TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 313, Number 1, May 1989

LOCAL PROPERTIES OF SECANT VARIETIES IN SYMMETRIC PRODUCTS. PART II

TRYGVE JOHNSEN

ABSTRACT. Let V be a linear system on a curve C. In Part I we described a method for studying the secant varieties V_d^r locally. The varieties V_d^r are contained in the d-fold symmetric product $C^{(d)}$.

In this paper (Part II) we apply the methods from Part I. We give a formula for local tangent space dimensions of the varieties V_d^1 valid in all characteristics (Theorem 2.4).

Assume $\operatorname{rk} V = n+1$ and $\operatorname{char} K = 0$. In §§3 and 4 we describe in detail the projectivized tangent cones of the varieties V_n^1 for a large class of points. The description is a generalization of earlier work on trisecants for a space curve.

In §5 we study the curve in $C^{(2)}$ consisting of divisors D such that $2D \in V_4^1$. We give multiplicity formulas for all points on this curve in $C^{(2)}$ in terms of local geometrical invariants of C. We assume $\operatorname{char} K = 0$.

1. INTRODUCTION

Let C be a nonsingular curve over a field K, and let $V \subset H^0(C, L)$ be a linear system on C, where L is a line bundle. Denote by $C^{(d)}$ the dth symmetric product of C. The subschemes V_d^r of $C^{(d)}$ consist of those divisors that impose at most $d-r$ independent conditions on V. The V_d^r are secant varieties.

As an example consider the case where $\operatorname{rk} V = 4$ and V is very ample. Then V defines an embedding of C into P^3. The sections of V are then thought of as hyperplanes of P^3. The variety V_3^1 parametrizes those divisors of degree 3 that consist of 3 collinear points on C in P^3. Roughly speaking: V_3^1 parametrizes the trisecant lines of the embedded curve.

It is a well-known fact that the V_d^r can be defined scheme-theoretically as the zero schemes

$$Z\left(\bigwedge^{d-r+1} \sigma\right) \quad \text{for } r = 1, \dots, d,$$

where σ is a canonical $C^{(d)}$-bundle map

$$\sigma \colon V \otimes \mathcal{O}_{C^{(d)}} \to E_L,$$

Received by the editors September 10, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14M15, 14H45, 14B12.
Key words and phrases. Secant varieties of curves, local geometry.

and E_L is a vector bundle of rank d on $C^{(d)}$ obtained from L by a so-called symmetrization process. See, for example, [ACGH, p. 340].

In Part I we constructed a computational device for studying the map σ and the varieties V_d^r locally. Our main results were given in Theorem 4.2 and Proposition 4.4 of Part I. We constructed a local matrix description of σ and described the formal completion $\hat{O}_{V_d^r, D}$ of the local ring of V_d^r at a point (divisor) D. Such a local description is often trivial when D consists of d distinct points. The main purpose with our results is to study the V_d^r at points on the diagonal in $C^{(d)}$. In Part II we will use the information from Part I to obtain some geometrical results.

In §2 we give a formula for the tangent space dimension of the variety V_d^1 at a point D. The formula is valid in any characteristic.

In §3 we study a large class of points on the variety V_n^1, where $\operatorname{rk} V = n+1$. We describe the tangent cones of V_n^1 at such points, and in particular we give a formula for the multiplicity of V_n^1 at these points.

In §4 we find further properties of the tangent cones described in §3. We will indicate when the projectivized tangent cones are singular. This is a generalization of a result in [J] concerning trisecant lines for a space curve.

In §5 we study stationary bisecants for a nonsingular space curve. A stationary bisecant is a bisecant line, where the curve tangents at the points of secancy meet, or a tangent line at a point where the osculating plane of the curve is hyperosculating. We define a curve in $C^{(2)}$ that parametrizes these situations, and we describe the local structure of this curve. We find out how the tangent cone of the curve in $C^{(2)}$ at a secant divisor is determined by the local geometry of C at the points of secancy. The study of stationary bisecants was proposed to the author by Ignacio Sols. Similar results to those of our §5 have been found independently by Miguel Gonzalez [G]. He uses bundles of principal parts. Global properties of the varieties V_d^r and of the various diagonals in $C^{(d)}$ are described in [ACGH, Chapter VIII], in the case $K = \mathbf{C}$. Our paper can be read as a local supplement of that chapter.

Remark. References to results and formulas in Part I will be marked by the prefix I. Theorem I.4.2. means Theorem 4.2. of Part I, equation I.(4.1) means equation (4.1) of Part I and so on.

2. The tangent space dimension of V_d^1 at $D \in C^{(d)}$

The varieties V_d^1 are interesting since they parametrize divisors that are "special" with respect to the linear systems V.

Let $D = \sum_{i=1}^k d_i P_i$, where $D \in V_d^1$, and the P_i are distinct points on C. We will use Theorem I.4.2. to compute the tangent space dimension of V_d^1 at D. The Brill-Noether matrix BN (see I.(4.2)) consists of k groups of consecutive rows, where the ith group (consisting of d_i rows) corresponds to the point P_i, for $i = 1, \dots, k$.

Definition 2.1. l_i is the maximal integer $s \in \{0, \dots, d_i - 1\}$ such that the matrix consisting of all rows of BN except the $(s+1)$st row in the ith group has rank $d-1$. If no such integer exists, set $l_i = -1$.

Explanation. Assume for simplicity that V is base-point free and thus maps C into some P^n. For a chosen set of local parameters of C at the P_i we can talk about derivative vectors of C at the P_i. Call the point P_i itself the 0th derivative vector of C at P_i. Then l_i is the maximal integer $s \in \{0, \dots, d_i - 1\}$ such that the union of the 0th, ..., $\hat{s}$th, ..., $(d_i - 1)$st derivative vectors of C at P_i and the 0th, ..., $(d_j - 1)$st derivative vectors at P_j, for $j \neq i$, span a $d-2$ plane in P^n. If no such s exists, then $l_i = -1$.

Observation 2.2. $D \in V_d^2 \Leftrightarrow l_i = -1$ for $i = 1, \dots, k$.

Definition 2.3. Assume $D' \in C^{(d')}$ for some $d' \in N$. Denote by $V(-D')$ the linear system $V \cap H^0(C, L(-D'))$. We now give the main result of this section (valid in any characteristic).

