TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 313, Number 1, May 1989

AN HNN-EXTENSION WITH CYCLIC ASSOCIATED SUBGROUPS AND WITH UNSOLVABLE CONJUGACY PROBLEM

JODY MEYER LOCKHART

Dedicated to memory of William Werner Boone

ABSTRACT. In this paper, we consider the conjugacy problem for HNN-extensions of groups with solvable conjugacy problem for which the associated subgroups are cyclic. An example of such a group with unsolvable conjugacy problem is constructed. A similar construction is given for free products with amalgamation.

1. INTRODUCTION

It is known that an HNN-extension of a group with solvable conjugacy problem may have unsolvable conjugacy problem. Some restrictions placed on the type of HNN-extensions force the conjugacy problem to be solvable. In this paper, we investigate HNN-extensions with infinite cyclic associated subgroups. That is, we consider HNN-extensions of the form

$$G = \langle H , t ; t^{-1}at = b\rangle ,$$

where $a , b \in H$ are of infinite order. We also consider the analogous situation of free products with cyclic amalgamated subgroups.

The conjugacy problem for HNN-extensions and free products with amalgamation of this type has been considered by M. Anshel and P. Stebe [1], L. P. Comerford and B. Truffault [2], R. D. Hurwitz [3], and S. Lipschutz [4], among others. All obtained results giving conditions which guarantee that such groups have solvable conjugacy problem.

In this paper, we approach the problem from the opposite direction. An example of an HNN-extension $G = \langle H , t ; t^{-1}at = b\rangle$ where H has solvable conjugacy problem, G has solvable word problem, but G has unsolvable conjugacy problem is constructed. A similar example is given involving free products.

In §2, we show the existence of a one-to-one recursive function $f: N \to N$ with nonrecursive range such that $S_n = \{f(kn): k = 1 , 2 , \dots\}$ is recursive for

Received by the editors January 4, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 03D40, 20F05, 20F10; Secondary 03020.

©1989 American Mathematical Society
0002-9947/89 $1.00 + $.25 per page

each $n = 2, 3, \ldots$. In §3, we use the function f to construct group presentations H and G, each with a recursive set of generators and a recursively enumerable set of defining relators. H has the following properties.

(i) H has solvable conjugacy problem;

(ii) H has an infinite cycle subgroup $\langle d \rangle$ such that

(a) there is no algorithm to decide if an arbitrary W in H is conjugate to an element of $\langle d \rangle$;

(b) the problem of membership in $\langle d \rangle$ is decidable;

(iii) If $W \in H$ then W is not conjugate in H to W^{-1}.

Let $G = \langle H, t; t^{-1}dt = d^{-1} \rangle$. The group G has solvable word problem by (ii)(b). Straightforward arguments show that for any $W \in H$, W is conjugate to W^{-1} in G if and only if W is conjugate in H to an element of $\langle d \rangle$. Thus G has unsolvable conjugacy problem.

We then use the standard HNN embedding into a two-generator group to obtain the result for recursively presented groups. Free products with amalgamation are also considered.

§4 contains the proofs of some assertions used but not proved in §3.

2. The function f

In this section, we prove the existence of a recursive function f with certain special properties.

Lemma 1. *There is a one-to-one recursive function $f: N \to N$ with nonrecursive range such that $S_n = \{f(kn): k = 1, 2, \ldots\}$ is recursive for each $n = 2, 3, 4, \ldots$.*

Proof. Let g be any one-to-one recursive function with nonrecursive range. Let p_j be the jth prime number. Define f by

$$f(1) = 1, f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n}) = p_{g(i_1)}^{\alpha_1} p_{g(i_1)+i_2}^{\alpha_2} p_{g(i_1)+i_3}^{\alpha_3} \cdots p_{g(i_1)+i_n}^{\alpha_n}$$

where $i_1 < i_2 < \cdots < i_n$. The function f is clearly recursive.

We next show that f is one-to-one. Suppose

$$f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n}) = f(p_{j_1}^{\beta_1} p_{j_2}^{\beta_2} \cdots p_{j_m}^{\beta_m}).$$

Then

$$p_{g(i_1)}^{\alpha_1} p_{g(i_1)+i_2}^{\alpha_2} \cdots p_{g(i_1)+i_n}^{\alpha_n} = p_{g(j_1)}^{\beta_1} p_{g(j_1)+j_2}^{\beta_2} \cdots p_{g(j_1)+j_m}^{\beta_m}.$$

Since $g(i_1) < g(i_1) + i_2 < g(i_1) + i_3 < \cdots < g(i_1) + i_n$ and

$$g(j_1) < g(j_1) + j_2 < g(j_1) + j_3 < \cdots < g(j_1) + j_m,$$

it follows that

$$g(i_1) = g(j_1), \qquad g(i_1) + i_2 = g(j_1) + j_2, \ldots, g(i_1) + i_n = g(j_1) + j_n.$$

Therefore, $m = n$ and $i_k = j_k$ and $\alpha_k = \beta_k$ for all $k = 1, \ldots, n$. Hence,

$$p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \cdots p_{i_n}^{\alpha_n} = p_{j_1}^{\beta_1} p_{j_2}^{\beta_2} \cdots p_{j_m}^{\beta_m}.$$

Now consider the range of f.

$$p_i \in \text{range } f \begin{cases} \text{iff } p_i = f(p_j) \text{ for some } j, \\ \text{iff } p_i = p_{g(j)} \text{ for some } j, \\ \text{iff } i \in \text{range } g. \end{cases}$$

Since the range of g is nonrecursive, the range of f is nonrecursive.

