CELL-LIKE MAPPINGS AND NONMETRIZABLE COMPACTA OF FINITE COHOMOLOGICAL DIMENSION

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ABSTRACT. Compact Hausdorff spaces X of cohomological dimension $\dim_{\mathbb{Z}} X \leq n$ are characterized as cell-like images of compact Hausdorff spaces Z with covering dimension $\dim Z \leq n$. The proof essentially uses the newly developed techniques of approximate inverse systems.

1. INTRODUCTION

In 1978, R. D. Edwards [2] announced the result that every metrizable compact space X of cohomological dimension $\dim_{\mathbb{Z}} X \leq n$ (integer coefficients) is the image of a cell-like mapping $f: Z \rightarrow X$ of a compact metric space Z with $\dim Z \leq n$. A proof of this result was published in 1981 by J. J. Walsh [16]. By the classical Vietoris-Begle theorem (see, e.g. [15, Chapter 6, §9, Theorem 15]), the converse also holds, and one thus has a characterization of metrizable compacta X with $\dim_{\mathbb{Z}} X \leq n$.

More recently, L. R. Rubin and P. J. Schapiro [14] have succeeded in generalizing the Edwards-Walsh theorem to the case of metrizable spaces X and Z .

The purpose of our present paper is to generalize the Edwards-Walsh theorem in another direction, i.e., to establish the result for compact Hausdorff spaces (see §11, Theorem 3).

In generalizing the theorem to compact Hausdorff spaces, one encounters a difficulty which was not present in the two previous cases. In the case of metric compacta [16], the space X was represented as the limit of an inverse sequence X of polyhedra. This sequence led to another sequence Z of polyhedra of dimension $\leq n$, and the space Z was obtained as the limit of Z .

Dealing with a noncompact situation, Rubin and Schapiro [14] had to overcome many obstacles. Still they were able to obtain Z as the inverse limit

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of a sequence $\mathbf{Z}$ of noncompact polyhedra because the uniform structure of a metric space has a countable basis of uniform coverings.

In the case of compact Hausdorff spaces X , one cannot avoid using partially ordered inverse systems $\mathbf{X}$ of polyhedra. Upon applying the process of approximating the bonding maps of $\mathbf{X}$ on n -skeleta as in [16, 14], one invariably obtains a system $\mathbf{Z}$ whose bonding maps $r_{a_1 a_2}$ do not satisfy the commutativity condition $r_{a_1 a_2} r_{a_2 a_3} = r_{a_1 a_3}$, $a_1 \leq a_2 \leq a_3$. The best one can achieve is to keep the distances $d(r_{a_1 a_2} r_{a_2 a_3}, r_{a_1 a_3})$ small. Therefore, the attempt to prove the theorem for compact Hausdorff spaces had to be preceded by the development of a theory of approximately commutative inverse systems. This was initiated by the authors in [6], and it was continued in [8, 9, 10].

An interesting phenomenon, which adds additional difficulty, is that the approximate system yielding $\mathbf{Z}$, constructed herein, is a system of metric compacta Z_a^* (see §6) which are not polyhedra.

Recently, A. N. Dranishnikov [1] solved a classical problem of P. S. Aleksandrov by exhibiting a metric compactum X with $\dim_Z X \leq 3$ and $\dim X = \infty$. This shows that $\dim_Z$ and $\dim$ differ and in our main result (Theorem 3) one cannot take for f the identity map 1_X .

2. APPROXIMATE INVERSE SYSTEMS

We quote from [6] the basic definition of an approximate system and of its limit.

Definition 1. An approximate (inverse) system of metric compacta $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ consists of the following: A directed ordered set $(A, \leq)$ with no maximal element; for each $a \in A$, a compact metric space X_a with metric $d = d_a$ and a real number $\varepsilon_a > 0$; for each pair $a \leq a'$ from A , a mapping $p_{aa'}: X_{a'} \rightarrow X_a$, satisfying the following conditions:

$$(A1) \quad d(p_{a_1 a_2} p_{a_2 a_3}, p_{a_1 a_3}) \leq \varepsilon_{a_1}, \quad a_1 \leq a_2 \leq a_3; \quad p_{aa} = \text{id}.$$

$$(A2) \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a_2 \geq a_1 \geq a') \quad d(p_{aa_1} p_{a_1 a_2}, p_{aa_2}) \leq \eta.$$

$$(A3) \quad (\forall a \in A)(\forall \eta > 0)(\exists a' \geq a)(\forall a'' \geq a')(\forall x, x' \in X_{a''}) \quad d(x, x') \leq \varepsilon_{a''} \Rightarrow d(p_{aa''}(x), p_{aa''}(x')) \leq \eta.$$

We refer to the numbers ε_a as the meshes of $\mathbf{X}$. We say $\mathbf{X}$ is cofinite if A is cofinite, i.e., every element $a \in A$ has only finitely many predecessors.

If $\pi_a: \prod_{a \in A} X_a \rightarrow X_a$, $a \in A$, denote projections, we define the limit space $X = \lim \mathbf{X}$ and the natural projections $p_a: X \rightarrow X_a$ as follows.

Definition 2. A point $x = (x_a) \in \prod X_a$ belongs to $X = \lim \mathbf{X}$ provided for every $a \in A$,

$$x_a = \lim_{a_1} p_{aa_1}(x_{a_1}).$$

The natural projection $p_a = \pi_a|X: X \rightarrow X_a$.

We now quote (as propositions) several results from [6 and 8] needed in this paper.

Proposition 1. *If $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ is an approximate system and $X_a \neq \emptyset$ for each $a \in A$, then $X = \lim \mathbf{X}$ is a compact Hausdorff space and $X \neq \emptyset$ (see [6, Theorems 1 and 2]).*

Proposition 2. *If $\mathcal{V}_a$ is a basis for X_a , $a \in A$, then the sets $p_a^{-1}(V_a)$, $V_a \in \mathcal{V}_a$, $a \in A$, form a basis for $X = \lim \mathbf{X}$ (see [6, Lemma 3]).*

Proposition 3. *For any $a \in A$, $\lim_{a_1} d(p_a, p_{aa_1} p_{a_1}) = 0$, where $d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ (see [6, Lemma 4]).*

Proposition 4. *Every approximate system $\mathbf{X}$ has the following two properties:*

- (B1) *Let $a \in A$ and let $U \subseteq X_a$ be an open set which contains $p_a(X)$. Then there exists an $a' \geq a$ such that $p_{aa''}(X_{a''}) \subseteq U$ for any $a'' \geq a'$.*
- (B2) *For every open covering $\mathcal{U}$ of X , there exists an $a \in A$ such that for any $a_1 \geq a$ there exists an open covering $\mathcal{V}$ of X_{a_1} for which $(p_{a_1})^{-1}(\mathcal{V})$ refines $\mathcal{U}$ (see [8, Theorem 3] and [6, Theorem 1]).*

Proposition 5. *Let $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system with limit X . If $\dim X_a \leq n$ for all $a \in A$, then the covering dimension $\dim X \leq n$ (see [6, Theorem 4]).*

Proposition 6. *Let $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system with limit X , and let $B \subseteq A$ be a cofinal subset of A . Then $\mathbf{Y} = (X_b, \varepsilon_b, p_{bb'}, B)$ is also an approximate system. Moreover, the restriction $p = \pi|_X$ of the projection $\pi: \prod_{a \in A} X_a \rightarrow \prod_{b \in B} X_b$ is a homeomorphism $p: X \rightarrow Y$ (see [8, Proposition 2]).*

We will now add several new propositions.

Proposition 7. *Every approximate system $\mathbf{X}$ has the following property:*

- (R1) *For every $\varepsilon > 0$, every compact ANR P , and every mapping $h: X \rightarrow P$, there is an $a \in A$ such that for any $a' \geq a$ there is a mapping $f: X_{a'} \rightarrow P$ which satisfies $d(fp_{a'}, h) \leq 2\varepsilon$.*

Proof. (This proof follows closely that given for the analogous theorem for commutative systems given in [7, Chapter I, §5.2, Theorem 8]. We first embed P in the Hilbert cube Q and choose a closed neighborhood G of P in Q which admits a retraction $r: G \rightarrow P$. Then we choose $\delta > 0$ so small that $\delta \leq \varepsilon/2$, the δ -neighborhood of P is contained in G , and

$$(1) \quad y, y' \in G, \quad d(y, y') \leq \delta \Rightarrow d(r(y), r(y')) \leq \varepsilon/2.$$

We then choose an open covering $\mathcal{U}$ of X so fine that each $h(U)$, $U \in \mathcal{U}$, is contained in a convex set $B \subseteq G$ with $\text{diam } B < \delta$.

By property (B2) (Proposition 4) there is an $a_0 \in A$ such that there is an open covering $\mathcal{W}$ of X_{a_0} for which $(p_{a_0})^{-1}(\mathcal{W})$ refines $\mathcal{U}$. Let $\mathcal{W}_1$ be a finite

open covering of X_{a_0} such that $\mathcal{W}_1$ is a star-refinement of $\mathcal{W}$. Let $\mathcal{W}_2 \subseteq \mathcal{W}_1$ consist of all $W \in \mathcal{W}_1$ with $W \cap p_{a_0}(X) \neq \emptyset$. For each $W \in \mathcal{W}_2$ we choose a point $y_w \in h((p_{a_0})^{-1}(W))$. Let N be a closed neighborhood of $p_{a_0}(X)$ in X_{a_0} covered by $\mathcal{W}_2$, and let $(\varphi_w, W \in \mathcal{W}_2)$ be a partition of unity on N subordinated to the cover $\mathcal{W}_2|N$. We define a map $g: N \rightarrow Q \subseteq \mathbf{R}^\infty$ by

$$(2) \quad g(z) = \sum_{W \in \mathcal{W}_2} \varphi_w(z) y_w, \quad z \in N.$$

It is not difficult to show that

$$(3) \quad g(N) \subseteq G.$$

Moreover,

$$(4) \quad d(rgp_{a_0}, h) \leq \varepsilon.$$

(see [7, pp. 63–64]).