Theorem 2.4. *The tangent space dimension of V_d^1 at D is*

$$\min\left(d, \operatorname{rk} V\left(-\sum_{i=1}^{k}(d_i + l_i + 1)P_i\right) + 2d - n - 2\right), \text{ where } \operatorname{rk} V = n + 1.$$

Proof. It is enough to study the constant and linear parts of the matrix M (I.(4.1)). Since $\operatorname{rk} BN \leq d-1$, we may assume that only the $d-1$ first columns of BN are nonzero. Since we will only study the linear parts of the d-minors, we may assume that the entries in the $d-1$ first columns are constant. Assume first $D = dP$. We may drop the index i in M, and we have

$$M = \begin{bmatrix} a_{0,0} & \cdots & a_{d-2,0} & (-1)^{d-1}a_{d-1,d}s_d & \cdots & (-1)^{d-1}a_{n,d}s_d \\ a_{0,1} & \cdots & a_{d-2,1} & (-1)^{d-2}a_{d-1,d}s_{d-1} + (-1)^{d-1}a_{d-1,d+1}s_d & \cdots & (-1)^{d-2}a_{n,d}s_{d-1} + (-1)^{d-1}a_{n,d+1}s_d \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{0,d-1} & \cdots & a_{d-2,d-1} & a_{d-1,d}s_1 + \cdots + (-1)^{d-1}a_{d-1,2d-1}s_d & \cdots & a_{n,d}s_1 + \cdots + (-1)^{d-1}a_{n,2d-1}s_d \end{bmatrix}.$$

Here we used that the linear part of $W_j(s_1, \dots, s_d)$ is $(-1)^{j-1}s_j$ for $j = 1, \dots, d$. See I.(3.4). Let D_{j-1} be the $d-1$ minor formed by the $d-1$ first columns of M (or BN) minus the jth row. We see that l is the largest integer j such that $D_j \neq 0$ if such an integer exists (see Definition 2.1).

The linear parts of the $n+2-d$ relations cutting out V_d^1 are (up to signs):

$$(a_{i,d}D_l)s_{d-l} - (a_{i,d+1}D_l + a_{i,d}D_{l-1})s_{d-l+1} + \cdots$$
$$+ (-1)^l(a_{i,d+l}D_l + \cdots + a_{i,d}D_0)s_d$$

for $i = d-1, \dots, n$.

The coefficient matrix of these relations in $s_1, \ldots, s_d$ is easily seen to have the same rank as

$$N = \begin{bmatrix} a_{d-1,d} & \cdots & a_{d-1,d+l} \\ \vdots & & \vdots \\ a_{n,d} & \cdots & \alpha_{n,d+l} \end{bmatrix}$$

Hence the tangent space dimension of V_d^1 at D is $d - \operatorname{rk} N$ if $l \geq 0$ and d otherwise. Assume first $l \geq 0$. Let us find

$$\operatorname{rk} V(-(l+d+1)P).$$

Since $l \geq 0$, Observation 2.2. gives that the matrix BN has rank exactly $d-1$, and therefore a section contained in $V(-dP)$ must be of the form $c_{d-1}X_{d-1} + \cdots + c_n X_n$, where the $c_j \in K$, and where X_j is the section corresponding to the $(j+1)$st column of M. The conditions that such a section should be contained in $V(-(l+d+1)P)$ are

$$\begin{array}{ccc} a_{d-1,d}c_{d-1} & + \cdots + & a_{n,d}c_n = 0, \\ \vdots & & \vdots \\ a_{d-1,d+l}c_{d-1} & + \cdots + & a_{n,d+l}c_n = 0. \end{array}$$

These equations in the variables $c_{d-1}, \ldots, c_n$ give rise to a coefficient matrix, which is the transpose of N.

Hence $\operatorname{rk} V(-(l+d+1)P) = n-d+2-\operatorname{rk} N$, and we deduce that the tangent space dimension of V_d^1 at D is

$$d - \operatorname{rk} N = 2d - n - 2 + \operatorname{rk} V(-(l+d+1)P).$$

Since $\operatorname{rk} V(-(l+d+1)P) \leq \operatorname{rk} V(-dP) = n-d+2$, our tangent space dimension is at most $(2d-n-2)+(n-d+2) = d$. Hence the theorem holds when $D = dP$, and $l \geq 0$.

When $D = dP$ and $l = -1$, the tangent space dimension is d since all the D_j are zero. On the other hand,

$$\begin{aligned} 2d - n - 2 + \operatorname{rk} V(-(l+d+1)P) &= 2d - n - 2 + \operatorname{rk} V(-dP) \\ &= (2d - n - 2) + (n + 1 - \operatorname{rk} BN) \geq d + 1 \end{aligned}$$

since $\operatorname{rk} BN \leq d-2$. Hence d is the minimum of d and $2d - n - 2 + \operatorname{rk} V(-(l+d+1)P)$. Our proof is now complete in the case $D = dP$. The general case follows easily using the same argument for each group of d_i rows of M.

3. A local study of V_n^1 where $\operatorname{rk} V = n+1 \geq 4$

In [J, Theorem 2.3.1], we gave a multiplicity formula for trisecant lines to a space curve. In this section we will generalize this formula.

Let $D \in C^{(n)}$ be a point of V_n^1, where $\operatorname{rk} V = n+1 \geq 4$. Assume:

(1) for each $D' \in C^{(n-1)}$, such that $D' \leq D$, we have $D' \notin V_{n-1}^1$;

(2) if $D=\sum_{i=1}^{k} n_i P_i$ (all $n_i>0$), then $D+P_i \notin V_{n+1}^2$, for $i=1,\dots,k$;
(3) $\operatorname{char} K = 0$ and $K=\overline{K}$.

Proposition 3.1. *Under assumptions* (1), (2) *and* (3), *we have*:

(a) *the tangent space dimension of* V_n^1 *at* D *is* $\operatorname{rk} V(-2D)+n-2$, *where* $\operatorname{rk} V(-2D)$ *is* 0 *or* 1;

(b) $\dim O_{V_n^1,D} = n-2$;

(c) *the multiplicity of* V_n^1 *at* D *is the largest integer* s *such that* $\operatorname{rk} V(-sD) \geq 1$ (*with equality if* V_n^1 *is singular at* D).

