

THE CONNECTEDNESS OF SYMMETRIC AND SKEW-SYMMETRIC DEGENERACY LOCI: EVEN RANKS

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ABSTRACT. A degeneracy locus is the set of points where a vector-bundle map has rank at most a given integer. Such a set is symmetric or skew-symmetric according as whether the vector-bundle map is symmetric or skew-symmetric. We prove a connectedness result, first conjectured by Fulton and Lazarsfeld, for skew-symmetric degeneracy loci and for symmetric degeneracy loci of even ranks.

INTRODUCTION

In this paper we prove a conjecture of Fulton and Lazarsfeld on the connectedness of symmetric and skew-symmetric degeneracy loci, when the rank is even. Since skew-symmetric maps always have even ranks, this settles the conjecture in the skew-symmetric case. The odd-rank symmetric case remains open.

Let E and F be complex vector bundles over an irreducible variety X , and $u: E \rightarrow F$ a vector-bundle morphism between them. For a given nonnegative integer r , the *degeneracy locus of rank r* of u is defined to be

$$(0.1) \quad D_r(u) = \{x \in X \mid \text{rank } u(x) \leq r\}.$$

The concept of a degeneracy locus is quite general, and includes as special cases all projective hypersurfaces, complete intersections, zero sets of sections of a vector bundle, and dependency loci of sections. However, in algebraic geometry one often encounters not only general vector-bundle morphisms, but morphisms satisfying symmetry conditions. If L is a line bundle over X , a morphism $u: E \otimes E \rightarrow L$ is said to be *symmetric* if it is symmetric on each fiber; similarly for a skew-symmetric morphism. Their degeneracy loci are defined as in (0.1). As examples of symmetric bundle morphisms we cite (1) the second fundamental form of a smooth hypersurface in a projective space, and (2) the differential of the period map of a family of varieties. For more background information on degeneracy loci the reader is referred to [T].

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These three types of vector-bundle morphisms may be viewed as sections of the bundles $\mathrm{Hom}(E, F)$, $(\mathrm{Sym}^2 E^*) \otimes L$, and $(\wedge^2 E^*) \otimes L$ respectively, where E^* denotes the dual of E . In [FL1] Fulton and Lazarsfeld proved that if $\mathrm{Hom}(E, F)$ is ample and u is a section of $\mathrm{Hom}(E, F)$, then $D_r(u)$ is connected provided its expected dimension, $\dim X - (e-r)(f-r)$, is at least one. They conjectured the following statements for symmetric and skew-symmetric degeneracy loci [FL2, Remark (2), p. 50].

Conjecture 0.2. *Let E be a vector bundle of rank e and L a line bundle over the irreducible variety X . Suppose $u: E \otimes E \rightarrow L$ is a symmetric bundle map and r a nonnegative integer. If $(\mathrm{Sym}^2 E^*) \otimes L$ is ample and $\dim X - \binom{e-r+1}{2} \geq 1$, then $D_r(u)$ is connected.*

Conjecture 0.3. *Suppose $u: E \otimes E \rightarrow L$ is a skew-symmetric bundle map and r a nonnegative even integer. If $(\wedge^2 E^*) \otimes L$ is ample and $\dim X - \binom{e-r}{2} \geq 1$, then $D_r(u)$ is connected.*

We prove both conjectures for even values of r , thus settling Conjecture 0.3 completely. The strategy of the proof is to represent a symmetric degeneracy locus of even rank as the image of a zero locus on a Grassmann bundle, as in Pragacz [P], using the characterization of the rank of a symmetric matrix by the dimensions of its isotropic subspaces. It then suffices to prove the connectedness of the zero locus. If the zero locus were that of a section of an ample vector bundle, its connectedness would follow immediately from the well-known Lefschetz-type theorem for ample vector bundles [S]. Unfortunately, the relevant vector bundle on the Grassmann bundle is no longer ample, and so we try to prove the connectedness by computing cohomology groups. This is done using a construction by which the cohomology of the complement of the zero locus is related to the cohomology of an affine variety. One hitch here is that the crucial cohomology comparison lemma of Fulton and Lazarsfeld [FL1, Lemma 1.6] applies only to birational maps with very restrictive fiber dimensions and is not applicable to our situation. We formulate and prove a generalization (Lemma 3.6) of the Fulton-Lazarsfeld lemma that compares the cohomology of a variety M with that of an affine image of M , allowing arbitrary positive-dimensional fibers and thereby completing the proof.