We now apply property (B1) (Proposition 4) and find an $a \geq a_0$ such that for any $a' \geq a$ one has

$$(5) \quad p_{a_0 a'}(X_{a'}) \subseteq N.$$

Using Proposition 3, we can also assume that

$$(6) \quad d(p_{a_0}, p_{a_0 a'} p_{a'}) \leq \omega,$$

where $\omega > 0$ is such that

$$(7) \quad z, z' \in N, \quad d(z, z') \leq \omega \Rightarrow d(rg(z), rg(z')) \leq \varepsilon.$$

Then, (6) and (7) yield

$$(8) \quad d(rgp_{a_0}, rgp_{a_0 a'} p_{a'}) \leq \varepsilon.$$

Putting

$$(9) \quad f = rgp_{a_0 a'}: X_{a'} \rightarrow P,$$

one obtains

$$(10) \quad d(fp_{a'}, h) \leq 2\varepsilon, \quad a' \geq a.$$

Proposition 8. *Let $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system of metric compacta with the following property:*

$$(\forall a_1)(\exists a'_1 \geq a_1)(\forall a_2 \geq a'_1)(\exists a'_2 \geq a_2)(\forall a_3 \geq a'_2) \\ p_{a_1 a_2} p_{a_2 a_3} \simeq 0.$$

Then $X = \lim \mathbf{X}$ has the shape of a point, $sh(X) = 0$.

Proof. Let P be an ANR and $f: X \rightarrow P$ be a map. It suffices to prove that $f \simeq 0$. Choose $\delta > 0$ such that δ -near maps into P are homotopic. By

property (R1) (Proposition 7), there is an $a_1 \in A$ and a map $g: X_{a_1} \rightarrow P$ such that $d(gp_{a_1}, f) \leq \delta$ and therefore $gp_{a_1} \simeq f$. It therefore suffices to prove that

$$(11) \quad gp_{a_1} \simeq 0.$$

Let $\eta > 0$ be such that

$$(12) \quad d(x, x') \leq 2\eta \Rightarrow d(g(x), g(x')) \leq \delta.$$

Then for any map $p'_{a_1}: X \rightarrow X_{a_1}$,

$$(13) \quad d(p'_{a_1}, p_{a_1}) \leq 2\eta \Rightarrow d(gp'_{a_1}, gp_{a_1}) \leq \delta$$

and therefore,

$$(14) \quad d(p'_{a_1}, p_{a_1}) \leq 2\eta \Rightarrow gp'_{a_1} \simeq gp_{a_1}.$$

We now choose an $a'_1 \geq a_1$ according to the assumption of the proposition. Next we choose $a_2 \geq a'_1$ in such a way that

$$(15) \quad d(p_{a_1 a_3} p_{a_3 a_4}, p_{a_1 a_4}) \leq \eta,$$

for all $a_4 \geq a_3 \geq a_2$ (property (A2)). Clearly, for any $a_3 \geq a_2$, (15) implies

$$(16) \quad d(p_{a_1 a_2} p_{a_2 a_3}, p_{a_1 a_3}) \leq \eta.$$

Moreover, (15) and Proposition 3 imply

$$(17) \quad d(p_{a_1 a_3} p_{a_3}, p_{a_1}) \leq \eta.$$

(16) and (17) yield

$$(18) \quad d(p_{a_1 a_2} p_{a_2 a_3} p_{a_3}, p_{a_1}) \leq 2\eta,$$

which, by (14), implies

$$(19) \quad gp_{a_1} \simeq gp_{a_1 a_2} p_{a_2 a_3} p_{a_3}, \quad \text{for any } a_3 \geq a_2.$$

We now choose an $a'_2 \geq a_2$ according to the assumption of the proposition. Therefore, if we choose $a_3 \geq a'_2$, we have

$$(20) \quad p_{a_1 a_2} p_{a_2 a_3} \simeq 0.$$

Now, (11) follows from (19) and (20).

Remark 1. An analogous proposition holds for the following property and any $n \geq 1$:

$$(\forall a_1)(\exists a'_1 \geq a_1) \cdots (\forall a_n \geq a'_{n-1})(\exists a'_n \geq a_n)(\forall a_{n+1} \geq a'_n) \\ p_{a_1 a_2} p_{a_2 a_3} \cdots p_{a_n} p_{a_{n+1}} \simeq 0.$$

Proposition 9. Let $X = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system with limit X and projections p_a . Let $<'$ be a binary relation on A satisfying the following conditions:

- (i) $a_1 <' a_2 \Rightarrow a_1 < a_2$,
- (ii) $a_1 <' a_2$ and $a_2 \leq a_3 \Rightarrow a_1 <' a_3$,
- (iii) $(\forall a \in A)(\exists a' \in A) a <' a'$.

Write $a_1 \leq' a_2$ if $a_1 <' a_2$ or $a_1 = a_2$, and let A' be the set A provided with the relation $\leq'$. Then A' is a directed set with no maximal element and $\mathbf{X}' = (X_a, \varepsilon_a, p_{aa'}, A')$ is an approximate system with limit X' and projections p'_a . Moreover, $X' = X$ and $p'_a = p_a$.

Proof. If $a_1 <' a_2$ and $a_2 <' a_3$, then by (i), $a_2 < a_3$ and by (ii), $a_1 <' a_3$. Therefore $\leq'$ is transitive. For any $a_1, a_2 \in A$, by (iii) there exist indexes $a'_1, a'_2 \in A$ such that $a_1 <' a'_1, a_2 <' a'_2$. Since $(A, \leq)$ is directed, there is an $a'' \in A$ such that $a'_1 \leq a'', a'_2 \leq a''$. Now (ii) implies $a_1 <' a'', a_2 <' a''$, which proves that A' is directed.

We now verify that $\mathbf{X}'$ is an approximate system. (A1) is an immediate consequence of (i). For given $a \in A$ and $\eta > 0$, choose $a' \geq a$ in accordance with (A2) for $\mathbf{X}$. By directedness of A' , there is an $a'_1 \in A$ such that $a \leq' a'_1$ and $a' \leq' a'_1$. If $a'_1 \leq' a_1 \leq' a_2$, then $a' \leq a_1 \leq a_2$ and therefore

$$(21) \quad d(p_{aa_1} p_{a_1 a_2}, p_{aa_2}) \leq \eta,$$

as required by (A2) for $\mathbf{X}'$.

If $a' \geq a$ satisfies (A3) for $\mathbf{X}$, then we choose a'_1 so that $a \leq' a'_1$ and $a' \leq a'_1$. For any a'' with $a'_1 \leq' a''$ we have $a'_1 \leq a''$ and therefore,

$$(22) \quad d(x, x') \leq \varepsilon_{a''} \Rightarrow d(p_{aa''}(x), p_{aa''}(x')) \leq \eta,$$

as required by (A3) for $\mathbf{X}'$.

Finally, for any $a \in A$,

$$(23) \quad A'_a = \{a_1 \in A : a \leq' a_1\} \subseteq A_a = \{a_1 \in A : a \leq a_1\}$$

and the set A'_a is cofinal in A_a . Therefore,

$$(24) \quad \lim_{a_1 \in A'_a} p_{aa_1}(x_{a_1}) = \lim_{a_1 \in A_a} p_{aa_1}(x_{a_1}) \quad \text{for any } (x_a) \in \prod X_a.$$

By Definition 2, this shows that $X = X'$ and thus also $p_a = p'_a$.

Proposition 10. Let $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system of metric compacta X_a with metrics d_a . Then there exist metrics $d'_a \leq 1$ on X_a , defining the same topology on X_a , and there exist numbers $\varepsilon'_a > 0$ such that $\mathbf{X}' = (X_a, \varepsilon'_a, p_{aa'}, A)$ is also an approximate system. Moreover, $X' = X$ and $p'_a = p_a$, $a \in A$.

Proof. For each $a \in A$ we put

$$(25) \quad d'_a = \frac{d_a}{1 + d_a},$$

$$(26) \quad \varepsilon'_a = \frac{\varepsilon_a}{1 + \varepsilon_a}.$$

Note that $d'_a \leq 1$ and d'_a is a metric compatible with d_a . Moreover,

$$(27) \quad d'_a \leq d_a,$$

$$(28) \quad d_a(x, x') \leq \varepsilon_a \Leftrightarrow d'_a(x, x') \leq \varepsilon'_a.$$

The verification of (A1)–(A3) for $\mathbf{X}'$ is now straightforward. Moreover, $X' = X$, $p'_a = p_a$, because the limit space depends only on the topology of the spaces X_a and on the maps $p_{aa'}$.

3. REPRESENTING COMPACT SPACES AS APPROXIMATE LIMITS

It is well known that every compact Hausdorff space X is the limit of a (commutative) inverse system of compact polyhedra $\mathbf{X} = (X_a, p_{aa'}, A)$ (with PL bonding maps $p_{aa'}$) (see, e.g., [7, I, §5.2, Theorem 7]). However, B. A. Pasynkov has shown [11, 12] that there exist compact Hausdorff spaces X which are not obtainable as limits of inverse systems $\mathbf{X}$ of polyhedra with surjective bonding mappings $p_{aa'}$. This difficulty vanishes if one considers approximate inverse systems as we show in the following theorem (needed later).

Theorem 1. *Every compact Hausdorff space X is the limit of an approximate (cofinite) inverse system $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$, where the spaces X_a are polyhedra and all the bonding maps $p_{aa'}$ are (irreducible) surjective PL-maps. Moreover, the cardinal $\text{card}(A) \leq w(X)$, the weight of X .*

We recall some notions and simple facts needed in the proof. By a polyhedron we always mean a compact polyhedron and by a complex, a finite simplicial complex. If K is a complex, then $|K|$ denotes its carrier, i.e., the corresponding polyhedron.

Let K be a complex and let $f, g: X \rightarrow |K|$ be mappings. We say that g is a K -modification of f if for every point $x \in X$ and simplex $\sigma \in K$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. Note that a simplicial approximation $\varphi: K_1 \rightarrow K_2$ of a mapping $\pi: |K_1| \rightarrow |K_2|$ is a K_2 -modification of π . Moreover, if K' is a subdivision of K and $g: X \rightarrow |K'|$ is a K' -modification of $f: X \rightarrow |K'| = |K|$, then g is also a K -modification of f .

We say that a mapping $f: X \rightarrow |K|$ is K -irreducible if for every K -modification g of f one has $g(X) = |K|$. Since f is its own K -modification, a K -irreducible map f is onto. A mapping $f: X \rightarrow P$ into a polyhedron is called *irreducible* if it is K -irreducible for some triangulation K of P . Note that every irreducible map $f: X \rightarrow P$ is onto.