Next consider S_n where n is prime. We claim that

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} \in S_{p_i}$$

if and only if

(i) $j_1 = g(i)$ and $j_l > j_{l-1} > \cdots > j_2 > j_1 + i$; or

(ii) $j_1 = g(i')$ for some $i' < i, j_l > \cdots > j_2 > j_1 + i'$, and $j_s - j_1 = i$ for some s between 2 and l.

It follows immediately from the claim that S_n is recursive if n is prime.

We now prove the claim. First assume that $p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l}$ is in S_{p_i}. Then

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} = f(p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_l}^{\beta_l})$$

for some $p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_l}^{\beta_l}$ where $i_1 < i_2 < \cdots < i_l$ and $i = i_t$ for some $t = 1, \dots, l$. There are two cases to consider: (1) $t = 1$ and (2) $t \geq 2$.

Case (1). If

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} = f(p_{i}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i_l}^{\beta_l})$$

then

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} = p_{g(i)}^{\beta_1} p_{g(i)+i_2}^{\beta_2} \cdots p_{g(i)+i_l}^{\beta_l}.$$

Therefore, $j_1 = g(i)$ and $j_2 > j_1 + i$; that is, (i) holds.

Case (2). If

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} = f(p_{i_1}^{\beta_1} p_{i_2}^{\beta_2} \cdots p_{i}^{\beta_t} \cdots p_{i_l}^{\beta_l}),$$

then

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} = p_{g(i_1)}^{\beta_1} p_{g(i_1)+i_2}^{\beta_2} \cdots p_{g(i_1)+i}^{\beta_t} \cdots p_{g(i_1)+i_l}^{\beta_l}.$$

Therefore, $j_1 = g(i_1)$ for some $i_1 < i$, $j_2 > j_1 + i_1$, and $i = j_t - j_1$ for some t between 2 and l. That is, (ii) holds.

Now suppose $n = p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l}$ satisfies (i) or (ii). We will show that $n \in S_{p_i}$.

Suppose n satisfies (i). Let $t_m = j_m - j_1$ for $m = 2, \dots, l$. Therefore, (i) gives $t_l > t_{l-1} > \cdots > t_2 > i$. We have

$$\begin{aligned} f(p_i^{\alpha_1} p_{t_2}^{\alpha_2} p_{t_3}^{\alpha_3} \cdots p_{t_l}^{\alpha_l}) &= p_{g(i)}^{\alpha_1} p_{g(i)+t_2}^{\alpha_2} p_{g(i)+t_3}^{\alpha_3} \cdots p_{g(i)+t_l}^{\alpha_l} \\ &= p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} p_{j_3}^{\alpha_3} \cdots p_{j_l}^{\alpha_l}. \end{aligned}$$

Hence, $n \in S_{p_i}$.

Suppose n satisfies (ii). Let $t_m = j_m - j_1$ for $m = 2, 3, \dots, l$. Then $t_l > t_{l-1} > \cdots > t_3 > t_2 > i'$ where $g(i') = j_1$. Therefore,

$$\begin{aligned} f(p_{i'}^{\alpha_1} p_{t_2}^{\alpha_2} p_{t_3}^{\alpha_3} \cdots p_{t_l}^{\alpha_l}) &= p_{g(i')}^{\alpha_1} p_{g(i')+t_2}^{\alpha_2} \cdots p_{g(i')+t_l}^{\alpha_l} \\ &= p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_l}^{\alpha_l} \end{aligned}$$

and since $j_s - j_1 = i$ for some s, we have $t_s = i$ for some s. Therefore, $n \in S_{p_i}$. This completes the proof of the claim.

To complete the proof of the lemma, we need to show that S_n is recursive if $n > 1$ and n is not prime. Let p be any prime divisor of n. Then, $x \in S_n$ implies $x \in S_p$. To determine if $x \in S_n$, first determine if $x \in S_p$. If not, then $x \notin S_n$. If so, we can effectively find the unique number a such that $x = f(ap)$. If n divides ap then $x \in S_n$. Otherwise, $x \notin S_n$. □

3. The groups

In this section, we give an example of a recursive group presentation N with solvable conjugacy problem and with infinite cyclic subgroups A_1 and A_2 such that the HNN-extension of G associating A_1 and A_2 has unsolvable conjugacy problem.

If W_1 and W_2 are words, we use $W_1 \equiv W_2$ to mean that W_1 and W_2 are identical words, $W_1 \underset{G}{=} W_2$ to mean that W_1 and W_2 are equal as elements of the group G, $W_1 \underset{Fr}{=} W_2$ to mean that W_1 and W_2 are freely equal, and $W_1 \underset{G}{\sim} W_2$ to mean that W_1 and W_2 are conjugate as elements of G.

Let f be the function of Lemma 1 and let

$$H = \langle x_1, z_1, x_2, z_2, \ldots ;\ z_i^{-1} x_{f(i)} z_i = z_1^{-1} x_1^i z_1 ;\ (i = 2, 3, 4, \ldots)\rangle.$$

We begin with some definitions.

Definition. If $W \equiv x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n}$, where α_k, β_k are integers, then W is said to be in *condensed form* if $\beta_k = 0$ implies $i_k \neq i_{k+1}$ and $\alpha_k = 0$ implies $j_{k-1} \neq j_k$.

Definition. If $W \equiv x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n}$, where α_k, β_k are integers, then W is said to be *reduced* if

(i) W is freely reduced;
(ii) $i_s \neq f(j_{s-1})$ for $s = 2, 3, \ldots, n$, $i_s \neq f(j_s)$ for $s = 1, 2, \ldots, n$ and $i_1 \neq f(j_n)$; and
(iii) no nontrivial subword of W or of any cyclic permutation of W equals 1 in H.