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Notational convention. Throughout this paper, by $\dim X$ we mean the complex dimension of X .

1. ISOTROPIC SUBSPACES

In this section we establish the characterization of the rank of a symmetric map by the dimensions of its isotropic subspaces. The result is undoubtedly

well-known to the experts and its inclusion is justified only by the lack of a suitable reference. So let E be a complex vector space of dimension e , and let E^* be its dual $\text{Hom}(E, \mathbb{C})$. A linear map $f: E \rightarrow E^*$ is said to be *symmetric* if $f = f^*$, where $f^*: E \rightarrow E^*$ is the dual of f . Equivalently, a linear map $f: E \rightarrow E^*$ is symmetric if and only if its associated bilinear form $\phi_f: E \times E \rightarrow \mathbb{C}$ is symmetric. Since there is a bijection between the set of all symmetric maps $f: E \rightarrow E^*$ and the set of all symmetric bilinear forms $\phi: E \times E \rightarrow \mathbb{C}$, we will speak interchangeably of the two sets. For example, we define the kernel of a symmetric bilinear form to be the kernel of its associated linear map. The following characterization of the rank of a linear map is a standard fact from linear algebra.

Proposition 1.1. *A linear map $f: E \rightarrow E^*$ has rank $\leq r$ if and only if $\dim(\ker f) \geq e - r$, where $e = \dim E$.*

Recall that an *isotropic subspace* of a symmetric bilinear form $\phi: E \times E \rightarrow \mathbb{C}$ is a subspace V of E such that $\phi|_V: V \times V \rightarrow \mathbb{C}$ is identically zero. For a symmetric map $f: E \rightarrow E^*$ there is a more interesting characterization of the rank in terms of isotropic subspaces (Proposition 1.4).

Proposition 1.2. *Let $\phi: E \times E \rightarrow \mathbb{C}$ be a symmetric bilinear form on a vector space E of dimension e . Then the dimension of any isotropic subspace of ϕ is at most $e - ((\text{rank } \phi)/2)$.*

Proof. Denote by K the kernel of the associated linear map $f: E \rightarrow E^*$ and let V be a maximal isotropic subspace. Clearly, by the symmetry of f , $K \subset V$ and ϕ induces a bilinear map $\phi': (V/K) \times (E/V) \rightarrow \mathbb{C}$. By the definition of K , the associated linear map $V/K \rightarrow (E/V)^*$ of ϕ' is injective. Therefore,

$$\dim V - \dim K \leq e - \dim V.$$

Since $\dim K = e - \text{rank } \phi$, the proposition follows. \square

A symmetric bilinear map $\phi: E \times E \rightarrow \mathbb{C}$ is said to be *nondegenerate* if $\phi(x, y) = 0$ for all y in E implies that $x = 0$.

Proposition 1.3. *Let $\phi: E \times E \rightarrow \mathbb{C}$ be a symmetric bilinear form on a vector space E of dimension e . Then*

- (i) *All maximal isotropic subspaces of ϕ have the same dimension.*
- (ii) *If $\text{rank } \phi$ is even, say $2p$, then a maximal isotropic subspace of ϕ has dimension $e - p$.*
- (iii) *If $\text{rank } \phi$ is odd, say $2p + 1$, then a maximal isotropic subspace of ϕ has dimension $e - p - 1$.*

Proof. (i) Denote by K the kernel of ϕ . Then ϕ induces a nondegenerate symmetric bilinear map $\phi': E/K \times E/K \rightarrow \mathbb{C}$. Any maximal isotropic subspace V of ϕ must contain K , and V/K is a maximal isotropic subspace of ϕ' . By [L, Corollary 2, p. 362] all maximal isotropic subspaces of the nondegenerate ϕ' have the same dimension. So if V_1 and V_2 are two maximal isotropic

subspaces of ϕ , then $\dim V_1/K = \dim V_2/K$. It follows that V_1 and V_2 have the same dimension.