For every complex K and mapping $f: X \rightarrow |K|$ there is a subcomplex $L \subseteq K$ and a K -modification $g: X \rightarrow |L| \subseteq |K|$ which is L -irreducible and, therefore $g: X \rightarrow |L|$ is irreducible and onto. If f is already K -irreducible, we put $L = K$, $g = f$. If not, there is a K -modification f_1 of f with $f_1(X) \neq |K|$. Clearly $f_1(X)$ can be “pushed off” some principal simplex $\sigma \in K$ (σ is not a proper face of some $\tau \in K$). f_1 is a K -modification of f and the

carrier of $f_1(X)$ (i.e., the minimal subcomplex of K containing $f_1(X)$) has fewer simplexes than the carrier of $f(X)$. After finitely many steps the process stops and we obtain the desired subcomplex L and the desired map g .

If $f: X \rightarrow |K|$ is K -irreducible and K' is a subdivision of K , then f is also K' -irreducible. Indeed, every K' -modification $g: X \rightarrow |K'|$ of f is also a K -modification and, therefore, $g(X) = |K| = |K'|$.

Our proof of Theorem 1 is based on the following lemma.

Lemma 1. *Let X be a compact Hausdorff space, let $f_i: X \rightarrow P_i$ be maps to polyhedra P_i and $\varepsilon_i > 0$, $i = 1, \dots, k$. Then there exist a polyhedron Q , an irreducible (onto) map $g: X \rightarrow Q$ and PL-mappings $p_i: Q \rightarrow P_i$ such that $d(f_i, p_i g) \leq \varepsilon_i$, $i = 1, \dots, k$. Moreover, if for a given index i the mapping f_i is irreducible, then the corresponding mapping p_i is also irreducible and therefore onto.*

Proof of Lemma 1. For each $i \in \{1, \dots, k\}$ choose a triangulation K_i of P_i . If f_i is irreducible, let f_i be K_i -irreducible. Let L_i be a subdivision of K_i with

$$(1) \quad \text{mesh } L_i \leq \varepsilon_i/2.$$

Note that f_i is L_i -irreducible if it is irreducible.

Let $P = P_1 \times \dots \times P_k$, let $f = f_1 \times \dots \times f_k: X \rightarrow P$ and let $\pi_i: P \rightarrow P_i$, $i = 1, \dots, k$, be the projections. Choose $\delta > 0$ so small that

$$(2) \quad d(x, x') \leq \delta \Rightarrow d(\pi_i(x), \pi_i(x')) \leq \varepsilon_i/2, \quad i = 1, \dots, k.$$

Let K be a triangulation of P so fine that

$$(3) \quad \text{mesh } K \leq \delta,$$

and the projections $\pi_i: |K| \rightarrow |L_i|$ admit simplicial approximations $p_i: K \rightarrow L_i$, $i = 1, \dots, k$. Since p_i is an L_i -modification of π_i , we have

$$(4) \quad d(p_i, \pi_i) \leq \text{mesh } L_i \leq \varepsilon_i/2.$$

There exists a subcomplex $L \subseteq K$ and a K -modification $g: X \rightarrow |L|$ of f such that g is L -irreducible. Putting $Q = |L|$, we see that $g: X \rightarrow Q$ is irreducible (and onto). Note that $d(f, g) \leq \text{mesh } K \leq \delta$, and therefore,

$$(5) \quad d(\pi_i f, \pi_i g) \leq \varepsilon_i/2, \quad i = 1, \dots, k.$$

Since $\pi_i f = f_i$, (4) and (5) yield

$$(6) \quad d(f_i, p_i g) \leq \varepsilon_i, \quad i = 1, \dots, k.$$

We will now show that $p_i g$ is an L_i -modification of $\pi_i f = f_i$. Let $x \in X$ and let $\sigma \in K$ be the carrier of $f(x)$. Let $\sigma_i = p_i(\sigma) \in L_i$. Then σ_i is the carrier of $p_i f(x)$. Since p_i is an L_i -modification of π_i , we conclude that σ_i is a face of the carrier $\tau_i \in L_i$ of $\pi_i f(x) = f_i(x)$. Since g is a K -modification of f , we have $g(x) \in \sigma$ and therefore $p_i g(x) \in p_i(\sigma) = \sigma_i \leq \tau_i$. This shows

that $p_i g(x)$ belongs to the carrier τ_i of $f_i(x)$ in $|L_i|$ and therefore $p_i g$ is indeed an L_i -modification of f_i .

We will now show that $p_i: |L| \rightarrow |L_i|$ is L_i -irreducible if $f_i: X \rightarrow P_i$ is irreducible. In this case we already know that f_i is L_i -irreducible. Moreover, for any L_i -modification $q_i: |L| \rightarrow |L_i|$ of p_i , the mapping $q_i g$ is an L_i -modification of $p_i g$ and therefore also an L_i -modification of f_i . This then implies $q_i(|L|) = q_i g(X) = |L_i|$.

Proof of Theorem 1. Repeat (with obvious modifications) the proof of Theorem 5 of [6] or the proof of Theorem 3 of [8]. Use Lemma 1 instead of Lemma 5 (in the first case) and Lemma 2 (in the second case). Note that in Theorem 5 of [6] the bonding maps are not required to be onto. In Theorem 3 of [8] the bonding maps are onto but need not be PL-maps. Moreover, this result does not apply to the class of all polyhedra.

4. COHOMOLOGICAL DIMENSION OF LIMITS OF APPROXIMATE SYSTEMS

For compact Hausdorff spaces X , one can define the cohomological dimension $\dim_{\mathbb{Z}} X$ (integer coefficients) by putting $\dim_{\mathbb{Z}} X \leq n$, $n \geq 1$, provided every map $f: A \rightarrow K(\mathbb{Z}, n)$ from a closed subset A of X to an Eilenberg-Mac Lane complex $K(\mathbb{Z}, n)$ admits an extension $\tilde{f}: X \rightarrow K(\mathbb{Z}, n)$ (see, e.g., [4, Remark 5 and Theorem 26] or [3]). In [5], $\dim_{\mathbb{Z}} X \leq n$ was characterized by an approximate factorization property, which we will now describe.

Definition 3. A map $p: Q \rightarrow P$ between polyhedra is called (n, ε) -approximable, $\varepsilon > 0$, $n \geq 1$, provided for every triangulation M of Q there is a PL-mapping $p': |M^{(n+1)}| \rightarrow P$ of the $(n+1)$ -skeleton of M such that

- (1) $d(p', p| |M^{(n+1)}|) \leq \varepsilon$,
- (2) $\dim p'(|M^{(n+1)}|) \leq n$.

The following proposition was proved in [5] as Theorem 1.

Proposition 11. A compact Hausdorff space X has cohomological dimension $\dim_{\mathbb{Z}} X \leq n$, $n \geq 1$, if and only if for every polyhedron P , every map $f: X \rightarrow P$, and every $\varepsilon > 0$, there is a polyhedron Q and there are maps $g: X \rightarrow Q$, $p: Q \rightarrow P$ such that

- (3) $d(pg, f) \leq \varepsilon$,

and p is (n, ε) -approximable.

Using Proposition 11 we will now give a criterion for determining whether $\dim_{\mathbb{Z}} X \leq n$, when X is the limit of an approximate system of polyhedra.

Theorem 2. Let $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system of polyhedra. The limit $X = \lim \mathbf{X}$ satisfies $\dim_{\mathbb{Z}} X \leq n$, $n \geq 1$, if and only if for every $a \in A$

and every $\eta > 0$, there is an $a' \geq a$ such that for every $a'' \geq a'$ the mapping $p_{aa''}$ is (n, η) -approximable.

Proof of necessity. Let $\dim_Z X \leq n$ and let $a \in A$, $\eta > 0$ be given. By (A2) there is an $a_1 \geq a$ such that for any $a' \geq a_1$ one has

$$(4) \quad d(p_{aa'}, p_{a'a''}, p_{aa''}) \leq \eta/7, \quad a'' \geq a' \geq a_1.$$

Note that (4) implies

$$(5) \quad d(p_{aa'} p_{a'a''} p_{a''}, p_{aa''} p_{a''}) \leq \eta/7, \quad a'' \geq a'.$$

Passing to the limit with a'' and taking into account Proposition 3, one obtains

$$(6) \quad d(p_{aa'} p_{a'}, p_a) \leq \eta/7, \quad a' \geq a_1.$$

By Proposition 11, there is a polyhedron Q and there are maps $g: X \rightarrow Q$, $p: Q \rightarrow X_a$ such that

$$(7) \quad d(p_a, pg) \leq \eta/7$$

and p is $(n, \eta/7)$ -approximable.

Choose $\delta > 0$ so small that

$$(8) \quad d(x, x') \leq \delta \Rightarrow d(p(x), p(x')) \leq \eta/7.$$

By Property (R1) (Proposition 7), there is an $a' \geq a_1$ such that there is a mapping $p': X_{a'} \rightarrow Q$ satisfying

$$(9) \quad d(p' p_{a'}, g) \leq \delta.$$

Now (8) and (9) imply

$$(10) \quad d(pp' p_{a'}, pg) \leq \eta/7.$$

Note that (10), (7) and (6) yield

$$(11) \quad d(pp' p_{a'}, p_{aa'} p_{a'}) \leq 3\eta/7.$$

By (11) there is a neighborhood U of $p_{a'}(X)$ in $X_{a'}$ such that

$$(12) \quad d(pp'|U, p_{aa'}|U) \leq 4\eta/7.$$

By Property (B1) (Proposition 4), there is an $a'_1 \geq a'$ such that for any $a'' \geq a'_1$ one has

$$(13) \quad p_{a'a''}(X_{a''}) \subseteq U$$

and therefore,

$$(14) \quad d(pp' p_{a'a''}, p_{aa'} p_{a'a''}) \leq 4\eta/7.$$

Note that (14) and (4) yield

$$(15) \quad d(pp' p_{a'a''}, p_{aa''}) \leq 5\eta/7.$$

We will show that $p_{aa''}$ is (n, η) -approximable for any $a'' \geq a'_1$.