Notice that the free group presented by

$$B = \langle x_j (j \notin \operatorname{range} f), x_1, z_1, z_2, \ldots ;\ \rangle$$

is isomorphic to the group presented by H.

We first show that H has solvable word problem from which it follows that there is an effective procedure to determine if a word is reduced.

Lemma 2. *H has solvable word problem.*

Proof. Assume $W \equiv x_{i_1}^{\alpha_1} z_{j_1}^{\beta_1} \cdots x_{i_n}^{\alpha_n} z_{j_n}^{\beta_n}$ is freely reduced and in condensed form. If $i_1, \ldots, i_n$ are not in the range of f, then $W \underset{H}{=} 1$ if and only if $W \underset{B}{=} 1$

generality, that $\sigma' \neq 1$. By Lemma 1, $S_{\sigma'}$ is recursive. Determine if $i \in S_{\sigma'}$. If not, then (ii) is not satisfied. If so, effectively find $a = m\sigma'$ with $i = f(a)$. The only possible value for b then is $(a\alpha)/\sigma$. If $j = f(a\alpha/\sigma)$, then (ii) is satisfied. Otherwise, (ii) is not satisfied. □

Theorem 1. *There is a group presentation H with a recursive set of generators and a recursively enumerable set of defining relators with solvable conjugacy problem and an HNN-extension G of H associating infinite cyclic subgroups of H that has unsolvable conjugacy problem.*

Proof. Let $H = \langle x_1, z_1, x_2, z_2, \ldots; z_i^{-1}x_{f(i)}z_i = z_1^{-1}x_1^i z_1 \quad (i = 2, 3, \ldots)\rangle$ be the group of Lemma 4 and let

$$G = \langle H, t; t^{-1}z_1^{-1}x_1z_1t = z_1^{-1}x_1^{-1}z_1\rangle.$$

Note that $z_1^{-1}x_1z_1$ has infinite order. We claim that $x_j \underset{G}{\sim} x_j^{-1}$ if and only if $j \in$ range of f, from which it immediately follows that G has unsolvable conjugacy problem.

We now prove the claim. If $j \in$ range of f, then in G

$$\begin{aligned} x_j \equiv x_{f(n)} &= z_n z_1^{-1}x_1^n z_1 z_n^{-1} \sim z_1^{-1}x_1^n z_1 \sim z_1^{-1}x_1^{-n}z_1 \\ &\sim z_n z_1^{-1}x_1^{-n}z_1 z_n^{-1} = x_{f(n)}^{-1} \equiv x_j^{-1}. \end{aligned}$$

If $x_j \underset{G}{\sim} x_j^{-1}$, then using the facts that x_j, $x_j^{-1} \in H$ and G is an HNN-extension of H, either

(i) $x_j \underset{H}{\sim} x_j^{-1}$, or

(ii) there is a sequence of words $V_1, V_2, \ldots, V_{2p}$ with $x_j \equiv V_1$ and $x_j^{-1} \equiv V_{2p}$ such that $V_{2j+1} \underset{H}{\sim} V_{2j+2}$ for $j = 0, \ldots, p-1$ and, for each $j = 1, \ldots, p-1$, there is an $\varepsilon = \pm 1$ with $t^{-\varepsilon}V_{2j}t^{\varepsilon} = V_{2j+1}$.

We will first show that (i) is impossible. If $j \notin$ range$f$, then $x_j \underset{H}{\sim} x_j^{-1}$ if and only if $x_j \underset{Fr}{\sim} x_j^{-1}$ which is clearly impossible. If $j = f(n)$, then

$$x_j \underset{H}{\sim} x_j^{-1} \begin{cases} \text{iff } z_n z_1^{-1}x_1^n z_1 z_n^{-1} \underset{Fr}{\sim} z_n z_1^{-1}x_1^{-n}z_1 z_n^{-1}, \\ \text{iff } x_1^n \underset{Fr}{\sim} x_1^{-n} \end{cases}$$

which is impossible since $n \neq 0$. Therefore, if $x_j \underset{G}{\sim} x_j^{-1}$, (ii) must occur and so x_j must be conjugate in H to an element of the form $z_1^{-1}x_1^n z_1$ where n is some integer. If $j \notin$ range f and $x_j \underset{H}{\sim} z_1^{-1}x_1^n z_1$, then $x_j \underset{Fr}{\sim} z_1^{-1}x_1^n z_1$ which is impossible. Therefore, $x_j \underset{G}{\sim} x_j^{-1}$ implies that $j \in$ range f. □

We mention without proof that G has solvable word problem.

We now extend the above result to recursive presentations by using the standard HNN embedding into a two-generator group and checking that the appropriate properties are preserved.