(ii) Suppose $\text{rank } \phi = 2p$. Relative to some basis of E , the bilinear form ϕ is represented by the matrix

$$\begin{bmatrix} 0_p & I_p & 0 \\ I_p & 0_p & 0 \\ 0 & 0 & 0_{e-2p} \end{bmatrix},$$

where I_p is the $p \times p$ identity matrix and 0_p is the $p \times p$ zero matrix. This shows that ϕ has an isotropic subspace of dimension $e - p$, which is the maximum possible by Proposition 1.2.

(iii) Suppose $\text{rank } \phi = 2p + 1$. Relative to some basis of E , the bilinear form ϕ is represented by the matrix

$$\begin{bmatrix} 1 & & & & & & & & & \\ & 0_p & I_p & & & & & & & \\ & & & & & & & & & \\ & I_p & 0_p & & & & & & & \\ & & & & 0_{e-2p-1} & & & & & \end{bmatrix},$$

which shows that ϕ has an isotropic subspace of dimension $e - p - 1$, the maximum possible by Proposition 1.2. \square

Using this proposition we can construct a table of the dimensions of the maximal isotropic subspaces V of a symmetric bilinear map $\phi: E \times E \rightarrow \mathbb{C}$.

rank ϕ	0	1	2	3	4	5	6	7	8	...
$\dim V$	e	$e - 1$	$e - 1$	$e - 2$	$e - 2$	$e - 3$	$e - 3$	$e - 4$	$e - 4$...

Proposition 1.4. *A symmetric bilinear map $\phi: E \times E \rightarrow \mathbb{C}$ has rank $\leq 2p$ if and only if it has an isotropic subspace of dimension $e - p$.*

Proof. This follows directly from the table above. \square

Remarks. (1) As Proposition 1.4 shows, the dimensions of the isotropic subspaces characterize only the bilinear forms of ranks at most an even integer.

(2) Analogously, Proposition 1.4 remains true for a skew-symmetric bilinear map.

2. A DUALITY CONSTRUCTION

Let E be a vector space with coordinates z_1, \dots, z_e . We think of the dual space E^* , with coordinates a_1, \dots, a_e , as the space of all hyperplanes in E . Then there is a classic duality:

$$\{\text{points in } E\} \longleftrightarrow \{\text{linear forms on } E^*\} = \Gamma(\mathbf{P}(E^*), \mathcal{O}(1)),$$

$$z = (z_1, \dots, z_e) \longleftrightarrow z^* = \sum_{i=1}^e z_i a_i.$$

Globalizing this simple observation, if now E is a vector bundle over a variety X , then a section s of E over X induces a section s^* of $\mathcal{O}(1)$ over the projective bundle $\mathbf{P}(E^*)$, where $s^*(x, [e^*]) = e^*(s(x))$ for $e^* \in E_x^*$.

$$\begin{array}{ccc} & \mathcal{O}(1) & \\ & \downarrow s^* & \\ E & \mathbf{P}(E^*) & \\ \downarrow s & \swarrow & \\ X & & \end{array}$$

Proposition 2.1. *If E is a rank e vector bundle over a variety X and s is a section of E over X , then $\mathbf{P}(E^*) - Z(s^*)$ is a \mathbf{C}^{e-1} -bundle over $X - Z(s)$, where $Z(\cdot)$ denotes the zero locus of a section.*

Proof. We examine the zero locus of s^* on $\mathbf{P}(E^*)$ fiber by fiber. If $s(x) = 0$, then s^* vanishes on the entire fiber $\mathbf{P}(E_x^*)$. At a point x in X where $s(x) \neq 0$, the restriction of s^* to the fiber $\mathbf{P}(E_x^*)$ vanishes precisely at the point (x, V) , where V is the hyperplane in E_x defined by $(s(x))^* = 0$. Therefore, $\mathbf{P}(E^*) - Z(s^*)$ fibers over $X - Z(s)$ with fiber \mathbf{C}^{e-1} .