Proof. Let l_i, for $i=1,\dots,k$, be the integers described in Definition 2.1. Assumption (1) gives $l_i = n_i - 1$ for all i. The tangent space dimension formula in (a) is then a special case of Theorem 2.4, and it holds also when $\operatorname{char} K > 0$.

Assumption (2) gives that $\operatorname{rk} V(-2D)$ is 0 or 1, because if $\operatorname{rk} V(-2D) \geq 2$, then $2D \in V_{2n}^{n+1}$, and then $D+P_i \in V_{n+1}^2$ for *all* $i \in \{1,\dots,k\}$. Hence (a) holds.

By general facts about determinantial varieties, we have $\dim O_{V_n^1,D} \geq n-2$.

If $\operatorname{rk} V(-2D)=0$, then the tangent space dimension of V_n^1 at D is $n-2$ by (a). Hence, $\dim \hat{O}_{V_n^1,D} \leq n-2$, and (b) follows. Furthermore V_n^1 is nonsingular at D in this case. Hence the multiplicity of V_n^1 at D is 1. Since $\operatorname{rk} V(-2D) = 0$, and $\operatorname{rk} V(-1\cdot D) \geq 2 \geq 1$, the number given in (c) is also 1. Hence (c) follows when $\operatorname{rk} V(-2D)=0$.

It remains to prove (b) and (c) when $\operatorname{rk} V(-2D)=1$. Let V be generated by the sections $\{X_0,\dots,X_n\}$. $\operatorname{rk} V(-D) \geq 2$ since $D \in V_n^1$, and $\operatorname{rk} V(-D) \leq 2$ since $D \notin V_n^2$ by (1).

Hence $\operatorname{rk} V(-D) = 2$, and we may assume that X_{n-1} and X_n generate $V(-D)$. This means that the entries in the two last columns of the BN matrix (see I.(4.2)) are zero.

We may assume that X_n generates $V(-2D)$ since $\operatorname{rk} V(-2D)=1$. The matrix M of I.(4.1) will have constant terms only in the entries of the $n-1$ first columns, corresponding to $X_0,\dots,X_{n-2}$ (since X_{n-1}, X_n are in $V(-D)$). On the other hand, each of these $n-1$ columns will have at least one entry with a nonzero constant term. This implies that the ideal generated by the n-minors of M is, in fact, generated by the two n-minors obtained by disregarding each of the two last columns, corresponding to X_{n-1} and X_n. We call these minors R_{n-1} and R_n, respectively.

The strategy now is as follows: (1) and (2) imply that the leading form of R_n is linear in the variables $s_{1,1},\dots,s_{k,n_k}$. One uses R_n to express one of the $s_{i,j}$, say $s_{i0,j0}$, as a power series in the remaining $s_{i,j}$ modulo the ideal (R_n) in $K[[\underline{s}]]$. Then one substitutes this power series for $s_{i0,j0}$ in R_{n-1} to get a new relation $\overline{R}_{n-1}$ in the remaining $s_{i,j}$. Using I.(3.4) it is easy to show that the leading form of R_{n-1} is of degree m, where m is

$$\max\{s|\operatorname{rk} V(-sD) = 1\} \geq 2.$$

It is less obvious, but still true, that the leading form of $\overline{R}_{n-1}$ is of degree m also. The verification of this is done in detail in [Prpr] for the case $D = nP$. There we also indicate how this verification can be modified to apply to the general case $D = \sum_{i=1}^{k} n_i P_i$. This gives the conclusion of our proposition.

Definition 3.2. For a variety X and a point P in X the tangent cone $\mathcal{T}_P(X)$ of X at P is

$$\operatorname{Spec}\left(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\right),$$

where m is the maximal ideal of the local ring $O_{X,P}$.

The projectivized tangent cone $P\mathcal{T}_P(X)$ of X at P is

$$\operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\right).$$

Corollary 3.3. *Under* (1), (2) *and* (3), *the projectivized tangent cone* $P\mathcal{T}_D(V_n^1)$ *is a hypersurface of degree* m *in* P^{n-2}, *where*

$$m = \max\{s|\operatorname{rk} V(-sD) \geq 1\}.$$

Proof. Corollary 3.3 follows from the proof of Proposition 3.1.

4. The tangent cone $\mathcal{T}_D(V_n^1)$, where $\operatorname{rk} V = n + 1 \geq 4$

In this section we will not always prove our assertions. Our goal is to give a geometrical interpretation of $\mathcal{T}_D(V_n^1)$ (or $P\mathcal{T}_D(V_n^1)$) described at the end of §3.

In §3 we studied a point D in V_n^1, where $\operatorname{rk} V = n + 1 \geq 4$. Under (1), (2) and (3) of §3 we gave a description of the dimension, embedding dimension and multiplicity of V_n^1 at D.

A question which then arises naturally is: When is the projectivized tangent cone $P\mathcal{T}_D(V_n^1)$ singular? If $n = 3$ and V_n^1 is a curve, then $P\mathcal{T}_D(V_n^1)$ is singular if V_n^1 does not have normal crossings at D; we also say that V_n^1 possesses a nonordinary singularity at D in this case. In [J] we gave necessary and sufficient local conditions on C for determining whether the trisecant curve (essentially V_3^1) possesses nonordinary singularities or not. We want to generalize these conditions to apply to any V_n^1, $n \geq 3$, where $\operatorname{rk} V = n+1$. In order to do this we assume:

(2′) V is base-point free and $D + P \notin V_{n+1}^2$ for any point $P \in C$.

Assumption (2′) is, of course, a strengthening of (2) of §3; but this strengthening is of no importance for the local geometry of V_n^1 at D. Whatever local result we prove for V_n^1 at D under (1), (2$'$) and (3) will also hold under Assumptions (1), (2) and (3). This is true because the matrix M (see I.(4.1)) is only dependent on the behaviour of V at the points $P_1, \dots, P_k$, and because any base point of V is outside $\{P_1, \dots, P_k\}$ by (1).

Under (2$'$) V defines a map $\phi\colon C \to \overline{C} \subset P^n$. Let $G = G(n-2, n)$ be the Grassmannian, which parametrizes the $n-2$ planes in P^n.

