TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 313, Number 1, May 1989

Δ-CLOSURES OF IDEALS AND RINGS

LOUIS J. RATLIFF, JR.

ABSTRACT. It is shown that if R is a commutative ring with identity and Δ is a multiplicatively closed set of finitely generated nonzero ideals of R, then the operation $I \to I_\Delta = \bigcup_{K \in \Delta}(IK : K)$ is a closure operation on the set of ideals I of R that satisfies a partial cancellation law, and it is a prime operation if and only if R is Δ-closed. Also, if none of the ideals in Δ is contained in a minimal prime ideal, then $I_\Delta \subseteq I_a$, the integral closure of I in R, and if Δ is the set of all such finitely generated ideals and I contains an ideal in Δ, then $I_\Delta = I_a$. Further, R has a natural Δ-closure R^Δ, $A \to A^\Delta$ is a closure operation on a large set of rings A that contain R as a subring, $A \to A^\Delta$ behaves nicely under certain types of ring extension, and every integral extension overring of R is R^Δ for an appropriate set Δ. Finally, if R is Noetherian, then the associated primes of I_Δ are also associated primes of $I_\Delta K$ and $(IK)_\Delta$ for all $K \in \Delta$.

1. INTRODUCTION

Integral dependence is one of the most widely used concepts in commutative algebra. From several points of view, integrally closed ideals and rings are nicer and easier to work with than their nonclosed counterparts. The subject itself is classical, and a list of mathematicians who have helped develop its current foundational place includes Cohen, Grell, Krull, Mori, Nagata, Noether, Northcott, Rees, Samuel, Seidenberg, and Zariski, to mention only a few.

The current paper is concerned, somewhat indirectly, with integral dependence; actually, the "closure" part of the subject. Closure operations (see (2.3) for the definitions) are of some interest in themselves, and they have been studied both in relation to specific ideal-closures (such as integral closure) and abstractly; for example, see [2, 3, 4, §43, 6, 8, 10, Appendix 4]. In this paper a new family of closure operations is introduced and several of the classical properties of integral closure are extended to these new closures. (Actually, for this paper I used as a working guide: *a result for integral closure should have a valid analogue for Δ-closure.* This is probably not a meta-theorem, but it is certainly

Received by the editors September 23, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 13A15, 13B20, 13C99; Secondary 13-01.
Key words and phrases. ACC on colon ideals, cancellation law, closure operation, directed set, flat extension ring, integral closure of an ideal, integral extension ring, Noetherian ring, polynomial extension ring, prime divisor, prime operation, reduction of an ideal, Rees ring, semiprime operation.
Research on this paper was supported by National Science Foundation Grant DMS-8521058.

a good precept to keep in mind while working with Δ-closures.) Specifically, for a multiplicatively closed set Δ of finitely generated nonzero ideals of a ring R (commutative with identity) we define the Δ-closure I_Δ of an ideal I of R and then show that the operation $I \to I_\Delta$ is a closure operation that satisfies the Δ-cancellation law: if $(IK)_\Delta \subseteq (JK)_\Delta$ and $K \in \Delta$, then $I_\Delta \subseteq J_\Delta$. Also, $I \to I_\Delta$ is a prime operation on the set of ideals I of R if and only if R is Δ-closed, and $I_\Delta \subseteq I_a$ for most such sets Δ and ideals I. Further, there is a natural Δ-closure R^Δ of R in its total quotient ring, $A \to A^\Delta$ is a closure operation on a large set of rings A that contain R as a subring, and $A \to A^\Delta$ behaves nicely under certain types of ring extension. Moreover, R^Δ plays a role relative to the ideals I_Δ that is analogous to the role played by the integral closure R' of R relative to the ideals I_a, and every integral extension overring of R is an R^Δ for an appropriate set Δ. Finally, if R is Noetherian, then it is shown that several recent results concerning the asymptotic prime divisors of I are special cases of results concerning the associated prime ideals of I_Δ.

Throughout this paper the terminology is standard, the methods are generally elementary, most proofs are short and straightforward, and some of them are patterned after their classical (integral dependence) analogues. Simply stated, this is a rather nice piece of "almost classical" (1930–1960) commutative algebra that seems to have been overlooked until now. The working guide (mentioned above) made the results seem natural, and I think they are potentially very useful; for example, in regard to certain of its ideals, R^Δ has some of the nice properties that R' does, and R^Δ may be a finite R-module when R' is not.

A brief summary of this paper will now be given. In §2 the Δ-closure I_Δ of I is introduced and it is shown that $I \to I_\Delta$ is a semiprime operation that satisfies the Δ-cancellation law. The relationships between I_Δ and I_a are developed in §3, and characterizations are given (in terms of I_Δ) for I and R to be integrally closed. §4 contains a few miscellaneous properties of the ideals I_Δ, and the results in §5 are concerned with the relationship between the ideals I_Δ and the analogous closure of the ideals IB, where B is an R-algebra. In §6 it is shown that R has a Δ-closure R^Δ, that $A \to A^\Delta$ is a closure operation on a large set of rings A containing R as a subring, and that R^Δ is a good analogue of the integral closure R' of R. The results in §7 show that $A \to A^\Delta$ behaves nicely under certain types of ring extension, and in §8 it is shown that if A is an integral extension ring of R that is contained in the total quotient ring of R, then $A = R^\Delta$ for an appropriate set Δ. Finally, it is shown in §9 that if R is Noetherian, then the associated prime ideals of I_Δ are also associated primes of $I_\Delta K$ and $(IK)_\Delta$ for all $K \in \Delta$, and it is then shown that this implies some of the recent results in the literature concerning asymptotic prime divisors.

2. The Δ-closure of an ideal

The Δ-closure I_Δ of an ideal I is introduced in this section, and it is shown that $I \to I_\Delta$ is a semiprime operation that satisfies the Δ-cancellation law.

Throughout this section, R is a commutative ring with identity and Δ is an arbitrary (but fixed) multiplicatively closed set of finitely generated nonzero ideals of R.

(2.1) **Theorem.** *If I is an ideal in R, then $\mathbf{D}(I) = \{IK : K; K \in \Delta\}$ is a directed set and $I_\Delta = \bigcup\{IK : K; K \in \Delta\} = \sum_{K\in\Delta}(IK : K)$ is an ideal in R such that $I_\Delta = I_\Delta K : K$ for all $K \in \Delta$.*

Proof. Let H and K be two ideals in Δ. Then $IH : H$ and $IK : K$ are both contained in $IHK : HK$, and $IHK : HK \in \mathbf{D}(I)$, since $HK \in \Delta$, so $\mathbf{D}(I)$ is a directed set of ideals of R. Therefore it is readily checked that $\bigcup\{IK : K; K \in \Delta\} = \sum_{K\in\Delta}(IK : K)$, so I_Δ is an ideal in R. And, if $K = (b_1, \dots, b_n)R \in \Delta$ and $x \in I_\Delta K : K$, then

$$\begin{aligned} xK \subseteq I_\Delta K &= \left(\bigcup_{H\in\Delta}(IH : H)\right)K = \left(\sum_{H\in\Delta}(IH : H)\right)K \\ &= \sum_{H\in\Delta} K(IH : H) \subseteq \sum_{H\in\Delta}(IKH : H) = \bigcup_{H\in\Delta}(IKH : H), \end{aligned}$$

so for $i = 1, \dots, n$ there exists an ideal $H_i \in \Delta$ such that $xb_i \in IKH_i : H_i$. Let $L = H_1 \cdots H_n$, so $L \in \Delta$, and $xb_i \in IKL : L$ for $i = 1, \dots, n$. Therefore $xK \subseteq IKL : L$, so $x \in (IKL : L) : K = IKL : KL \subseteq \bigcup_{H\in\Delta}(IH : H) \subseteq I_\Delta$. Thus $I_\Delta K : K \subseteq I_\Delta$, and the opposite inclusion is clear. Q.E.D.

It is clear from (2.1) that if $IK : K \in \mathbf{D}(I)$, then $IK : K \subseteq I_\Delta$. And it is also clear from (2.1) that an element x is in I_Δ if and only if $x \in IK : K$ for some $K \in \Delta$ if and only if $xK \subseteq IK$ for some $K \in \Delta$. These facts will often be used implicitly in what follows.

(2.2) **Definition.** The Δ-*closure* of I is the ideal $I_\Delta = \bigcup_{K\in\Delta}(IK : K) = \sum_{K\in\Delta}(IK : K)$ given by (2.1).

The main reason for this terminology is that I_Δ is usually a small (incremental) enlargement of I. Specifically, it is shown in (3.2.1) that if no ideal in Δ is contained in any minimal prime ideal, then I_Δ is contained in the integral closure of I.

If Δ is a multiplicatively closed subset of the set of regular principal ideals bR of R (an ideal is regular in case it contains a nonzero divisor), then $IbR : bR = I$, so it follows from (2.1) that $I_\Delta = I$ for all ideals I. Also, if $\Delta_1 \subseteq \Delta_2$ are two muiltiplicatively closed sets of finitely generated nonzero ideals of R, then it is clear that the Δ_1-closure of each ideal is contained in its Δ_2-closure. (4.1) contains a few more miscellaneous properties of the ideals I_Δ.

In (2.4) it is shown that $I \to I_\Delta$ is a closure operation and, more specifically, a semiprime operation, so we now give the appropriate definitions.

(2.3) **Definition.** Let $I \to I_x$ be an operation on the set of ideals I of a ring R, and consider the following rules, where I and J are ideals of R and b is a regular nonunit in R: (a) $I \subseteq I_x$; (b) if $I \subseteq J$, then $I_x \subseteq J_x$; (c) $(I_x)_x = I_x$;

(d) $I_x J_x \subseteq (IJ)_x$; and (e) $(bI)_x = bI_x$. Then $I \to I_x$ is a *closure operation* if (a)–(c) hold for all ideals I and J in R, it is a *semiprime operation* if (a)–(d) hold for all ideals I and J in R, and it is a *prime operation* if (a)–(e) hold for all ideals I and J and regular nonunits b of R.

It should be noted that the definition of a prime operation in (2.3) differs from that given by Krull in [4, §43] and also from that given by Zariski and Samuel in [10, Appendix 4]. In both [4] and [10] a prime operation satisfies (a)–(e). However, in [4], $I \to I_x$ is applied more generally to fractional ideals I, but it is assumed that R is an integrally closed integral domain. And in [10], $I \to I_x$ is applied to the R-modules I contained in an extension field K of an integral domain R, and a prime operation must satisfy the cancellation law and also be such that $R_x = R'$, the integral closure of R in K. However, in both [4] and [10] it is pointed out that all prime operations that satisfy these additional properties are essentially known. Since we want to derive a new family of prime operations in this paper, we chose the definition in (2.3), which does agree with that given in [6] and [8].

(2.4) **Theorem.** *The operation $I \to I_\Delta$ is a semiprime operation on the set of ideals I of R; that is, for all ideals I and J of R:*

(2.4.1) $$I \subseteq I_\Delta,$$

(2.4.2) $$I \subseteq J \Rightarrow I_\Delta \subseteq J_\Delta,$$

(2.4.3) $$(I_\Delta)_\Delta = I_\Delta,$$

(2.4.4) $$I_\Delta J_\Delta \subseteq (IJ)_\Delta.$$

Proof. If $K \in \Delta$, then $I \subseteq IK : K \in \mathbf{D}(I)$, so (2.4.1) holds by (2.1). Also, if $I \subseteq J$ are ideals in R and $K \in \Delta$, then $IK \subseteq JK$, so $IK : K \subseteq JK : K \in \mathbf{D}(J)$, hence it follows from (2.1) that (2.4.2) holds. Further, it follows from $I_\Delta = I_\Delta K : K$ for all $K \in \Delta$ (see (2.1)) that $(I_\Delta)_\Delta = I_\Delta$. Finally, if $x \in I_\Delta$ and $y \in J_\Delta$, then there exist ideals H_1 and H_2 in Δ such that $x \in IH_1 : H_1$ and $y \in JH_2 : H_2$. Therefore $xyH_1H_2 \subseteq IJH_1H_2$, so $xy \in IJH_1H_2 : H_1H_2 \in \mathbf{D}(IJ)$, so it follows that (2.4.4) holds. Q.E.D.

(2.5) *Remark.* Let Γ be an index set. Then it is known (and readily verified) that for any semiprime operation $I \to I_x$ on the set of ideals I of R it holds that $(I_x J_x)_x = (IJ)_x$, $(\sum_{i\in\Gamma}(I_i)_x)_x = (\sum_{i\in\Gamma} I_i)_x$, and $(\bigcap_{i\in\Gamma}(I_i)_x)_x = \bigcap_{i\in\Gamma}(I_i)_x$. Therefore it follows from (2.4) that

(a) $(I_\Delta J_\Delta)_\Delta = (IJ)_\Delta$,

(b) $(\sum_{i\in\Gamma}(I_i)_\Delta)_\Delta = (\sum_{i\in\Gamma} I_i)_\Delta$, and

(c) $(\bigcap_{i\in\Gamma}(I_i)_\Delta)_\Delta = \bigcap_{i\in\Gamma}(I_i)_\Delta$.

