\sigma_k w_k$, and $\langle w_k, {}^t\xi_{k+1}\rangle = (-1)^k(\mathbf{1}_s + \sigma_k)$.

Since all these formulas are "decomposable" it suffices to consider the case when

$$\sigma_0 = \lambda \begin{pmatrix} 1 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{pmatrix}.$$

The general case follows by an obvious "block" argument.

Now we want to show that in this particular case we have $|\lambda| = 1$, σ_0 is of rank at most equal to 2, and that the equality holds only if $\lambda = -1$.

(1) *Case when rank* $(\sigma_0) = 1$. In this case we have $\hat{\pi}_0^n(w_0^{(1)}) = (-\lambda)^n w_0^{(1)}$ and $\hat{\pi}_1^n(w_1^{(1)}) = (-\lambda^{-1})^n w_1^{(1)}$.

Since the sequences $(\hat{\pi}_0^n(w_0^{(1)}))_{n\geq 0}$ and $(\hat{\pi}_1^n(w_1^{(1)}))_{n\geq 0}$ are bounded, we have

$$(|\lambda| \leq 1 \text{ or } w_0^{(1)} = 0) \quad \text{and} \quad (|\lambda| \geq 1 \text{ or } w_1^{(1)} = 0).$$

So if $w_0^{(1)} \neq 0$ and $w_1^{(1)} \neq 0$, we have $|\lambda| = 1$.

On the other hand we have $\lambda = -1$ if $w_0^{(1)} = 0$ or $w_1^{(1)} = 0$. Indeed:

$$w_0^{(1)} = 0 \text{ implies } 0 = \langle w_0^{(1)}, \xi_1^{(1)}\rangle = 1 + \lambda, \quad \text{and}$$
$$w_1^{(1)} = 0 \text{ implies } 0 = \langle w_1^{(1)}, \xi_2^{(1)}\rangle = -(1 + \lambda^{-1}).$$

This shows that we have always $|\lambda| = 1$.

(2) *Case when rank* $(\sigma_0) = s \geq 2$. Easy computations show that we have

$$\hat{\pi}_0^n(w_0^{(1)}) = (-\lambda)^n w_0^{(1)}, \quad \hat{\pi}_1^n(w_1^{(s)}) = (-\lambda^{-1})^n w_1^{(s)},$$
$$\hat{\pi}_0^n(w_0^{(2)}) = (-\lambda)^n [n w_0^{(1)} + w_0^{(2)}],$$

and

$$\hat{\pi}_1^n(w_1^{(s-1)}) = (-\lambda^{-1})^n [w_1^{(s-1)} - n w_0^{(s)}].$$

An argument similar to the above shows that $|\lambda| = 1$. And using the boundedness of $(\hat{\pi}_0^n(w_0^{(2)}))_{n\geq 0}$ we deduce that $w_0^{(1)} = 0$, and hence $\lambda = -1$, since $\langle w_0^{(1)}, \xi_1^{(1)}\rangle = 1 + \lambda$.

We have also $w_1^{(s)} = 0$ by the boundedness of $(\hat{\pi}_1^n(w_1^{(s-1)}))_{n\geq 0}$.

If we assume that rank $(\sigma_0) = s \geq 3$, an easy computation leads to $\hat{\pi}_0^n(w_0^{(3)}) = n w_0^{(2)} + w_0^{(3)}$ (we use that $w_0^{(1)} = 0$, and $\lambda = -1$); this implies that $w_0^{(2)} = 0$, which is a contradiction since $\langle w_0^{(2)}, \xi_1^{(1)}\rangle = (\mathbf{1}_s + \sigma_0)_{(2,1)} = -1$.

Let us recapitulate what we have proved. If

$$\sigma_0 = \lambda \begin{pmatrix} 1 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{pmatrix},$$

then only the following cases hold:

(1) $\operatorname{rank}(\sigma_0) = 1$ and $|\lambda| = 1$.

(2) $\operatorname{rank}(\sigma_0) = 2$ and $\lambda = -1$. In this case we have $w_0^{(1)} = 0$, $w_0^{(2)} \neq 0$, $w_1^{(1)} \neq 0$, and $w_1^{(2)} = 0$.

This proves 5.1(ii) in the complex case.

2nd case: the real case. The arguments will be similar to the ones used in the complex case.

By a suitable bases change we can assume that $\sigma_0 = \rho_1 \oplus \cdots \oplus \rho_r \oplus \tau_1 \oplus \cdots \oplus \tau_t$, where each matrix ρ_i, $1 \leq i \leq r$, is of the form

$$\lambda_i \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 1 & & \ddots & & \\ & \ddots & & \ddots & \\ 0 & & & 1 & 1 \end{pmatrix},$$

and each matrix τ_i, $1 \leq i \leq t$ is of the form

$$\mu_i \begin{pmatrix} R(\varphi_i) & & & & 0 \\ R(\varphi_i) & \ddots & & & \\ & \ddots & \ddots & & \\ 0 & & & R(\varphi_i) & R(\varphi_i) \end{pmatrix},$$

where

$$R(\varphi_i) = \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{pmatrix},$$

and μ_i is a real scalar. The angles φ_i must satisfy $\varphi_i \not\equiv 0\ [\pi]$, since the τ_i's are the Jordan matrices corresponding to the *pure complex* eigenvalues of the matrix σ_0.

As in the complex case it suffices to consider the case when σ_0 is a "ρ-matrix", which was done before, and the case when σ_0 is a "τ-matrix", which is the remaining case.

We are going to prove that if σ_0 is a "τ-matrix", then we have necessarily $\sigma_0 = R(\varphi)$ for some φ (satisfying $\varphi \not\equiv 0\ [\pi]$).