For an $n-2$ plane H denote by $[H]$ the corresponding point in G. Denote by F the incidence variety

$$\{([H], P) \in G \times P^n \mid P \in H\}.$$

Consider the following diagram:

$$\begin{tikzcd}
 & F \arrow[ddl, "q"'] \arrow[ddr, "p"] & \\
 & \mathscr{C}_F \arrow[u, phantom, "\cup"] \arrow[dl] \arrow[dr] & \\
G & & \overline{C} \subseteq P^n
\end{tikzcd}$$

Here p and q are the natural projection maps from F to P^n and G, respectively, and $\mathscr{C}_F = p^{-1}(\overline{C})$.

Let Sec be the subvariety of G cut out by the sheaf of O_G-ideals: $F^{n-1}(q_* O_{\mathscr{C}_F})$, that is, the sheaf of $(n-1)$st. Fitting ideals of the O_G-sheaf $q_* O_{\mathscr{C}_F}$. Then Sec parametrizes $n-2$ planes that are n-secant to C. This definition of Sec is taken from [GP], where the case $n = 3$ is treated. As in §3 set $D = \sum n_i P_i$, where $\sum n_i = n$.

Assume $D \in V_n^1$, and that (1), (2$'$) and (3) hold. Then D spans a unique $n-2$ plane; that is, $P_1, \dots, P_k$ and the $n_i - 1$ first derivative vectors of $\overline{C}$ at P_i for $i = 1, \dots, k$, span a unique $n-2$ plane H.

We make the following claim:

(4.1) $$\mathscr{T}_D(V_n^1) \simeq \mathscr{T}_{[H]}(\mathrm{Sec}).$$

In fact, we strongly believe

(4.2) $$\hat{O}_{V_n^1, D} \simeq \hat{O}_{\mathrm{Sec}, [H]}.$$

We have not made any attempts to prove (4.2), but we have proved (4.1) when D consists of n distinct points.

To find $\mathscr{T}_D(V_n^1)$ one simply calculates the leading forms of the relations $R_{n-1}(\underline{s})$ and $R_n(\underline{s})$ described in the proof of Proposition 3.1. In [J] an explicit description of $\mathscr{T}_{[H]}(\mathrm{Sec})$ is given in the case where $n = 3$, whether D consists of three distinct points or not.

It is easy, but a little painstaking, to generalize this explicit description to arbitrary $n \geq 3$, when the n points of D are distinct. Comparing the two tangent cones one sees that they are isomorphic.

We omit the very technical calculations here. In principle the same method should work when the n points are not distinct.

Definition 4.1. For a curve C and a hypersurface M in P^n, denote by $I(P, C \cap M)$ the usual intersection number between C and M at P.

In the following we will only use (4.1) in Corollary 4.4. Assume that (1), (2′) and (3) hold for D. From Proposition 3.1(a) and from local results about Sec we have: Sec is singular at $[H] \Leftrightarrow V_n^1$ is singular at $D \Leftrightarrow$ There exists a unique hyperplane $\mathscr{H}$ in P^n with

$$I(P_i, \overline{C} \cap \mathscr{H}) \geq 2n_i \quad \text{for } i = 1, \dots, k.$$

We have $\mathrm{Sec} \subset G \subset P^S$ for some large S. Making explicit calculations analogous to those in [GP and J] one finds that the embedded (compactified) tangent space in P^S to Sec at $[H]$ is $\check{\mathscr{H}} \subset G \subset P^S$, where $\check{\mathscr{H}}$ is the $n-1$ plane in G, which parametrizes the $n-2$ planes in the hyperplane $\mathscr{H} \subset P^n$.

Hence the embedded tangent cone in P^S to Sec at $[H]$ is a union of an $(n-3)$-dimensional family of lines in $\check{\mathscr{H}}$. Each point of the projectivized tangent cone $P\mathscr{T}_{[H]}(\mathrm{Sec})$ or $P\mathscr{T}_D(V_n^1)$ corresponds to one such line.

A line L in $\check{\mathscr{H}}$ through $[H]$ is a nesting of a one-dimensional family of $n-2$ planes in $\mathscr{H}$ containing a fixed $n-3$ plane h_L contained in H.

Hence each point of $P\mathscr{T}_{[H]}(\mathrm{Sec})$ and $P\mathscr{T}_D(V_n^1)$ corresponds to an $n-3$ plane h_L in the $n-2$ plane H. Denote by $[h]$ the point in $\check{H}$ corresponding to an $n-3$ plane h, where $\check{H}$ is the $n-2$ plane which parametrizes the $n-3$ planes in H.

By Corollary 3.3, $P\mathscr{T}_D(V_n^1)$ is a hypersurface of degree

$$m = \max\{s \mid \mathrm{rk}\, V(-sD) \geq 1\}$$

in P^{n-2}. From the above discussion it is clear that a natural geometrical interpretation of this P^{n-2} is $\check{H}$, and that

$$P\mathscr{T}_{[H]}(\mathrm{Sec}) \cong \{[h_L] \mid L \text{ is a line in } \check{\mathscr{H}} \text{ through } [H], \text{ such that } L \text{ is contained in the embedded tangent cone to Sec at } [H]\}.$$

Two problems now arise in a natural way:

(i) Find those $n-3$ planes h in H such that $[h] \in P\mathscr{T}_{[H]}(\mathrm{Sec})$.

(ii) Find those $n-3$ planes h in H such that $[h]$ is a singular point of $P\mathscr{T}_{[H]}(\mathrm{Sec})$.

We state without proofs the solutions to problems (i) and (ii) (Results 4.2 and 4.3, respectively). Result 4.2 is a generalization of Theorem 2.3.2 of [J], and Result 4.3 is a generalization of Theorem 2.3.3 in [J].

We have proved Results 4.2 and 4.3 in the case where D consists of n distinct points, but we omit the technical details here.

Result 4.2. Under (1), (2′) and (3) we have $[h] \in P\mathscr{T}_{[H]}(\mathrm{Sec})$ if and only if there exists a hypersurface M in P^n such that:

(a) $\deg M = m + 1$, and M has a singularity of multiplicity at least m at all points of h;

(b) $I(P_i, M \cap \overline{C}) \geq (m+1)n_i$ for all $P_i \in H \cap C$;

(c) $m \cdot H \subseteq M \cap \mathscr{H}$, i.e. $I(M) \subseteq (I(\mathscr{H}) + I(H)^m)$, and $H \not\subseteq \mathrm{Sing}(M)$;

(d) the equation defining M in P^n is equal to the equation of a cone of degree $m+1$ with h contained in its vertex set, modulo the square of the ideal defining H.