Let M be any triangulation of $X_{a''}$. Choose a triangulation N of Q so fine that $\text{mesh}(N) \leq \delta$. Let M' be a subdivision of M so fine that $p'p_{a'a''}: X_{a''} \rightarrow Q$ admits a simplicial approximation $q: M' \rightarrow N$. Note that the $(n+1)$ -skeleton $|M^{(n+1)}| \subseteq |M'^{(n+1)}|$ and

$$(16) \quad q(|M^{(n+1)}|) \subseteq q(|M'^{(n+1)}|) \subseteq |N^{(n+1)}|.$$

Moreover,

$$(17) \quad d(q, p'p_{a'a''}) \leq \text{mesh}(N) \leq \delta.$$

Therefore (8) implies

$$(18) \quad d(pq, pp'p_{a'a''}) \leq \eta/7.$$

(15) and (18) imply

$$(19) \quad d(pq, p_{aa''}) \leq 6\eta/7.$$

Since p is $(n, \eta/7)$ -approximable, there is a PL-mapping $p^*: |N^{(n+1)}| \rightarrow X_a$ such that

$$(20) \quad d(p^*, p| |N^{(n+1)}|) \leq \eta/7,$$

$$(21) \quad \dim p^* (|N^{(n+1)}|) \leq n.$$

By (16), $p^*q| |M^{(n+1)}|$ is a well-defined PL-mapping $|M^{(n+1)}| \rightarrow X_a$. Formulas (19) and (20) imply

$$(22) \quad d(p_{aa''}| |M^{(n+1)}|, p^*q| |M^{(n+1)}|) \leq \eta.$$

Moreover, (16) and (21) imply

$$(23) \quad \dim p^*q (|M^{(n+1)}|) \leq n.$$

This shows that $p^*q| |M^{(n+1)}|$ is an (n, η) -approximation of $p_{aa''}$.

Proof of sufficiency. Let $f: X \rightarrow P$ be a mapping into a polyhedron P and let $\eta > 0$. By Proposition 11 it suffices to exhibit an $a'' \in A$ and a mapping $p: X_{a''} \rightarrow P$ such that

$$(24) \quad d(pp_{a''}, f) \leq \eta$$

and p is (n, η) -approximable.

By Property (R1) (Proposition 7), there is a mapping $g: X_a \rightarrow P$ such that

$$(25) \quad d(f, gp_a) \leq \eta/2.$$

By simplicial approximation we can achieve that g is a PL-mapping. Let $\delta > 0$ be such that

$$(26) \quad d(x, x') \leq \delta \Rightarrow d(g(x), g(x')) \leq \eta/2.$$

By Proposition 3, there is an $a' \geq a$ such that for any $a'' \geq a'$ one has

$$(27) \quad d(p_{aa''}, p_{a''}, p_a) \leq \delta$$

and therefore,

$$(28) \quad d(gp_{aa''}, p_{a''}, gp_a) \leq \eta/2.$$

By assumption there is an $a'' \geq a'$ for which $p_{aa''}$ is (n, δ) -approximable. If we put $p = gp_{aa''}: X_{a''} \rightarrow P$, (28) and (25) imply (24). It remains to show that p is (n, η) -approximable.

Let M be a triangulation of $X_{a''}$. Since $p_{aa''}$ is (n, δ) -approximable, there is a PL-mapping $p': |M^{(n+1)}| \rightarrow X_a$ such that

$$(29) \quad d(p', p_{aa''}| |M^{(n+1)}|) \leq \delta,$$

$$(30) \quad \dim p'(|M^{(n+1)}|) \leq n.$$

Note that $gp': |M^{(n+1)}| \rightarrow P$ is a PL-map. (29) and (26) imply

$$(31) \quad d(gp', p| |M^{(n+1)}|) \leq \eta.$$

Moreover, (30) and the fact that g is a PL-map imply

$$(32) \quad \dim gp'(|M^{(n+1)}|) \leq n.$$

(31) and (32) prove that p is indeed (n, η) -approximable.

Remark 2. Theorem 2 is a generalization of R. D. Edwards' criterion for the limit X of an inverse sequence of polyhedra to satisfy $\dim_{\mathbb{Z}} X \leq n$ (see [16, Theorem 4.2]).

5. THE n -DIMENSIONAL CORE OF A COMPLEX

In this section we will associate with every complex K and every integer $n \geq 0$ a compact metric space $Z_K = Z_K^{(n)}$ with $\dim Z_K \leq n$, called the n -dimensional core of K .

Let $K, K', K^2, \dots, K^k, \dots$ denote the iterated barycentric subdivisions of K . For each $k \geq 0$ choose a simplicial approximation $q_{k, k+1}: K^{k+1} \rightarrow K^k$ of the identity map $1: |K^{k+1}| \rightarrow |K^k|$ and let $q_{k, k+j} = q_{k, k+1} \cdots q_{k+j-1, k+j}: K^{k+j} \rightarrow K^k$. Note that $q_{k, k+j}$ is a simplicial approximation of the identity $1: |K^{k+j}| \rightarrow |K^k|$ and is therefore a K^k -modification of 1 . Consequently,

$$(1) \quad d(q_{k, k+j}, 1) \leq \text{mesh}(K^k), \quad j \geq 0.$$

Since the maps $q_{k, k+1}$ are simplicial we have

$$(2) \quad q_{k, k+1}((K^{k+1})^{(n)}) \subseteq (K^k)^{(n)},$$

where $L^{(n)}$ denotes the n -skeleton of L . Therefore we have an inverse sequence of polyhedra

$$(3) \quad \mathbf{K} = (|K^k|^{(n)}, q_{k, k+1}).$$

The n -dimensional core of K is defined as the inverse limit

$$(4) \quad Z_K = \lim K.$$

Since $\dim |(K^k)^{(n)}| \leq n$, we have

$$(5) \quad \dim Z_K \leq n.$$

We denote the natural projections from Z_K to $|(K^k)^{(n)}|$ by q_k . Since the maps q_{k+k+1} are onto (Sperner's lemma, see, e.g., [15, Chapter 3, Ex. D3]), so are the maps q_{k+k+j} and q_k .

We now define a mapping $f_K: Z_K \rightarrow |K|$ by putting

$$(6) \quad f_K = \lim_k q_k.$$

Since $q_{k+k+j}q_{k+j} = q_k$, (1) implies

$$(7) \quad d(q_k, q_{k+j}) \leq \text{mesh}(K^k), \quad j \geq 0,$$

and since $\lim_k \text{mesh}(K^k) = 0$, we see that (q_k) is a Cauchy sequence of maps $Z_K \rightarrow |K|$. Therefore f_K exists and is continuous. Moreover, (7) implies

$$(8) \quad d(q_k, f_K) \leq \text{mesh}(K^k).$$

Since q_k is onto and $\text{mesh}(K^k) \rightarrow 0$, we see that $f_K(Z_K)$ is dense in $|K|$ and therefore $f_K: Z_K \rightarrow |K|$ is also an onto mapping.

Remark 3. The sequence $(|K^k|, \text{mesh}(K^k), \text{id}, \mathbb{N})$ is actually an approximate inverse sequence. Since $\mathbf{K}$ is a commutative cofinite sequence, one can provide it with meshes ε_k and view $\mathbf{K}$ also as an approximate sequence. (7) shows that the inclusion maps $|(K^k)^{(n)}| \rightarrow |K^k|$ define a map of approximate systems. The existence of f_K and its properties now follow from the general theory of maps between approximate systems (see [10 or 17]).

In our constructions in §7 we need to associate with every complex K an n -dimensional metric compactum Z_K^* which is a compactification of the topological sum of the n -skeleta $|(K^k)^{(n)}|$ of all the barycentric subdivisions K^k of K , with remainder Z_K . We call Z_K^* the *stacked n -dimensional core* of K .

$$(9) \quad Z_K^* = \left(\bigoplus_{k \geq 0} |(K^k)^{(n)}| \right) \cup Z_K.$$

To describe precisely the topology of Z_K^* we form a new inverse sequence $\mathbf{K}^* = (|K^{*k}|, q_{k+k+1}^*)$, where

$$(10) \quad K^{*k} = K^{(n)} \oplus (K')^{(n)} \oplus \cdots \oplus (K^k)^{(n)}.$$

Note that

$$(11) \quad |K^{*k+1}| = |K^{*k}| \oplus |(K^{k+1})^{(n)}|.$$

The bonding maps $q_{k \ k+1}^*: |K^{**k+1}| \rightarrow |K^{**k}|$ are defined by

$$(12) \quad q_{k \ k+1}^* | (K^{k+1})^{(n)} | = q_{k \ k+1},$$

$$(13) \quad q_{k \ k+1}^* | K^{**k} | = \text{id}.$$

Finally, we put

$$(14) \quad Z_K^* = \lim \mathbf{K}^*$$

and denote the natural projections by $q_k^*: Z_K^* \rightarrow |K^{**k}|$. We have

$$(15) \quad \dim Z_K^* \leq n,$$

because $\dim |K^{**k}| \leq n$. Moreover,

$$(16) \quad Z_K \subseteq Z_K^*, \quad |K^{**k}| \subseteq Z_K^*, \quad k \geq 0,$$

$$(17) \quad q_k^* | (K^{k+j})^{(n)} | = q_{k \ k+j}, \quad j \geq 0,$$

$$(18) \quad q_k^* | Z_K = q_k.$$

We now extend the mapping $f_K: Z_K \rightarrow |K|$ to $f_K^*: Z_K^* \rightarrow |K|$ by

$$(19) \quad f_K^* = \lim_k q_k^*.$$

Note that (12), (13), (7) imply

$$(20) \quad d(q_k^*, q_{k+j}^*) \leq \text{mesh}(K^k), \quad j \geq 0,$$

so that f_K^* exists. Moreover, from (17) and (18)

$$(21) \quad f_K^* | (K^k)^{(n)} | \text{ is inclusion into } |K|, \quad k \geq 0,$$

$$(22) \quad f_K^* | Z_K = f_K.$$

Note that

$$(23) \quad f_K^*(Z_K^*) = f_K(Z_K) = |K|.$$

Also note that (20) implies

$$(24) \quad d(q_k^*, f_K^*) \leq \text{mesh}(K^k).$$

In the applications of these constructions in §7 we will also need a metric on Z_K^* . If we have a metric d on $|K|$ such that the diameter $\text{diam}|K| \leq 1$, then we can choose metrics d^* on Z_K^* and d^k on $|K^{**k}|$ such that $\text{diam} Z_K^* \leq 1$, $\text{diam}|K^{**k}| \leq 1$, and

$$(25) \quad d^k(q_k^*(x), q_k^*(x')) \leq d^*(x, x'), \quad x, x' \in Z_K^*, \quad k \geq 0.$$

Indeed, we first define a metric d^k on $|K^{**k}| = |K^{(n)}| \oplus \cdots \oplus |(K^k)^{(n)}|$ by putting $d^k(x, y) = \frac{1}{2^k} d(x, y)$ if x, y belong to the same summand $|(K^i)^{(n)}| \subseteq |K|$, and

$d^k(x, y) = \frac{1}{2^k}$ otherwise. Clearly $d^k \leq \frac{1}{2^k}$. We now define a metric d^* on $\prod_{k \geq 0} |K^{*k}|$. If $x = (x^k)$, $x' = (x'^k) \in \prod_{k \geq 0} |K^{*k}|$, we put

$$(26) \quad d^*(x, x') = \sup_k d^k(x^k, x'^k) \leq 1.$$

Since $Z_K^* \subseteq \prod_{k \geq 0} |K^{*k}|$, d^* is also a metric on Z_K^* and $\text{diam } Z_K^* \leq 1$. Moreover, $x^k = q_k^*(x)$, $x'^k = q_k^*(x')$, and we see that (26) implies (25).