Corollary 2.2. *Under the hypothesis of the proposition,*

$$H^q(\mathbf{P}(E^*) - Z(s^*); \mathbf{Z}) \simeq H^q(X - Z(s); \mathbf{Z}).$$

The vector bundle E over X is said to be *ample* if the tautological line bundle $\mathcal{O}(1)$ over $\mathbf{P}(E^*)$ is ample. This implies immediately that $\mathbf{P}(E^*) - Z(s^*)$ is an affine variety.

3. SYMMETRIC DEGENERACY LOCI

Given a rank e vector bundle E and a line bundle L over a variety X and a symmetric bundle map $u: E \otimes E \rightarrow L$ over X , recall that the *rank r degeneracy locus* of u is defined to be

$$D_r(u) = \{x \in X \mid \text{rank } u(x) \leq r\}.$$

For the sake of simplicity we assume in this section that X is smooth and L is trivial. Our goal now is to prove the following theorem.

Theorem 3.1. *Let $u: E \otimes E \rightarrow \mathbf{C}$ be a symmetric bundle map over a smooth variety X , and p a nonnegative integer. If $\text{Sym}^2 E^*$ is ample and $\dim X - \binom{e-2p+1}{2} \geq 1$, then $D_{2p}(u)$ is connected.*

By the correspondence between symmetric linear maps and quadratic forms, u may be regarded as a section of the vector bundle $\text{Sym}^2 E^*$ over X . Using the characterization of the rank in Proposition 1.4 we can represent the degeneracy locus $D_{2p}(u)$ as the image of a zero locus, as follows.

Let $\pi: G = G(e-p, E) \rightarrow X$ be the bundle whose fiber at each point x in X is the Grassmannian $G(e-p, E_x)$ of $(e-p)$ -dimensional subspaces of the fiber E_x , and let S be the universal subbundle over G . The inclusion map

$S \hookrightarrow \pi^* E$ over G induces the restriction map $\text{Sym}^2 \pi^* E^* \rightarrow \text{Sym}^2 S^*$. The section $u: X \rightarrow \text{Sym}^2 E^*$ gives rise naturally to a section $(\pi^* u)(x, V) = u(x)$ of $\pi^* \text{Sym}^2 E^*$ over $G(e - p, E)$ and hence by restriction to a section of $\text{Sym}^2 S^*$ over G :

$$t(x, V) = u(x)|_V: V \rightarrow V^*.$$

$$\begin{array}{ccc}
 \pi^* \text{Sym}^2 E^* & \xrightarrow{\quad} & \text{Sym}^2 S^* \\
 \swarrow \pi^* u & & \nearrow t \\
 Z(t) \subset G = G(e - p, E) & & \\
 \searrow \pi & & \uparrow u \\
 D_{2p}(u) \subset X & & \text{Sym}^2 E^*
 \end{array}$$

Proposition 3.2. *The degeneracy locus $D_{2p}(u)$ is the image under π of the zero locus $Z(t)$.*

Proof. This proposition follows from the following series of equivalences:

$$\begin{aligned}
 x \in D_{2p}(u) & \text{ iff rank } u(x) \leq 2p \\
 & \text{ iff } u(x) \text{ has an } (e - p)\text{-dimensional isotropic subspace } V \subset E_x \\
 & \text{ iff } t(x, V) = 0 \\
 & \text{ iff } (x, V) \in Z(t). \quad \square
 \end{aligned}$$

Hence, it suffices to prove the connectedness of the zero locus $Z(t)$.

Lemma 3.3. *Let M be a connected compact orientable manifold of (real) dimension n and A a subset of M . Then*

- (i) A is nonempty iff $H^n(M - A; \mathbf{Z}) = 0$;
- (ii) A is connected if $H^n(M - A; \mathbf{Z}) = H^{n-1}(M - A; \mathbf{Z}) = 0$.