The following proposition is quite useful in its own right, and it will be used in the proof of the Δ-cancellation law (2.7).

(2.6) **Proposition.** *If I is an ideal in R and if either $K \in \Delta$ or K is a regular principal ideal, then $(IK)_\Delta : K = I_\Delta K : K = I_\Delta$.*

Proof. It is clear that $I_\Delta \subseteq I_\Delta K : K$, and it follows from (2.4.1) and (2.4.4) that $I_\Delta K : K \subseteq (IK)_\Delta : K$, so it remains to show that $(IK)_\Delta : K \subseteq I_\Delta$. For this, let Ω be the multiplicatively closed set of ideals generated by Δ and the set of all regular principal ideals of R. Then it is readily seen that $I_\Delta = I_\Omega$ for all ideals I in R, so it suffices to consider the case when $K \in \Delta$. For this, let $x \in (IK)_\Delta : K$. Then $xK \subseteq (IK)_\Delta = \bigcup_{H\in\Delta}(IKH : H)$, so it follows as in the proof of (2.1) that $x \in I_\Delta$, so $(IK)_\Delta : K \subseteq I_\Delta$. Q.E.D.

By (2.4) the following ideals lie between I_Δ and $(IK)_\Delta : K$, so by (2.6) they are all equal when $K \in \Delta$: $I_\Delta, I_\Delta K_\Delta : K_\Delta, I_\Delta K_\Delta : K, I_\Delta K : K, (IK)_\Delta : K_\Delta$, and $(IK)_\Delta : K$.

(2.7) **Theorem** (Δ-Cancellation Law). *If I, J, and K are ideals in R such that either $K \in \Delta$ or K is a regular principal ideal, and if $IK \subseteq (JK)_\Delta$, then $I \subseteq J_\Delta$. Therefore if $(IK)_\Delta = (JK)_\Delta$, then $I_\Delta = J_\Delta$.*

Proof. If $IK \subseteq (JK)_\Delta$, then $I \subseteq (JK)_\Delta : K = J_\Delta$, by (2.6). Therefore, if $(IK)_\Delta = (JK)_\Delta$, then $IK \subseteq (JK)_\Delta$, by (2.4.1), so $I \subseteq J_\Delta$, by what was just proved, hence $I_\Delta \subseteq J_{\Delta\Delta} = J_\Delta$, by (2.4.2) and (2.4.3). Therefore it follows by symmetry that $I_\Delta = J_\Delta$. Q.E.D.

By ignoring the first statement in (2.7) and instead using (2.6) on both sides of $(IK)_\Delta = (JK)_\Delta$, we get the following shorter proof of the last statement in (2.7): $I_\Delta = (IK)_\Delta : K = (JK)_\Delta : K = J_\Delta$.

The converse of (2.7) also holds: if $I_\Delta = J_\Delta$, then $(IK)_\Delta = (JK)_\Delta$ for all ideals K in R (not just those in Δ). (For, $I_\Delta = J_\Delta$ implies $I_\Delta K_\Delta = J_\Delta K_\Delta$, so the conclusion follows from (2.5)(a).) A stronger converse is true if R satisfies the ACC on colon ideals, as will be shown in (2.9). However, to prove (2.9) we need the following remark.

(2.8) *Remark.* Assume that R satisfies the ACC on colon ideals. Then by this ACC (but without assuming that the ideals in Δ are finitely generated) it is easy to show that (a) for each ideal I in R there exists an ideal K in Δ such that $I_\Delta = IK : K \in \mathbf{D}(I)$ and (b) if $I_\Delta = IK : K$, then $I_\Delta = I_\Delta H : H = IKH : KH$ for all $H \in \Delta$ (since $IK : K \subseteq ((IK : K)H) : H \subseteq IKH : KH \in \mathbf{D}(I))$.

It is only in the proofs that $I_\Delta K : K = I_\Delta$ (in (2.1)) and $(IK)_\Delta : K = I_\Delta$ (in (2.6)) that the finite generation of the ideals in Δ has been directly used. (The first of these was used to show that $I_{\Delta\Delta} = I_\Delta$, and the second was used to prove (2.7).) Using (2.8), it is easy to show that (2.1) and (2.6) hold without assuming that the ideals in Δ are finitely generated, so if R satisfies the ACC on colon ideals, then all the previous results in this section (and in the remainder of the paper, except for (5.1.2), (5.2.1), (6.6.3), and the results following (6.6.3) that reference one of these three results) hold for arbitrary multiplicatively closed sets of nonzero ideals of such a ring R.

(2.9) **Theorem.** *Assume that R satisfies the ACC on colon ideals and let I and J be ideals in R. Then the following are equivalent*:

(2.9.1) $$ID = JD \quad \textit{for some } D \in \Delta,$$

(2.9.2) $$(IK)_\Delta = (JK)_\Delta \quad \textit{for some } K \in \Delta,$$

(2.9.3) $$I_\Delta = J_\Delta.$$

Proof. (2.9.1) $\Rightarrow$ (2.9.2), since I_Δ is well defined for all ideals I, and (2.9.2) $\Rightarrow$ (2.9.3) by the comment preceding this theorem. Finally, if (2.9.3) holds, then by (2.8)(a) there exist ideals K and H in Δ such that $IK : K = I_\Delta = J_\Delta = JH : H$. Therefore (2.8)(b) shows that $IKH : KH = I_\Delta = J_\Delta = JKH : KH$. Let $D = KH$. Then $D \in \Delta$ and $ID = D(ID : D) = D(JD : D) = JD$, hence (2.9.3) $\Rightarrow$ (2.9.1). Q.E.D.

3. The Δ-closure and the integral closure of an ideal

The notation in this section will be the same as in §2. Therefore Δ is a multplicatively closed set of finitely generated nonzero ideals of a ring R, but often Δ will be at least partly specified, and to save repetition, Λ *will consistently be used to denote the multiplictively closed set of all finitely generated ideals of R that are not contained in any minimal prime ideal in R*. In this section we consider the relationship between the Δ-closure of an ideal and its integral closure. The main results show that generally (but not always) the Δ-closure is contained in the integral closure, and equality holds for most ideals when $\Delta = \Lambda$. We begin with the definition of, and some facts concerning the integral closure of an ideal.

(3.1) **Definition.** If I is an ideal in R, then the *integral closure I_a of I in R* is the set $I_a = \{x \in R;\ x$ satisfies an equation of the form $x^n + b_1x^{n-1} + \cdots + b_n = 0$, where $b_i \in I^i$ for $i = 1, \ldots, n\}$.

It is well known that I_a is an ideal in R such that $I \subseteq I_a \subseteq \mathrm{Rad}(I)$. Also, it is shown in [6, §6] that $I \to I_a$ is a semiprime operation on the set of ideals I of R such that the following cancellation law holds: if $(IK)_a \subseteq (JK)_a$, if $K_a = F_a$ for some finitely generated ideal F in R, and if I is contained in every minimal prime ideal in R that contains K, then $I_a \subseteq J_a$. This will be used in the proof of (3.2.1).

(3.2) **Theorem.** *Let Λ be the set of all finitely generated ideals in R that are not contained in any minimal prime ideal. Then*:

(3.2.1) *If $\Delta \subseteq \Lambda$, then $I_\Delta \subseteq I_a$ for all ideals I in R.*

(3.2.2) *If $\Lambda \subseteq \Delta$, then $I_a \subseteq I_\Delta$ for all ideals I in R that contain an ideal in Λ.*

(3.2.3) *If $\Delta = \Lambda$, then $I_a = I_\Delta$ for all ideals I in R that contain an ideal in Λ.*

Proof. For property (3.2.1) let I be an ideal in R and let $x \in I_\Delta$, so there exists an ideal K in Δ such that $x \in IK : K$. Now $(IK : K)K = IK$, so $((IK : K)K)_a = (IK)_a$, so by the hypothesis on the ideals in Δ it follows from [6, §6] (see the comment preceding this theorem) that $(IK : K)_a \subseteq I_a$. Therefore $x \in IK : K \subseteq (IK : K)_a \subseteq I_a$, hence $I_\Delta \subseteq I_a$.

For (3.2.2) assume that $H \in \Lambda$ is contained in I and let $x \in I_a$. Then to show that $x \in I_\Delta$, by (2.4.1) it may be assumed that $x \notin I$, so $x^n + b_1x^{n-1} + \cdots + b_n = 0$ for some integer $n > 1$ and for some elements $b_i \in I^i$. Now I has a (possibly infinite) basis, and each element in a power I^i of I is a (finite) linear combination of products of i factors from these basis elements, so it follows that there exists a finitely generated ideal J contained in I such that $b_i \in J^i$ for $i = 1, \ldots, n$, so $x^n \in J(x, J)^{n-1}$. Let $K = (x, H, J)R$. Then K is finitely generated and is not contained in any minimal prime ideal, so $K \in \Lambda \subseteq \Delta$. Also, $K^n = x^nR + (H, J)K^{n-1} = (H, J)K^{n-1}$, since $x^n \in J(x, J)^{n-1} \subseteq JK^{n-1}$. Therefore $x \in K \subseteq K^n : K^{n-1} = (H, J)K^{n-1} : K^{n-1} \subseteq IK^{n-1} : K^{n-1} \subseteq I_\Delta$. Thus $x \in I_\Delta$, so $I_a \subseteq I_\Delta$.

(3.2.3) follows immediately from (3.2.1) and (3.2.2). Q.E.D.

(3.6) gives some additional relations between I_a and I_Δ when I is contained in at least one minimal prime ideal.

(3.3) *Remark.* Let Λ be as in (3.2) and let I be an ideal in R. Then:

(3.3.1) If I is integrally closed (for example, if I is a prime ideal in R), then $IK : K = I$ for all ideals K in Λ, hence $I_\Delta = I$ for all multiplicatively closed subsets Δ of Λ.

(3.3.2) If I contains an ideal H in Λ and if $IK : K = I$ for all finitely generated ideals K of the form $(x, I_0)^n$, where $x \in I_a$, I_0 is a finitely generated ideal that is contained in I and that contains H, and $n \geq 1$, then I is integrally closed.

Proof. For (3.3.1), if $K \in \Lambda$, then $IK : K \subseteq I_\Lambda$. Also, $I_\Lambda \subseteq I_a$ (by (3.2.1) applied to $\Delta = \Lambda$) and $I_a = I$, by hypothesis, and it is clear that $I \subseteq IK : K$. Therefore $I = IK : K$, so if Δ is a multiplicatively closed subset of Λ, then $I_\Delta = I$ by (2.1). (For the parenthetical statement, it is readily checked that a prime ideal is integrally closed.)

For (3.3.2), the proof of (3.2.2) shows that if $x \in I_a$, $\notin I$, then $x \in IK : K$, where $x^n + b_1x^{n-1} + \cdots + b_n = 0$ and $K = (x, H, J)^{n-1}$ for some finitely generated ideal J contained in I and $n > 1$. Therefore the hypothesis implies that $I_a \subseteq I$, and the opposite inclusion holds since $I \to I_a$ is a semiprime operations, by [6, §6]. Q.E.D.

The following corollary, applied to the case when R is an integral domain, gives an interesting characterization of integrally closed nonzero ideals in R, and this characterization also holds for $I = (0)$, since $(0)K : K = (0)$ and $(0) = (0)_a$ in an integral domain.

(3.4) **Corollary.** *Let I be an ideal in R that is not contained in any minimal prime ideal and assume either that I is finitely generated or that there are only finitely many minimal prime ideals in R. Then $I = I_a$ if and only if $IK : K = I$ for all ideals K in Λ.*

Proof. If $I = I_a$, then the conclusion is given by (3.3.1).

For the converse, it is well known that an ideal contained in a finite union of prime ideals is contained in one of them. Therefore, since I is not contained in any minimal prime ideal in R, it follows that if there are only finitely many minimal prime ideals in R, then I contains an ideal H in Λ. And if I is finitely generated, then $I \in \Lambda$. Therefore, in either case it follows from (3.3.2) that $I_a = I$. Q.E.D.

The following corollary gives an interesting characterization of integrally closed rings.

(3.5) **Corollary.** *R is integrally closed if and only if $bK : K = bR$ for all regular principal ideals bR and for all finitely generated regular ideals K in R.*

Proof. It is readily checked that a ring R is integrally closed if and only if $bR = (bR)_a$ for all regular principal ideals bR of R. Therefore the corollary follows immediately from (3.3.1) and (3.3.2). Q.E.D.