(1) *Case when rank* $(\sigma_0) = 2$. In this case we have

$$R(-n\varphi)\hat{\pi}_0^n \begin{pmatrix} w_0^{(1)} \\ w_0^{(2)} \end{pmatrix} = (-\mu)^n \begin{pmatrix} w_0^{(1)} \\ w_0^{(2)} \end{pmatrix}$$

and

$$R(n\varphi)\hat{\pi}_1^n \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \end{pmatrix} = (-\mu^{-1})^n \begin{pmatrix} w_1^{(1)} \\ w_1^{(2)} \end{pmatrix}.$$

This implies that we have ($|\mu| \leq 1$ or $w_0^{(1)} = w_0^{(2)} = 0$) and ($|\mu| \geq 1$ or $w_1^{(1)} = w_1^{(2)} = 0$).

But we cannot have $w_0^{(1)} = 0$, since if it were the case we would have $0 = \langle w_0^{(1)}, \xi_1^{(2)} \rangle = -\mu \sin\varphi$, and this equality implies $\varphi \equiv 0\ [\pi]$ (since $\mu \neq 0$) which was excluded. Similarly we cannot have $w_1^{(1)} = 0$.

So we have $|\mu| = 1$, and hence $\sigma_0 = R(\varphi)$ for some φ (since $-R(\varphi) = R(\varphi + \pi)$).

Observe that in this case we have $\dim W_0 = 2$, since if not, we would have $0 = \det(\langle w_0, {}^t\xi_4 \rangle) = 2(1 + \cos\varphi)$ which is impossible. A similar argument gives $\dim W_1 = 2$.

(2) *Case when rank* $(\sigma_0) = 2s \geq 4$. We are going to prove that this case cannot hold. If it were the case, easy computations would lead to

$$R(-n\varphi)\hat{\pi}_0^n \begin{pmatrix} w_0^{(1)} \\ w_0^{(2)} \end{pmatrix} = (-\mu)^n \begin{pmatrix} w_0^{(1)} \\ w_0^{(2)} \end{pmatrix},$$

$$R(n\varphi)\hat{\pi}_1^n \begin{pmatrix} w_1^{(2s-1)} \\ w_1^{(2s)} \end{pmatrix} = (-\mu^{-1})^n \begin{pmatrix} w_1^{(2s-1)} \\ w_1^{(2s)} \end{pmatrix}$$

and

$$R(-n\varphi)\hat{\pi}_0^n \begin{pmatrix} w_0^{(3)} \\ w_0^{(4)} \end{pmatrix} = (-\mu)^n \left[n \begin{pmatrix} w_0^{(1)} \\ w_0^{(2)} \end{pmatrix} + \begin{pmatrix} w_0^{(3)} \\ w_0^{(4)} \end{pmatrix} \right].$$

An argument similar to the above shows that $|\mu| = 1$, and $w_0^{(1)} \neq 0$. But the boundedness of

$$\left(\hat{\pi}_0^n \begin{pmatrix} w_0^{(3)} \\ w_0^{(4)} \end{pmatrix} \right)_{n \geq 0}$$

implies that $w_0^{(1)} = 0$, which is a contradiction.

This proves 5.1(ii) in the real case.

PROOF OF 5.1(iii). We will give the proof only in the complex case. In the real case the proof is exactly the same.

Let

$$\sigma_0 = (-\mathbf{1}_a) \oplus \overline{\sigma} \oplus \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}^{\otimes c},$$

where $\overline{\sigma}$ is a diagonal unitary (b, b)-matrix whose diagonal terms are all different from -1.

The first result we are going to prove is:

Claim 1: If σ_0 has the form described before, then every $k \in \mathbf{Z}$:

$$\dim W_k = \operatorname{Card}\{j \in [a+1, s] \colon w_k^{(j)} \neq 0\} + \dim \operatorname{sp}[w_k^{(j)} \colon 1 \leq j \leq a].$$

PROOF. Let α be a 1-row matrix such that $\alpha w_0 = 0$. Lemma 2.9(iv) implies that $\alpha(\mathbf{1}_s + \sigma_0)\xi_0 = 0$, and since the vectors $(\xi_0^{(j)})_{1 \leq j \leq s}$ are linearly independent, this implies $\alpha(\mathbf{1}_s + \sigma_0) = 0$.

Using the special form of the matrix σ_0, we deduce that $\alpha^{(a+j)} = 0$ for every $j \in [1, b]$ and $\alpha^{(a+b+2j)} = 0$ for every $j \in [1, c]$, and hence $\sum_{j=1}^{a} \alpha^{(j)} w_0^{(j)} = 0$ (since $w_0^{(a+b+2j-1)} = 0$ for every $j \in [1, c]$).

This shows that $\dim \operatorname{sp}[w_0^{(j)} \colon a+1 \leq j \leq s] = \operatorname{Card}\{j \in [a+1, s] \colon w_0^{(j)} \neq 0\}$ and that the spaces $\operatorname{sp}[w_0^{(j)} \colon a+1 \leq j \leq s]$ and $\operatorname{sp}[w_0^{(j)} \colon 1 \leq j \leq a]$ are linearly independent.

This proves the claim for the even indices. A similar argument holds for the odd indices.

The proof of 5.1(iii) will be complete if we prove the following:

Claim 2. It is possible to choose the bases ξ_3 and ξ_4 in such a way that σ_0 keeps the special form described above and such that $\dim \operatorname{sp}[w_k^{(j)} \colon 1 \leq j \leq a] = \operatorname{Card}\{j \in [1, a] \colon w_k^{(j)} \neq 0\}$ for every $k \in \mathbf{Z}$.