Proof. Both statements are consequences of Lefschetz duality

$$H^{n-q}(M - A) = H_q(M, A)$$

and the long exact sequence of a pair:

$$\begin{array}{ccccccc}
 \rightarrow H_1(M, A) & \rightarrow & H_0(A) & \rightarrow & H_0(M) & \rightarrow & H_0(M, A) \rightarrow 0. \\
 & & & & \parallel & & \\
 & & & & \mathbf{Z} & &
 \end{array}$$

□

By this lemma, the connectedness of the locus $Z(t)$ in the Grassmann bundle $G = G(e - p, E)$ follows from the vanishing of the two cohomology groups

$H^{2\dim G}(G - Z(t); \mathbf{Z})$ and $H^{2\dim G-1}(G - Z(t); \mathbf{Z})$. Instead of computing the cohomology of $G - Z(t)$ directly, we can apply Corollary 2.2 and compute the cohomology of $\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*)$ first. The advantage here is that one can map $\mathbf{P}(\mathrm{Sym}^2 S) - Z(t^*)$ to the affine variety $\mathbf{P}(\mathrm{Sym}^2 E) - Z(u^*)$, and use the Leray spectral sequence to compare the cohomology of these two spaces. As an affine variety the cohomology of $\mathbf{P}(\mathrm{Sym}^2 E) - Z(u^*)$ vanishes above its middle dimension. So the computation of $H^*(G - Z(t))$ will be based on the following diagram:

$$\begin{array}{c}
 \begin{array}{ccc}
 P - Z(t^*) \subset P = \mathbf{P}(\mathrm{Sym}^2 S) & & \\
 \downarrow \text{same cohomology} & \downarrow \pi^* \mathrm{Sym}^2 E^* \rightarrow \mathrm{Sym}^2 S^* & \\
 G - Z(t) \subset G = G(e - p, E) & \searrow t & \mathrm{Sym}^2 S^*
 \end{array} \\
 (3.4) \quad \begin{array}{ccc}
 & & \mathcal{O}(1) \\
 & & \downarrow \uparrow u^* \\
 & \mathrm{Sym}^2 E^* & \mathbf{P}(\mathrm{Sym}^2 E) = P' \supset P' - Z(u^*) \\
 \pi \swarrow & \downarrow u & \nwarrow \\
 D_{2p}(u) \subset X & &
 \end{array}
 \end{array}$$

h

Let $P = \mathbf{P}(\mathrm{Sym}^2 S)$ and $P' = \mathbf{P}(\mathrm{Sym}^2 E)$. In this diagram $P - Z(t^*)$ is an affine-space bundle over $G - Z(t)$ by Proposition 2.1 and therefore has the same cohomology as $G - Z(t)$. By hypothesis $\mathrm{Sym}^2 E^*$ is an ample vector bundle over X , so $\mathcal{O}_{P'}(1)$ is an ample line bundle over P' , and $P' - Z(u^*)$ is an affine variety. The map $h: \mathbf{P}(\mathrm{Sym}^2 S) \rightarrow \mathbf{P}(\mathrm{Sym}^2 E)$ is induced from the inclusion of $\mathrm{Sym}^2 V$ in $\mathrm{Sym}^2 E_x$:

$$h(x, V \subset E_x, \phi \in \mathrm{Sym}^2 V) = (x, \phi \in \mathrm{Sym}^2 E_x).$$

Proposition 3.5. *The map $h: P \rightarrow P'$ sends $P - Z(t^*)$ to $P' - Z(u^*)$.*

Proof. By the definition of t^* ,

$$\begin{aligned}
 t^*(x, V \subset E_x, \phi \in \mathrm{Sym}^2 V) &= (u(x)|_V)^*(\phi) = u(x)^*(\phi) \\
 &= u^*(x, \phi \in \mathrm{Sym}^2 E_x) = u^*(h(x, V, \phi)).
 \end{aligned}$$

So if $t^*(x, V, \phi) \neq 0$, then $u^*(h(x, V, \phi)) \neq 0$ and h sends $P - Z(t^*)$ to $P' - Z(u^*)$. \square

To compute $H^*(P - Z(t^*))$ we will apply the following cohomology lemma, whose proof is postponed to the next section. Denote by N the set of natural numbers, $\{0, 1, 2, 3, \dots\}$.