It follows from (3.2.3) that if R is an integral domain, then $I \to I_a$ is a (very important) special case of $I \to I_\Delta$. Unfortunately, this need no longer be true when R is not a domain; this, among other things, is shown in (3.6).

(3.6) *Remark.* (3.6.1) If every ideal in Δ is regular (and finitely generated), then $(0)_\Delta = (0)$ and $(0)_a = \mathrm{Rad}(R)$ (so $I_\Delta < I_a$ can hold when I is contained in a minimal prime ideal (see (3.2.1); it will be shown in (4.2) that is also can hold for regular ideals).

(3.6.2) If $\Delta \subseteq \Lambda$, then $(0)_\Delta \subseteq (\mathrm{Rad}(R))_\Delta = \mathrm{Rad}(R)$.

(3.6.3) If z is a minimal prime ideal in R, if there exists an ideal K in Δ such that $K \subseteq z$, and if I is an ideal in R such that $I \subseteq z$, then $I_a \subseteq z$ and $I_\Delta \nsubseteq z$. Therefore $(I_a)_a = I_a < (I_a)_\Delta$, so $I_a < I_\Delta$ can hold (with $I = I_a$) when some $K \in \Delta$ is contained in a minimal prime ideal (see (3.2.2)).

(3.6.4) If I contains a nonnilpotent element and Δ contains all finitely generated nonnilpotent ideals L contained in I_a, then $I_a \subseteq I_\Delta$ and the equality holds only if I is not contained in any minimal prime ideal (see (3.2.2)).

(3.6.5) If I contains a nonnilpotent element, if I is contained in some minimal prime ideal z in R, if $IR_z \neq zR_z$, and if the zero ideal in R has a finite primary decomposition, then $I_a \neq I_\Delta$ for all multiplicatively closed sets Δ of finitely generated nonzero ideals of R.

Proof. For (3.6.1), if K is regular, then $(0)K : K = (0) : K = (0)$, so $(0)_\Delta = (0)$ by (2.1), and it is readily checked that $(0)_a = \mathrm{Rad}(R)$.

For (3.6.2), $(0)_\Delta \subseteq ((\mathrm{Rad}(R))_\Delta$ by (2.4.2). Also, if z is a minimal prime ideal in R, then $zK : K = z$ (since $K \nsubseteq z$ implies that $z : K = z$, so

$z \subseteq zK : K \subseteq z : K = z)$, so $z_\Delta = z$, by (2.1). Therefore, if Γ is the set of minimal prime ideals in R, then

$$(\mathrm{Rad}(R))_\Delta = \left(\bigcap_{z\in\Gamma} z\right)_\Delta = \left(\bigcap_{z\in\Gamma} z_\Delta\right)_\Delta = \bigcap_{z\in\Gamma} z_\Delta, \quad \text{by (2.5)(c)},$$
$$= \bigcap_{z\in\Gamma} z = \mathrm{Rad}(R).$$

For (3.6.3), it has already been noted that prime ideals are integrally closed and that $I \to I_a$ is a semiprime operation, so $I \subseteq z$ implies that $I_a \subseteq z_a = z$. Also, since K is finitely generated it follows that $K^n R_z = (0)$ for some $n \geq 1$, so it follows that $(IK^n : K^n)R_z = R_z$, hence $I_\Delta \not\subseteq z$. Therefore, applying what has already been shown to I_a in place of I it follows that $(I_a)_a = I_a < (I_a)_\Delta$.

For (3.6.4), the proof that $I_a \subseteq I_\Delta$ is similar to the proof of (3.2.2), but with the nonnilpotent element playing the role of H. For the last statement, assume that I is contained in some minimal prime ideal z and let b be a nonnilpotent element in I. Then $bR \in \Delta$, by hypothesis, so $(bR)_\Delta \not\subseteq z$, by (3.6.3). However, $(bR)_\Delta \subseteq I_\Delta$, so $I_\Delta \not\subseteq z$, and $I_a \subseteq z_a = z$, so $I_\Delta \neq I_a$.

Finally, for (3.6.5) suppose that there exists a multiplicatively closed set Δ of finitely generated nonzero ideals of R such that $I_a = I_\Delta$. It will be shown that this leads to a contradiction by considering the two cases: (a) there exists an ideal K in Δ such that $K \subseteq z$, and (b) $K \not\subseteq z$ for all $K \in \Delta$. If (a) holds, then (3.6.3) implies that $I_\Delta \not\subseteq z$ and $I_a \subseteq z$, and this contradicts the supposition. Therefore (b) must hold, so $KR_z = R_z$ for all $K \in \Delta$, hence it follows from (5.1.2) below that $I_\Delta R_z = IR_z$. Now the zero ideal in R has a finite primary decomposition, so it follows that $I_a R_z = (IR_z)_a$, and $(IR_z)_a = zR_z$, since z is a minimal prime ideal. Finally, $IR_z < zR_z$, by hypothesis, so the supposition implies that $zR_z = I_a R_z = I_\Delta R_z = IR_z < zR_z$, and this is a contradiction, so it follows that there does not exist such a set Δ. Q.E.D.

With the comment preceding (3.6) in mind, it seems worthwhile to give the following two examples: (a) shows that $I \to I_a$ and $I \to I_\Delta$ can coincide when R is not an integral domain, and (b) is a specific example of (3.6.5).

(a) Let $R = Z_2 \oplus Z_3$, where Z_k denotes the integers modulo k. Then R has exactly four ideals (including R) and each is integrally closed, so $I \to I_a$ and $I \to I_\Delta$ coincide for $\Delta = \{R\}$. (A similar result holds whenever R is a finite direct sum of fields.)

(b) Let $R = Z_4 \oplus Z_3$ and let $I = 4Z_4 \oplus Z_3$. Then it is readily checked that R and I satisfy the hypotheses of (3.6.5), so $I_a \neq I_\Delta$ for all choices of Δ.

It is worth noting that essentially the same proof as for (3.6.2) shows that if no ideal K in Δ is contained in any prime ideal in R that is minimal with respect to containing a given ideal I, then $I_\Delta \subseteq (\mathrm{Rad}(I))_\Delta = \mathrm{Rad}(I)$.

(3.7) (together with (2.4)) shows that $I \to I_\Delta$ is a prime operation on the set of ideals I of R when R is integrally closed and $\Delta \subseteq \Lambda$.

(3.7) **Theorem.** *Assume that* R *is integrally closed and that* $\Delta \subseteq \Lambda$. *Then* $(bI)_\Delta = bI_\Delta$ *for all regular nonunits* b *in* R *and for all ideals* I *in* R.

Proof. By (2.4.1) and (2.4.4) it suffices to show that $(bI)_\Delta \subseteq bI_\Delta$. For this, let $x \in (bI)_\Delta$, so $x \in (bI)_a$, by (3.2.1), and $(bI)_a \subseteq (bR)_a = bR$, since R is integrally closed and b is regular, so it follows that $x/b \in R$. Also, there exists an ideal K in Δ such that $x \in bIK : K$, so $xK \subseteq bIK$, hence $(x/b)K \subseteq IK$. Thus it follows that $x/b \in IK : K \subseteq I_\Delta$, and so $x \in bI_\Delta$. Q.E.D.

(3.7) will be sharpened in (6.10), where it is shown that $I \to I_\Delta$ is a prime operation on the set of ideals I of R if and only if R is Δ-closed (see (6.1)).

4. Some related results

This section contains a few additional results concerning the ideals I_Δ, where, as usual, Δ is a multiplicatively closed set of finitely generated nonzero ideals of R.

(4.1) **Proposition.** *Let* I *be an ideal in* R. *Then*:

(4.1.1) *If* J *is an ideal in* R *such that* $I \subseteq J \subseteq I_\Delta$ *(in particular, if* $J \in \mathbf{D}(I)$*), then* $J_\Delta = I_\Delta$.

(4.1.2) $(0)_\Delta = \bigcup_{K\in\Delta}((0) : K)$ *and* $(0)_\Delta : K = (0)_\Delta$ *for all* $K \in \Delta$.

(4.1.3) *If* $I : K = I$ *for all* $K \in \Delta$, *then* $I_\Delta = I$. *The converse is false.*

(4.1.4) $(I_\Delta : J)_\Delta = I_\Delta : J$ *for all ideals* J *in* R.

(4.1.5) *If* $b_1, \dots, b_n$ *are regular nonunits in* R, *then* $(b_1, \dots, b_n R)_\Delta = b_1 \cdots b_n R$ *if and only if* $(b_i R)_\Delta = b_i R$ *for* $i = 1, \dots, n$.

(4.1.6) *If* b *is a regular nonunit in* R *such that* $(b^k R)_\Delta = b^k R$ *for some* $k \geq 1$, *then* $(b^n R)_\Delta = b^n R$ *for all* $n \geq 1$.

(4.1.7) *If no ideal in* Δ *is contained in a minimal prime ideal, then* (a) *if* $I \neq R$, *then* $I_\Delta \neq R$, (b) $(I_a)_\Delta = I_a$, *and* (c) *if* $I \in \Delta$, *then* $I \subseteq I^* \subseteq I_\Delta \subseteq I_a$, *where* $I^* = \bigcup\{I^{n+1} : I^n ; n \geq 1\}$.

(4.1.8) *If* $B_1, \dots, B_n$ *are R-algebras such that* $(0) \notin \Omega_i = \{KB_i ; K \in \Delta\}$ *for* $i = 1, \dots, n$, *if each* B_i *satisfies the ACC on colon ideals, and if* $\mathbf{C}_i$ *is a finite collection of ideals in* B_i *for* $i = 1, \dots, n$, *then there exists an ideal* K *in* Δ *such that* $JKB_i : KB_i = J_{\Omega_i}$ *for all* $J \in \mathbf{C}_i$ $(i = 1, \dots, n)$.

Proof. (4.1.1) is clear by (2.4.2) and (2.4.3).

For (4.1.2) note that $(0)K : K = (0) : K$ for all ideals K, so $(0)_\Delta = \bigcup_{K\in\Delta}((0) : K)$ by (2.1). Also, if $K \in \Delta$, then $(0)_\Delta : K = ((0)K)_\Delta : K = (0)_\Delta$ by (2.6).

For (4.1.3), if $I : K = I$, then $I \subseteq IK : K \subseteq I : K = I$. Thus if $I : K = I$ for all $K \in \Delta$, then it follows from (2.1) that $I_\Delta = I$. To see that the converse is false let b be a regular nonunit in I and let $\Delta = \{b^n R ; n \geq 1\}$. Then $I_\Delta = I$ and $I : K = R$ for all $K \in \Delta$.

For (4.1.4) let $x \in (I_\Delta : J)_\Delta$, so there exists an ideal $K \in \Delta$ such that $xK \subseteq (I_\Delta : J)K$. Therefore $x \in (K(I_\Delta : J)) : K \subseteq ((KI_\Delta) : J) : K = (KI_\Delta) :$

$JK = ((I_\Delta K) : K) : J = (I_\Delta) : J$ by (2.6). Therefore $(I_\Delta : J)_\Delta \subseteq I_\Delta : J$, and the opposite inclusion is given by (2.4.1).

For (4.1.5) assume that $(b_1 \cdots b_n R)_\Delta = b_1 \cdots b_n R$, fix $i = 1, \ldots, n$, and let $x \in (b_i R)_\Delta$. Then there exists an ideal K in Δ such that $xK \subseteq b_i K$, so $xb_1 \cdots b_{i-1} b_{i+1} \cdots b_n K \subseteq b_1 \cdots b_n K$. Therefore $xb_1 \cdots b_{i-1} b_{i+1} \cdots b_n \in b_1 \cdots b_n K : K = b_1 \cdots b_n R$, by the hypothesis and (2.1), so since each b_i is regular it follows that $x \in b_i R$. Thus $(b_i R)_\Delta \subseteq b_i R$, and the opposite inclusion is given by (2.4.1).

Now assume that $(b_i R)_\Delta = b_i R$ for $i = 1, \cdots, n$. The proof that $(b_1 \cdots b_n R)_\Delta = b_1 \cdots b_n R$ will be by induction on n. The case $n = 1$ is clear, so assume that $n > 1$ and that the result holds for $n-1$ regular nonunits. Now $(b_1 \cdots b_n R)_\Delta \subseteq (b_n R)_\Delta = b_n R$, by (2.4.2) and the hypothesis, so

$$\begin{aligned}(b_1 \cdots b_n R)_\Delta &= (b_1 \cdots b_n R)_\Delta \cap b_n R = b_n[(b_1 \cdots b_n R)_\Delta : b_n R] \\ &= b_n (b_1 \cdots b_{n-1} R)_\Delta \quad \text{by (2.6)} \\ &= b_n (b_1 \cdots b_{n-1} R) \quad \text{by induction} \\ &= b_1 \cdots b_n R.\end{aligned}$$

(4.1.6) is a special case of (4.1.5).

