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THE GROUP OF AUTOMORPHISMS OF $L^1(0,1)$ IS CONNECTED

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ABSTRACT. It is shown that the group of the automorphisms of the radical convolution algebra $L^1(0,1)$ is connected in the operator norm topology, and thus every automorphism is of the form $e^{\lambda d}e^q$, where λ is a complex number, d is the derivation $df(x) = xf(x)$ and q is a quasinilpotent derivation.

Suppose in the Banach space $L^1(0,1)$ we define the "convolution" product $*$ by

$$(f * g)(x) = \int_0^x f(x-y)g(y)\,dy \qquad (f, g \in L^1(0,1),\ \text{a.e. } x \in (0,1)).$$

Then $V = L^1(0,1)$ with this product becomes a radical Banach algebra [6], called the Volterra algebra. Kamowitz and Scheinberg in [6] investigated the structure of the automorphisms and derivations on V. There was one problem left open: is the automorphism group of V connected, in the operator norm topology? We answer this question in the affirmative. We will use the fact that the automorphisms and the derivations on V are continuous [4, Remark (3a)]. Every automorphism of V has an extension to an automorphism of the measure algebra $M[0,1)$, which we will denote by the same symbol [5, §8]. On the space $B(V)$ of all bounded linear operators on V, we consider strong operator topology (SO) defined by: a net (T_α) of operators tends to an operator T in (SO) if, and only if, $T_\alpha f \to Tf$ in norm, for every $f \in V$. Since $M[0,1)$ can be identified with the multiplier algebra of V [6, Remark 10] the topology (SO) induces to $M[0,1)$. We denote the induced topology by (so). Let $C_0[0,1)$ be the space of continuous functions f on $[0,1)$ with $\lim_{x\to 1^-} f(x) = 0$. Then $M[0,1) = C_0[0,1)^*$. Let $w^* = \sigma[M[0,1),\ C_0[0,1)]$. Then if (μ_α) is a bounded net and $\mu_\alpha \xrightarrow{(so)} \mu$, then $\mu_\alpha \xrightarrow{w^*} \mu$ [1, Lemma 1-1].

In [1] we have shown that if an automorphism θ of V is extended to $M[0,1)$, then there exists a complex number z, such that for every $x \in [0,1)$,

$$\theta(\delta_x) = e^{zx}\delta_x + \mu_x, \tag{1}$$

where $\alpha(\mu_x) \geq x$ and $\mu_x(\{x\}) = 0$ (for every measure μ, we denote the infimum of the support of μ by $\alpha(\mu)$). Following the terminology used by

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S. Grabiner [2] for the automorphisms of the power series algebra we call an automorphism θ of V *principal* if $z = 0$ in (1). If an automorphism θ of V satisfies (1), then

$$e^{-zd}\theta(\delta_x) = \delta_x + e^{-zd}\mu_x \qquad (x \in [0,1)), \tag{2}$$

with $\alpha(e^{-zd}\mu_x) \geq x$ and $(e^{-zd}\mu_x)(\{x\}) = 0$. Therefore $e^{-zd}\theta$ is a principal automorphism and in order to show that every automorphism is of the form $e^{zd}e^{q}$, it suffices to show that every principal automorphism is in the component of the identity.

Proposition 1. *Suppose T is a bounded linear operator on V which can be extended to a bounded linear operator $\overline{T}$ on $M[0,1)$. If $\overline{T}$ is* (so)-(so) *continuous, then*

$$\|T\| = \|\overline{T}\| = \sup\{\|\overline{T}(\delta_x)\| : x \in [0,1)\}.$$

Proof. Let $\mu \in M[0,1)$, with $\|\mu\| \leq 1$. By identifying $\overline{T}(\mu)$ with a multiplier on V, given $\varepsilon > 0$, there exists $f \in V$ with $\|f\| \leq 1$ and with

$$\|\overline{T}(\mu)\| < \|\overline{T}(\mu) * f\| + \varepsilon. \tag{1}$$

By [1, Lemma 1.2] and (so)-(so) continuity of $\overline{T}$ there exists a linear combination $\alpha_1\lambda_1\delta_{x_1} + \alpha_2\lambda_2\delta_{x_2} + \cdots + \alpha_n\lambda_n\delta_{x_n}$, with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, $\alpha_i > 0$, $|\lambda_i| = 1$, $i = 1, 2, \ldots, n$, and with

$$\|\overline{T}(\mu) * f\| < \|\overline{T}(\alpha_1\lambda_1\delta_{x_1} + \cdots + \alpha_n\lambda_n\delta_{x_n}) * f\| + \varepsilon. \tag{2}$$

From (1), (2) and $\|f\| \leq 1$ it follows that

$$\begin{aligned} \|\overline{T}(\mu)\| &\leq \alpha_1\|\overline{T}(\delta_{x_1})\| + \cdots + \alpha_n\|\overline{T}(\delta_{x_n})\| \\ &\leq \sup\{\|\overline{T}(\delta_x)\| : x \in [0,1)\}. \end{aligned} \tag{3}$$

Thus

$$\|\overline{T}\| = \sup\{\|\overline{T}(\delta_x)\| : x \in [0,1)\}. \tag{4}$$

To prove $\|\overline{T}\| = \|T\|$, let (e_n) be a bounded approximate identity of V bounded by 1. Then $\delta_x = (\text{so})\text{-}\lim \delta_x * e_n$. Hence $\overline{T}(\delta_x) = (\text{so})\text{-}\lim T(\delta_x * e_n)$. Hence by [1, Lemma 1.1] $\overline{T}(\delta_x) = w^*\text{-}\lim T(\delta_x * e_n)$.