Lemma 3.6. *Let $h: M \rightarrow Y$ be a surjective proper morphism from any variety M to an affine variety Y . Suppose there is a strictly increasing function $d: N \rightarrow N$ and a sequence of closed subvarieties*

$$\cdots \subset Y_{k+1} \subset Y_k \subset Y_{k-1} \subset \cdots \subset Y_0 = Y$$

such that for all x in $Y_k - Y_{k+1}$,

$$d(k) = \dim_{\mathbb{C}} h^{-1}(x).$$

Define

$$R = \max_{k \geq 0} \{ \dim_{\mathbb{C}} Y_k + 2d(k) \}.$$

Then $H^q(M, \mathbb{Z}) = 0$ for all $q > R$.

To apply this lemma, let $\mathbf{P}(D_k(\text{Sym}^2 E))$ be the subvariety of $\mathbf{P}(\text{Sym}^2 E)$ consisting of elements of rank at most k , and let

$$U_k = \mathbf{P}(D_k(\text{Sym}^2 E)) \cap (P' - Z(u^*)).$$

The image of $h: P - Z(t^*) \rightarrow P' - Z(u^*)$ lies in U_{e-p} . Define $Y_k = U_{e-p-k}$. Then

$$\cdots \subset Y_{k+1} \subset Y_k \subset Y_{k-1} \subset \cdots \subset Y_0$$

and $\dim Y_k = \dim P' - \binom{p+k+1}{2}$. For $(x, \phi \in \text{Sym}^2 E_x) \in Y_k - Y_{k+1}$, ϕ may be viewed as a symmetric map $\phi: E_x^* \rightarrow E_x$ of rank exactly $e - p - k$, so that $\text{im } \phi$ is an $(e - p - k)$ -dimensional subspace of E_x . Thus,

$$\begin{aligned} h^{-1}(x, \phi) &\simeq \{V \in G(e - p, E_x) \mid \text{im } \phi \subset V \subset E_x\} \\ &\simeq G((e - p) - (e - p - k), E_x / \text{im } \phi) \\ &\simeq G(k, p + k). \end{aligned}$$

Therefore, for $(x, \phi) \in Y_k - Y_{k+1}$, $\dim_{\mathbb{C}} h^{-1}(x, \phi) = pk$. In the cohomology lemma (3.6) set $d(k) = pk$. Then

$$\begin{aligned} R &= \max_{k \geq 0} \{ \dim Y_k + 2pk \} \\ &= \max_{k \geq 0} \left\{ \dim P' - \binom{p+k+1}{2} + 2pk \right\} \\ &= \max_{k \geq 0} \left\{ \dim P' - p - \frac{(k-p+1)(k-p)}{2} \right\}. \end{aligned}$$

For integer values of p and k , the minimum of $(k - p + 1)(k - p)/2$ is 0, when $k = p$ or $p - 1$. Therefore,

$$R = \dim P' - p = \dim X + \binom{e+1}{2} - 1 - p.$$

By Lemma 3.6,

$$(3.7) \quad H^q(P - Z(t^*); \mathbb{Z}) = 0 \quad \text{for } q \geq \dim X + \binom{e+1}{2} - p.$$

A straightforward computation shows that

$$\dim X \geq \binom{e-2p+1}{2} + 1 \Leftrightarrow 2 \dim G - 1 \geq \dim X + \binom{e+1}{2} - p.$$

Therefore, if $\dim X \geq \binom{e-2p+1}{2} + 1$, then

$$H^{2 \dim G}(G - Z(t); \mathbf{Z}) = H^{2 \dim G-1}(G - Z(t); \mathbf{Z}) = 0.$$

By Proposition 3.2 and the remark following Lemma 3.3, $Z(t)$ and hence $D_{2p}(u)$ is connected.

4. PROOF OF THE COHOMOLOGY LEMMA

We prove here Lemma 3.6, which allows one to compare the cohomology of a variety M with the cohomology of an affine image Y of M . For technical reasons that will be apparent shortly, we extend the definition of the fiber-dimension function d to $\{-1, 0, 1, 2, \dots\}$ by defining $d(-1) = -1$. This makes sense, for the fibers over the empty set $Y_{-1} - Y_0$ are of course the empty sets. In the notations of Lemma 3.6, applying the Leray spectral sequence to the surjective map $h: M \rightarrow Y$, we find that $H^j(Y, R^i h_* \mathbf{Z})$ abuts to $H^{i+j}(M, \mathbf{Z})$. So it suffices to prove that $H^j(Y, R^i h_* \mathbf{Z}) = 0$ for all $i + j > R$.