For (4.1.7)(a), $I_\Delta \subseteq I_a$, by (3.2.1), and it is readily seen that $1 \in I_a$ if and only if $I = R$. For (4.1.7)(b), $I_\Delta \subseteq I_a$, as already noted, so $I_a \subseteq (I_a)_\Delta \subseteq (I_a)_a = I_a$, since $I \to I_a$ is a semiprime operation. Finally, if $I \in \Delta$, then the first two containments in (4.1.7)(c) are clear, and $I_\Delta \subseteq I_a$ by (3.2.1).

Finally, for (4.1.8), the notation KB_i denotes the ideal $f_i(K)B_i$, where f_i is the natural homomorphism from R into B_i. Therefore it is readily seen that each $\mathbf{\Omega}_i$ is a multiplicatively closed set of finitely generated nonzero ideals of B_i. Therefore fix $J_1, J_2 \in \bigcup\{\mathbf{C}_i ; i = 1, \ldots, n\}$, say $J_1 \in \mathbf{C}_1$ and $J_2 \in \mathbf{C}_2$ (and possibly $\mathbf{C}_1 = \mathbf{C}_2$). Then the hypothesis and (2.8)(a) (applied to each B_i) imply that there exist ideals $H_i \in \Delta$ such that $J_{i\Delta} = JH_iB_i : H_iB_i$ for $i = 1, 2$. Then $L = H_1H_2 \in \Delta$ and $J_{i\Delta} = J_iLB_i : LB_i$, since $J_{i\Delta} = J_iH_iB_i : H_iB_i \subseteq J_iH_1H_2B_i : H_1H_2B_i \subseteq J_{i\Delta}$ for $i = 1, 2$. The conclusion readily follows from this by using induction on the cardinality of $\mathbf{C}_1 \cup \cdots \cup \mathbf{C}_n$. Q.E.D.

Concerning (4.1.7)(a), note that if z is a finitely generated minimal prime ideal in R that is also a maximal ideal, and if $z \in \Delta$, then $z_\Delta = R$ (since $z \subseteq z_\Delta \not\subseteq z$ by (2.4.1) and (3.6.3)).

The next two results are concerned with $I^* = \bigcup\{I^{n+1} : I^n ; n \geq 1\}$. For (4.2), note that it is shown in (4.1.7)(c) that $I \subseteq I^* \subseteq I_a$. (4.2) is an example to show that both containments may be proper; in particular, if $\Delta = \{I^n ; n \geq 1\}$, then (4.2) shows that $I < I_\Delta = I^* < I_a$ can hold even for a regular ideal in a Noetherian domain (see (3.6.1)).

(4.2) **Example.** Let R be the polynomial ring $F[X, Y, Z]$, where X, Y, Z are indeterminates and F is a field. Let

$$K = (X^3, Y^3, Z^3)(X^3, Y^3, Z^3, X^2Y)R,$$

and let $\Delta = \{K^n ; n \geq 1\}$. Then $K < K^* = K_\Delta < K_a$.

Proof. It is clear that $K^* = K_\Delta$. Also, it is shown in [4, (3.3)] that $K < \bigcup\{K^{n+1} : K^n ; n \geq 1\} = K^*$. (The proof is not difficult, but it involves some deep concepts.) Further, it is readily seen that $K_a = (X, Y, Z)^6 R$ and that $(X^2Y^2Z^2)X^{6n} \notin K^{n+1}$ for all $n \geq 1$. But $X^2Y^2Z^2 \in K_a$ and $X^{6n} \in K^n$. Therefore it follows that $K_a \nsubseteq K^{n+1} : K^n$ for each $n \geq 1$, so $K_a \nsubseteq K_\Delta$, hence $K_\Delta < K_a$ by (4.1.7)(c). Q.E.D.

(4.3) is concerned with the following question that I was not able to answer: if I is an ideal in R that is not contained in any minimal prime ideal of R, then for what ideals J between I and I_a does there exist a Δ such that $J = I_\Delta$? Taking $\Delta = \{R\}$ gives $I_\Delta = I$, and (3.2.3) specifies a set Δ for $I_\Delta = I_a$. Also, for each J such that there exists an ideal K in R such that $J : K^n = J$ for all $n \geq 1$, it follows from (4.1.3) that $J = J_\Delta$, where $\Delta = \{K^n ; n \geq 1\}$. (4.3) gives a little more information on this (for if J in (4.3) is not contained in any minimal prime ideal in R, then $I \subseteq J \subseteq J^* = I_\Delta \subseteq I_a$, by (4.3) and (3.2.1)).

(4.3) **Proposition.** *Let I be a nonnilpotent ideal in R, assume that I is a reduction of a finitely generated ideal J (so $I \subseteq J$ and $IJ^n = J^{n+1}$ for some positive integer n), and let $\Delta = \{J^n ; n \geq 1\}$. Then $I_\Delta = J^*$.*

Proof. Note first that $(0) \notin \Delta$, since I (and hence also J) is nonnilpotent. Therefore $I_\Delta = \bigcup\{IJ^n : J^n ; n \geq 1\}$ and $J^* = \bigcup\{J^{n+1} : J^n ; n \geq 1\} = J_\Delta$. However, since I is a reduction of J it follows that $IJ^n = J^{n+1}$ for all large n, so since $IJ^n : J^n \subseteq IJ^{n+1} : J^{n+1}$ for all $n \geq 0$, it follows that $I_\Delta = J^*$. Q.E.D.

This section will be closed by giving a negative answer to the following question: if Δ_1 and Δ_2 are distinct multiplicatively closed sets of finitely generated nonzero ideals of R, then is there at least one ideal I in R such that $I_{\Delta_1} \neq I_{\Delta_2}$? Concerning this, it was noted following (2.2) that $I_\Delta = I$ for every multiplicatively closed set of the regular principal ideals of R, and it was noted in the proof of (2.6) that $I_{\Delta_1} = I_{\Delta_2}$ if Δ_2 is the multiplicatively closed set generated by Δ_1 and any subset of the set of regular principal ideals of R, so let us rule out these cases (so R cannot be a valuation ring). But even now the answer is still no. For example, let $R = F[X, Y]$ and $K = (X, Y)R$, fix an integer $k > 1$, and let $\Delta_1 = \{K^n ; n \geq 1\}$ and $\Delta_2 = \{K^{kn} ; n \geq 1\}$, so Δ_2 is a proper subset of Δ_1. Then by (2.8)(a) for each ideal I in R there exist positive integers m and n such that $I_{\Delta_1} = IK^m : K^m$ and $I_{\Delta_2} = IK^{kn} : K^{kn}$. Then it follows from (2.8)(b) that $I_{\Delta_1} = IK^{mkn} : K^{mkn} = I_{\Delta_2}$.

5. The Δ-closure in R-algebras

In this section we consider some rings B related to R and some multiplicatively closed sets Ω of finitely generated nonzero ideals of B such that

$\{KB;K \in \Delta\} \subseteq \Omega$, and several relationships between the ideals I_Δ in R and the ideals $(IB)_\Omega$ are proved. (Here, as in the proof of (4.1.8), $KB = f(K)B$, where f is the natural homomorphism from R into B.)

(5.1) **Theorem.** *Let R be a ring and let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R. Then:*

(5.1.1) *If B is an R-algebra such that $KB \neq (0)$ for all ideals $K \in \Delta$ (this holds if $R \subseteq B$), and if Ω is a multiplicatively closed set of finitely generated nonzero ideals of B such that $\{KB;B \in \Delta\} \subseteq \Omega$, then $I_\Delta B \subseteq (IB)_\Omega$ for all ideals I in R, so $I_\Delta \subseteq (IB)_\Omega \cap R$.*

(5.1.2) *If B is a flat R-algebra such that $KB \neq (0)$ for all $K \in \Delta$, and if $\Omega = \{KB;K \in \Delta\}$, then $I_\Delta B = (IB)_\Omega$ for all ideals I in R. Therefore, if B is a faithfully flat R-algebra, then $I_\Delta B = (IB)_\Omega$ and $(IB)_\Omega \cap R = I_\Delta$.*

(5.1.3) *If H is an ideal in R such that $H \not\supseteq K$ for all $K \in \Delta$, and if Ω is a multiplicatively closed set of finitely generated nonzero ideals of R/H such that $\{(K+H)/H;K \in \Delta\} \subseteq \Omega$, then $(I_\Delta + H)/H \subseteq ((I+H)/H)_\Omega$ for all ideals I in R.*

(5.1.4) *Let $Z = (0)_\Delta$, let H be an ideal in R such that $H \subseteq Z$, let an overbar denote residue class modulo H, and let $\Omega = \{\overline{K};K \in \Delta\}$. Then* (a) $\overline{(0)} : \overline{K} \subseteq \overline{Z}$ *for all $K \in \Delta$, and* (b) $H \subseteq I_\Delta$ *and $\overline{I_\Delta} = (\overline{I})_\Omega$ for all ideals I in R.*

(5.1.5) *Let I be an ideal in R, let $\mathbf{R} = R[u,tI]$, where t is an indeterminate and $u = 1/t$, and let $\Omega = \{K\mathbf{R};K \in \Delta\}$. Then $(u^n\mathbf{R})_\Omega \cap R = (I^n)_\Delta$ for all $n \geq 1$.*

Proof. For (5.1.1),

$$I_\Delta B = \left(\sum_{K\in\Delta}(IK : K)\right) B = \sum_{K\in\Delta}((IK : K)B) \subseteq \sum_{K\in\Delta}((IB)(KB) : KB) \subseteq (IB)_\Omega,$$

by (2.1) applied to IB and Ω. Therefore, since $J \subseteq JB \cap R$ holds for all ideals J of R, it follows that $I_\Delta \subseteq I_\Delta B \cap R \subseteq (IB)_\Omega \cap R$. (For the parenthetical statement, if $R \subseteq B$, then $K \subseteq KB \cap R$ for all ideals K in R and $(0)B \cap R = (0)R$, so $KB \neq (0)$ for all $K \in \Delta$.)

For (5.1.2), the hypothesis implies that Ω is a multiplicatively closed set of finitely generated nonzero ideals in B. Therefore by (5.1.1), to prove the first statement it suffices to show that $(IB)_\Omega \subseteq I_\Delta B$. For this if $x \in (IB)_\Omega$, then by the definition of Ω there exists an ideal K in Δ such that $x \in (IB)(KB) : KB = (IK : K)B$, since K is finitely generated and B is a flat R-algebra. Also, $IK : K \subseteq I_\Delta$, so $x \in (IK : K)B \subseteq I_\Delta B$, so $(IB)_\Omega \subseteq I_\Delta B$, as desired. The last statement in (5.1.2) is clear from this.

For (5.1.3) let an overbar denote residue class modulo H. Then $\overline{I_\Delta} = \{\overline{x};xK \subseteq IK$ for some $K \in \Delta\}$. Therefore it follows that if $\overline{x} \in \overline{I_\Delta}$, then $\overline{x} \in \overline{IK} : \overline{K} \subseteq (\overline{I})_\Omega$, since $\overline{K} \in \Omega$, hence it follows that $\overline{I_\Delta} \subseteq (\overline{I})_\Omega$.

For (5.1.4), it was noted in (4.1.2) that $(0)_\Delta : K = (0)_\Delta$ for all $K \in \Delta$, so $H \subseteq Z = (0)_\Delta$ implies that $\overline{(0)} : \overline{K} \subseteq \overline{Z}$ for all $K \in \Delta$. Therefore Ω is a multiplicatively closed set of finitely generated nonzero ideals in $\overline{R} = R/H$.

Also, $Z = (0)_\Delta \subseteq I_\Delta$ for all ideals I in R, by (2.4.2), so $H \subseteq I_\Delta$ for all ideals I. Finally, by (5.1.3) it remains to show that $(\overline{I})_\Omega \subseteq \overline{I_\Delta}$. For this let $\overline{x} \in (\overline{I})_\Omega$, so by the definition of Ω there exists an ideal K in Δ such that $\overline{x}\overline{K} \subseteq \overline{IK}$, and so $xK \subseteq IK + H \subseteq (IK)_\Delta + Z = (IK)_\Delta$, since $Z = (0)_\Delta \subseteq (IK)_\Delta$. Therefore $x \in (IK)_\Delta : K = I_\Delta$ by (2.6), hence $\overline{x} \in \overline{I_\Delta}$ as desired.