Remark. M can be taken to be a union of a one-dimensional family of $n-2$ planes containing H. Thus M gives rise to a curve $[$ in G. The tangent line to $[$ at $[H]$ is L, where $h = h_L$.

Result 4.3. Under (1), (2′) and (3) we have: $[h]$ is a singular point of $P\mathscr{T}_{[H]}(\mathrm{Sec})$ if and only if there exists a hypersurface N in P^n such that:

(a) N is a cone of degree m, and h is contained in the vertex set of N;

(b) $I(P_i, \overline{C} \cap N) \geq (m+1)n_i$ for $i = 1, \dots, k$;

(c) $H \not\subseteq \mathrm{Sing}(N)$.

Corollary 4.4. *Assume that* (1), (2), (3), *and* (4.1) *hold for* D. *We have:* $P\mathscr{T}_D(V_n^1)$ *is singular if and only if there exists a cone* N *and an* $n-3$ *plane* h *as described in Result* 4.3(a), (b), (c).

5. Stationary bisecants for a space curve

In §5 we assume $\mathrm{char}\, K = 0$, and $K = \overline{K}$. Let C be a nonsingular curve in P^3, and let P_1 and P_2 be points on C. The line $\overline{P_1P_2}$ is usually called a stationary bisecant if the tangents to C at P_1 and P_2 meet. In general there is a one-dimensional family of stationary bisecants for a space curve. We will define a scheme in $C^{(2)}$, which essentially parametrizes divisors $P_1 + P_2$ with P_1 and P_2 as described. Some divisors $2P$ may also occur as points on this scheme in $C^{(2)}$ since tangent lines are in some sense bisecants.

Let C be mapped into P^3 by evaluating sections of some linear system V of rank 4. Consider the map:

$$i\colon C^{(2)} \to C^{(4)}, \quad \text{where } i(D) = 2D \tag{5.1}$$

for divisors D in $C^{(2)}$.

Definition 5.1. The scheme of stationary bisecants for C with respect to V is $i^{-1}(V_4^1)$.

Remark 5.2. Clearly $D \in i^{-1}(V_4^1) \Leftrightarrow 2D \in V_4^1$. If $P_1 \neq P_2$, then $P_1 + P_2 \in i^{-1}(V_4^1) \Leftrightarrow$ the tangent lines to C at P_1 and P_2 meet.

We also have:

$2P \in i^{-1}(V_4^1) \Leftrightarrow P$ is a flex on C, or the osculating plane of C at P is hyperosculating.

It will follow from the proofs of Propositions 5.3 and 5.5 that $i^{-1}(V_4^1)$ is either a curve or empty.

We will use Theorem I.4.2. to determine the multiplicity of $i^{-1}(V_4^1)$ at an arbitrary point D (in $C^{(2)}$) in terms of the local geometry of C at the secant points in P^3. The cases $D = 2P$ and $D = P_1 + P_2$ $(P_1 \neq P_2)$ will be treated separately. As before we denote by $I(Q, C \cap F)$ the intersection multiplicity between a curve C and a surface F at a point Q in P^3.

The multiplicity of the curve $i^{-1}(V_4^1)$ *at* $D = P_1 + P_2$. Assume $P_1 \neq P_2$, and let L be the line $\overline{P_1P_2}$. Set $n_i = I(P_i, C \cap H)$ for $i = 1, 2$, where H is a general member of the pencil of planes containing L. We may assume $n_1 \geq n_2$.

Let r be the maximal integer such that there exists a plane H with

$$I(P_i, C \cap H) \geq n_i + r \quad \text{for } i = 1, 2.$$

Let r_2 be the maximal integer such that there exists a plane H_2 containing L with

$$I(P_2, C \cap H_2) = n_2 + r_2.$$

Proposition 5.3. *The multiplicity of the curve* $i^{-1}(V_4^1)$ *at* $P_1 + P_2$ *is*

$$\min(n_1 + n_2 + r - 2, 2n_2 + r_2 - 1).$$

Proof. Choose coordinates X_0, X_1, X_2, X_3 for P^3, and let t_i be a local parameter at P_i for $i = 1, 2$. Without loss of generality we choose

$$\begin{aligned} X_0 &= 1, \\ X_1 &= t_i + k_i, \\ X_2 &= \sum_{j \geq n_i} \alpha_{i,j} t_i^j, \\ X_3 &= \sum_{j \geq n_i + r} \beta_{i,j} t_i^j \end{aligned}$$

as local parametrizations at P_i for $i = 1, 2$.

By the definitions of n_1, n_2, and r, we may assume that α_{i,n_1} and α_{2,n_2} are nonzero, and that β_{1,n_1+r} or β_{2,n_2+r} is nonzero.

We see that the line $L = \overline{P_1P_2}$ has equations $X_2 = X_3 = 0$, and that $P_i = (1, k_i, 0, 0)$ for $i = 1, 2$, with $k_1 \neq k_2$.

The unique plane (if any) that intersects C at least $n_i + 1$ times at P_i for $i = 1, 2$, has equation $X_3 = 0$. This is also the equation of H_2.