Lemma 5. *Let* $H = \langle h_1, h_2, \ldots; s_1, s_2, \ldots \rangle$ *be a presentation with a recursive set of generators and a recursively enumerable set of defining relators such that*

(i) H *has solvable conjugacy problem*;
(ii) $h_i \neq 1$ *for all* $i = 1, 2, 3, \ldots$;
(iii) $h_i \neq h_j$ *for all* $i \neq j$, $i, j = 1, 2, 3, \ldots$; and
(iv) *there is an algorithm which given a word* W *of* H *decides if* $W = h_i^{\varepsilon} h_j^{\delta}$ *for some* i, $j = 1, 2, \ldots$ *and some* ε, $\delta = \pm 1$ *or* 0 *and, if so, which ones.*

Let $F = H^* \langle a, b \rangle$ *and*

$$N = \langle F, t; t^{-1}at = b, t^{-1}b^{-i}ab^{i}t = h_i a^{-i} b a^{i} \ (i = 1, 2, \ldots) \rangle.$$

Then N *has solvable conjugacy problem.*

Proof. It is known that N has solvable word problem. Let G_1 be the group generated by $\{a, b^{-i}ab^{i} \ (i = 1, 2, \ldots)\}$ and let G_2 be the group generated by $\{b, h_i a^{-i} b a^{i} \ (i = 1, 2, \ldots)\}$. Let ϕ be the isomorphism from G_1 to G_2 defined by $\phi(a) = b$ and $\phi(b^{-i}ab^{i}) = h_i a^{-i} b a^{i}$ for $i = 1, 2, \ldots$. We need the following assertions whose proofs will be delayed until §4.

(1) There is an algorithm to decide if elements of $G_1 \cup G_2$ are conjugate in N.
(2) There is an algorithm to decide, given $V \in F$, if V is conjugate in F to an element of $G_1 \cup G_2$ and, if so, to find such an element.
(3) There is an effective procedure to determine, for arbitrary X, $Y \in F$, if there is a $Z \in G_2$ such that $XZY \in G_1$ and, if so, to produce the finite number of possible Z.
(4) If $X, Y \in F - G_2$, $D \in G_2$ is reduced, and $XDY \in G_2$, then $\#_b(D) \leq \#_b(X) + \#_b(Y) + 2$, where $\#_b(A)$ is the number of occurrences of the symbol b in the word A.

Since t-reduction is effective, we may assume that we are dealing with cyclically t-reduced words and we may consider the base and nonbase cases separately.

Base Case. We first consider conjugacy in N of elements of F. Suppose U, $V \in F$. By Collins' Lemma, $U \underset{N}{\sim} V$ if and only if

(a) $U \underset{F}{\sim} V$; or
(b) there exist words $W_1, \ldots, W_k$ such that $U \underset{F}{\sim} W_1 \underset{t}{\sim} W_2 \underset{F}{\sim} W_3 \underset{t}{\sim} \cdots \sim W_k \underset{F}{\sim} V$ where $W_1, \ldots, W_k \in G_1 \cup G_2$ and $\underset{t}{\sim}$ indicates conjugation by t or t^{-1}.

Case (a) can be decided since F has solvable conjugacy problem. Case (b) holds if and only if there exist $W_1, W_k \in G_1 \cup G_2$ such that $U \underset{F}{\sim} W_1$, $V \underset{F}{\sim} W_k$, and $W_1 \underset{G}{\sim} W_k$. By assertions (1) and (2) this can be decided.

Nonbase Case. We now consider conjugacy in N of elements of $N-F$. Assume that

$$U \equiv x_1 t^{\varepsilon_1} \cdots x_m t^{\varepsilon_m} x_{m+1} \quad \text{and } V \equiv y_1 t^{\varepsilon_1} \cdots y_m t^{\varepsilon_m} y_{m+1}$$

both cyclically t-reduced. We consider three cases:

(i) $m = 1$;
(ii) $m > 1$ and the ε_i do not cyclically alternate; i.e. $\varepsilon_m = \varepsilon_1$ or there is an i, $1 \le i \le m-1$, such that $\varepsilon_i = \varepsilon_{i+1}$;
(iii) $m > 1$ and the ε_i do cyclically alternate.

First consider (i); $m = 1$. By Collins' Lemma, we may assume that $U \equiv xt$; $V \equiv yt$; and $U \underset{N}{\sim} V$ if and only if there is a $C \in G_2$ such that $C^{-1}xtC \underset{N}{=} yt$. But $C^{-1}xtC \underset{N}{=} yt$ implies that $t^{-1}y^{-1}C^{-1}xtC \underset{N}{=} 1$. Therefore, x and $y \in F$, $C^{-1} \in G_2$, and $y^{-1}C^{-1}x \in G_1$. By assertion (3), there are only finitely many possible C and they can be effectively found. Since N has solvable word problem, $C^{-1}xtC \underset{N}{=} yt$ can be effectively tested for each possible C and so $U \underset{N}{\sim} V$ can be decided in case (i).

Now consider (ii); $m > 1$ and the ε_i do not cyclically alternate. We may, by Collins' Lemma and by using U^{-1} and V^{-1} instead of U and V if necessary, assume that $\varepsilon_{m-1} = \varepsilon_m = -1$ and $x_{m+1} = y_{m+1} = 1$. Then $U \underset{N}{\sim} V$ if and only if there is a $C \in G_1$ such that

$$(*) \qquad C^{-1}x_1 t^{\varepsilon_1} \cdots x_{m-1} t^{-1} x_m t^{-1} C t y_m^{-1} t y_{m-1}^{-1} \cdots t^{-\varepsilon_1} y_1^{-1} \underset{N}{=} 1.$$

By considering the first two t-pinches on the left-hand side of $(*)$, one obtains that $C^{-1}UC \underset{N}{=} V$ implies $x_m \phi(C) y_m^{-1} \in G_1$. Therefore, any conjugating element C has the properties

(a) $\phi(C) \in G_2$ and (b) $x_m \phi(C) y_m^{-1} \in G_1$.

By assertion (3), there is an effective procedure that will either determine that no such C exists or produce a finite list containing all C satisfying (a) and (b). As in case (i), $U \underset{N}{\sim} V$ can be decided.