To prove this claim we will need the following lemma:

LEMMA 5.3. *If σ_0 has the form described above then, for every $k \in \mathbf{Z}$, and every $x \in E_k$ we have $\sum_{j=1}^{a}\langle x, w_{k+1}^{(j)}\rangle w_k^{(j)} = 0$.*

PROOF. Applying Lemma 5.2(ii) and an easy inductive argument we obtain that

$$\hat{\pi}_k^n(x) = P_k(x) - (-1)^k \langle x, {}^t w_{k+1}\rangle \left(\sum_{p=0}^{n-1}(-\sigma_k)^p\right) w_k$$

(use the fact that $\langle P_k(x), {}^t w_{k+1}\rangle = \langle x, {}^t w_{k+1}\rangle$: Lemma 3.3).

Now using the special form of the matrix σ_0 and nullity properties of the vectors $w_k^{(j)}$ for $j > a+b$, the above equation can be written

$$\begin{aligned}\hat{\pi}_k^n(x) &= P_k(x) - (-1)^k \sum_{j=1}^{a+b}\langle x, w_{k+1}^{(j)}\rangle \left[\sum_{p=0}^{n-1}(-\sigma_k(j,j))^p\right] w_k^{(j)} \\ &= P_k(x) - (-1)^k \left[n\sum_{j=1}^{a}\langle x, w_{k+1}^{(j)}\rangle w_k^{(j)} \right. \\ &\qquad \left. + \sum_{j=a+1}^{b}\langle x, w_{k+1}^{(j)}\rangle \frac{1-(-\sigma_k(j,j))^n}{1+\sigma_k(j,j)} w_k^{(j)} \right].\end{aligned}$$

This implies the desired result, since the sequence $(\hat{\pi}_k^n(x))_{n\geq 0}$ is bounded. □

PROOF OF CLAIM 2. This claim will be proved in two steps.

Step 1. Without loss of generality we can assume that $(w_0^{(j)})_{1\leq j\leq a_0}$ is a maximal linearly independent subsystem of $(w_0^{(j)})_{1\leq j\leq a}$. Then there exists a matrix T_0 such that $(w_0^{(1+a_0)}, \ldots, w_0^{(a)}) = (w_0^{(1)}, \ldots, w_0^{(a_0)})T_0$.

Let

$$C = \left(\begin{array}{c|c} \mathbf{1}_{a_0} & 0 \\ \hline {}^tT_0 & \mathbf{1}_{a-a_0}\end{array}\right) \oplus \mathbf{1}_{s-a} \quad (\text{and } C_* = {}^tC^{-1}).$$

If we define the new bases ξ_3' and ξ_4' by $\xi_3' = C_*\xi_3$ and $\xi_4' = C\xi_4$ it is easy to check that we have $\sigma_0' = \sigma_0$, $w_0'^{(j)} = w_0^{(j)}$ for every $j \in [1, a_0]$, and $w_0'^{(j)} = 0$ for every $j \in [a_0+1, a]$.

Step 2. By the preceding step, we can suppose without loss of generality that the vectors $(w_0^{(j)})_{1\leq j\leq a_0}$ are linearly independent and that $w_0^{(j)} = 0$ for every $j \in [a_0+1, a]$.

By Lemma 5.3, for every $x \in E_{-1}$, we have $\sum_{j=1}^{a_0}\langle x, w_0^{(j)}\rangle w_{-1}^{(j)} = 0$, and by linear independence and since $(E_{-1})^* = E_0$, the above equations imply that $w_{-1}^{(j)} = 0$, and hence $w_1^{(j)} = 0$, for every $j \in [1, a_0]$.

Without loss of generality we can also assume that $(w_1^{(a+1-j)})_{1\leq j\leq a_1}$ is a maximal linearly independent subsystem of $(w_1^{(j)})_{1\leq j\leq a}$. Then there exists a matrix T_1 such that

$$(w_1^{(1+a_0)}, \ldots, w_1^{(a-a_1)}) = (w_1^{(a+1-a_1)}, \ldots, w_1^{(a)})T_1.$$

Let

$$C_* = \mathbf{1}_{a_0} \oplus \left(\begin{array}{c|c} \mathbf{1}_{a-a_0-a_1} & -{}^tT_1 \\ \hline 0 & \mathbf{1}_{a_1}\end{array}\right) \oplus \mathbf{1}_{s-a} \quad (\text{and } C = {}^tC_*^{-1}).$$

If we consider the new bases $\xi_3' = C_*\xi_3$ and $\xi_4' = C\xi_4$ it is easy to check that we have $\sigma_0' = \sigma_0$, $w_0'^{(j)} = w_0^{(j)}$ for every $j \in [1, a]$, $w_1'^{(j)} = 0$ for every $j \in [1, a - a_1]$, and $w_1'^{(j)} = w_1^{(j)}$ for every $j \in [a + 1 - a_1, a]$.

This concludes the proof of Claim 2 and of Theorem 5.1. □

In view of Theorem 5.1 it is quite natural to consider the following definitions.

DEFINITION 5.4 The real case: Let E be a real Banach space which is isometric to its bidual E^{**}. We will say that E is of type $(s; a_0, a_1, b, c, d)$ if $\dim(E^{**}/E) = s$, and if, after the reduction of the parameters we have:

(i)

$$\sigma_0 = (-\mathbf{1}_a) \oplus \overline{\sigma} \oplus \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}^{\otimes c} \oplus R(\varphi_1) \oplus \cdots \oplus R(\varphi_d),$$

where $\overline{\sigma}$ is a diagonal unitary (b, b)-matrix not having -1 as an eigenvalue, and $\varphi_i \not\equiv 0\ [\pi]$ for every $i \in [1, d]$.