By Theorem I.4.2, we have

$$\hat{O}_{V_4^1, 2P_1 + 2P_2} \cong K[[s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}]]/(\det M),$$

where

$$M = \begin{bmatrix} 1 & k_1 + s_{1,1} & \sum_{j \geq n_1} \alpha_{1,j} W_j(s_{1,1}, s_{1,2}) & \sum_{j \geq n_1 + r} \beta_{1,j} W_j(s_{1,1}, s_{1,2}) \\ 0 & 1 & \sum_{j \geq n_1} \alpha_{1,j} W_{j-1}(s_{1,1}, s_{1,2}) & \sum_{j \geq n_1 + r} \beta_{1,j} W_{j-1}(s_{1,1}, s_{1,2}) \\ 1 & k_2 + s_{2,1} & \sum_{j \geq n_2} \alpha_{2,j} W_j(s_{2,1}, s_{2,2}) & \sum_{j \geq n_2 + r} \beta_{2,j} W_j(s_{2,1}, s_{2,2}) \\ 0 & 1 & \sum_{j \geq n_2} \alpha_{2,j} W_{j-1}(s_{2,1}, s_{2,2}) & \sum_{j \geq n_2 + r} \beta_{2,j} W_{j-1}(s_{2,1}, s_{2,2}) \end{bmatrix}.$$

The map $i\colon C^{(2)} \to C^{(4)}$, where $i(D) = 2D$ induces a map

$$i^*\colon \hat{O}_{C^{(4)},2P_1+2P_2} \to \hat{O}_{C^{(2)},P_1+P_2}.$$

Now

$$\hat{O}_{C^{(4)},2P_1+2P_2} \simeq \hat{O}_{C^{(2)},2P_1} \underset{K}{\otimes} \hat{O}_{C^{(2)},2P_2}$$
$$\simeq K[[s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}]],$$

where the $s_{l,j}$ can be regarded as formal, algebraically independent, variables.

$s_{l,j}$ can also be regarded as the jth elementary function in two replicas $t_{l,1}, t_{l,2}$ of the local parameter t_l of C at P_l for $l = 1, 2$, $j = 1, 2$.

Furthermore,

$$\hat{O}_{C^{(2)},P_1+P_2} \simeq \hat{O}_{C,P_1} \underset{K}{\otimes} \hat{O}_{C,P_2} \simeq K[[t_1, t_2]].$$

Hence we regard i^* as a map

$$i^*\colon K[[s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}]] \to K[[t_1, t_2]].$$

We have $\hat{O}_{i^{-1}(V_4^1),P_1+P_2} \simeq K[[t_1, t_2]]/\det M(i^*s_{1,1}, \dots, i^*s_{2,2})$. Clearly $i^*s_{l,j} = s_{l,j}(t_l, t_l)$, $l = 1, 2$, $j = 1, 2$. From I.(3.5) we then obtain

$$i^*W_j(t_l) = W_j(i^*s_{l,1}, i^*s_{l,2}) = (j+1)t_l^j.$$

This implies that

$$\hat{O}_{i^{-1}(V_4^1),P_1+P_2} \simeq K[[t_1, t_2]]/(R),$$

where R is the determinant of the matrix obtained from M by substituting $W_j(s_{l,1}, s_{l,2})$ by $(j+1)t_l^j$ for $l = 1, 2$, $j \geq 0$.

Calculation gives that the leading form of R is

$$(5.2)\quad (k_1 - k_2)[n_1(n_2+r)\alpha_{1,n_1}\beta_{2,n_2+r}t_2^r - n_2(n_1+r)\alpha_{2,n_2}\beta_{1,n_1+r}t_1^r]\cdot t_1^{n_1-1}\cdot t_2^{n_2-1},$$

or

$$(5.3)\quad r_2\alpha_{2,n_2}\beta_{2,n_2+r_2}\cdot t_2^{2n_2+r_2-1},$$

or the sum of these forms.

One must check that neither of the forms vanishes identically as a polynomial in t_1, t_2, and that the forms do not cancel each other. Clearly (5.3) does not vanish. (5.3) cancels (5.2) only if $n_1 = 1$, but then $n_2 = 1$ also, and the forms have different degrees. Hence they do not cancel each other. For the form (5.2) we have two cases:

(a) $r = 0$. Then the form vanishes iff

$$\alpha_{1,n_1}\beta_{2,n_2} - \alpha_{2,n_2}\beta_{1,n_1} = 0.$$

But the last expression is zero if and only if there is a plane H, with

$$I(P_i, C \cap H) \geq n_i + 1 \quad \text{for } i = 1, 2.$$

This would contradict the definition of r, so the form does not vanish.

(b) $r > 0$. The form does not vanish since
(i) $k_1 \neq k_2$,
(ii) α_{1,n_1} and α_{2,n_2} are nonzero, and
(iii) β_{1,n_1+r} or β_{2,n_2+r} is nonzero.

Hence the multiplicity of $i^{-1}(V_4^1)$ at $P_1 + P_2$ is equal to the degree of the leading form of R:

$$\min(n_1 + n_2 - 2 + r, 2n_2 + r_2 - 1).$$

This gives the proposition.

Corollary 5.4. *If a stationary secant* $\overline{P_1P_2}$ *is not a tangent to* C *at any of the points* P_1, P_2, *then the multiplicity of* $i^{-1}(V_4^1)$ *at* $P_1 + P_2$ *is*

$$r = \min(I(P_1, C \cap H), I(P_2, C \cap H)) - 1,$$

where H *is the plane spanned by the tangent lines to* C *at* P_1 *and* P_2.

The multiplicity of $i^{-1}(V_4^1)$ *at* $D = 2P$. Let L be the tangent line to C at the point P. Set $m_2 = \max\{l | lP \in V_l^{l-2}\}$ or, equivalently, $m_2 = I(P, C \cap H)$ for a general member H of the pencil of planes containing L. If P is not a flex on C, then $m_2 = 2$. Set $m_3 = \max\{l | lP \in V_l^{l-3}\}$ or, equivalently, $m_3 = \max_{H \supseteq L}\{I(P, C \cap H)\}$. Clearly $m_3 \geq m_2 + 1$.

We now give our main result in the case $D = 2P$.

Proposition 5.5. *The multiplicity of* $i^{-1}(V_4^1)$ *at* $2P$ *is* $[(m_2 + m_3)/2] - 2$, *where* $[x]$ *means the integral part of the real number* x.

Proof. Let t be a local parameter for C at P. Without loss of generality we may assume that C is parametrized locally at P as

$$\begin{aligned} X_0 &= 1, \\ X_1 &= t, \\ X_2 &= \sum_{j \geq m_2} \alpha_j t^j, \qquad \alpha_{m_2} \neq 0, \\ X_3 &= \sum_{j \geq m_3} \beta_j t^j, \qquad \beta_{m_3} \neq 0. \end{aligned}$$

Let s_1, s_2, s_3, s_4 be local parameters for $C^{(4)}$ at $4P$, where the s_k are the k th elementary functions in t_1, t_2, t_3, t_4; four replicas of t.