Now consider (iii); the ε_i cyclically alternate. We may, without loss of generality, assume that

$$U \equiv x_1 t x_2 t^{-1} x_3 t x_4 t^{-1} \cdots x_{n-1} t x_n t^{-1};$$

$$V \equiv y_1 t y_2 t^{-1} y_3 t y_4 t^{-1} \cdots y_{n-1} t y_n t^{-1};$$

n is even; and $U \underset{N}{\sim} V$ if and only if there is a $C \in G_1$ such that $C^{-1}UC \underset{N}{=} V$.

Define the homomorphism $\rho: F \to \langle a, b \rangle$ by $\rho(h) = 1$ for all $h \in H$, $\rho(a) = a$ and $\rho(b) = b$. Define the homomorphism $\tau: F \to F$ by $\tau(h) = h$

for all $h \in H$, $\tau(a) = b$, and $\tau(b) = a$. Now,

$$C^{-1}UC \underset{N}{=} V \Rightarrow C^{-1}x_1tx_2t^{-1}\cdots x_nt^{-1}Cty_n^{-1}t^{-1}y_{n-1}^{-1}\cdots t^{-1}y_1^{-1} \underset{N}{=} 1$$

$$\Rightarrow$$

$$C^{-1}x_1\phi^{-1}(x_2\phi(x_3\cdots\phi(x_{n-1}\phi^{-1}(x_n\phi(C)y_n^{-1})y_{n-1}^{-1}\cdots y_3^{-1})y_2^{-1})y_1^{-1} \underset{F}{=} 1$$

$$\Rightarrow$$

$$\rho(c^{-1}x_1\phi^{-1}(x_2\phi(x_3\cdots\phi(x_{n-1}\phi^{-1}(x_n\phi(c)y_n^{-1})y_{n-1}^{-1}\cdots y_3^{-1})y_2^{-1})y_1^{-1}) \underset{\langle a,b\rangle}{=} 1$$

$$\Rightarrow$$

$C^{-1}U_1C \underset{\langle a,b\rangle}{=} U_2$, where $U_1 = \rho(x_1)(\tau\circ\rho)(x_2)\rho(x_3)(\tau\circ\rho)(x_4)\cdots(\tau\circ\rho)(x_n)$ and $U_2 = \rho(y_1)(\tau\circ\rho)(y_2)\rho(y_3)(\tau\circ\rho)(y_4)\cdots(\tau\circ\rho)(y_n)$

$$\Rightarrow$$

$C \underset{\langle a,b\rangle}{=} W^mD$, where D and W are words on $\{a,b\}$ such that $D^{-1}U_1D \underset{\langle a,b\rangle}{=} U_2$ and $U_1 \underset{\langle a,b\rangle}{=} W^k$ for some k.

Since such W and D can be effectively found, it remains only to find a bound for m. For then we will have effectively found the finite number of possible conjugating elements and each can be tested. We know that $x_n, y_n \notin G_2$ for otherwise U, V would not be reduced. If $C^{-1}UC = V$, then $\phi(C) \in G_2$ and $x_n\phi(C)y_n^{-1} \in G_2$. By assertion (4), $\#_b(\phi(C)) \le \#_b(x_n) + \#_b(y_n) + 2$. Hence, a bound for m can be found completing the proof for case (iii). □

Lemma 6. *Let H and N be as in Lemma 5 and suppose $G = \langle H, v; v^{-1}dv = e\rangle$ has solvable conjugacy problem where d and e are elements of H of infinite order. Let $L = \langle N, v; v^{-1}dv = e\rangle$. Then L has unsolvable conjugacy problem.*

Proof. We first show that, for $y_1, y_2 \in H$, $y_1 \underset{H}{\sim} y_2$ if and only if $y_1 \underset{N}{\sim} y_2$. The statement is clear from left to right. If $y_1 \underset{N}{\sim} y_2$, then either

(1) $y_1 \underset{F}{\sim} y_2$ or
(2) $y_1 \underset{F}{\sim} W_1 \underset{t}{\sim} W_2 \underset{F}{\sim} W_3 \underset{t}{\sim} \cdots \underset{t}{\sim} W_n \underset{F}{\sim} y_2$.

If (1), then clearly $y_1 \underset{H}{\sim} y_2$. If (2), then $y_1 \underset{F}{\sim} U \in G_1 \cup G_2$. This is impossible unless $y_1 = 1$ in which case $y_2 = 1$ and $y_1 \underset{H}{\sim} y_2$.

We will next show that, for $V_1, V_2 \in G$, $V_1 \underset{G}{\sim} V_2$ if and only if $V_1 \underset{L}{\sim} V_2$, thus showing that L has unsolvable conjugacy problem. The statement from left to right is clear. Suppose that $V_1 \underset{L}{\sim} V_2$. We may, without loss of generality, assume that V_1 and V_2 are v-reduced in G.

If $V_1, V_2 \in H$, then either

(i) $V_1 \underset{N}{\sim} V_2$; or
(ii) $V_1 \underset{N}{\sim} (d^m \text{ or } e^m) \underset{v}{\sim} (e^m \text{ or } d^m) \underset{N}{\sim} (d^k \text{ or } e^k) \sim \cdots \sim V_2$.