(ii) $w_0^{(j)} = 0$ for every $j \in [a_0 + 1, a]$, and $w_1^{(j)} = 0$ for every $j \in [1, a - a_1]$.

In the complex case we define similarly the spaces of type $(s; a_0, a_1, b, c)$ [the parameter d disappears].

Examples of each type, both in the real and the complex case, could be constructed (by suitable direct sums) if we construct examples of spaces of special types. To make this more precise let us introduce the following terminology:

DEFINITION 5.5 Let E be a Banach space which is isometric to its bidual E^{**}. We will say that:

(1) E is of type (I) if $\dim(E^{**}/E) = 1$, and if $w_0^{(1)} = 0$, and $w_1^{(1)} = 0$. (In this case we have $\sigma_0 = -1$.)

(2) E is of type (II_0) (resp. of type (II_1)) if $\dim(E^{**}/E) = 1$ and if $w_0^{(1)} \neq 0$, and $w_1^{(1)} = 0$ (resp. $w_0^{(1)} = 0$, and $w_1^{(1)} \neq 0$). (In this case we have also $\sigma_0 = -1$).

(3) E is of type (III) if $\dim(E^{**}/E) = 1$, and if $\sigma_0 \neq -1$. (In this case we have $w_0^{(1)} \neq 0$, and $w_1^{(1)} \neq 0$.)

(4) E is of type (IV) if $\dim(E^{**}/E) = 2$, and if $\sigma_0 = \left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$. (In this case we have $w_0^{(1)} = 0$, $w_0^{(2)} \neq 0$, $w_1^{(1)} \neq 0$, and $w_1^{(2)} = 0$.)

(5) (In the real case only) E is of type (V) if $\dim(E^{**}/E) = 2$ and if $\sigma_0 = \left(\begin{smallmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{smallmatrix}\right)$ for some $\varphi \not\equiv 0\ [\pi]$. (In this case we have $\dim W_0 = \dim W_1 = 2$.)

It is now clear that by direct sums we can produce examples of any type if we can construct examples of the above special types. This will be the subject of the next section.

REMARK. It is easy to see that E is of type (I) [resp. (III), (IV), (V)] if and only if E^* is of the same type, and that E is of type (II_0) [resp. (II_1)] if and only if E^* is of type (II_1) [resp. (II_0)].

6. Examples. In what we have done before, we have imposed no normalization conditions on the vectors $\xi_k^{(j)}$. We are going to prove that these restrictions hold automatically in the case $s = 1$.

Recall that we always assume that the parameters have been reduced (so that they satisfy Theorem 5.1).

LEMMA 6.1. *In the case $s = 1$ we have*

(i) $\mathrm{Max}(|1 + \sigma_0|, 1) \leq \|\xi_0\| \cdot \|\xi_1\| \leq 2$,

(ii) $\|\xi_0\| \cdot \|\xi_1\| = 1$ *if* $w_0 = 0$ *and* $w_1 = 0$.

PROOF. (i) Since $\langle \xi_1, \xi_0 \rangle = 1$ we have $\|\xi_0\| \cdot \|\xi_1\| \geq 1$.

To see that $\|\xi_0\| \cdot \|\xi_1\| \geq |1+\sigma_0|$ recall that we have $\|w_0\| \leq \|\xi_0\|$ and $\|w_1\| \leq \|\xi_1\|$ (Corollary 2.7). Then by Lemmas 2.9 and 3.2, we have $|1+\sigma_0| = |\langle w_0, w_1 \rangle| \leq \|\xi_0\| \cdot \|\xi_1\|$.

On the other hand, for every $\varepsilon > 0$, let $\alpha\xi_0 + x_{-2}$, $x_{-2} \in E_{-2}$, be a *norm one* vector such that $\alpha = \langle \alpha\xi_0 + x_{-2}, \xi_1 \rangle \geq (1-\varepsilon)\|\xi_1\|$, then $(1-\varepsilon)\|\xi_1\| \cdot \|\xi_0\| \leq \|\alpha\xi_0\| \leq 2$ by Lemma 2.2, and since $\varepsilon > 0$ is arbitrary we have $\|\xi_0\| \cdot \|\xi_1\| \leq 2$.

(ii) If $w_0 = 0$ and $w_1 = 0$, we have $H_{2k} = F$ and $H_{2k+1} = F_*$ for every $k \in \mathbf{Z}$.

By Theorem 2.1(v), there exists $f_* \in F_*$ such that $\|f_* + \|\xi_0\|\xi_1\| = 1$ (indeed if $f_* + \sum_{n=0}^{\infty} \alpha_{-1-2n}\xi_{-1-2n} + \|\xi_0\|\xi_1$ is a norm one supporting vector of ξ_0, then $f_* + \|\xi_0\|\xi_1$ also supports ξ_0, and hence has to be of norm one), and there exists $f \in F$ such that $\|f + \|\xi_1\|\xi_2\| = 1$.

By Lemma 2.6 we have $\|f + \|\xi_1\|(\xi_2 + \xi_0)\| \leq 1$, and by Lemma 2.2 $\|f + \|\xi_1\|\xi_0\| \leq 1$.

Then

$$|\langle f_* + \|\xi_0\|\xi_1, f + \|\xi_1\|\xi_2 \rangle| = |\langle f_*, f \rangle - \|\xi_0\| \cdot \|\xi_1\|| \leq 1$$

and

$$|\langle f_* + \|\xi_0\|\xi_1, f + \|\xi_1\|\xi_0 \rangle| = |\langle f_*, f \rangle + \|\xi_0\| \cdot \|\xi_1\|| \leq 1.$$

These two inequalities lead clearly to $\|\xi_0\| \cdot \|\xi_1\| = 1$. □

In all the following examples we exhibit an isometry A for which $M_k = (-1)^k \mathbf{1}_s$. (We leave the verification to the reader.)