Finally, if $r \in (u^n\mathbf{R})_\Omega \cap R$, then $r \in (u^n\mathbf{R})_\Omega$, so there exists an ideal K in Δ such that $r(K\mathbf{R}) \subseteq u^n(K\mathbf{R})$. Now $\mathbf{R} \subseteq R[u,t] = R[t,1/t]$ and t is an indeterminate, so $rK\mathbf{R} \cap R = rK$, and since $u^n(K\mathbf{R})$ is a homogeneous ideal it is readily checked that $u^n(K\mathbf{R}) \cap R = I^nK$, so $r \in I^nK : K \subseteq (I^n)_\Delta$. For the opposite inclusion, if $r \in (I^n)_\Delta$, then there exists an ideal $K \in \Delta$ such that $rK \subseteq I^nK$, so $r(K\mathbf{R}) \subseteq I^n(K\mathbf{R}) \subseteq u^n(K\mathbf{R})$, hence $r \in (u^n(K\mathbf{R}) : (K\mathbf{R})) \cap R \subseteq (u^n\mathbf{R})_\Omega \cap R$. Q.E.D.

In (5.1.4), even if $H = Z$ (so $\overline{(0)} : \overline{K} = \overline{(0)}$ for each $K \in \Delta$) and K is finitely generated, it may not be assumed that $\overline{K}$ is a regular ideal. For example, it is shown in [1, Example 3, p. 63] that there can exist a nonregular ideal K that is finitely generated and that satisfies $(0) : K = (0)$.

The ring $R[u, tI]$ in (5.1.5) is called the *Rees ring of R with respect to I*. One of the properties of such rings is $u^nR[u,tI] \cap R = I^n$ for all $n \geq 1$, and quite often this can be used to extend to arbitrary ideals a result that is known for principal ideals. They will be used in this manner in the proof of (6.8.2).

(5.2) **Corollary.** *With the notation of* (5.1) *let I be an ideal in R such that $I = I_\Delta$. Then*:

(5.2.1) *If B is a flat R-algebra such that $KB \neq (0)$ for all $K \in \Delta$ and if $\Omega = \{KB; K \in \Delta\}$, then $IB = (IB)_\Omega$.*

(5.2.2) *If H is an ideal in R such that $H \subseteq Z = (0)_\Delta$ and if $\Omega = \{(K+H)/H; K \in \Delta\}$, then $(I+H)/H = ((I+H)/H)_\Omega$.*

Proof. (5.2.1) follows immediately from (5.1.2), and (5.2.2) follows immediately from (5.1.4). Q.E.D.

6. The Δ-closure of a ring

In this section it is shown that if Δ is a multiplicatively closed set of finitely generated nonzero ideals of a ring R, then R has a natural Δ-closure. And it is then shown that $A \to A^\Delta$ is a closure operation on the set $\{A; R$ is a subring of A and if B is an intermediate ring between R and A, then regular elements in B remain regular in $A\}$. It is also shown that R^Δ plays a role relative to the ideals I_Δ that is analogous to the role played by the integral closure of R relative to the ideals I_a, and that $I \to I_\Delta$ is a prime operation on the set of ideals I of R if and only if R is Δ-closed. We begin with the definition of R^Δ. (It should be noted that the notation R_Δ would not be appropriate for the ring defined in (6.1), since $R = R_\Delta$ is the Δ-closure of the ideal R.)

(6.1) **Definition.** Let Δ be a multiplicatively closed set of finitely generated nonzero ideals of a ring R and let T be the total quotient ring of R. Then a

Δ-extension ring of R is an extension ring A of R such that $bA \cap R \subseteq (bR)_{\Delta}$ for all regular nonunits b in R and some nonzero multiple of each nonzero element in A is integral over R. The *Δ-closure* R^{Δ} *of* R is the largest Δ-extension ring of R that is contained in T, and R is said to be *Δ-closed* if $R = R^{\Delta}$.

Concerning (6.1), it is known that if A is an integral extension ring of R, then $bA \cap R \subseteq (bR)_a$ for all regular nonunits b in R, so it follows from (3.2.3) that every integral extension ring of R is a Δ-extension ring of R (for $\Delta = \Lambda$, where Λ is as in (3.2)). Also, if A is an extension ring of R that is contained in T, then it is clear that some nonzero multiple of each nonzero element in A is integral over R (in fact, is in R), so in this case, to see if A is a Δ-extension it is only necessary to show that $bA \cap R \subseteq (bR)_{\Delta}$ for each regular nonunit b in R; this will be implicitly used in the remainder of this paper.

It is clear that R satisfies the conditions on A in (6.1), so there exist maximal such rings contained in T by Zorn's lemma. We now show that there is, in fact, a largest such ring, as the last part of (6.1) suggests.

(6.2) **Theorem.** *Let* Δ *be a multiplicatively closed set of finitely generated nonzero ideals of a ring* R *and let* $S = \{c/b;\ b$ *is a regular nonunit in* R *and* $c \in (bR)_{\Delta}\}$. *Then* $R^{\Delta} = S = R[S]$ *is the Δ-closure of* R.

Proof. It will first be shown that:

(6.2.1) if $x = v/u \in S$ with $u, v \in R$ such that u is regular, then $v \in (uR)_{\Delta}$. For this, by the definition of S there exist b, c in R such that b is regular, $x = c/b$, and $c \in (bR)_{\Delta}$, so there exists an ideal K in Δ such that $cK \subseteq bK$, so $(v/u)K = (c/b)K \subseteq K$, hence $vK \subseteq uK$, and so $v \in uK : K \subseteq (uR)_{\Delta}$. Therefore (6.2.1) holds.

Also, if $x = c/b$ and $y = v/u$ are in S (with b, c, u, v in R such that b and u are regular), then $x + y = (cu + bv)/(bu)$ and $xy = (cv)/(bu)$ are in S, since $c \in (bR)_{\Delta}$ and $v \in (uR)_{\Delta}$ imply that $cu + bv$ and cv are in $(buR)_{\Delta}$ by (2.4.1) and (2.4.4). Therefore S is a ring, and it is clear that $R \subseteq S \subseteq T$, so $S = R[S]$.

Further, if b is a regular nonunit in R and $c \in bS \cap R$, then $c/b \in S$, so $c \in (bR)_{\Delta}$ by (6.2.1). Therefore $bS \cap R \subseteq (bR)_{\Delta}$, so S is a Δ-extension of R.

Finally, let A be a ring between R and T such that $bA \cap R \subseteq (bR)_{\Delta}$ for all regular nonunits b in R. Let $x \in A$. Then there exist b, c in R such that b is regular and $x = c/b$, and it then follows that $c \in bA \cap R \subseteq (bR)_{\Delta}$. Therefore $x = c/b \in S$, so $A \subseteq S$, hence S is the Δ-closure of R. Q.E.D.

(6.3) **Corollary.**

(6.3.1) $bR^{\Delta} \cap R = (bR)_{\Delta}$ *for all regular nonunits* b *in* R.

(6.3.2) $R = R^{\Delta}$ *(that is,* R *is Δ-closed) if and only if* $bR = (bR)_{\Delta}$ *for all regular nonunits* b *in* R.

(6.3.3) *If no ideal in* Δ *is contained in a minimal prime ideal in* R*, then* $R^{\Delta} \subseteq R'$*, the integral closure of* R*, and the equality holds if* Δ *is the set of all* *finitely generated nonzero ideals of R that are not contained in any minimal prime ideal.*

Proof. (6.3.1) and (6.3.2) follow easily from (6.2) and the formula therein for R^Δ. For (6.3.3), the hypothesis and (3.2.1) imply that $(bR)_\Delta \subseteq (bR)_a$ for all regular nonunits b in R, and by (3.2.3) the equality holds if Δ is the set of all finitely generated nonzero ideals of R that are not contained in any minimal prime ideal. Therefore (6.3.3) follows from the formula for R^Δ in (6.2) and the fact that $R' = R[\{c/b\,; b$ is a regular nonunit in R and $c \in (bR)_a\}]$. Q.E.D.

Concerning (6.3.3), it will be shown in (8.1) that every ring between R and R' is the Δ-closure of R for an appropriate set Δ.

There is now an obvious question: is R^Δ Δ-closed? It will be shown in (6.6) that the answer is yes, and to prove this we need the following characterization of R^Δ.

(6.4) **Theorem.** *Let R be a ring, let T be the total quotient ring of R, and let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R. Then R^Δ is the largest ring A such that* (a) *$R \subseteq A \subseteq T$, and* (b) *for each subring B of A that is finitely generated over R there exists an ideal K in Δ that is an ideal in B.*

Proof. It follows from (6.1) that R^Δ satisfies (a). To see that R^Δ satisfies (b) let $B = R[x_1, \dots, x_n]$ be a finitely generated subring of R^Δ. Then each x_i is in R^Δ, so there exist b_i, c_i in R such that b_i is regular, $x_i = c_i/b_i$, and $c_i \in (b_iR)_\Delta$ by the formula for R^Δ in (6.2). Therefore for $i = 1, \dots, n$ there exists an ideal K_i in Δ such that $c_i \in b_iK_i : K_i$. Let $K = K_1 \cdots K_n$. Then $c_i \in b_iK : K$ for $i = 1, \dots, n$, so it follows that $(c_i/b_i)K \subseteq K$. It now readily follows that $x_1^{e_1} \cdots x_n^{e_n}K = (c_1/b_1)^{e_1} \cdots (c_n/b_n)^{e_n}K \subseteq K$ for all nonnegative integers $e_1, \dots, e_n$, hence K is an ideal in B.

Finally, if A is a ring that satisfies (a) and (b), and if $x \in A$, then there exists an ideal K in Δ such that K is an ideal in $R[x]$. Let $x = c/b$ with b and c in R. Then it follows that $(c/b)K \subseteq K$, so $c \in bK : K \subseteq (bR)_\Delta$. Therefore $x = c/b \in R^\Delta$, so it follows that $A \subseteq R^\Delta$. Q.E.D.

(6.5) *Remark.* It is interesting to note that for each finitely generated nonnilpotent ideal I in R there exists a uniquely determined largest ring A containing R and contained in T such that, for all finitely generated subrings B of A, I^n is an ideal in B for some positive integer n. (This follows from (6.4) by taking $\Delta = \{I^n\,; n \geq 1\}$, and then it follows that $A = R^\Delta$.) It follows from (4.1.3) that if $\Delta = \{I^n\,; n \geq 1\}$ and if $bR : I^n = bR$ for all regular principal ideals bR and positive integers n, then $A = R$. However, the converse is false, since if $\Delta = \{b^nR\,; n \geq 1\}$ and $I = bR$ for some regular nonunit b in R, then it is clear that $A = R$, but $bR : I^n = R$ for all $n \geq 1$.

It will now be shown that $A \to A^\Delta$ is a closure operation on a large set of rings A that contain R as a subring (see (6.7)). To prove this, *the following* *notational convention will be adopted for the remainder of this paper*: if Δ is a mulitplicatively closed set of finitely generated nonzero ideals of R, if B is an R-algebra such that $(0) \notin \Delta B = \{f(K)B; K \in \Delta\}$ (where f is the natural homomorphism from R into B), and if I is an ideal in B, then I_Δ will be used to denote the closure in B of I relative to ΔB and B^Δ will be used to denote the ΔB-closure of B. (That is, I_Δ will be used to denote $I_{\Delta B}$ and B^Δ will be used to denote $B^{\Delta B}$.) In this regard, note that the preceding results for R and Δ hold for B and ΔB (since only the hypotheses that R is a ring and Δ is a multiplicatively closed set of finitely generated nonzero ideals of R were used to prove these results).

(6.6) **Theorem.** *Let R be a ring, let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R, and let A and B be R-algebras such that $(0) \notin \Delta A$ and $(0) \notin \Delta B$. Then*

(6.6.1) $A \subseteq A^\Delta$.

(6.6.2) *If $A \subseteq B$ and if regular elements in A remain regular in B, then $A^\Delta \subseteq B^\Delta$.*

(6.6.3) $(A^\Delta)^\Delta = A^\Delta$.

(6.6.4) *If $A \subseteq B \subseteq A^\Delta$, then $B^\Delta = A^\Delta$.*

Proof. (6.6.1) is clear by the definition of A^Δ (see (6.1)), and if $A \subseteq B$, then it follows from (5.1.1) that $(bA)_\Delta \subseteq (bB)_\Delta$, so (6.6.2) follows from the formula for R^Δ (applied to A and B) in (6.2) (since regular elements in A remain regular in B).