Note that d^0 coincides with the metric d on $|K|$ and therefore (25) yields

$$(27) \quad d(q_0^*(x), q_0^*(x')) \leq d^*(x, x'), \quad x, x' \in Z_K^*.$$

6. CONSTRUCTION OF THE APPROXIMATE SYSTEM Z

The following is an easy consequence of results already established.

Proposition 12. *Let X be a compact Hausdorff space with $\dim_Z X = n \geq 1$. Then there exists an approximate inverse system $\mathbf{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ with $\lim \mathbf{X} = X$ such that*

- (i) X_a is a polyhedron with a metric $d = d_a \leq 1$,
- (ii) $\dim X_a \geq n \geq 1$,
- (iii) $p_{aa'}: X_{a'} \rightarrow X_a$ is a surjective PL-mapping,
- (iv) $\text{card } A \leq w(X)$.

Proof. Theorem 1 yields a system $\mathbf{X}$ with $\lim \mathbf{X} = X$, $\text{card } A \leq w(X)$, where X_a are polyhedra and $p_{aa'}$ are PL-surjections. By Proposition 10, one can assume that $d_a \leq 1$.

There is an index $a_1 \in A$ such that for every $a \geq a_1$ one has $\dim X_a \geq n$. If this were not the case, then the set $B \subseteq A$ of all indexes $b \in A$ with $\dim X_b \leq n - 1$ would be cofinal in A . Then Propositions 6 and 5 would imply $\dim X \leq n - 1$. Since $\dim_Z X \leq \dim X$ (see [4]), we would have a contradiction with the assumption $\dim_Z X = n$. If we now restrict A to the set of all $a \geq a_1$, we obtain an approximate system which satisfies all conditions (i)–(iv) (use Proposition 6 again).

From now on we assume that we have chosen a system $\mathbf{X}$ as in Proposition 12. We will now define an approximate system $\mathbf{Z} = (Z_a^*, \varepsilon_a, r_{aa'}, A')$.

We first choose a triangulation K_a for X_a , $a \in A$, such that

$$(1) \quad 6 \text{ mesh}(K_a) \leq \varepsilon_a, \quad a \in A.$$

We next define the directed set A' . As a set A' equals A , but A' has a new ordering $\leq'$. In order to define it, we consider for any $a_1 < a_2$ and any integer $k \geq 0$ the following three conditions:

- (2) $d(p_{a_1 a'} p_{a' a''}, p_{a_1 a''}) \leq \text{mesh}(K_{a_1}^k)$, for $a'' \geq a' \geq a_2$,
- (3) $d(x, x') \leq \varepsilon_{a''} \Rightarrow d(p_{a_1 a''}(x), p_{a_1 a''}(x')) \leq \text{mesh}(K_{a_1}^k)$, for $a'' \geq a_2$,
- (4) $p_{a_1 a''}: X_{a''} \rightarrow X_{a_1}$ is $(n, \text{mesh}(K_{a_1}^k))$ -approximable, for $a'' \geq a_2$.

We put $a_1 <' a_2$ provided $a_1 < a_2$ and conditions (2), (3), (4) hold for $k = 0$.

Lemma 2. *The binary relation $<'$ on A has properties (i)–(iii) from Proposition 9, and therefore A' is a directed set with no maximal element. Moreover, for any $a_1 \in A$ and integer $k \geq 0$ there exists an $a_2 > a_1$ such that (2), (3) and (4) hold.*

Proof. (i) and (ii) are obviously true and (iii) follows from the last assertion for $k = 0$. To verify the latter, put

$$(5) \quad \eta = \text{mesh}(K_{a_1}^k) > 0$$

and apply Theorem 2. We obtain an $a_2 > a_1$ such that for any $a'' \geq a_2$ the mapping $p_{a_1 a''}$ satisfies (4). However, by (A2) and (A3), one can assume that a_2 also satisfies (2) and (3).

Lemma 3. *If $a_1 < a_2$, the set of all integers $k \geq 0$, which satisfy (3) is finite.*

Proof. Assume that there is an infinite sequence $k_1 < k_2 < \dots$ of integers satisfying (3). Then for any two points $x, x' \in X_{a_2}$

$$(6) \quad d(x, x') \leq \varepsilon_{a_2} \Rightarrow p_{a_1 a_2}(x) = p_{a_1 a_2}(x').$$

This is so because, by (3),

$$(7) \quad d(p_{a_1 a_2}(x), p_{a_1 a_2}(x')) \leq \text{mesh}(K_{a_1}^{k_i}), \quad i = 1, 2, \dots,$$

and $\text{mesh}(K_{a_1}^{k_i}) \rightarrow 0$ as $i \rightarrow \infty$. Consequently, $p_{a_1 a_2}$ maps every component of X_{a_2} to a single point. Since $X_{a_1} = p_{a_1 a_2}(X_{a_2})$, it follows that X_{a_1} is a finite set of points, which contradicts (ii) of Proposition 12.

Whenever $a_1 <' a_2$, by definition of $<'$ and Lemma 3, there is a maximal integer $k \geq 0$ such that (2), (3) and (4) hold. We denote it by $k(a_1, a_2)$.

Lemma 4. *If $a_1 <' a_2$, then (2), (3) and (4) hold for $k = k(a_1, a_2)$ and also*

$$(8) \quad d(p_{a_1 a'} p_{a'} p_{a_1}, p_{a_1}) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}), \quad \text{for } a' \geq a_2.$$

If $a_1 <' a_2$ and $a_2 \leq a_3$, then

$$(9) \quad k(a_1, a_2) \leq k(a_1, a_3).$$

Furthermore, for $a_1 \in A$ and any integer $k \geq 0$ there is an $a_2 \in A$ such that $a_1 <' a_2$ and

$$(10) \quad k \leq k(a_1, a_2).$$

Proof. (2), (3) and (4) hold for $k = k(a_1, a_2)$ by the very definition of $k(a_1, a_2)$. By (2), one has

$$(11) \quad d(p_{a_1 a'} p_{a' a''} p_{a''}, p_{a_1 a''} p_{a''}) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}), \quad \text{for } a'' \geq a' \geq a_2.$$

Passing to the limit with a'' (using Proposition 3), one obtains (7).

$k = k(a_1, a_2)$ satisfies (2), (3) and (4) also for a_1 and a_3 . Therefore, (8) follows from the maximality of $k(a_1, a_3)$. By the last assertion of Lemma 2, for any $k \geq 0$ there exists an $a_2 > a_1$ such that (2), (3) and (4) hold. Clearly, $a_1 <' a_2$. Now (9) follows from the maximality of $k(a_1, a_2)$.

Lemma 5. *Let K and L be complexes and let $p: |K| \rightarrow |L|$ be an $(n, \text{mesh}(L))$ -approximable mapping. Then there exists a PL-mapping $g: |K^{(n+1)}| \rightarrow |L^{(n)}|$ such that*

$$(12) \quad d(g, p|_{|K^{(n+1)}|}) \leq 2 \text{mesh}(L).$$

Proof. By assumption, there is a PL-mapping $p': |K^{(n+1)}| \rightarrow |L|$ such that

$$(13) \quad d(p', p|_{|K^{(n+1)}|}) \leq \text{mesh}(L),$$

$$(14) \quad \dim(p'(|K^{(n+1)}|)) \leq n.$$

Let $\varphi: p'(|K^{(n+1)}|) \rightarrow |L|$ be a simplicial approximation to the inclusion of $p'(|K^{(n+1)}|)$ into $|L|$ relative to L . Then, $g = \varphi p': |K^{(n+1)}| \rightarrow |L^{(n)}|$ is a PL-mapping and

$$(15) \quad d(g, p') \leq \text{mesh}(L).$$

Now (13) and (15) yield (12).

For $a_1 <' a_2$, we define a PL-mapping

$$(16) \quad g_{a_1 a_2}: |K_{a_2}^{(n+1)}| \rightarrow |(K_{a_1}^k)^{(n)}|, \quad k = k(a_1, a_2),$$

by applying Lemma 5 to $K = K_{a_2}$, $L = K_{a_1}^k$, and $p = p_{a_1 a_2}$. Lemma 5 is applicable because of (4) and Lemma 4 and yields the following conclusion.

Lemma 6. *If $a_1 <' a_2$, then $g_{a_1 a_2}$ satisfies*

$$(17) \quad d(g_{a_1 a_2}, p_{a_1 a_2}|_{|K_{a_2}^{(n+1)}|}) \leq 2 \text{mesh}(K_{a_1}^k), \quad k = k(a_1, a_2).$$

For $a \in A$ we now put

$$(18) \quad Z_a^* = Z_{K_a}^*$$

(see §5). For $a_1 <' a_2$ we define $r_{a_1 a_2}: Z_{a_2}^* \rightarrow Z_{a_1}^*$ by

$$(19) \quad r_{a_1 a_2} = g_{a_1 a_2} q_{0 a_2}^*.$$

Here $q_{0 a_2}^*: Z_{a_2}^* \rightarrow |K_{a_2}^{(n)}|$ is the mapping $q_0^*: Z_K^* \rightarrow |K^{(n)}|$ from §5. Note that

$$(20) \quad r_{a_1 a_2}(Z_{a_2}^*) \subseteq |(K_{a_1}^k)^{(n)}|, \quad k = k(a_1, a_2).$$

Lemma 7. $Z = (Z_a^*, \varepsilon_a, r_{aa'}, A')$ is an approximate system of nonempty metric compacta Z_a^* with $\dim Z_a^* \leq n$. The limit $Z = \lim Z$ is a nonempty compact Hausdorff space with $\dim Z \leq n$ and $w(Z) \leq \text{card}(A') = \text{card}(A) \leq w(X)$.