Unfortunately we were not able to produce examples of type (IV).

EXAMPLE 6.2. *Spaces of type* (I). We are going to show that the spaces v_p^0 are of type (I) for every p, $1 < p < \infty$. (v_2^0 is the usual James space usually denoted by J.)

Let us first recall the definitions of the space v_p^0 [**J**, **LT**].

For $1 < p < \infty$ and a sequence of scalars $(\alpha_n)_{n \geq 0}$, let

$$(*) \qquad \|(\alpha_n)\|_{v_p} = \frac{1}{2^{1/p}} \operatorname*{Sup}_{\substack{k \geq 0 \\ 0 \leq n_0 \leq \cdots \leq n_k}} \left[|\alpha_{n_k} - \alpha_{n_0}|^p + \sum_{i=0}^{k-1} |\alpha_{n_{i+1}} - \alpha_{n_i}|^p \right]^{1/p}$$

$\left(\|(\alpha_n)\|_{v_p} < \infty \Rightarrow \lim_{n=\infty} \alpha_n \text{ exists}\right)$.

The spaces v_p^0 are defined by

$$v_p^0 = \{(\alpha_n)_{n \geq 0} \in c_0 \colon \|(\alpha_n)\|_{v_p} < \infty\}$$

equipped with the norm given by $(*)$.

Since the natural basis of v_p^0 is bimonotone and shrinking for every $p \in]1, \infty[$, its bidual $(v_p^0)^{**}$ is given by

$$(v_p^0)^{**} = \{(\alpha_n)_{n \geq 0} \colon \|(\alpha_n)\|_{v_p} < \infty\}$$

equipped with the norm

$$\|(\alpha_n)\|_{v_p}^{**} = \operatorname*{Sup}_N \|(\alpha_0, \alpha_1, \ldots, \alpha_N, 0, 0, \ldots)\|_{v_p}.$$

With this norm, v_p^0 is isometric to $(v_p^0)^{**}$, and the isometry $A\colon v_p^0 \to (v_p^0)^{**}$ is given by

$$A(\alpha_0, \alpha_1, \alpha_2, \dots) = (\alpha_1 - \alpha_0, \alpha_2 - \alpha_0, \dots).$$

To compute A^*, we need some information on $(v_p^0)^*$ and $(v_p^0)^{(3)}$.

If $(\alpha_0^*, \alpha_1^*, \dots) \in (v_p^0)^*$, then $\sum_{n=1}^{\infty} \alpha_n^*$ exists. Indeed, if $(\delta_n)_{n\geq 0}$ denotes the natural basis of v_p^0, and $(\delta_n^*)_{n\geq 0}$ the corresponding biorthogonal system (which is a basis of $(v_p^0)^*$ since $(\delta_n)_{n\geq 0}$ is shrinking), then

$$\left|\sum_{n=N}^{M} \alpha_n^*\right| = \left|\left\langle \sum_{n=N}^{M} \alpha_n^*\delta_n^*, \sum_{n=N}^{M} \delta_n \right\rangle\right| \leq \left\|\sum_{n=N}^{M} \alpha_n^*\delta_n^*\right\|.$$

This implies that $\sum_{n=0}^{\infty} \alpha_n^*$ exists.

Every element of $(v_p^0)^{(3)}$ can be represented as a sequence $(\alpha_0^*, \alpha_1^*, \dots; \hat{\alpha})$, where $(\alpha_0^*, \alpha_1^*, \dots) \in (v_p^0)^*$. Such an element acts on $(v_p^0)^{**}$ as follows

$$\langle(\alpha_0^*, \alpha_1^*, \dots; \hat{\alpha}); (\alpha_0, \alpha_1, \dots)\rangle = \sum_{n=0}^{\infty} \alpha_n \alpha_n^* + \hat{\alpha} \lim_{n=\infty} \alpha_n.$$

With this convention an easy computation leads to

$$A^*(\alpha_0^*, \alpha_1^*, \dots; \hat{\alpha}) = (-\hat{\alpha} - \sum_{n=0}^{\infty} \alpha_n^*, \alpha_0^*, \alpha_1^*, \dots).$$

Now it is easy to check that we can take $\xi_2 = -\sum_{n=0}^{\infty} \delta_n$ (the convergence holds in the ω^*-sense). Indeed if $e^* = (\alpha_0^*, \alpha_1^*, \dots) \in (v_p^0)^*$ (i.e.: $\hat{\alpha} = 0$), then $\langle A^*(e^*), \sum_{n=0}^{\infty} \delta_n\rangle = 0$.

With this choice of ξ_2, and by the definition of A, it is easily seen that $\xi_{-2n} = \delta_n$ for every $n \geq 0$. This shows that $G_0 = v_p^0$, and then $H_0 = \{0\}$. Hence $H_1 = G_0^{\perp} = \{0\}$, and then $G_1 = (v_p^0)^*$.

This concludes the proof of the fact that the spaces v_p^0, $1 < p < \infty$, are of type (I).

EXAMPLE 6.3.: *Spaces of type* (II_0) *and* (II_1). Notice first that it suffices to construct examples of type (II_0). Examples of type (II_1) are then obtained by taking the duals of the first ones.

For spaces of type (II) we have to consider the parameter $\theta = \|\xi_0\| \cdot \|\xi_1\|$ which belongs to [1,2].