Using the fact that, for $y_1, y_2 \in H$, $y_1 \underset{N}{\sim} y_2$ implies $y_1 \underset{H}{\sim} y_2$, both (i) and (ii) yield $V_1 \underset{G}{\sim} V_2$. Now suppose $V_1 \underset{L}{\sim} V_2$ and $V_1, V_2 \in G - H$. If $V_1 \underset{L}{\sim} V_2$ then there are cyclic permutations V_1' and V_2' of V_1 and V_2, respectively, with the same v-structures such that $d^{-k}V_1'd^k = V_2'$ or $e^{-k}V_1'e^k = V_2'$ for some k. Since all words involved are words of G and since G is embedded in L, one of the equalities must hold in G. Hence, $V_1 \underset{G}{\sim} V_2$. Therefore, for $V_1, V_2 \in G$, $V_1 \underset{L}{\sim} V_2$ if and only if $V_1 \underset{G}{\sim} V_2$ and so L has unsolvable conjugacy problem. □

Theorem 2. *There is a finitely generated group N with solvable conjugacy problem and an HNN-extension $L = \langle N, v; v^{-1}dv = e\rangle$ where d and e are elements of N of infinite order such that L has unsolvable conjugacy problem.*

Proof. Take H and G to be the infinitely generated groups of Theorem 1 and N to be the group of Lemma 5 constructed from this H. By Lemma 6, L has unsolvable conjugacy problem. By Lemma 5, to show that N has solvable conjugacy problem it suffices to verify conditions (ii), (iii), and (iv). (Condition (i) is already known to be true.)

Condition (ii). Generators in H are not equal to 1 in H: If $i \notin \operatorname{range} f$, then $x_i \underset{H}{=} 1$ implies $x_i \underset{Fr}{=} 1$ which is impossible. If $i = f(j)$, then

$$x_i \underset{H}{=} 1 \Rightarrow z_j z_1^{-1} x_1^j z_1 z_j^{-1} \underset{Fr}{=} 1 \Rightarrow x_1^j \underset{Fr}{=} 1 \Rightarrow j = 0$$

which is impossible. Therefore, $x_i \underset{H}{\neq} 1$ for all $i = 1, 2, \ldots$. Since $z_i \underset{H}{=} 1$ implies $z_i \underset{Fr}{=} 1$, it is clear that $z_i \underset{H}{\neq} 1$ for all $i = 1, 2, \ldots$.

Condition (iii). Distinct generators of H are not equal in H: If $i, j \notin \operatorname{range} f$, then

$$x_i \underset{H}{=} x_j \Rightarrow x_i \underset{Fr}{=} x_j \Rightarrow i = j.$$

If $i \notin \operatorname{range} f$ and $j = f(k)$, then

$$x_i \underset{H}{=} x_j \Rightarrow x_i \underset{Fr}{=} z_k z_1^{-1} x_1^k z_1 z_k^{-1}$$

which is impossible. If $i, j \in \operatorname{range} f$, it is easy to see that $x_i \underset{H}{=} x_j$ implies $i = j$. It is also easy to see that $z_i \underset{H}{=} z_j$ implies $i = j$ and that $x_i \underset{H}{=} z_j$ is impossible.

Therefore, distinct generators of H are not equal in H.

Condition (iv). Given a word W of H, we must decide if W is a product of one or two generators or inverses of generators.

Semireduce W to obtain W' (see proof of Lemma 2 for definition of semireduction). Then

(1) $W \underset{H}{=} z_{i_1}^{\varepsilon_1} z_{i_2}^{\varepsilon_2} \Leftrightarrow W' \underset{Fr}{=} z_{i_1}^{\varepsilon_1} z_{i_2}^{\varepsilon_2}$.

(2) $W \underset{H}{=} x_i^{\varepsilon} z_j^{\delta} \Leftrightarrow W' \underset{Fr}{=} x_i^{\varepsilon} z_j^{\delta}$ or $W' \underset{Fr}{=} z_k z_1^{-1} x_1^k z_1 z_k^{-1} z_j$ where $f(k) = i$.

(3) $W \underset{H}{=} x_i^{\varepsilon} x_j^{\delta}$ can be dealt with similarly.

Therefore, H satisfies conditions (ii), (iii), and (iv) of Lemma 5. $\square$

Theorem 3. *There exist group presentations N_1 and N_2 each with a finite number of generators and a recursively enumerable set of defining relators and each having solvable conjugacy problem such that there is a free product with amalgamation M of N_1 and N_2 with unsolvable conjugacy problem that amalgamates cyclic subgroups.*

Proof. Let $N_1 = \langle a, t; r_1, r_2, \ldots \rangle$ and $N_2 = \langle \hat{a}, \hat{t}; \hat{r}_1, \hat{r}_2, \ldots \rangle$ be copies of the group N of Theorem 2. Recall that

$$H = \langle x_1, z_1, x_2, z_2, \ldots; z_i^{-1} x_{f(i)} z_i = z_1^{-1} x_1^i z_1 \ (i = 2, 3, \ldots) \rangle.$$

Let α and β be the usual embeddings of H into N_1 and H into N_2, respectively. Let $W_j = \alpha(x_j)$ and $\hat{W}_j = \beta(x_j)$. Let

$$M = \langle N_1 * N_2; W_1 = \hat{W}_1 \rangle.$$

We claim that $W_j \underset{M}{\sim} \hat{W}_j$ if and only if $j \in \operatorname{range} f$. If $j \in \operatorname{range} f$, then $j = f(k)$ for some k and

$$\begin{aligned} W_j = \alpha(z_k z_1^{-1} x_1^k z_1 z_k^{-1}) &\underset{N_1}{\sim} \alpha(x_1^k) \underset{N_1}{=} W_1^k \underset{M}{=} \hat{W}_1^k \\ &\underset{N_2}{=} \beta(x_1^k) \underset{N_2}{\sim} \beta(z_k z_1^{-1} x_1^k z_1 z_k^{-1}) = \hat{W}_j. \end{aligned}$$

Therefore, if $j \in \operatorname{range} f$ then $W_j \underset{M}{\sim} \hat{W}_j$.