The proof of Lemma 7 is given in the next section.

7. VERIFYING (A1)–(A3) FOR $\mathbf{Z}$. THE SPACE $\mathbf{Z}$

Proof of Lemma 7. For each $a \in A$, $\dim X_a \geq n$ and therefore $|(K_a^k)^{(n)}| \neq \emptyset$, $k \geq 0$. It follows by §5(4), that $Z_a \neq \emptyset$ and $Z_a^* \neq \emptyset$. Moreover, by §5(15), $\dim Z_a^* \leq n$.

We will now verify conditions (A1)–(A3) for $\mathbf{Z}$. Let $a_1 <' a_2 <' a_3$. (A1) and (A2) require certain upper bounds for $d(r_{a_1 a_2} r_{a_2 a_3}, r_{a_1 a_3})$. By (19) of §6, it suffices to find the appropriate bounds for $d(r_{a_1 a_2} g_{a_2 a_3} | |K_{a_3}^{(n)}|, g_{a_1 a_3} | |K_{a_3}^{(n)}|)$. By §5(17) and §5(1) we have

$$(1) \quad q_{0a_2}^* | |(K_{a_2}^k)^{(n)}| = q_{0k a_2} | |(K_{a_2}^k)^{(n)}|,$$

$$(2) \quad d(q_{0a_2}^* | |(K_{a_2}^k)^{(n)}|, 1 | |(K_{a_2}^k)^{(n)}|) \leq \text{mesh}(K_{a_2}),$$

and therefore,

$$(3) \quad d(q_{0a_2}^* g_{a_2 a_3}, g_{a_2 a_3}) \leq \text{mesh}(K_{a_2}).$$

On the other hand, §6(17) yields

$$(4) \quad d(g_{a_2 a_3}, p_{a_2 a_3} | |K_{a_3}^{(n+1)}|) \leq 2 \text{mesh}(K_{a_2}^k) \leq 2 \text{mesh}(K_{a_2}), \quad k = k(a_2, a_3).$$

Now (3), (4) and §6(1) yield

$$(5) \quad d(q_{0a_2}^* g_{a_2 a_3}, p_{a_2 a_3} | |K_{a_3}^{(n+1)}|) \leq 3 \text{mesh}(K_{a_2}) \leq \varepsilon_{a_2}.$$

By §6(3), (5) implies

$$(6) \quad d(p_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3}, p_{a_1 a_2} p_{a_2 a_3} | |K_{a_3}^{(n+1)}|) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \text{mesh}(K_{a_1}).$$

Moreover, by §6(17),

$$(7) \quad d(g_{a_1 a_2}, p_{a_1 a_2} | |K_{a_2}^{(n+1)}|) \leq 2 \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \leq 2 \text{mesh}(K_{a_1}),$$

and therefore

$$(8) \quad d(g_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3}, p_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3}) \leq 2 \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \leq 2 \text{mesh}(K_{a_1}).$$

Now (6) and (8) yield

$$(9) \quad d(g_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3}, p_{a_1 a_2} p_{a_2 a_3} | |K_{a_3}^{(n+1)}|) \leq 3 \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \leq 3 \text{mesh}(K_{a_1}).$$

Furthermore, by §6(2) we have

$$(10) \quad d(p_{a_1 a_2} p_{a_2 a_3}, p_{a_1 a_3}) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \text{mesh}(K_{a_1}),$$

and by §6(17) we have

$$(11) \quad d(g_{a_1 a_3}, p_{a_1 a_3} | |K_{a_3}^{(n+1)}|) \leq 2 \text{mesh}(K_{a_1}^{k(a_1, a_3)}) \leq 2 \text{mesh}(K_{a_1}^{k(a_1, a_2)}) \\ \leq 2 \text{mesh} K_{a_1}$$

(use $a_2 \leq a_3$ and §6(9)).

Finally, (9), (10), (11) and §6(1) yield

$$(12) \quad d(g_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3}, g_{a_1 a_3}) \leq 6 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq 6 \text{ mesh}(K_{a_1}) \leq \varepsilon_{a_1}.$$

Since $r_{a_1 a_2} r_{a_2 a_3} = g_{a_1 a_2} q_{0a_2}^* g_{a_2 a_3} q_{0a_3}^*$ and $r_{a_1 a_3} = g_{a_1 a_3} q_{0a_3}^*$, (12) yields

$$(13) \quad d(r_{a_1 a_2} r_{a_2 a_3}, r_{a_1 a_3}) \leq 6 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \varepsilon_{a_1},$$

which verifies (A1).

To obtain (A2), for given a_1 and $\eta > 0$ choose an integer k so large that

$$(14) \quad 6 \text{ mesh}(K_{a_1}^k) \leq \eta.$$

By Lemma 2, there is an index $a_2 > a_1$ such that §6(2)–(4) hold for k, a_1, a_2 . This proves that $a_1 < a_2$. Since $k(a_1, a_2)$ is the maximal k with these properties, we have $k \leq k(a_1, a_2)$ and therefore

$$(15) \quad 6 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \eta.$$

Now let a_3 be any index with $a_2 < a_3$. Then (13) and (15) imply

$$(16) \quad d(r_{a_1 a_2} r_{a_2 a_3}, r_{a_1 a_3}) \leq \eta,$$

which establishes (A2).

We now prove (A3). As above, for a given $a_1 \in A$ and $\eta > 0$ there is an $a_2 \in A$, $a_1 < a_2$, such that (15) holds. By (A3) for X , we can assume that for any $a_3 > a_2$ and any $y, y' \in |K_{a_3}^{(n)}|$,

$$(17) \quad d(y, y') \leq \varepsilon_{a_3} \Rightarrow d(p_{a_1 a_3}(y), p_{a_1 a_3}(y')) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}).$$

(17) and (11) (applied to y and y') and (15) yield

$$(18) \quad d(y, y') \leq \varepsilon_{a_3} \Rightarrow d(g_{a_1 a_3}(y), g_{a_1 a_3}(y')) \leq 5 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \eta, \\ y, y' \in |K_{a_3}^{(n)}|.$$

Now let $x, x' \in Z_{a_3}^*$. Then $y = q_{0a_3}^*(x)$, $y' = q_{0a_3}^*(x') \in |K_{a_3}^{(n)}|$. Moreover, by §5(27),

$$(19) \quad d(y, y') \leq d^*(x, x').$$

Therefore, $d^*(x, x') \leq \varepsilon_{a_3}$ implies $d(y, y') \leq \varepsilon_{a_3}$, and (18) yields

$$(20) \quad d(r_{a_1 a_3}(x), r_{a_1 a_3}(x')) \leq 5 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \eta.$$

To complete the proof of Lemma 7, note that Propositions 1 and 5 imply that $Z = \lim Z$ is a nonempty compact Hausdorff space with $\dim Z \leq n$. Moreover, by Propositions 2 and 12, $w(Z) \leq \text{card}(A') = \text{card}(A) \leq w(X)$. We will denote the natural projections from Z to Z_a^* , $a \in A$, by $r_a: Z \rightarrow Z_a^*$.

Remark 4. For given $a \in A$ and integer $k \geq 0$ consider the sets

$$C_a^k = |K_a^{(n)}| \oplus \cdots \oplus |(K_a^k)^{(n)}|, \quad D_a^k = Z_a \oplus |(K_a^{k+1})^{(n)}| \oplus \cdots.$$

They are disjoint and are closed in Z_a^* , $Z_a^* = C_a^k \cup D_a^k$. Let $\eta > 0$ be such that $d(C_a^k, D_a^k) > \eta$. For sufficiently large a'' one has $d(r_{aa''}r_{a''}, r_a) \leq \eta$ and $k(a, a'') \geq k+1$. Therefore, $r_{aa''}r_{a''}(Z_a^*) \subseteq |(K_a^{k(a, a'')})^{(n)}| \subseteq D_a^k$ and thus also $r_a(Z_a^*) \subseteq D_a^k$. This implies that $r_a(Z_a^*) \cap |(K_a^k)^{(n)}| = \emptyset$ for all k and therefore, $r_a(Z_a^*) \subseteq Z_a$. Nevertheless, $Z_a \cap r_{aa'}(Z_{a'}) \subseteq Z_a \cap r_{aa'}(Z_{a'}^*) = \emptyset$ for every $a <' a'$.

8. THE MAPPING $f: Z \rightarrow X$

In order to define the mapping f we first establish a lemma about maps $f_a^*: Z_a^* \rightarrow X_a$, defined by $f_a^* = f_{K_a}^*$ (see §5(19)).

Lemma 8. If $a_1 <' a_2$, then

$$(1) \quad d(f_{a_1}^* r_{a_1 a_2}, p_{a_1 a_2} f_{a_2}^*) \leq 3 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \varepsilon_{a_1}.$$

Proof. By §5(24) (for $k=0$), we have

$$(2) \quad d(f_{a_2}^*, q_{0a_2}^*) \leq \text{mesh}(K_{a_2}) \leq \varepsilon_{a_2}$$

and therefore, by §6(3),

$$(3) \quad d(p_{a_1 a_2} f_{a_2}^*, p_{a_1 a_2} q_{0a_2}^*) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}).$$

Using §6(17), we also have

$$(4) \quad d(g_{a_1 a_2} q_{0a_2}^*, p_{a_1 a_2} q_{0a_2}^*) \leq 2 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}).$$

Since $r_{a_1 a_2} = g_{a_1 a_2} q_{0a_2}^*$, (3) and (4) yield

$$(5) \quad d(r_{a_1 a_2}, p_{a_1 a_2} f_{a_2}^*) \leq 3 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq \varepsilon_{a_1}.$$

Since $r_{a_1 a_2}: Z_{a_2}^* \rightarrow |(K_{a_1}^{k(a_1, a_2)})^{(n)}| \subseteq X_{a_1}$ and $f_{a_1}^*: |(K_{a_1}^k)^{(n)}|$ is the inclusion $|(K_{a_1}^k)^{(n)}| \rightarrow X_{a_1}$ (see §5(21)), (5) is the desired formula (1).