Let $\theta \in [1,2]$, and consider on the space $\mathbf{K} \oplus v_2^0$ the norm given by

$$\|(\beta; \alpha_0, \alpha_1, \dots)\| = \mathrm{Max}\left\{\frac{1}{\theta}\|(\alpha_0, \alpha_1, \dots)\|_{v_2}; \mathrm{Sup}_n\, |\beta - \alpha_n|\right\}.$$

If $(\delta_n)_{n\geq 0}$ is the natural basis of v_2^0, and if $\nu = (1; 0, 0, \dots)$, it is not difficult to see that the sequence $(\nu, \delta_0, \delta_1, \dots)$ is a monotone shrinking basis of $\mathbf{K} \oplus v_2^0$. Then the bidual norm on $(K \oplus v_2^0)^{**}$ is given by

$$\|(\beta; \alpha_0, \alpha_1, \dots)\|^{**} = \mathrm{Sup}_N\, \|(\beta; \alpha_0, \dots, \alpha_N, 0, 0, \dots)\|.$$

With this norm the space $\mathbf{K} \oplus v_2^0$ is isometric to its bidual, and the isometry is given by

$$A(\beta; \alpha_0, \alpha_1, \alpha_2, \dots) = (\beta - \alpha_0; \alpha_1 - \alpha_0, \alpha_2 - \alpha_0, \dots).$$

Reproducing the computations of Example 6.2, we get

$$A^*(\beta^*, \alpha_0^*, \alpha_1^*, \dots; \hat{\alpha}) = \left(\beta^*; -\hat{\alpha} - \beta^* - \sum_{n=0}^{\infty} \alpha_n^*, \alpha_0^*, \alpha_1^*, \dots\right).$$

This shows that we can take $\xi_3 = (0; 0, 0, \dots; -1)$ and $\xi_2 = \nu - \sum_{n=0}^{\infty} \delta_n$. Then $\xi_{-2n} = \delta_n$ and $\xi_{1-2n} = \delta_n^* - \delta_{n-1}^*$ for every $n \geq 0$ (we have put $\delta_{-1}^* = 0$). This leads to $\sigma_0 = -1$ $(= \sigma_1)$, $G_0 = \overline{\mathrm{sp}}[\delta_n : n \geq 0]$, $G_1 = \overline{\mathrm{sp}}[\delta_n^* : n \geq 0]$, $w_0 = \nu$, and then $w_1 = 0$, $H_0 = F = \mathbf{K} \cdot \nu$, $H_1 = G_0^\perp = \mathbf{K} \cdot \nu^*$, and $F_* = \{0\}$.

Let us prove now that $\|\xi_0\| \cdot \|\xi_1\| = \theta$ (notice that $\langle \xi_1, \xi_0 \rangle = 1$).

$$\begin{aligned}
\|\xi_0\| &= \mathrm{Max}(1, \tfrac{1}{\theta}) = 1. \\
\|\xi_1\| &= \mathrm{Sup}\{|\alpha_0| : \|(\beta; \alpha_0, \alpha_1, \dots)\| \leq 1\} \\
&= \mathrm{Sup}\{|\alpha_0| : \|(\beta; \alpha_0, 0, 0, \dots)\| \leq 1\} \\
&= \mathrm{Sup}\{|\alpha_0| : \mathrm{Max}(|\beta|, |\beta - \alpha_0|, \tfrac{1}{\theta}|\alpha_0|) \leq 1\} \\
&= \mathrm{Min}(2, \theta) = \theta.
\end{aligned}$$

EXAMPLE 6.4. *Spaces of type* (III). In this case the parameter $\theta = \|\xi_0\| \cdot \|\xi_1\|$ has to satisfy $\mathrm{Max}(1, |1 + \sigma_0|) \leq \theta \leq 2$. We will give examples for all the possible values of σ_0 and θ except for the cases $(\theta = 2, \sigma_0 \neq 1)$ and $(\theta = 1,\ |1 + \sigma_0| \leq 1)$.

Let σ be a scalar of modulus one, $\sigma \neq -1$, and $p \in]1, \infty[$. On $\mathbf{K} \oplus v_p^0$ we consider a norm defined by

$$\|(\beta; \alpha_0, \alpha_1, \dots)\| = \sup_{\substack{k \geq 0 \\ 0 \leq n_0 \leq \cdots \leq n_k}} \left[|(\beta + \alpha_{n_0}) + \sigma(\beta + \alpha_{n_k})|^p + \sum_{i=0}^{k-1} |\alpha_{n_{i+1}} - \alpha_{n_i}|^p \right]^{1/p}.$$

If $(\delta_n)_{n \geq 0}$ is the natural basis of v_p^0 and if $\nu = (1; 0, 0, \dots)$ the sequence $(\nu, \delta_0, \delta_1, \dots)$ is a monotone shrinking basis of $\mathbf{K} \oplus v_p^0$. Then the bidual norm on $(\mathbf{K} \oplus v_p^0)^{**}$ is given by

$$\|(\beta; \alpha_0, \alpha_1, \dots)\|^{**} = \sup_N \|(\beta; \alpha_0, \dots, \alpha_N, 0, 0, \dots)\|.$$

With this norm the space $\mathbf{K} \oplus v_p^0$ is isometric to its bidual and the isometry is given by

$$\begin{aligned}
&A(\beta; \alpha_0, \alpha_1, \alpha_2, \dots) \\
&\quad = \left(-\frac{\beta + \alpha_0}{\sigma}; \alpha_1 + \frac{\alpha_0 + (1 + \sigma)\beta}{\sigma}, \alpha_2 + \frac{\alpha_0 + (1 + \sigma)\beta}{\sigma}, \dots\right).
\end{aligned}$$