By Theorem I.4.2, we have

$$\hat{O}_{V_4^1, 4P} = K[[s_1, s_2, s_3, s_4]]/(\det M),$$

where

$$M = \begin{bmatrix} 1 & s_1 & \sum_j \alpha_j W_j(\underline{s}) & \sum_j \beta_j W_j(s) \\ 0 & 1 & \sum_j \alpha_j W_{j-1}(\underline{s}) & \sum_j \beta_j W_{j-1}(s) \\ 0 & 0 & \sum_j \alpha_j W_{j-2}(\underline{s}) & \sum_j \beta_j W_{j-2}(s) \\ 0 & 0 & \sum_j \alpha_j W_{j-3}(\underline{s}) & \sum_j \beta_j W_{j-3}(s) \end{bmatrix}.$$

We see that

$$\det M = \sum_{j\geq m_2} \alpha_j W_{j-2}(\underline{s}) \cdot \sum_{j\geq m_3} \beta_j W_{j-3}(\underline{s}) - \sum_{j\geq m_2} \alpha_j W_{j-3}(\underline{s}) \cdot \sum_{j\geq m_3} W_{j-2}(\underline{s}). \tag{5.4}$$

Let S_1 and S_2 be local parameters of $C^{(2)}$ at $2P$, where the S_k are the k th symmetric functions in T_1, T_2; two formal replicas of t.

The map (5.1) induces a map

$$i^*: K[[s_1, s_2, s_3, s_4]] \to K[[S_1, S_2]].$$

Clearly $\hat{O}_{i^{-1}(V_4^1), 2P} \cong K[[S_1, S_2]]/(R)$, where R is the power series obtained by substituting $i^* s_k$ for s_k in (5.2), for $k = 1, 2, 3, 4$. The multiplicity $\mathrm{mult}_{2P}(i^{-1}(V_4^1))$ is the lowest value $e_1 + e_2$ for any term $S_1^{e_1} S_2^{e_2}$ occurring in R. We will first find the $i^* s_k$. Let $s_k = s_k(t_1, t_2, t_3, t_4)$; that is: Regard s_k as the k th elementary symmetric function in four replicas of t for $k = 1, \dots, 4$.

We define

$$\psi_k(T_1, T_2) = s_k(T_1, T_1, T_2, T_2).$$

Clearly $\psi_k(T_1, T_2)$ is symmetric in T_1, T_2 for $k = 1, \dots, k$. Hence there are unique functions $\phi_k(S_1, S_2)$ such that

$$\phi_k(S_1(T_1, T_2), S_2(T_1, T_2)) = \psi_k(T_1, T_2)$$

for $i = 1, \dots, k$.

One sees that $i^* s_k(S_1, S_2) = \phi_k(S_1, S_2)$ for all k.

We then obtain

$$i^* s_1 = 2S_1, \qquad i^* s_2 = S_1^2 + 4S_2,$$
$$i^* s_3 = 2S_1 S_2, \qquad i^* s_4 = S_2^2.$$

We have

$$R = \sum_{j\geq m_2} \alpha_j (i^* W_{j-2}) \cdot \sum_{j\geq m_3} \beta_j (i^* W_{j-3}) - \sum_{j\geq m_2} \alpha_j (i^* W_{j-3}) \cdot \sum_{j\geq m_3} \beta_j (i^* W_{j-2}), \tag{5.5}$$

where $i^* W_l = W_l(i^* s_1, \dots, i^* s_4)$ for all l.

We now must verify that the leading form of R in the variables S_1 and S_2 is as stated in Proposition 5.5. This was done in detail in [Prpr]. We omit the very technical calculations here.

Corollary 5.6. *If P is not a flex on C, then the multiplicity of $i^{-1}(V_4^1)$ at $2P$ is $[m_3/2] - 1$, where $m_3 = I(P, C \cap H)$, for the osculating plane H of C at P.*

Comment 5.7. Assume:

(a) No plane intersects C more than four times at any point;

(b) C has no bitangents;

(c) C has no flexes;

(d) no plane is osculating at more than one point of C;

(e) for each tangential trisecant line to C tangent to C at say P_1 and intersecting C transversally at say P_2, the osculating plane at P_1 does not contain the tangent to C at P_2.

Then it follows from Propositions 5.3 and 5.5 that the curve $i^{-1}(V_4^1)$ is nonsingular.

A nonsingular space curve has only finitely many tangential trisecants, flexes, bitangents, and hyperosculating or biosculating planes. Hence it follows that the curve (scheme) $i^{-1}(V_4^1)$ is always reduced.

This curve however, might, be reducible. As an example of this, take C as the complete intersection of two quadric surfaces. Then C is contained in four quadric cones, and each generatrix of each such cone is a stationary bisecant line. Hence $i^{-1}(V_4^1)$ has (at least) four components in this case.

A geometrical interpretation of the tangent cone $\mathcal{T}_D(i^{-1}(V_4^1)$. In Definition 3.2 we described the (projectivized) tangent cone of a variety at a point. The tangent cone of the curve $i^{-1}(V_4^1)$ at a point D is determined by the leading form of the relation R as given in (5.5) in the case $D = 2P$, or as in (5.2) and (5.3) where the leading form is given explicitly in the case $D = P_1 + P_2$, $P_1 \neq P_2$.

In both cases the tangent cone is determined by a homogeneous polynomial of degree m in two variables, where m is the multiplicity of $i^{-1}(V_4^1)$ at D. This polynomial splits into m linear factors. It turns out that in many cases each linear factor in the leading form corresponds to a point on the secant line L with a certain geometrical significance. Clearly each linear factor corresponds to a point of the projectivized tangent cone $P\mathcal{T}_D(i^{-1}(V_4^1))$. Hence we have an analogy to Result 4.2 in these cases. We would like to explain this more closely.

As usual we denote by $l(L)$ the point in the Grassmannian $G = G(1,3)$ corresponding to a line L. Set

$$B = \{l(L) | L \text{ satisfies (a) or (b) below}\}.$$

(a) $L \cap C = \{P_1, P_2\}$, and L is not a tangent line to C.

(b) $L \cap C = \{P\}$, and L is a tangent, but not a flex tangent line to C at P.

By the Trisecant Lemma the closure $\overline{B}$ is a surface in G. It is a standard fact that $\overline{B}$ is locally isomorphic to $C^{(2)}$ at points of B under the map that sends the secant (tangent) line $l(L)$ to the divisor $P_1 + P_2 (2P)$. Moreover $\overline{B}$ is nonsingular at points of B.