Now suppose $W_j \underset{M}{\sim} \hat{W}_j$. Since M is a free product with amalgamation, there is a sequence of words $V_1, V_2, \ldots, V_k$ such that

$$W_j \equiv V_1 \underset{\alpha_1}{\sim} V_2 \underset{\alpha_2}{\sim} V_3 \underset{\alpha_3}{\sim} \cdots \underset{\alpha_{k-1}}{\sim} V_k \equiv \hat{W}_j$$

where $V_2, \ldots, V_{k-1}$ are in the amalgamated part and the α_i alternate between N_1 and N_2. Therefore,

$$W_j \underset{N_1}{\sim} W_1^n \text{ for some } n \Rightarrow \alpha(x_j) \underset{N_1}{\sim} \alpha(x_1^n) \text{ for some } n \Rightarrow$$
$$x_j \underset{H}{\sim} x_1^n \text{ for some } n \Rightarrow j \in \operatorname{range} f.$$

The second implication follows from the fact that, for $y_1, y_2 \in H$, $y_1 \underset{H}{\sim} y_2$ if and only if $y_1 \underset{N}{\sim} y_2$ (see proof of Lemma 6). Since $W_j \underset{M}{\sim} \hat{W}_j$ if and only if $j \in \operatorname{range} f$ and range f is nonrecursive, M has unsolvable conjugacy problem. $\square$

4. Proof of assertions

We now prove the assertions needed in Lemma 5.

(1) There is an algorithm to decide if elements of $G_1 \cup G_2$ are conjugate in N.

Proof. Suppose U, $V \in G_1 \cup G_2$; then $U \underset{N}{\sim} V$ if and only if

(i) $U \underset{F}{\sim} V$; or

(ii) there are words $W_1, \ldots, W_m$ such that $U \underset{t}{\sim} W_1 \underset{F}{\sim} W_2 \underset{t}{\sim} W_3 \sim \cdots \sim W_m \sim V$.

Since (i) can be decided we need only consider (ii). By considering cyclic reductions with respect to $*$ (i.e. with respect to the free product $F = H * \langle a, b\rangle$), it can be seen that if $1 \neq A \in G_1$ and $1 \neq B \in G_2$ then A and B are not conjugate in F. Therefore, W_{2k-1} and W_{2k} must be in the same G_i.

Suppose first that $U, V \in G_1$. Then $W_1, W_2 \in G_2$ and $W_3, W_4 \in G_1$. It can be seen that $U \underset{G_1}{\sim} W_3$ and $U \underset{G_1}{\sim} W_4$. Continuing in this fashion, we get

$$U \underset{\langle a,b\rangle}{\sim} W_{4k-1} \underset{\langle a,b\rangle}{\sim} W_{4k} \quad \text{for all } k = 1, \ldots, [m/4].$$

Since $V \in G_1$, V is W_{4k-1} or W_{4k} for some k and so $U \underset{\langle a,b\rangle}{\sim} V$. Therefore, (ii) occurs if and only if $U \underset{F}{\sim} V$ and so conjugacy in N of elements of G_1 is decidable.

Now suppose $U, V \in G_2$. Then $W_1 \in G_1$, $W_m \in G_1$, and $W_1 \underset{N}{\sim} W_m$. By the above, $W_1 \underset{\langle a,b\rangle}{\sim} W_m$. So, for $U, V \in G_2$, (ii) occurs if and only if $\phi^{-1}(U) \underset{\langle a,b\rangle}{\sim} \phi^{-1}(V)$.

The case $U \in G_1$ and $V \in G_2$ is similar. □

(2) There is an algorithm to decide, given $V \in F$, if V is conjugate in F to an element of $G_1 \cup G_2$ and, if so, to find such an element.

Proof. Let $V \in F$. We may assume that V is cyclically reduced with respect to $*$. First (effectively) decide if $V \in \langle a, b\rangle$. If so, by assumptions (ii) and (iii), $V \sim U \in G_1 \cup G_2$ if and only if $V \sim U \in G_1$ or $V \sim b^m$ which can be decided. If $V \notin \langle a, b\rangle$ then $V \sim U \in G_1 \cup G_2$ if and only if $V \sim U \in G_2$. Suppose that $V \equiv A_1 H_1 \cdots A_m H_m$ where $A_i \in \langle a, b\rangle$ and $H_i \in H$. If $V \sim U \in G_2$ then by assumptions (ii) and (iii) each A_i is a product of at most three words of the forms $a^{-j} b^{\pm 1} a^j$ or b^k (k an integer). It can be effectively determined if each A_i is such a product and, if so, each A_i can be effectively written as such a product. The only possible cyclically reduced U (up to cyclic permutations) can then be determined and tested. □

(3) There is an effective procedure to determine, for arbitrary $X, Y \in F$, if there is a $Z \in G_2$ such that $XZY \in G_1$ and, if so, to produce the finite number of possible Z.