We have also

$$\begin{aligned}
A^*(\beta^*; \alpha_0^*, \alpha_1^*, \dots; \hat{\alpha}) = &\left(-\frac{\beta^*}{\sigma} + \frac{1 + \sigma}{\sigma}\left(\hat{\alpha} + \sum_{n=0}^{\infty} \alpha_n^*\right);\right. \\
&\left. -\frac{\beta^*}{\sigma} + \frac{1}{\sigma}\left(\hat{\alpha} + \sum_{n=0}^{\infty} \alpha_n^*\right), \alpha_0^*, \alpha_1^*, \dots\right).
\end{aligned}$$

This shows that we can take $\xi_3 = (0; 0, 0, \dots; 1)$ and $\xi_2 = -\nu + \sum_{n=0}^{\infty} \delta_n$. Then

$$\xi_1 = \frac{1 + \sigma}{\sigma} \nu^* + \frac{1}{\sigma} \delta_0^*,$$

and for every $n \geq 0$,

$$\xi_{-1-2n} = \frac{1}{\sigma}(\delta^*_{n+1} - \delta^*_n), \quad \text{and} \quad \xi_{-2n} = \sigma\delta_n.$$

This implies $\sigma_0 = \sigma$, $w_0 = \sigma\nu$, $w_1 = -(1+\sigma)\nu^*/\sigma$,

$$G_0 = \overline{\text{sp}}[\delta_n : n \geq 0], \quad G_1 = \overline{\text{sp}}\left[\delta^*_n + \frac{1+\sigma}{\sigma}\nu^* : n \geq 0\right],$$

$$H_0 = \mathbf{K}\cdot\nu, \quad H_1 = \mathbf{K}\cdot\nu^*, \quad F = \{0\}, \quad \text{and} \quad F_* = \{0\}.$$

Let us show now that $\theta = \|\xi_0\|\cdot\|\xi_1\| = \text{Max}(|1+\sigma|; 2^{1/p})$.

$$\begin{aligned}
\|\xi_0\| &= \text{Max}(|1+\sigma|; 2^{1/p}),\\
\|\xi_1\| &= \text{Sup}\{|(1+\sigma)\beta + \alpha_0| : \|(\beta;\alpha_0, 0, \dots)\| \leq 1\}\\
&= \text{Sup}\{|(1+\sigma)\beta + \alpha_0| :\\
&\qquad \text{Sup}[|1+\sigma|^p|\beta|^p; |1+\sigma|^p|\beta+\alpha_0|^p;\ |(1+\sigma)\beta+\alpha_0|^p + |\alpha_0|^p] \leq 1\}\\
&= 1.
\end{aligned}$$

EXAMPLE 6.5. *Spaces of type* (V). The example we give is formally identical to the above one. We have only to replace σ by the matrix

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

For $a, b \in \mathbf{R}$, let $|\binom{a}{b}|^2 = a^2 + b^2$, and define on the space $\mathbf{R}^2 \oplus v^0_2 \oplus v^0_2$ a norm by

$$\|(\beta;\alpha_0,\alpha_1,\dots)\| = \operatorname*{Sup}_{\substack{k\\0\leq n_0\leq\cdots\leq n_k}} \left[|(\beta+\alpha_0) + (\mathbf{1}_2 + R(\varphi))(\beta+\alpha_{n_k})|^2 + \sum_{i=0}^{k-1}|\alpha_{n_{i+1}} - \alpha_{n_i}|^2\right]^{1/2}$$

where we have put

$$\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix}, \quad \text{and} \quad \alpha_n = \begin{pmatrix} \alpha^1_n \\ \alpha^2_n \end{pmatrix}.$$

The isometry A is given formally by the same expression as in the above example, and we can check that $\sigma_0 = R(\varphi)$.

EXAMPLE 6.6. In this example we will construct a type (I) space E for which the decomposition $E = G_0 \oplus H_0$ is not bicontractive.

Let $(u_l)_{0\leq l\leq L}$ be scalars satisfying $\sum_{l=0}^{L} u_l = 0$, and define the scalars $(u_l)_{l\geq 0}$ by periodicity: $u_{l+L+1} = u_l$.

On the space $\mathbf{K}\oplus v^0_2$ we define a norm by

$$\|(\beta;\alpha_0,\alpha_1,\dots)\| = \text{Max}\{\Phi_{\beta,u}((\alpha_0,\alpha_1,\dots)); \|(\alpha_0,\alpha_1,\dots)\|_{v_2}\}$$

where

$$\Phi_{\beta,u}((\alpha_0,\alpha_1,\dots)) = \operatorname*{Sup}_{\substack{k\geq 0\\0\leq n_0\leq\cdots\leq n_L}} \left|\beta - \sum_{l=0}^{L} u_{k+l}\alpha_{n_l}\right|.$$

One can check that the operator $A\colon \mathbf{K}\oplus J \to (\mathbf{K}\oplus J)^{**}$ defined by

$$A(\beta;\alpha_0,\alpha_1,\alpha_2,\dots) = (\beta;\alpha_1-\alpha_0,\alpha_2-\alpha_0,\dots)$$

is an onto isometry.

Reproducing the computations of Example 6.2, we can see that E is of type (I) and characterize its parameters: $G_0 = \overline{\mathrm{sp}}[\delta_n\colon n\ge 0]$, $G_1 = \overline{\mathrm{sp}}[\delta_n^*, n\ge 0]$, $H_0 = F = \mathbf{K}\cdot\nu$, and $H_1 = F_* = \mathbf{K}\cdot\nu^*$.