For (6.6.3), by (6.6.1) and (6.6.2) it suffices to show that $(A^\Delta)^\Delta \subseteq A^\Delta$. For this, let $c/b \in (A^\Delta)^\Delta$ (with b, c in A such that b is regular), so $c \in (bA^\Delta)_\Delta$. Therefore there exists an ideal K in Δ such that $c(KA^\Delta) \subseteq b(KA^\Delta)$. By hypothesis, KA has a finite basis (since K does), say $x_1, \dots, x_n$, so it follows that for $i = 1, \dots, n$ there exist elements $f_{ij} \in A^\Delta$ such that $cx_i = \sum_{j=1}^n bf_{ij}x_j$. Let $C = A[\{f_{ij}\}_{i,j=1,\dots,n}]$. Then it follows that $c(KC) \subseteq b(KC)$. Also, C is finitely generated over A and $C \subseteq A^\Delta$, so by (6.4) (applied to A in place of R) there exists an ideal H in Δ such that $HC = (HA)C \subseteq HA$, so $c(KHA) \subseteq c(KHC) \subseteq b(KHC) \subseteq b(KHA)$. Therefore $c \in b(KHA) : (KHA) \subseteq (bA)_\Delta$, so $c/b \in A^\Delta$, hence $(A^\Delta)^\Delta \subseteq A^\Delta$.

Finally, (6.6.4) follows immediately from (6.6.2) and (6.6.3). Q.E.D.

(6.7) **Corollary.** *$A \to A^\Delta$ is a closure operation on the set $\mathbf{A} = \{A; A$ is a ring, R is a subring of A, and if C is a ring between R and A, then regular elements in C remain regular in $A\}$.*

Proof. If $A \in \mathbf{A}$, then $(0) \notin \Delta A$ as noted in (5.1.1). Also, if C is a ring between A and its total quotient ring, then $C \in \mathbf{A}$, so $A \in \mathbf{A}$ implies that $A^\Delta \in \mathbf{A}$, so the conclusion readily follows from (6.6). Q.E.D.

It is known that if I is an ideal in a ring R and R' is the integral closure of R, then $(IR')_a \cap R = I_a$. (6.8.2) shows that the analogous result holds for R^Δ and I_Δ; this verifies the comment concerning R^Δ in the first paragraph of this section.

(6.8) **Theorem.** *If R is a ring and Δ is a multiplicatively closed set of finitely generated nonzero ideals of R, then*:

(6.8.1) $(bR^\Delta)_\Delta = bR^\Delta$ *for all regular nonunits b in R^Δ.*

(6.8.2) $(IR^\Delta)_\Delta \cap R = I_\Delta$ *for all ideals I in R.*

Proof. (6.8.1) follows immediately from (6.6.3) and (6.3.2).

For (6.8.2) let I be an ideal in R and let $\mathbf{R} = R[u, tI]$, where t is an indeterminate and $u = 1/t$. Also, let $\mathbf{S} = R^\Delta[u, tIR^\Delta]$, so $\mathbf{R} \subseteq \mathbf{S}$. Now it follows from (5.1.5) that $I_\Delta = (u\mathbf{R})_\Delta \cap R$ and $(IR^\Delta)_\Delta = (u\mathbf{S})_\Delta \cap R^\Delta$. Also, it follows from (6.6.2) that $R^\Delta \subseteq \mathbf{R}^\Delta$, so it follows that $\mathbf{S} \subseteq \mathbf{R}^\Delta$, hence $\mathbf{S}^\Delta = \mathbf{R}^\Delta$ by (6.6.4). Finally, $u\mathbf{S}^\Delta \cap \mathbf{S} = (u\mathbf{S})_\Delta$ and $u\mathbf{R}^\Delta \cap \mathbf{R} = (u\mathbf{R})_\Delta$ by (6.3.1), so it follows that $(IR^\Delta)_\Delta \cap R = I_\Delta$. Q.E.D.

Note that it follows from (6.8.2) that if I is an ideal in R, then $I \subseteq IR^\Delta \cap R \subseteq I_\Delta R^\Delta \cap R \subseteq (I_\Delta R^\Delta)_\Delta \cap R = (IR^\Delta)_\Delta \cap R$ (using (5.1.1)) $= I_\Delta$.

In (6.4) R^Δ is characterized as being the largest ring A between R and its total quotient ring T that has a certain property. A characterization of R^Δ from the opposite extreme is given in (6.9), namely R^Δ is the smallest ring A between R and T such that $(bA)_\Delta = bA$ for all regular nonunits b in R.

(6.9) **Theorem.** *Let R be a ring, let T be the total quotient ring of R, and let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R. Then R^Δ is the smallest ring A such that* (a) $R \subseteq A \subseteq T$, *and* (b) $(bA)_\Delta = bA$ *for all regular nonunits b in R.*

Proof. It follows from (6.1) and (6.8.1) that R^Δ satisfies (a) and (b), so let A be another ring that satisfies (a) and (b) and let $c/b \in R^\Delta$ (with b, c in R such that b is regular). Then $c \in (bR)_\Delta$ by (6.2.1), and $(bR)_\Delta \subseteq (bA)_\Delta = bA$, by (5.1.1) and the hypothesis on A. Therefore $c/b \in A$, so $R^\Delta \subseteq A$. Q.E.D.

The final result in this section shows that the condition of R being Δ-closed is characterized by $I \to I_\Delta$ being a prime operation on the set of ideals I of R.

(6.10) **Theorem.** *The operation $I \to I_\Delta$ is a prime operation on the set of ideals I of R if and only if $R = R^\Delta$.*

Proof. If $(bI)_\Delta = bI_\Delta$ for all regular nonunits b and ideals I in R, then by taking $I = R$ it follows that $(bR)_\Delta = bR_\Delta = bR$, so $R = R^\Delta$ by (6.3.2).

For the converse, by (2.4) it suffices to show that $(bI)_\Delta \subseteq bI_\Delta$ for all regular nonunits b and ideals I in R. For this, if $x \in (bI)_\Delta$, then $x \in (bR)_\Delta$, by

(2.4.2) (since $bI \subseteq bR$), so $x/b \in R$, by hypothesis, hence it follows as in the proof of (3.7) that $(bI)_\Delta \subseteq bI_\Delta$. Q.E.D.

7. The Δ-closure of R-algebras

In this section it is shown that the Δ-closure analogues of several standard results concerning the integral closure of R-algebras are valid. Throughout this section, the notational convention specified between (6.5) and (6.6) will be used.

(7.1) **Theorem.** *Let Δ be a multiplicatively closed set of finitely generated nonzero ideals of a ring R and let T be the total quotient ring of R. Then:*

(7.1.1) *If S is a multiplicatively closed set in R such that $0 \notin \Delta R_S$, then $(R^\Delta)_S \subseteq (R_S)^\Delta$, and the equality holds if $R_S \subseteq T$.*

(7.1.2) *If B is a faithfully flat R-algebra, then $B^\Delta \cap T = R^\Delta$ and $(xB^\Delta)_\Delta \cap T = xR^\Delta$ for all regular nonunits $x \in R^\Delta$.*

(7.1.3) *If $\{X_i\}_{i\in\Gamma}$ is a set of indeterminates, then $R[\{X_i\}_{i\in\Gamma}]^\Delta \cap T[\{X_i\}_{i\in\Gamma}] = R^\Delta[\{X_i\}_{i\in\Gamma}]$.*

(7.1.4) *If H is an ideal in T such that $H \subseteq Z = (0T)_\Delta$, then $(R/(H\cap R))^\Delta \cap (T/H) = R^\Delta/H$.*

Proof. For (7.1.1) let f be the natural homomorphism from R into R_S and let $x \in (R^\Delta)_S$. Then there exist $c/b \in R^\Delta$ (with $b, c \in R$ such that b is regular) and $s \in S$ such that $x = (c/b)/s$. Now $c/b \in R^\Delta$ implies that $c \in (bR)_\Delta$ by (6.2.1), so $f(c) \in (f(b)R_S)_\Delta$ by (5.1.1), and $f(b)R_S = f(bs)R_S$, since $f(s)$ is a unit in R_S. Therefore $(c/b)/s = c/bs = f(c)/f(bs) \in (R_S)^\Delta$ by the formula for R^Δ in (6.2), hence $(R^\Delta)_S \subseteq (R_S)^\Delta$.

Now assume that $R_S \subseteq T$, so $R_S \subseteq (R^\Delta)_S \subseteq (R_S)^\Delta$ by (6.6.1) and the preceding paragraph, so $(R_S)^\Delta \subseteq ((R^\Delta)_S)^\Delta \subseteq ((R_S)^\Delta)^\Delta = (R_S)^\Delta$, by (6.6.2) and (6.6.3), so it suffices to show that $((R^\Delta)_S)^\Delta = (R^\Delta)_S$. For this, since the hypothesis implies that $((R^\Delta)_s)^\Delta \subseteq T$, let $c/b \in ((R^\Delta)_S)^\Delta$, where $b, c \in R$ with b regular. Then

$$\begin{aligned} c \in (b(R^\Delta)_S)_\Delta &= (bR^\Delta)_\Delta (R^\Delta)_S \quad \text{by (5.1.2) (applied to } R^\Delta \text{ in place of } R) \\ &= (bR^\Delta)(R^\Delta)_S \quad \text{by (6.8.1)} \\ &= b((R^\Delta)_S). \end{aligned}$$

Therefore $c/b \in (R^\Delta)_S$, so $((R^\Delta)_S)^\Delta \subseteq (R^\Delta)_S$, and the opposite inclusion is given by (6.6.1). Therefore $(R_S)^\Delta = (R^\Delta)_S$, so (7.1.1) holds.

For (7.1.2), if B is a faithfully flat R-algebra, then regular elements in R remain regular in B, so it follows from (6.6.2) that $R^\Delta \subseteq B^\Delta \cap T$, hence to prove that $R^\Delta = B^\Delta \cap T$ it suffices to show that $B^\Delta \cap T \subseteq R^\Delta$. For this let $c/b \in B^\Delta \cap T$, where $c \in R$ and b is a regular element in R. Then $c/b \in B^\Delta$, so $c \in (bB)_\Delta$ by (6.2.1), so $c \in (bB)_\Delta \cap R = (bR)_\Delta$ by (5.1.2). Thus $c/b \in R^\Delta$ by the formula for R^Δ in (6.2), so $B^\Delta \cap T \subseteq R^\Delta$.

Now let x be a regular nonunit in R^Δ and let $t \in (xB^\Delta)_\Delta \cap T$, so $t \in B^\Delta \cap T = R^\Delta$ by the preceding paragraph. Also, $(xB^\Delta)_\Delta = xB^\Delta$ by (6.8.1), so $t \in (xB^\Delta)_\Delta$ implies that $t/x \in B^\Delta \cap T = R^\Delta$. Therefore $t \in xR^\Delta$, so (7.1.2) holds.

For (7.1.3), $R[\{X_i\}_{i\in\Gamma}]$ is a faithfully flat R-algebra, so $R[\{X_i\}_{i\in\Gamma}]^\Delta \cap T = R^\Delta$ by (7.1.2). Therefore

$$R^\Delta[\{X_i\}_{i\in\Gamma}] = (R[\{X_i\}_{i\in\Gamma}]^\Delta \cap T)[\{X_i\}_{i\in\Gamma}] = R[\{X_i\}_{i\in\Gamma}]^\Delta \cap T[\{X_i\}_{i\in\Gamma}].$$

Finally, to prove (7.1.4) note first that $H \cap R \subseteq (0R)_\Delta$. (For, $H \cap R \subseteq (0T)_\Delta \cap R$, and if $r \in (0T)_\Delta \cap R$, then there exists an ideal $K \in \Delta$ such that $rKT = (0)$ in T, so $rK = (0)$ in R, and so $r \in (0) : K \subseteq (0R)_\Delta$.) Now let an overbar denote residue class modulo H and let $\overline{x} \in \overline{R^\Delta}$, so there exist $b, c \in R$ such that b is regular, $c/b \in R^\Delta$, and $\overline{x} = \overline{c/b}$. Now T is a localization of R and b is a unit in T, so it follows that $\overline{c/b} = \overline{c}/\overline{b}$. Also, $c \in (bR)_\Delta$ by (6.2.1), so $\overline{c} \in (\overline{bR})_\Delta$ by (5.1.4). Therefore $\overline{x} = \overline{c}/\overline{b} \in \overline{R}^\Delta$ by the formula for R^Δ in (6.2), so it follows that $\overline{R^\Delta} \subseteq \overline{R}^\Delta$. Therefore $\overline{R^\Delta} \subseteq \overline{R}^\Delta \cap \overline{T}$, so it remains to show that the opposite inclusion holds.