Lemma 9. For any $a_1 \in A$ and $\eta > 0$, there is an $a_2 \in A$, $a_1 <' a_2$, such that for any $a'' \geq a_2$, one has

$$(6) \quad d(f_{a_1}^* r_{a_1 a''}, p_{a_1 a''} f_{a''}^*) \leq \eta.$$

Proof. Choose an integer $k \geq 0$ such that §7(14) holds. Repeating the argument which led to §7(15), we find an $a_2 \in A$, $a_1 <' a_2$, such that §7(15) holds. Applying Lemma 8 to any $a'' \geq a_2$ we obtain

$$(7) \quad d(f_{a_1}^* r_{a_1 a''}, p_{a_1 a''} f_{a''}^*) \leq \eta, \quad a'' \geq a_2.$$

Lemma 10. There is a mapping $f: Z \rightarrow X$ such that

$$(8) \quad f_a^* r_a = p_a f, \quad a \in A,$$

where $r_a: Z \rightarrow Z_a^*$ are the natural projections.

Proof. For a given $\eta > 0$, choose $\delta > 0$ so small that

$$(9) \quad d(z, z') \leq \delta \Rightarrow d(f_a^*(z), f_a^*(z')) \leq \eta.$$

By Proposition 3 for Z^* , there is an $a' \in A$, $a <' a'$, such that for any $a'' \geq a'$

$$(10) \quad d(r_{aa''} r_{a''}, r_a) \leq \delta$$

and therefore,

$$(11) \quad d(f_a^* r_{aa''} r_{a''}, f_a^* r_a) \leq \eta.$$

On the other hand, by (6), for sufficiently large a'' ,

$$(12) \quad d(f_a^* r_{aa''} r_{a''}, p_{aa''} f_{a''}^* r_{a''}) \leq \eta.$$

(11) and (12) yield

$$(13) \quad d(f_a^* r_a, p_{aa''} f_{a''}^* r_{a''}) \leq 2\eta,$$

for a'' sufficiently large. We have thus proved,

$$(14) \quad \lim_{a''} p_{aa''} f_{a''}^* r_{a''} = f_a^* r_a.$$

Let $f: Z \rightarrow \prod_{a \in A} X_a$ be the mapping given by (8). Formula (14) and Proposition 3 show that $f(Z) \subseteq X = \lim X$, so that one can view f as a mapping $f: Z \rightarrow X$.

Remark 5. Lemma 10 also follows from more general results (see [10, Remark 4]).

9. FIBERS OF THE MAPPING $f: Z \rightarrow X$

For a given $x \in X$ we will now express the fiber $f^{-1}(x)$ in terms of the system Z . We put

- (1) $x_a = p_a(x)$, $a \in A$,
- (2) $N_a(x) = \{x' \in X_a : d(x', x_a) \leq \varepsilon_a\}$.

We often abbreviate $N_a(x)$ to N_a .

Lemma 11. *If $a_1 <' a_2$, then*

- (3) $x' \in N_{a_2} \Rightarrow d(p_{a_1 a_2}(x'), x_{a_1}) \leq 2 \text{ mesh}(K_{a_1}^{k(a_1, a_2)}) \leq 2 \text{ mesh}(K_{a_1}) \leq \varepsilon_{a_1}$.
- (4) $p_{a_1 a_2}(N_{a_2}) \subseteq N_{a_1}$.

Proof. $x' \in N_{a_2}$ implies $d(x', x_{a_2}) \leq \varepsilon_{a_2}$, and therefore by §6(3),

$$(5) \quad d(p_{a_1 a_2}(x'), p_{a_1 a_2}(x_{a_2})) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}).$$

Moreover, by (1) and §6(8),

$$(6) \quad d(p_{a_1 a_2}(x_{a_2}), x_{a_1}) \leq \text{mesh}(K_{a_1}^{k(a_1, a_2)}).$$

Now, (5) and (6) yield (3). Formula (4) is an immediate consequence of (3).

It is a consequence of Lemma 11 that

$$(7) \quad N(x) = (N_a(x), \varepsilon_a, p_{aa'}, A')$$

is an approximate system.

Lemma 12. *The limit of the approximate system $\mathbf{N}(x)$ is $\{x\}$.*

Proof. Since $p_a(x) = x_a \in N_a$, the point x belongs to $\lim \mathbf{N}(x)$. Now assume that $x' \in X$ belongs to $\lim \mathbf{N}(x)$. We will show that $p_a(x') = x_a$ for every $a \in A$ and therefore $x' = x$.

By (A3) for $\mathbf{X}$, there is an a' , $a <' a'$, such that for every $a'' \geq a'$,

$$(8) \quad d(y, y') \leq \varepsilon_{a''} \Rightarrow d(p_{aa''}(y), p_{aa''}(y')) \leq \eta.$$

Since $p_{a''}(x') \in N_{a''}(x)$, we have $d(p_{a''}(x'), x_{a''}) \leq \varepsilon_{a''}$, and therefore,

$$(9) \quad d(p_{aa''}p_{a''}(x'), p_{aa''}(x_{a''})) \leq \eta.$$

Passing to the limit with a'' we obtain (using Proposition 3)

$$(10) \quad d(p_a(x'), x_a) \leq \eta.$$

Since $\eta > 0$ was arbitrary, we conclude that indeed $p_a(x') = x_a$ for all $a \in A$.

For $a \in A$ and $x \in X$ put

$$(11) \quad M_a = M_a(x) = f_a^{*-1}(N_a).$$

Lemma 13. *If $a_1 <' a_2$, then*

$$(12) \quad r_{a_1 a_2}(M_{a_2}) \subseteq M_{a_1}.$$

Proof. Let $y \in M_{a_2} = f_{a_2}^{*-1}(N_{a_2})$. Then $x' = f_{a_2}^*(y) \in N_{a_2}$. By (3) we obtain

$$(13) \quad d(p_{a_1 a_2}(x'), x_{a_1}) \leq 2 \text{ mesh}(K_{a_1}).$$

On the other hand, by §8(1),

$$(14) \quad d(f_{a_1}^* r_{a_1 a_2}(y), p_{a_1 a_2} f_{a_2}^*(y)) \leq 3 \text{ mesh}(K_{a_1}),$$

so that we obtain

$$(15) \quad d(f_{a_1}^* r_{a_1 a_2}(y), x_{a_1}) \leq 5 \text{ mesh}(K_{a_1}) \leq \varepsilon_{a_1}.$$

This proves that $f_{a_1}^* r_{a_1 a_2}(y) \in N_{a_1}$, i.e., $r_{a_1 a_2}(y) \in M_{a_1}$.

It is a consequence of Lemma 13 that

$$(16) \quad \mathbf{M}(x) = (M_a(x), \varepsilon_a, r_{aa'}, A')$$

is an approximate system.

Lemma 14. *The limit of the approximate system $\mathbf{M}(x)$ is $f^{-1}(x)$.*

Proof. Let $z \in f^{-1}(x)$. By §8(8), we have

$$(17) \quad f_a^* r_a(z) = p_a f(z) = p_a(x) = x_a \in N_a(x)$$

so that $r_a(z) \in f_a^{*-1}(N_a(x)) = M_a(x)$. This shows that

$$(18) \quad f^{-1}(x) \subseteq \lim \mathbf{M}(x).$$

Conversely, assume that $z \in Z$ belongs to $\lim \mathbf{M}(x)$. Then $r_a(z) \in M_a = f_a^{*-1}(N_a(x))$ and therefore $f_a^* r_a(z) \in N_a(x)$. By §8(8), $p_a f(z) = f_a^* r_a(z) \in N_a(x)$, so that $f(z)$ belongs to $\lim \mathbf{N}(x) = \{x\}$ (Lemma 12). Consequently, $f(z) = x$, i.e., $z \in f^{-1}(x)$.

Lemma 15. *The mapping $f: Z \rightarrow X$ is onto.*

Proof. Let $x \in X$. We will first show that

$$(19) \quad M_a(x) \neq \emptyset, \quad a \in A.$$

We have shown in §5 that $f_K: Z_K \rightarrow |K|$ is an onto mapping. Since $Z_K \subseteq Z_K^*$ and $f_K^*|Z_K = f_K$, we see that also $f_K^*: Z_K^* \rightarrow |K|$ is onto. This means that all the maps $f_a^*: Z_a^* \rightarrow X_a$ are onto. For any $x \in X$, $N_a(x) \neq \emptyset$ because it contains $x_a = p_a(x)$. Therefore, also $M_a(x) = f_a^{*-1}(N_a(x)) \neq \emptyset$.

We now use Lemma 14 and Proposition 1 to conclude that $f^{-1}(x) \neq \emptyset$.

10. $f: Z \rightarrow X$ IS CELL-LIKE.

We first establish an important technical lemma.

Lemma 16. *Let $a_1 < a_2$ and $j \geq 0$ be any integer. Then the restriction of $r_{a_1 a_2}$ to $|(K_{a_2}^j)^{(n)}|$ admits an extension*

$$(1) \quad \bar{r}_{a_1 a_2}: |(K_{a_2}^j)^{(n+1)}| \rightarrow |(K_{a_1}^k)^{(n)}|, \quad k = k(a_1, a_2).$$

Moreover,

$$(2) \quad d(\bar{r}_{a_1 a_2}, p_{a_1 a_2} | |(K_{a_2}^j)^{(n+1)}|) \leq 3 \text{ mesh}(K_{a_1}).$$

Proof. Recall that $r_{a_1 a_2} = g_{a_1 a_2} q_{0 a_2}^*$ (§6(19)). Therefore, the considered restriction equals $g_{a_1 a_2} q_{0 j a_2}$ (§5(17)). However, $q_{0 j a_2}: K_{a_2}^j \rightarrow K_{a_2}$ is a simplicial approximation of the identity (§5). Therefore, $q_{0 a_2}^*$ maps $|(K_{a_2}^j)^{(n+1)}|$ into $|K_{a_2}^{(n+1)}|$ and for any $z \in |(K_{a_2}^j)^{(n+1)}|$, one has

$$(3) \quad d(q_{0 a_2}^*(z), z) \leq \text{mesh} K_{a_2} \leq \varepsilon_{a_2}.$$

Since $g_{a_1 a_2}$ was actually defined on $|(K_{a_2})^{(n+1)}|$ (§6(16)), we see that $r_{a_1 a_2}$ has an extension $\bar{r}_{a_1 a_2}$ as required by (1).