For $u=(2,-1,2,-3)$, and $\alpha=(1,-1,1,0,0,\dots)$, we have $-6\le\sum_{l=0}^3 u_{k+l}\alpha_{n_l}$ ≤ 7. (The extreme values -6 and 7 are attained.)

Then $\|(0;1,-1,1,0,\dots)\| = 7$, and $\|(\frac{1}{2};1,-1,1,0,\dots)\| = \frac{13}{2}$; hence the decomposition $E = G_0\oplus H_0$ is not bicontractive.

Finally, we will describe some isometric properties of type (I) spaces. We first have to recall some definitions (see [**Gr**] for details).

Let P be an n-dimensional Choquet simplex, and let $S_P^* = P-P = \{x-y\colon x,y\in P\}$. The set S_P^* has exactly $2(n+1)$-simplectic faces of dimension $(n-1)$. We define the set S_p^{**} to be the intersection of the $2(n+1)$ half spaces containing S_p^* and which are supported by the above-mentioned faces.

A symmetric convex set K is said to be a Leichtweiss compact if $S_P^*\subset K\subset S_P^{**}$ for some Choquet simplex P.

THEOREM 6.7. *Let E be a type* (I) *space. Then for every $k\in\mathbf{Z}$, every $n\ge 0$, the unit ball of the space* $\mathrm{sp}[\xi_{k-2p}\colon 0\le p\le n]$ *is a Leichtweiss compact.*

PROOF. Notice first that in view of Lemma 6.1 we can assume that $\|\xi_0\| = \|\xi_1\| = 1$. For convenience we put $\tilde\xi_{k,-1}=0$ for every $k\in\mathbf{Z}$.

For the Choquet simplex $P=\mathrm{cv}[\tilde\xi_{k,p}\colon -1\le p\le n]$, it is easy to check that $S_p^* = \mathrm{cv}[\tilde\xi_{k,p}-\tilde\xi_{k,q}\colon -1\le p,q\le n]$, and that the n-dimensional simplectic faces of S_P^* are the faces $\pm F_j$, $-1\le j\le n$, where $F_j = -\tilde\xi_{k,j}+\mathrm{cv}[\tilde\xi_{k,p}\colon -1\le p\ne j\le n]$.

To prove the theorem it suffices now to prove that the faces F_j, $-1\le j\le n$, are in the unit sphere of $\mathrm{sp}[\xi_{k-2p}\colon 0\le p\le n]$.

By Corollary 2.7, for $-1\le p\ne j\le n$, we have $\|\tilde\xi_{k,j}-\tilde\xi_{k,p}\| = 1$. Hence it remains only to prove that

$$\left\|-\tilde\xi_{k,j}+\frac{1}{n+1}\sum_{\substack{p=-1\\ p\ne j}}^{n}\tilde\xi_{k,p}\right\| = 1,$$

for every $-1\le j\le n$. We distinguish two cases:

$j=-1$:

$$(n+1)\ge\left\|\sum_{p=0}^{n}\tilde\xi_{k,p}\right\| = \left\|\sum_{p=0}^{n}(n+1-p)\xi_{k-2p}\right\|\ge\|(n+1)\xi_k\| = n+1.$$

$0 \leq j \leq n$:

$$\begin{aligned} n+1 &\geq \left\| (n+2)\tilde{\xi}_{k,j} - \sum_{p=0}^{n} \tilde{\xi}_{k,p} \right\| \\ &= \left\| \sum_{p=0}^{j} (p+1)\xi_{k-2p} - \sum_{p=j+1}^{n} (n+1-p)\xi_{k-2p} \right\| \\ &\geq \left| \left\langle \xi_{k-2j-1}, \sum_{p=0}^{j} (p+1)\xi_{k-2p} - \sum_{p=j+1}^{n} (n+1-p)\xi_{k-2p} \right\rangle \right| \\ &= n+1. \end{aligned}$$

This concludes the proof of the theorem. □

REMARK. Leichtweiss' compacts are exactly the unit balls of finite dimensional spaces with some extremal properties. Let us recall one of them.

For an n-dimensional Banach space X, we define $p(X)$ as the smallest constant K so that for every Banach space Y containing X and satisfying $\dim(Y/X) = 1$, there exists a projection $P: Y \to X$, with $\|P\| \leq K$.

It is known that $p(X) \leq 2n/(n+1)$, and that $p(X) = 2n/(n+1)$ if and only if the unit ball of X is a Leichtweiss compact.

We refer to [**Gr**] for other extremal properties which are connected to Leichtweiss' compacts.

REFERENCES

[**B**] S. Bellenot, *Transfinite duals of quasi reflexive Banach spaces*, Trans. Amer. Math. Soc. **273** (1982), 551–577.

[**G**] G. Godefroy, *Espaces de Banach, existence et unicité de certains préduaux*, Ann. Inst. Fourier **28** (1978), 87–105.

[**Gr**] B. Grümbaum, *On some covering and intersection properties in Minkowski spaces*, Pacific J. Math. **9** (1959), 487–494.

[**J**] R. C. James, *A non reflexive Banach space isometric with its second conjugate*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 174–177.

[**LT**] J. Lindenstrauss and L. Tzafriri, *Classical Banach space* I, Ergebnisse der Mathematik, no. 92, Springer-Verlag, 1977.

[**S**] A. Sersouri, *Structure theorems for some quasi-reflexive Banach spaces*, Doctorat de l'Université Paris VI, (November 86).

[**V**] M. Valdivia, *On a class of Banach spaces*, Studia Math. **60** (1977), 11–13.

EQUIPE D'ANALYSE, UNIVERSITÉ PARIS VI, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05, FRANCE