For this let $\overline{x} \in \overline{R}^\Delta \cap \overline{T}$. Now $\overline{x} \in \overline{T}$, so there exist $b, c \in R$ such that b is regular, $x = c/b \in T$, and $\overline{x} = \overline{c/b}$, so $\overline{x} = \overline{c}/\overline{b}$ (as in the preceding paragraph). Then $\overline{c}/\overline{b} \in \overline{R}^\Delta$, so $\overline{c} \in (\overline{bR})_\Delta$ by (6.2.1), hence it follows from (5.1.4) that $c \in (bR)_\Delta$. Thus $c/b \in R^\Delta$, since b is regular in R, so $\overline{x} = \overline{c}/\overline{b} = \overline{c/b} \in \overline{R^\Delta}$, hence $\overline{R^\Delta} \supseteq \overline{R}^\Delta \cap \overline{T}$. Q.E.D.

This section will be closed with the following two corollaries of (7.1). Concerning (7.2), it is well known that if R is an integral domain, then the hypothesis that $R'[X_1, \ldots, X_n] = R[X_1, \ldots, X_n]'$ holds.

(7.2) **Corollary.** *Let Δ be a multiplicatively closed set of finitely generated nonzero ideals of a ring R, let $X_1, \ldots, X_n$ be indeterminates, and assume that $R'[X_1, \ldots, X_n] = R[X_1, \ldots, X_n]'$ and that no ideal in Δ is contained in a minimal prime ideal in R. Then $R[X_1, \ldots, X_n]^\Delta = R^\Delta[X_1, \ldots, X_n]$.*

Proof. Let $A = R[X_1, \ldots, X_n]$. Since no ideal in Δ is contained in a minimal prime ideal in R, it readily follows that no ideal in ΔA is contained in a minimal prime ideal in A, so it follows from (6.3.3) that $A^\Delta \subseteq A'$. Therefore $A^\Delta \subseteq A' = R'[X_1, \ldots, X_n] \subseteq T[X_1, \ldots, X_n]$, so the conclusion follows from (7.1.3). Q.E.D.

(7.3) **Corollary.** *With the notation of (7.1) let I be an ideal in R and let B be a faithfully flat R-algebra. Then $(IB^\Delta)_\Delta \cap T = (IR^\Delta)_\Delta$.*

Proof. $(IB^\Delta)_\Delta \cap T \subseteq B^\Delta \cap T = R^\Delta$ by (7.1.2), so $(IB^\Delta)_\Delta \cap T = (IB^\Delta)_\Delta \cap R^\Delta$. Also, $(IR^\Delta)_\Delta \subseteq (IB^\Delta)_\Delta \cap R^\Delta$ by (5.1.1), so it remains to show that the opposite inclusion holds. For this let $c/b \in (IB^{\Delta})_{\Delta} \cap R^{\Delta}$, where $b, c \in R$ with b regular. Then $c \in b(IB^{\Delta})_{\Delta} \subseteq (bIB^{\Delta})_{\Delta}$ by (2.4.1) and (2.4.4). Therefore

$$\begin{aligned} c \in (bIB^{\Delta})_{\Delta} \cap B \cap R &= (bIB)_{\Delta} \cap R \quad \text{by (6.8.2)} \\ &= (bI)_{\Delta} \quad \text{by (5.1.2).} \end{aligned}$$

Thus

$$\begin{aligned} c \in (bI)_{\Delta} R^{\Delta} &\subseteq (bIR^{\Delta})_{\Delta} \quad \text{by (5.1.1)} \\ &= b(IR^{\Delta})_{\Delta} \quad \text{by (6.6.3) and (6.10),} \end{aligned}$$

so $c/b \in (IR^{\Delta})_{\Delta}$, hence $(IB^{\Delta})_{\Delta} \cap T \subseteq (IR^{\Delta})_{\Delta}$. Q.E.D.

8. Some examples

If R is a ring, then many specific types of extension rings of R have been studied in the literature. The main result in this section, (8.1), shows that each such extension ring that is contained in the integral closure R' of R is the Δ-closure of R for a suitable choice of Δ. In fact, every ring A such that $R \subseteq A \subseteq R'$ is R^{Δ} for an appropriate Δ.

(8.1) **Example.** Let R be a ring and let A be a ring between R and R'. For each regular nonunit b in R let $b^{\#} = bA \cap R$, and let Δ be the set of all finite products of the ideals in $\{(b, x)R; b$ is a regular nonunit in R and $x \in b^{\#}\}$. Then $(bR)_{\Delta} = b^{\#}$ for all regular nonunits b in R and $R^{\Delta} = A$.

Proof. It is clear that Δ is a multplicatively closed set of finitely generated nonzero ideals of R. Also, if b is a regular nonunit in R and $x \in b^{\#}$, then $bA \subseteq (b, x)A \subseteq b^{\#}A = (bA \cap R)A \subseteq bA$, so it follows that KA is a regular principal ideal for all $K \in \Delta$. Further, if $x \in (bR)_{\Delta}$, then there exists an ideal K in Δ such that $x \in bK : K$. Therefore $x \in bK : K \subseteq (bK : K)A \subseteq bKA : KA = bA$, since KA is a regular principal ideal, so $x \in bA \cap R = b^{\#}$, hence $(bR)_{\Delta} \subseteq b^{\#}$.

To see that the opposite inclusion holds, note that $A \subseteq R'$, so $b^{\#} = bA \cap R \subseteq bR' \cap R = (bR)_a$. Therefore if $x \in b^{\#}$, then $x^m + r_1 x^{m-1} + \cdots + r_m = 0$ for some $m \geq 1$ and for some elements $r_i \in b^i R$ for $i = 1, \ldots, m$, so it follows that $(b, x)^m R = b(b, x)^{m-1} R$. Therefore $(b, x)R \subseteq b(b, x)^{m-1} R : (b, x)^{m-1} R \subseteq (bR)_{\Delta}$ (by the definition of Δ), hence $b^{\#} \subseteq (bR)_{\Delta}$ for all regular nonunits b in R.

Finally, $A = R[\{c/b; b$ is a regular nonunit in R and $c \in b^{\#}\}]$, since the definition of $b^{\#}$ shows that A contains each such c/b, and if $v/u \in A$ (with $u, v \in R$ such that u is regular), then $v \in uA \cap R = u^{\#}$. Therefore, since $(bR)_{\Delta} = b^{\#}$, it follows from the formula for R^{Δ} in (6.2) that $R^{\Delta} = A$. Q.E.D.

If R is a Noetherian domain, then the following specific rings A as in (8.1) have appeared in many papers in the literature:

(a) $A = R^{(1)} = \bigcap\{R_p\,;p$ is a height one prime ideal in $R\}$. (To have $A \subseteq R'$ it must be assumed that height one prime ideals in R' lie over height one prime ideals in R.)

(b) $A = R^{(w)} = \bigcap\{R_p\,;p$ is a nonmaximal prime ideal in $R\}$. (To have $A \subseteq R'$ it must be assumed that height one maximal ideals in R' lie over height one maximal ideals in R.)

(c) I is an ideal R and $A = T(I) = \bigcap\{R_p\,;p$ is a prime ideal in R such that $I \not\subseteq p\}$, so A is the *I-transform* of R. (It must be assumed that $A \subseteq R'$, but this does hold for many choices of I.)

(d) B is an arbitrary extension ring of R and $A = B \cap R'$.

(e) $A = R'$.

In (a)–(e), since R is Noetherian, the ideals $b^{\#} = bA \cap R$ are finitely generated, and then essentially the same proof as for (8.1) shows that Δ can be chosen to be the set of all finite products of the ideals $b^{\#}$ (rather than the finite products of all $(b, x)R$ with $x \in b^{\#}$).

(8.2) *Remark.* With the notation of (8.1):

(8.2.1) $IA = I_\Delta A = (IA)_\Delta$ for all ideals I in R.

(8.2.2) If Ω is a multiplicatively closed set of finitely generated nonzero ideals of R such that $(bR)_\Omega = (bR)_\Delta$ for all regular principal ideals bR, then there may still exist ideals I in R such that $I_\Omega \neq I_\Delta$.

Proof. For (8.2.1), it follows from (2.4.1) that $IA \subseteq I_\Delta A$, and $I_\Delta A \subseteq (IA)_\Delta$ by (5.2.1). Finally, it was shown in the first paragraph of the proof of (8.1) that ΔA consists of principal ideals, so $(IA)_\Delta = IA$.

For (8.2.2) let $R = A = R'$ in (8.1) and assume that R is a Noetherian domain. Then it follows that Δ in (8.1) is the set of nonzero principal ideals of R, hence $I_\Delta = I$ for all ideals I in R. Fix a nonprincipal ideal K in R such that $K \neq K^{n+1} : K^n$ for some $n \geq 1$ and let $\Omega = \{K^n\,; n \geq 1\}$, so $K_\Omega \neq K$. Then (3.3.1) implies that $(bR)_\Omega = bR$ for each nonzero principal ideal bR (since $bR = (bR)_a$), so $(bR)_\Delta = bR = (bR)_\Omega$ and $K_\Delta = K \neq K_\Omega$. Q.E.D.

If b is a regular nonunit in R, then the ring $R' \cap R[1/b]$ has appeared in many research papers. It is characterized as the smallest ring B between R and its total quotient ring such that $(bB)_a = bB$, and then $(b^n B)_a = b^n B$ for all $n \geq 1$. (8.3) considers the related ring $R^\Delta \cap R_b$; as noted just above, special cases of R^Δ can include $R^{(1)}$, $R^{(w)}$, $T(I)$, R', and any ring between R and R'.

(8.3) **Example.** Let b be a regular nonunit in a ring R, let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R such that no K in Δ is contained in a minimal prime ideal in R, and let $A = R^\Delta \cap R[1/b]$. Then A is the smallest ring B between R and its total quotient ring T such that $(bB)_\Delta = bB$, and then $(b^n A)_\Delta = b^n A$ for all $n \geq 1$. Moreover, if Ω is the set of all finite products of the ideals $(b^n, x)R$, where $n \geq 0$ and $x \in (b^n R)_\Delta$, then $A = R^\Omega$.

Proof. It will first be shown that $(b^n A)_\Delta = b^n A$ for all $n \geq 1$. For this, if $x \in (b^n A)_\Delta$, then $x/b^n \in A^\Delta$ by the formula for R^Δ in (6.2), and $A^\Delta = R^\Delta$ by (6.6.4). Also, $x/b^n \in A[1/b] = R[1/b]$, so $x/b^n \in R^\Delta \cap R[1/b] = A$, so $x \in b^n A$. Therefore $(b^n A)_\Delta \subseteq b^n A$, and the opposite inclusion is given by (2.4.1).

To show that A is the smallest such ring let B be a ring between R and T such that $(bB)_\Delta = bB$. Let $x \in A = R^\Delta \cap R[1/b]$, so there exist an element r in R and a positive integer n such that $x = r/b^n$, and then $r \in (b^n R)_\Delta$ by (6.2.1). However, $(b^n R)_\Delta \subseteq (b^n B)_\Delta \cap R$ by (5.1.1), and it follows from the hypothesis on bB and (4.1.6) that $(b^n B)_\Delta = b^n B$, so $r \in b^n B \cap R$. Therefore $x = r/b^n \in B$, so $A \subseteq B$.

To prove the last statement, note first that $A \subseteq R^\Delta$, so $A^\Delta = R^\Delta$ by (6.6.4), so it follows from (6.3.1) that $(b^n R)_\Delta = b^n R^\Delta \cap R$ and $(b^n A)_\Delta = b^n R^\Delta \cap A$, and it has already been shown that $(b^n A)_\Delta = b^n A$, so $(b^n R)_\Delta A \subseteq b^n A$. Therefore it follows that $(b^n R)_\Delta A = b^n A$ for all $n \geq 1$, so it follows from the definition of Ω that KA is a regular principal ideal for all $K \in \Omega$. Also, if c is a regular nonunit in R and $r \in (cR)_\Omega$, then there exists an ideal K in Ω such that $r \in cK : K$. Then $r \in (cK : K)A \subseteq cKA : KA = cA$, since KA is a regular principal ideal, so it follows that $(cR)_\Omega \subseteq cA$. Therefore if $x \in R^\Omega$, then $x = v/u$ for some $u, v \in R$ such that $v \in (uR)_\Omega \subseteq uA$, so $x = v/u \in A$, hence $R^\Omega \subseteq A$. Thus it remains to show that $A \subseteq R^\Omega$.

For this, it follows from the hypothesis on the ideals K in Δ and (3.2.1) that $(bR)_\Delta \subseteq (bR)_a$ for all regular nonunits b in R. Therefore it follows as in the second paragraph of the proof of (8.1) that if $x \in (b^n R)_\Delta$, then there exists a positive integer m such that $(b^n, x)R \subseteq b^n(b^n, x)^{m-1}R : (b^n, x)^{m-1}R \subseteq (b^n R)_\Omega$ (by the definition of Ω), so $(b^n R)_\Delta \subseteq (b^n R)_\Omega$. Also, if $x \in A = R^\Delta \cap R[1/u]$, then as in the second paragraph of this proof $x = r/b^n$ for some $r \in (b^n R)_\Delta$. It therefore follows that $r \in (b^n R)_\Omega$, so $x = r/b^n \in R^\Omega$, by the formula for R^Δ in (6.2). Therefore $A \subseteq R^\Omega$, so $A = R^\Omega$. Q.E.D.