Let S be the subcurve of $\overline{B}$ corresponding to stationary bisecants in the sense described earlier. Then S is locally isomorphic to $i^{-1}(V_4^1)$ at points of $S \cap B$.

Consider the Plücker embedding $G \subseteq P^5$. It is a well-known fact (see, for example, [GP, p. 16]) that the points of $S \cap B$ are exactly those points of B such that the embedded tangent planes to $\overline{B}$ in P^5 are globally contained in G (in fact as β-planes). For a point $l(L)$ on $S \cap B$, this tangent plane is $\check{H}$, where H is the stationary plane in P^3 spanned by the divisor $2D$ on C.

This information implies that if C is not contained in a cone consisting of stationary bisecant lines, then the family of stationary bisecant lines envelope another curve $\mathscr{C}$ in P^3.

Considering the stationary bisecants as dual lines, the same family envelopes a curve [in $\check{P}^3$.

The following is easily verified.

(1) C is on a cone consisting of stationary bisecant lines $\Leftrightarrow A$ component of $\mathscr{C}$ degenerates to a point $\Leftrightarrow A$ component of [is plane.

(2) $\mathscr{C}$ and [are dual to each other, that is, [parametrizes the osculating planes of $\mathscr{C}$, and vice versa.

(3) [parametrizes the bitangent planes of C.

Since $i^{-1}(V_4^1)$ is locally isomorphic to S at points of $S \cap B$, we can study the tangent cone to S at $l(L)$ instead of that of $i^{-1}(V_4^1)$ at D. Since the embedded tangent space of B at $l(L)$ is the dual plane $\check{H}$, we can embed $\mathscr{T}_{l(L)}(S)$ as a union of m lines in $\check{H}$ through the point $l(L)$. But a line in $\check{H} \subset G$ through $l(L)$ corresponds to a pencil of lines in $H \subset P^3$ through some point Q of L. Such points Q of L are exactly the points of $L \cap \mathscr{C}$ arising from the local branch(es) of S. This means that the explicit calculations of the leading forms performed earlier in §5 tell us how the points of $L \cap \mathscr{C}$ are located in Cases a and b.

Case a. $L \cap C = \{P_1, P_2\}$, L is not a tangent line. Set

$$r = \min(I(P_1, C \cap H), I(P_2, C \cap H)) - 1$$

for the stationary plane H. By (5.2) the leading form in t_1, t_2 is (up to a constant)

$$\alpha_{1,1}\beta_{2,r+1}t_2^r - \alpha_{2,1}\beta_{1,r+1}t_1^r.$$

Hence the multiplicity m is r, and we get r distinct points of $L \cap \mathscr{C}$ outside C unless either $\beta_{1,r+1}$ or $\beta_{2,r+1}$ is zero. If, say, $\beta_{1,r+1} = 0$, which means $I(P_1, C \cap H) \geq r+1$, then all r points of $L \cap \mathscr{C}$ collapse to one point. It turns out that this single point is P_2. See Result 5.8. below, or Remark 5.9.

Case b. $L \cap C = \{P\}$, L is tangent to C at P, but P is not a flex. We recall the definition $m_3 = I(P, C \cap H)$, where H is the osculating (stationary) plane of C at P.

We recall that the leading form in S_1, S_2 is $S_2^{(m_3-3)/2}$ when m_3 is odd and $(S_1 + kS_2)S^{m_3/2-2}$ when m_3 is even. It turns out that the factor S_2 corresponds to the (secant) point P of $C \cap L$, while the factor $S_1 + kS_2$ corresponds to a point outside P. "In general," when $m_3 = 4$, we get only the last factor.

In Cases a and b, we have another description of the points of $L \cap \mathscr{C}$ arising from the local branch(es) of S. Denote by m the multiplicity of S at $l(L)$.

Result 5.8. $Q \in L$ is a point of $\mathscr{C}$ iff there exists a cone N of degree $m+1$ with vertex at Q such that $\operatorname{Sing}(N) \not\supset L$ and:

Case a. $I(P_i, C \cap N) \geq m+2$ for $i = 1, 2$.

Case b. $I(P, C \cap N) \geq 2m+4$.

Idea of proof. Let F be the surface in P^3 swept out by the stationary bisecant lines. Let C' be a dummy curve on F transversal to the ruling around L. Regard L as a singular *tri*secant to $C \cup C'$. The point $l(L)$ is contained in a nonreduced component of the trisecant curve in G. Then apply Result 4.3 in the case $n = 3$.

Remark 5.9. Recall the local parametrizations of C introduced in the proof of Proposition 5.3. Referring to these parametrizations, Result 5.8 translates in Case a to $Q = (1, k, 0, 0)$ is a point on $L \cap \mathscr{C}$ iff

$$\left(\frac{k_2 - k}{k_1 - k}\right)^r = \frac{\beta_{1,r+1}}{\beta_{2,r+1}} \cdot \frac{\alpha_{2,1}^{r+1}}{\alpha_{1,1}^{r+1}}.$$

A similar result can be obtained in Case b.

We might return to a more detailed study of the curves $\mathscr{C}, S, [$ in another paper. With the information we have now it is easy to compute the "expected" genera, degrees, and numbers of cusps of these curves.

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves*, Vol. I, Springer-Verlag, Berlin, Heidelberg, and New York, 1985.

[G] M. Gonzalez, *Singularités de la courbe des bisécantes stationnaires*, C. R. Acad. Sci. Paris Ser. I **305** (1987), 341–343.

[GP] L. Gruson and C. Peskine, *Courbes de l'espace projectif, variétés de sécantes*, Enumerative Geometry and Classical Algebraic Geometry, Progress in Math., vol. 24, Birkhäuser, 1982, pp. 1–31.

[J] T. Johnsen, *The singularities of the* 3*-secant curve associated to a space curve*, Trans. Amer. Math. Soc. **295**, (1986), 107–118.

[Prpr] ——, *Local properties of secant varieties in symmetric products*, Preprint No. 1, 1987, Univ. of Oslo.

Institute of Mathematical Sciences, University of Tromsø, P. O. Box 953, 9001 Tromsø, Norway