Proof. Suppose $X = A_1 H_1 \cdots A_m H_m A_{m+1}$ and $Y = B_1 K_1 \cdots B_l K_l B_{l+1}$ where $A_i, B_i \in \langle a, b\rangle$ and H_i, $K_i \in H$. Since $XZY \in G_1$, all H factors must cancel and either

(a) $Z = A_{m+1}^{-1} H_m^{-1} A_m^{-1} \cdots H_1^{-1} Z' K_l^{-1} B_l^{-1} \cdots K_1^{-1} B_1^{-1} \in G_2$ and $A_1 Z' B_{l+1} \in G_1$; or

(b) $Z = A_{m+1}^{-1} H_m^{-1} A_m^{-1} \cdots H_j^{-1} Z' K_q^{-1} B_q^{-1} \cdots K_1^{-1} B_1^{-1} \in G_2$ where $2 \leq j \leq m$, $1 \leq q \leq l-1$, Z' is h-free and $A_1 H_1 \cdots A_{j-1} H_{j-1} A_j Z' B_{q+1} K_{q+1} \cdots B_l K_{l+1} B_{l+1} \in G_1$.

If (a) then $H_1^{-1} = h_{i_1}^{\delta_1} h_{i_2}^{\delta_2}$ and $K_l^{-1} = h_{j_1}^{t_1} h_{j_2}^{t_2}$ for some $\delta_2, t_1 = \pm 1$ and $\delta_1, t_2 = 0$ or ± 1. By assumption (iv), there is an effective procedure to determine if H_1^{-1} and K_l^{-1} can be written as such and, if so, to so write them. Since Z' is h-free and $Z \in G_2$, there are four possibilities: $h_{i_2}^{\delta_2} Z' h_{j_1}^{t_1} \in G_2$, $Z' h_{j_1}^{t_1} \in G_2$, $h_{i_2}^{\delta_2} Z' \in G_2$, $Z' \in G_2$. Since $A_1 Z' B_{l+1} \in G_1$ implies that the exponent sum of b in $A_1 Z' B_{l+1}$ is zero, each of these four possibilities give rise to finitely many possible Z' which in turn give rise to finitely many possible Z.

If (b) then $A_j Z' B_{q+1} \underset{\langle a,b \rangle}{=} 1$ which implies that

$$Z = A_{m+1}^{-1} H_m^{-1} A_m^{-1} \cdots H_j^{-1} A_j^{-1} B_{q+1}^{-1} K_q^{-1} B_q^{-1} \cdots K_1^{-1} B_1^{-1},$$

again giving rise to only finitely many possible Z. $\square$

(4) If $X, Y \in F - G_2$, if $D \in G_2$ is reduced and if $XDY \in G_2$, then $\#_b(D) \leq \#_b(X) + \#_b(Y) + 2$.

Proof. Let $V \equiv v_1 \cdots v_m$ where $v_i \in \{b^{\pm 1}, (h_i a^{-i} b a^i)^{\pm 1}\ (i = 1, 2, \dots)\}$. Suppose V is freely reduced with respect to the free group G_2 and $\#_b(V) \geq 3$. It is easy to see that if W is obtained from V by freely reducing with respect to the free product $H * \langle a, b \rangle$, then the subword of V containing the three b can be uniquely recovered from W.

Suppose $D \in G_2$, $XDY \in G_2$ and $\#_b(D) \geq \#_b(X) + \#_b(Y) + 3$. Assume further that X, D, and Y are reduced with respect to $*$. It suffices to show that $X \in G_2$. Consider the cancellation that occurs in the product XDY. Let

$$X \equiv X_1 X_2;\ D \equiv X_2^{-1} D_1 Y_1^{-1};\ \text{and } Y \equiv Y_1 Y_2.$$

We have that $X_1 D_1 Y_2 \in G_2$ is reduced with respect to $*$ and $\#_b(D_1) \geq 3$. By the comment above, the shortest words X_1'' and Y_2' such that $X_1 \equiv X_1' X_1''$, $Y_2 \equiv Y_2' Y_2''$ and $X_1'' D_1 Y_2' \in G_2$ can be effectively found. Since X_1'' and Y_2' are unique, $X_1' X_1'' D_1 Y_2' Y_2'' \in G_2$, and $X_1'' D_1 Y_2' \in G_2$, it follows that X_1' and $Y_2'' \in G_2$. Since X_1'' is the shortest word such that $X_1'' D_1 Y_2' \in G_2$ and since $X_2^{-1} D_1 Y_1^{-1} \in G_2$, it follows that X_2^{-1} ends in X_1''. That is, $X_2^{-1} \equiv X_3 X_1''$ and $X_3 \in G_2$. Therefore, $X \equiv X_1' X_1'' X_2 = X_1' X_3^{-1} \in G_2$ since $X_1' \in G_2$ and $X_3 \in G_2$. $\square$

5. Conclusion

The existence of a finitely presented group H and an HNN-extension $G = \langle H, t; t^{-1} a t = b \rangle$ where H has solvable conjugacy problem but G has unsolvable conjugacy problem is open. The analogous question for free products with amalgamation is also open.

References

1. M. Anshel and P. Stebe, *The solvability of the conjugacy problem for certain HNN groups*, Bull. Amer. Math. Soc. **80** (1974), 266–270.
2. L. P. Comerford and B. Truffault, *The conjugacy problem for free products of sixth-groups with cyclic amalgamation*, Math. Z. **149** (1976), 169–181.
3. R. D. Hurwitz, *On cyclic subgroups and the conjugacy problem*, Proc. Amer. Math. Soc. **79** (1980), 1–8.
4. S. Lipschutz, *The conjugacy problem and cyclic amalgamations*, Bull. Amer. Math. Soc. **81** (1975), 114–116.

Department of Mathematics, Indiana University at South Bend, South Bend, Indiana 46634

Current address: Department of Mathematics, United States Naval Academy, Annapolis, Maryland 21402