(8.4) **Corollary.** *Let b be a regular nonunit in a ring R and let Δ be a multiplicatively closed set of finitely generated nonzero ideals of R such that no K in Δ is contained in a minimal prime ideal in R. Assume that $R^\Delta \subseteq R[1/b]$ and let Ω be the set of all finite products of the ideals $(b^n, x)R$, where $n \geq 0$ and $x \in (b^n R)_\Delta$. Then $(cR)_\Omega = (cR)_\Delta$ for all regular nonunits c in R and $R^\Omega = R^\Delta$.*

Proof. Since $A = R^\Delta \cap R[1/b] = R^\Delta$, it follows immediately from (8.2) that $R^\Omega = R^\Delta$. Therefore it follows from (6.3.1) that $(cR)_\Omega = cR^\Omega \cap R = cR^\Delta \cap R = (cR)_\Delta$. Q.E.D.

9. The Δ-closure and associated primes

During the past few years several dozen papers concerning the associated primes of large powers of ideals in Noetherian rings have appeared in the literature. Together, these papers show that there are (at least) three parallel theories concerning the prime ideals associated with ideals in Noetherian rings, namely the standard theory (with associated primes, R-sequences, classical grade, and Cohen-Macaulay rings), the asymptotic theory (with asymptotic prime divisors, asymptotic sequences, asymptotic grade, and locally quasi-unmixed Noetherian rings), and the essential theory (with essential prime divisors, essential sequences, essential grade, and locally unmixed Noetherian rings). These papers also show that a result for one of these theories usually has a valid analogous result for the other two.

In this last section, as one application of Δ-closures, it is shown that several important theorems in the asymptotic theory are special cases of Δ-prime divisors. As in the previous sections, Δ is an arbitrary multiplicatively closed set of finitely generated nonzero ideals of a ring R, but here we restrict attention to the case when R is Noetherian, and we consider the prime divisors (= associated primes) of I_Δ. Among the results, it is shown that the prime divisors of I_Δ remain prime divisors of $I_\Delta K$ and of $(IK)_\Delta$ for all ideals K in Δ. We begin with the following definition.

(9.1) **Definition.** If I is an ideal in a Noetherian ring R, then $\mathrm{Ass}(R/I) = \{P; P$ is a prime ideal in R, $I \subseteq P$, and there exists an element x in R such that $I : xR = P\}$. The members of P are called the *prime divisors* (or *associated primes*) of I.

It is well known that $\mathrm{Ass}(R/I)$ is a finite set, and $\mathrm{Ass}(R/I)$ is a one-point set if and only if I is a primary ideal. It is also known (and readily checked) that if there exists an ideal H in R such that either $I : H = P$ or P is a prime divisor of $I : H$, then $P \in \mathrm{Ass}(R/I)$. These results will be used implicitly in this section.

(9.2) is one of the main results in this section; it will be seen that it implies several known theorems in the literature concerning asymptotic prime divisors.

(9.2) **Theorem.** *If $I \subseteq P$ are ideals in a Noetherian ring R such that $P \in \mathrm{Ass}(R/I_\Delta)$, and if $K \in \Delta$ is such that $I_\Delta = IK : K$, then the following hold for all ideals H in Δ:*

(9.2.1) $P \in \mathrm{Ass}(R/I_\Delta H) \cap \mathrm{Ass}(R/(IH)_\Delta)$.

(9.2.2) $P \in \mathrm{Ass}(R/IKH)$.

Proof. (R satisfies the ACC on all its ideals, since R is Noetherian, so it follows that (2.8) is applicable, and (2.8)(a) shows that there exists an ideal K in Δ such that $I_\Delta = IK : K$.) Since $P \in \mathrm{Ass}(R/I_\Delta)$, there exists an element b in R such that $I_\Delta : bR = P$. Therefore $P = I_\Delta : bR \subseteq I_\Delta H : bH \subseteq (IH)_\Delta : bH$, by (2.4.1) and (2.4.4), $= ((IH)_\Delta : H) : bR = I_\Delta : bR$, by (2.6), $= P$. (9.2.1) follows immediately from this. Also, $P = I_\Delta : bR = (IKH : KH) : bR$, by (2.8)(b), $= IKH : bKH$, and so (9.2.2) holds. Q.E.D.

(9.3) **Corollary.** *If* $I \subseteq P$ *are ideals in a Noetherian ring* R *such that* height$(I) \geq 1$ *and* $P \in \mathrm{Ass}(R/I_a)$, *then* $P \in \mathrm{Ass}(R/I_aK) \cap \mathrm{Ass}(R/(IK)_a)$ *for all ideals* K *in* R *such that* height$(K) \geq 1$.

Proof. Let Δ be the (multiplicatively closed) set of ideals K in R such that height$(K) \geq 1$. Then $I_a = I_\Delta$ for all ideals I in R such that height$(I) \geq 1$ by (3.2.3), so the conclusion follows immediately from (9.2.1). Q.E.D.

It should be noted that two important results concerning asymptotic prime divisors are special cases of (9.3). Specifically: (a) [7, (2.4)] if $P \in \mathrm{Ass}(R/(I^n)_a)$ for some positive integer n, then $P \in \mathrm{Ass}(R/(I^m)_a)$ for all $m \geq n$; and (b) [9, (2.8)] if $P \in \mathrm{Ass}(R/(I^n)_a)$ for some positive integer n, then $P \in \mathrm{Ass}(R/(IK^m)_a)$ for all regular ideals K in R and for all large integers m.

The following remark gives two other results that follow from (9.2). (9.4.2) is another important result concerning asymptotic prime divisors. In several recent papers the ideal $(I^k)^*$ is called the relevant component of I^k. It is known that if I is regular, then $(I^k)^* = I^k$ for all large k, so in this case (9.4.2) implies Brodmann's theorem: the sets $\mathrm{Ass}(R/I^n)$ are equal for all large n.

(9.4) *Remark.* Let $I \subseteq P$ be ideals in a Noetherian ring R such that P is prime. Then:

(9.4.1) If $P \in \mathrm{Ass}(R/I)$, then $P \in \mathrm{Ass}(R/bI)$ for all regular nonunits b in R.

(9.4.2) If $P \in \mathrm{Ass}(R/(I^k)^*)$ for some $k \geq 1$, where

$$(I^k)^* = \bigcup \{I^{k+n} : I^n;\ n \geq 0\},$$

then $P \in \mathrm{Ass}(R/(I^k)^* I^n) \cap \mathrm{Ass}(R/(I^{k+n})^*)$ for all $n \geq 1$.

Proof. For (9.4.1) take $\Delta = \{bR; b$ is a regular nonunit in $R\}$ in (9.2) (so $I_\Delta = I$ and $(bI)_\Delta = bI$), and for (9.4.2) take $\Delta = \{I^n; n \geq 1\}$ in (9.2) (so $(I^k)_\Delta = (I^k)^*$ for all $k \geq 1$). Q.E.D.

(9.5) is a variation of (9.2).

(9.5) **Proposition.** *If* $I \subseteq P$ *are ideals in a Noetherian ring* R *such that* P *is a minimal prime divisor of* $I : bR$ *and of* $I_\Delta : bR$ *for some* $b \in R$, *then* $P \in \mathrm{Ass}(R/IK)$ *for all ideals* K *in* Δ.

Proof. If $K \in \Delta$, then $I : bR \subseteq IK : bK = (IK : K) : bR \subseteq I_\Delta : bR$. Therefore it follows from the hypothesis that P is a minimal prime divisor of $IK : bK$, so $P \in \mathrm{Ass}(R/IK)$. Q.E.D.

(9.6) **Corollary.** *Let $I \subseteq P$ be ideals in a Noetherian ring R such that P is prime. Then*:

(9.6.1) *If P is a minimal prime divisor of $I : bR$ and $I^* : bR$ for some $b \in R$, then $P \in \mathrm{Ass}(R/I^n)$ for all $n \geq 1$.*

(9.6.2) *If* $\mathrm{height}(I) \geq 1$ *and P is a minimal prime divisor of $I : bR$ and of $I_a : bR$ for some $b \in R$, then $P \in \mathrm{Ass}(R/IK)$ for all ideals K in R such that* $\mathrm{height}(K) \geq 1$.

Proof. For (9.6.1) let $\Delta = \{I^n ; n \geq 1\}$ in (9.5). Then $I_\Delta = I^*$, so (9.6.1) follows readily from (9.5). And for (9.6.2) let $\Delta = \{K ; \mathrm{height}(K) \geq 1\}$ in (9.5). Then $I_\Delta = I_a$ by (3.2.3), so (9.6.2) follows easily from (9.5). Q.E.D.

As noted in (3.2.3), if Δ is the set of all ideals K in R such that $\mathrm{height}(K) \geq 1$, then $I_\Delta = I_a$, and the associated primes of I_a have been studied in a number of papers, so many results are known concerning them. The final result, (9.7), gives several analogous results concerning the associated primes of I_Δ.

(9.7) **Proposition.** *If $I \subseteq P$ are ideals in a Noetherian ring R such that P is prime, then*:

(9.7.1) *If P is a minimal prime divisor of I, then $P \in \mathrm{Ass}(R/I_\Delta)$ if and only if $(0) \notin \Delta R_P$.*

(9.7.2) $\mathrm{Ass}(R/I_\Delta) \subseteq \mathrm{Ass}(R/(IK)_\Delta)$ *for all ideals $K \in \Delta$.*

(9.7.3) *If B is a Noetherian flat extension ring of R such that $(0) \notin \Delta B$, then $\mathrm{Ass}(B/(IB)_\Delta) = \mathrm{Ass}(B/(I_\Delta B)) = \{P^* ; P^* \in \mathrm{Ass}(B/PB)$ for some $P \in \mathrm{Ass}(R/I_\Delta)\}$ and $\mathrm{Ass}(R/I_\Delta) \supseteq \{P^* \cap R ; P^* \in \mathrm{Ass}(B/(IB)_\Delta)\}$ with the equality holding if B is faithfully flat.*

(9.7.4) *Let $Z = (0)_\Delta$, let H be an ideal in R such that $H \subseteq Z$, and let an overbar denote residue class modulo H. Then $\mathrm{Ass}(\overline{R}/\overline{I}_\Delta) = \{\overline{P} ; P \in \mathrm{Ass}(R/I_\Delta)\}$.*

Proof. For (9.7.1) let $K \in \Delta$ such that $I_\Delta = IK : K$ and let P be a prime ideal in R that contains I. Then $I_\Delta R_P = IR_P KR_P : KR_P$, so since R_P is local it follows that $I_\Delta R_P$ is proper if and only if $KR_P \neq (0)$. (9.7.1) follows from this and (2.8)(b).

(9.7.2) follows immediately from (9.2.1).

For (9.7.3), $\mathrm{Ass}(B/(IB)_\Delta) = \mathrm{Ass}(B/(I_\Delta B))$ by (5.1.2), and the remaining parts of (9.7.3) follow readily from this and known properties of flat extensions.

Finally, (9.7.4) follows readily from (5.1.4). Q.E.D.

References

1. I. Kaplansky, *Commutative rings*, Allyn & Bacon, Boston, Mass., 1970.
2. D. Kirby, *Closure operations on ideals and submodules*, J. London Math. Soc. **44** (1969), 283–291.
3. W. Krull, *Beitrage zur Arithmetik kommutativer Integritatsbereiche*, I, Math. Z. **41** (1936), 545–577.
4. ——, *Idealtheorie*, Chelsea, New York, 1948.

5. S. McAdam and L. J. Ratliff, Jr., *Sporadic and irrelevant prime divisors*, Trans. Amer. Math. Soc. **303** (1987), 311–324.

6. J. W. Petro, *Some results on the asymptotic completion of an ideal*, Proc. Amer. Math. Soc. **15** (1964), 519–524.

7. L. J. Ratliff, Jr., *On asymptotic prime divisors*, Pacific J. Math. **111** (1984), 395–413.

8. M. Sakuma, *On prime operations in the theory of ideals*, Hiroshima Math. J. **20** (1957), 101–106.

9. K. Whittington, *Asymptotic prime divisors and the altitude formula*, 20-page preprint.

10. O. Zariski and P. Samuel, *Commutative algebra*, Vol. II, Van Nostrand, New York, 1960.

Department of Mathematics, University of California, Riverside, California 92521