Claim. Whenever there is a $k \in N$ such that

$$(4.1) \quad i > 2d(k-1) \quad \text{and} \quad j > \dim Y_k,$$

then $H^j(Y, R^i h_ \mathbf{Z}) = 0$.*

Proof. Suppose $i > 2d(k-1)$. For dimension reasons, $R^i h_* \mathbf{Z}$ vanishes on $Y - Y_k$. Thus, $R^i h_* \mathbf{Z}$ is concentrated on Y_k and

$$H^j(Y, R^i h_* \mathbf{Z}) = H^j(Y_k, R^i h_* \mathbf{Z}).$$

As a direct image of a constructible sheaf, $R^i h_* \mathbf{Z}$ is again constructible on Y_k . By a standard theorem, the cohomology of a constructible sheaf on an affine variety vanishes above the dimension of the variety. So $H^j(Y_k, R^i h_* \mathbf{Z}) = 0$ for $j > \dim Y_k$. This concludes the proof of the claim.

Since $d: \{-1, 0, 1, 2, \dots\} \rightarrow \{-1, 0, 1, 2, \dots\}$ is a strictly increasing function, given any nonnegative integer i we can choose k such that

$$2d(k) \geq i > 2d(k-1).$$

(This inequality is the reason the domain and the image of the function d must contain -1 , since i may be 0 .) If $i + j > R$, then

$$j > R - i \geq R - 2d(k) \geq \dim Y_k,$$

where the last inequality follows from the definition of R . By (4.1),

$$H^j(Y_k, R^i h_* \mathbf{Z}) = 0 \quad \text{for all } i + j > R.$$

5. GENERALIZATIONS

In §3 we proved the connectedness conjecture (0.2) for an even-rank symmetric degeneracy locus under the assumptions that the ambient space X is smooth and that the line bundle L is trivial. It is not difficult to remove both restrictions.

In the proof of §3 the smoothness assumption was needed only to invoke Lefschetz duality and to conclude that if $H^q(G - Z(t); \mathbf{Z}) = 0$ for $q = 2 \dim G$, $2 \dim G - 1$, then $H_q(G, Z(t); \mathbf{Z}) = 0$ for $q = 0, 1$. Thus, to generalize from a smooth variety to an arbitrary irreducible variety X , what is needed is a substitute for Lefschetz duality. This is provided by an argument in [FL1], which in fact goes through verbatim in the present situation. Their idea is first to show that one may assume X normal. For if $\nu: \tilde{X} \rightarrow X$ is the normalization of the irreducible variety X , then \tilde{X} is also irreducible. Furthermore, the section u of $\text{Sym}^2 E^*$ over X pulls back to a section ν^*u of $\text{Sym}^2 E^*$ over \tilde{X} , and $D_r(u)$ is the image of $D_r(\nu^*u)$ under ν . Since the pullback of an ample vector bundle under a finite morphism is again ample, $\nu^* \text{Sym}^2 E^*$ is ample. So it suffices to prove the connectedness of $D_r(\nu^*u)$. Replacing X by \tilde{X} , we may assume X normal. Then the Grassmann bundle G over X is also normal, and the following lemma applies.

Lemma [FL1, Lemma 1.3]. *Let G be a normal projective variety of dimension m and let $Z \subset G$ be a closed algebraic subset. Then there is an injection*

$$H^1(G, Z; \mathbf{Q}) \hookrightarrow H_{2m-1}(G - Z; \mathbf{Z}).$$

By the universal coefficient theorem, $H^q(G - Z(t); \mathbf{Z}) = 0$ for $q = 2 \dim G$, $2 \dim G - 1$ implies that $H_{2 \dim G - 1}(G - Z(t); \mathbf{Q}) = 0$ [M, Corollary 56.4 and Theorem 53.5]. Hence, $H^1(G, Z(t); \mathbf{Z}) = 0$ and $Z(t)$ is connected. This concludes the proof when X is an irreducible variety.