For any $z \in |K_{a_2}^j|$, (3) and §6(3) yield

$$(4) \quad d(p_{a_1 a_2} q_{0 a_2}^*(z), p_{a_1 a_2}(z)) \leq \text{mesh}(K_{a_1}).$$

On the other hand, by §6(17), we have

$$(5) \quad d(p_{a_1 a_2} q_{0 a_2}^*(z), g_{a_1 a_2} q_{0 a_2}^*(z)) \leq 2 \text{ mesh}(K_{a_1}^k).$$

Now (4) and (5) yield (2).

Lemma 17. *For every $x \in X$, the fiber $f^{-1}(x)$ has trivial shape, and therefore $f: Z \rightarrow X$ is a cell-like mapping.*

Proof. Let $a_1 \in A$ be arbitrary. Choose any $a_2 \in A$, $a_1 <' a_2$. In view of Lemma 14 and Proposition 8, it suffices to exhibit an a'_2 , $a_2 <' a'_2$ such that for any $a_3 \geq a'_2$, there is a homotopy $G: M_{a_3}(x) \times I \rightarrow M_{a_1}(x)$ such that G_1 is constant and

$$(6) \quad G_0 = r_{a_1 a_2} r_{a_2 a_3} |M_{a_3}(x).$$

Note that $N_{a_2}(x)$ is a neighborhood of $x_{a_2} = p_{a_2}(x)$ in X_{a_2} (see §9). Since X_{a_2} is a polyhedron, there is a polyhedral neighborhood U of x_{a_2} which is contained in $N_{a_2}(x)$ and is contractible in itself. One can achieve that $U = |L|$ where L is a subcomplex of the j th barycentric subdivision $K_{a_2}^j$ of K_{a_2} for some sufficiently large j .

Choose $\eta > 0$ so small that

$$(7) \quad d(x', x_{a_2}) \leq 3\eta \Rightarrow x' \in |L|,$$

and choose $k \geq j$ so large that

$$(8) \quad 3 \text{ mesh } K_{a_2}^j \leq \eta.$$

By Lemma 2, there is an a'_2 , $a_2 <' a'_2$, such that $k(a_2, a'_2) \geq k$. Therefore, by Lemma 8, for any a_3 , $a'_2 < a_3$, one has

$$(9) \quad d(f_{a_2}^* r_{a_2 a_3}, p_{a_2 a_3} f_{a_3}^*) \leq \eta.$$

By Proposition 3 and (A3), one can also assume that for $a_3 > a'_2$

$$(10) \quad d(p_{a_2 a_3}(x_{a_3}), x_{a_2}) \leq \eta,$$

$$(11) \quad d(y, y') \leq \varepsilon_{a_3} \Rightarrow d(p_{a_2 a_3}(y), p_{a_2 a_3}(y')) \leq \eta.$$

We will now show that

$$(12) \quad r_{a_2 a_3}(M_{a_3}(x)) \subseteq |L|.$$

Indeed, if $z \in M_{a_3}(x) = f_{a_3}^{*-1}(N_{a_3}(x))$ (see §9(11)), then $f_{a_3}^*(z) \in N_{a_3}(x)$, and therefore,

$$(13) \quad d(f_{a_3}^*(z), x_{a_3}) \leq \varepsilon_{a_3}.$$

This and (11) yield

$$(14) \quad d(p_{a_2 a_3} f_{a_3}^*(z), p_{a_2 a_3}(x_{a_3})) \leq \eta.$$

Now (9), (10) and (14) yield

$$(15) \quad d(f_{a_2}^* r_{a_2 a_3}(z), x_{a_2}) \leq 3\eta.$$

Therefore (7) implies

$$(16) \quad f_{a_2}^* r_{a_2 a_3}(z) \in |L|.$$

This establishes (12) because

$$(17) \quad r_{a_2 a_3}(Z_{a_3}^*) \subseteq |(K_{a_2}^{k(a_2, a_3)})^{(n)}|$$

and the restriction of $f_{a_2}^*$ to the right side of (17) is an inclusion (§5(21)).

Since L is a subcomplex of $K_{a_2}^j$ and $k(a_2, a_3) \geq k(a_2, a'_2) \geq k \geq j$, there is an i such that the i th barycentric subdivision L^i of L is a subcomplex of $K_{a_2}^{k(a_2, a_3)}$. Therefore for any integer $m \geq 0$

$$(18) \quad |L^i| \cap |(K_{a_2}^{k(a_2, a_3)})^{(m)}| = |(L^i)^{(m)}|.$$

Since $|L| = |L^i|$ is contractible in itself, there is a homotopy $H: |L^i| \times I \rightarrow |L^i|$ such that

$$(19) \quad H_0 = \text{id},$$

$$(20) \quad H_1 = v,$$

where v is a vertex of L^i . We now restrict H to $|(L^i)^{(n)}| \times I$. Note that this is a polyhedron of dimension $\leq n+1$ and $H|_{|(L^i)^{(n)}| \times \partial I}$ is a PL-mapping. By the relative simplicial approximation theorem (see [15, Chapter 3, §4.1]), we can assume that $H|_{|(L^i)^{(n)}| \times I} \rightarrow |L^i|$ is a PL-mapping. Then

$$(21) \quad \dim H(|(L^i)^{(n)}| \times I) \leq n+1.$$

It is now possible to “push” H off the simplexes of L^i of dimensions $> n+1$. One can therefore assume that H also satisfies

$$(22) \quad H(|(L^i)^{(n)}| \times I) \subseteq |(L^i)^{(n+1)}|.$$

If $z \in M_{a_3}(x)$, then by (12), (17) and (18) (for $m = n$),

$$(23) \quad r_{a_2 a_3}(z) \in |(L^i)^{(n)}|.$$

Therefore, for $(z, t) \in M_{a_3}(x) \times I$, by (22) and (18) (for $m = n+1$), we have

$$(24) \quad x' = H(r_{a_2 a_3}(z), t) \in |(L^i)^{(n+1)}| \subseteq |(K_{a_2}^{k(a_2, a_3)})^{(n+1)}|.$$

However, Lemma 16 yields a mapping

$$(25) \quad \bar{r}_{a_1 a_2}: |(K_{a_2}^{k(a_2, a_3)})^{(n+1)}| \rightarrow |(K_{a_1}^{k(a_1, a_2)})^{(n)}| \subseteq Z_{a_1}^*.$$

We conclude that

$$(26) \quad G = \bar{r}_{a_1 a_2} H(r_{a_2 a_3} \times 1): M_{a_3}(x) \times I \rightarrow Z_{a_1}^*$$

is a well-defined mapping.

By (24), $x' \in |L| \subseteq N_{a_2}(x)$, and therefore by §9(3),

$$(27) \quad d(p_{a_1 a_2}(x'), x_{a_1}) \leq 2 \text{ mesh}(K_{a_1}).$$

By (25) and (2) we also have

$$(28) \quad d(\bar{r}_{a_1 a_2}(x'), p_{a_1 a_2}(x')) \leq 3 \text{ mesh}(K_{a_1}),$$

so that

$$(29) \quad d(G(z, t), x_{a_1}) \leq 5 \text{ mesh}(K_{a_1}) \leq \varepsilon_{a_1}.$$

This proves that $f_{a_1}^* G(z, t) = G(z, t) \in N_{a_1}(x)$. Therefore,

$$(30) \quad G(M_{a_3}(x) \times I) \subseteq M_{a_1}(x),$$

and G is a homotopy $G: M_{a_3}(x) \times I \rightarrow M_{a_1}(x)$.

The map G_1 is constant because H_1 is constant. $G_0 = r_{a_1 a_2} r_{a_2 a_3}$ follows from (19), (24) and the fact that $\bar{r}_{a_1 a_2}$ extends $r_{a_1 a_2}|(K_{a_2}^{k(a_2, a_3)})^{(n)}$ (see Lemma 16).

11. CELL-LIKE IMAGES OF n -DIMENSIONAL COMPACT SPACES.

We now state our main result.

Theorem 3. *Let X be a compact Hausdorff space whose cohomological dimension $\dim_Z X \leq n$, $n \geq 1$. Then there exist a compact Hausdorff space Z of covering dimension $\dim Z \leq n$ and weight $w(Z) \leq w(X)$ and a cell-like mapping $f: Z \rightarrow X$.*

Proof. If $\dim_Z X = n$, $n \geq 1$, we apply Proposition 12 and obtain an approximate system $\mathbf{X}$. Using $\mathbf{X}$, we construct the approximate system $\mathbf{Z}$ of §7. By Lemma 7, $Z = \lim \mathbf{Z}$ is a compact Hausdorff space with $\dim Z \leq n$ and $w(Z) \leq w(X)$. We define $f: Z \rightarrow X$ as in §8. By Lemma 17, f is a cell-like mapping.

Corollary 1. *Every compact metric space X with $\dim_Z X \leq n$, $n \geq 1$, is the image of a metric compactum Z , $\dim Z \leq n$, under a cell-like mapping $f: Z \rightarrow X$.*

The next result gives a converse to Theorem 3.

Proposition 13. *If a paracompact space X is the cell-like image of a normal space Z with $\dim Z \leq n$, then $\dim_Z X \leq n$.*

Proof. Let $f: Z \rightarrow X$ be a cell-like mapping. Since f is proper and X is paracompact, one concludes that Z also is paracompact (see [13, Chapter 2, Proposition 5.9]). By the standard definition of cohomological dimension (see [4]), we must show that for any closed subset $A \subseteq X$ the Čech cohomology $\check{H}^m(X, A; \mathbf{Z}) = 0$, for $m \geq n + 1$. Since $\dim Z \leq n$, we have $\check{H}^m(Z, B; \mathbf{Z}) = 0$, for any closed subset $B \subseteq Z$ and $m \geq n + 1$. In particular, this holds for $B = f^{-1}(A)$. Therefore, it suffices to conclude that for all m , f induces an isomorphism

$$f^*: \check{H}^m(X, A; \mathbf{Z}) \rightarrow \check{H}^m(Z, f^{-1}(A); \mathbf{Z}).$$

Since the fibers $f^{-1}(x)$ are of trivial shape, their cohomology vanishes. Therefore the Vietoris-Begle theorem applies (see [15, Chapter 6, §9, Theorem 15]) and yields the desired conclusion that f^* is an isomorphism.

Remark 6. We do not know whether paracompact spaces X with $\dim_Z X \leq n$ are cell-like images of paracompact spaces Z with $\dim Z \leq n$.

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