Finally, suppose L is an arbitrary line bundle. If L happens to be the square of a line bundle, say $L \simeq K \otimes K$, then $(\text{Sym}^2 E^*) \otimes L \simeq \text{Sym}^2(E^* \otimes K)$ and Conjecture 0.2 for even ranks would follow immediately from Theorem 3.1 (but with X an irreducible instead of a smooth variety). The following lemma of Bloch and Gieseker effectively allows one to assume that every bundle is a square.

Lemma 5.1 [BG, Lemma 2.1]. *Let L be a line bundle on a projective variety X and d a positive integer. Then there exists a projective variety Y , a finite surjective morphism $f: Y \rightarrow X$, and a line bundle K on Y such that $f^*L \simeq K^{\otimes d}$.*

Proof. See [BG, p. 114 or FL2, Lemma 1.1]. \square

Returning to the proof of Conjecture 0.2 for even ranks, we take $d = 2$ in Lemma 5.1. Then

$$f^*((\text{Sym}^2 E^*) \otimes L) \simeq (f^* \text{Sym}^2 E^*) \otimes K^2 \simeq \text{Sym}^2(f^* E^* \otimes K).$$

The degeneracy locus $D_r(f^*u)$ in Y maps surjectively onto $D_r(u)$ in X . Since the pullback of an ample vector bundle under a finite morphism is again ample, we conclude by Theorem 3.1 (again with X irreducible instead of smooth) that $D_r(f^*u)$ is connected. Therefore, $D_r(u)$ is also connected.

6. SKEW-SYMMETRIC DEGENERACY LOCI

By the argument of §5, Conjecture 0.3 can also be reduced to the case where X is a smooth variety and L is a trivial line bundle. To prove Conjecture 0.3 under these assumptions, replace Sym^2 by \wedge^2 in Diagram 3.4. This gives a map $h: P - Z(t^*) \rightarrow P' - Z(u^*)$, where $P = \mathbf{P}(\wedge^2 S)$ and $P' = \mathbf{P}(\wedge^2 E)$. Let $U_k = \mathbf{P}(D_k(\wedge^2 E)) \cap P' - Z(u^*)$. The proof proceeds as in the even-rank symmetric case, except now because the rank of a skew-symmetric matrix is always even, the image of h lies in U_{e-p} or U_{e-p-1} depending on whether $e-p$ is even or odd, and the rank drops by two at a time. To combine the two cases in one proof, let

$$l = \begin{cases} 2k, & \text{if } e-p \text{ is even,} \\ 2k+1, & \text{if } e-p \text{ is odd,} \end{cases}$$

and define $Y_k = U_{e-p-l}$. Then the image of h lies in Y_0 ,

$$\dim Y_k = \dim P' - \binom{p+l}{2}$$

and

$$\cdots \subset Y_{k+1} \subset Y_k \subset Y_{k-1} \subset \cdots \subset Y_0.$$

If $(x, \phi \in \wedge^2 E_x) \in Y_k - Y_{k+1}$, then

$$h^{-1}(x, \phi) \simeq G(l, p+l) \quad \text{and} \quad \dim_{\mathbb{C}} h^{-1}(x, \phi) = pl.$$

In the cohomology lemma (3.6),

$$\begin{aligned} R &= \max_{k \geq 0} \left\{ \dim P' - \binom{p+l}{2} + 2pl \right\} \\ &= \max_{k \geq 0} \left\{ \dim P' + p - \frac{(p-l+1)(p-l)}{2} \right\} \\ &= \dim P' + p = \dim X + \binom{e}{2} - 1 + p. \end{aligned}$$

By Lemma (3.6),

$$H^q(P - Z(t^*); \mathbf{Z}) = 0 \quad \text{for } q \geq \dim X + \binom{e}{2} + p.$$

Again one checks easily that

$$\dim X \geq \binom{e-2p}{2} + 1 \Leftrightarrow 2 \dim G - 1 \geq \dim X + \binom{e}{2} + p.$$

So under the hypotheses of Conjecture 0.3,

$$H^{2 \dim G}(G - Z(t); \mathbf{Z}) = H^{2 \dim G - 1}(G - Z(t); \mathbf{Z}) = 0$$

and the connectedness of $D_{2p}(u)$ follows as before.

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