Since

$$\|T(\delta_x * e_n)\| \leq \|T\|\,\|\delta_x * e_n\| \leq \|T\|,$$

we get $\|\overline{T}(\delta_x)\| \leq \|T\|$. This together with (4) implies $\|\overline{T}\| = \|T\|$.

Remark. If D is the extension of a derivation of V to $M[0,1)$ and θ is the extension of an automorphism of V to $M[0,1)$, then from $D(\mu) * f = D(\mu * f) - D(f) * \mu$ and $\theta(\mu) * f = \theta(\mu * \theta^{-1}(f))$ $(\mu \in M[0,1),\ f \in V)$ it follows that D and θ are (so)-(so) continuous.

Lemma 1. *Suppose θ is an automorphism of V with*

$$\theta(\delta_x) = e^{zx}\delta_x + \mu_x \qquad (x \in [0,1))$$

where $\mu_x(\{x\}) = 0$ and $\alpha(\mu_x) \geq x$. Then

(a) $\lim_{n\to\infty} e^{-nd}\theta e^{nd}(\delta_x) = e^{zx}\delta_x$.

(b) *For each positive integer n, $\|e^{-nd}\theta e^{nd}\| \leq \|\theta\|$.*

Proof. We have

(1) $e^{-nd}\theta e^{nd}(\delta_x) = e^{nx}e^{-nd}\theta(\delta_x) = e^{nx}e^{-nd}(e^{zx}\delta_x + \mu_x) = e^{zx}\delta_x + e^{nx}e^{-nd}\mu_x$

and

$$\|e^{nx}e^{-nd}\mu_x\| = \int_{(x,1)} e^{-n(y-x)}\, d|\mu_x|(y) \to 0, \quad \text{as } n \to \infty$$

by Lebesgue dominated convergence theorem. This proves (a).

To prove (b) we note that for $x \in [0,1)$

$$\begin{aligned}\|e^{-nd}\theta e^{nd}(\delta_x)\| &= \|e^{zx}\delta_x + e^{nx}e^{-nd}\mu_x\| \\ &= |e^{zx}| + \int_{(x,1)} e^{-n(y-x)}\, d|\mu_x|(y) \\ &\leq |e^{zx}| + \int_{(x,1)} d|\mu_x|(y) = \|\theta(\delta_x)\|.\end{aligned}$$

Thus by Proposition 1, $\|e^{-nd}\theta e^{nd}\| \leq \|\theta\|$, and the proof is complete.

Lemma 2. *If θ is a principal automorphism of V, then θ^{-1} is a principal automorphism.*

Proof. Since the connected component of the identity in a topological group is a normal subgroup [3, Theorem 7.1], from [6, Theorem 9] it follows that there exists a complex number z and a quasinilpotent derivation q such that

$$\theta e^{-d}\theta^{-1} = e^{zd}e^{q}, \tag{1}$$

or equivalently,

$$\theta e^{-d} = e^{zd}e^{q}\theta. \tag{2}$$

Now if we examine the image of δ_x under the two sides of (2) we will see that $z = -1$ (see the proof of Theorem 1.4 in [2]). Thus

$$\theta e^{-d}\theta^{-1} = e^{-d}e^{q}. \tag{3}$$

Since $(\theta e^{-d}\theta^{-1})^n = \theta(e^{-nd})\theta^{-1}$, from (3) it follows

$$\theta e^{-nd}\theta^{-1} = (e^{-d}e^{q})^n. \tag{4}$$

Hence

$$\theta e^{-nd}\theta^{-1}e^{nd} = (e^{-d}e^{q})^n e^{nd}. \tag{5}$$

Now suppose that θ is a principal automorphism and

$$\theta(\delta_x) = \delta_x + \mu_x \qquad (\mu_x(\{x\}) = 0, \alpha(\mu_x) \geq x), \tag{6}$$

and for some complex number λ,

$$\theta^{-1}(\delta_x) = e^{\lambda x}\delta_x + \nu_x, \qquad x \in [0,1),$$

where $\alpha(\nu_x) \geq x$ and $\nu_x(\{x\}) = 0$.

Then by Lemma 1(a) and (5) we will have

$$e^{\lambda x}\delta_x + e^{\lambda x}\mu_x = e^{\lambda x}\theta(\delta_x) = \lim(e^{-d}e^{q})^n e^{nd}(\delta_x). \tag{7}$$

However for each n, the measure $(e^{-d}e^{q})^n e^{nd}(\delta_x)$ has mass 1 at x. Thus $\lambda = 0$ and θ^{-1} is a principal automorphism.

Lemma 3. *If θ is a principal automorphism of V then* (so)-$\lim e^{-nd}\theta e^{nd} = I$.

The proof of this lemma is implicit in the proof of Theorem 1.2 of [1] and is therefore omitted.

Lemma 4. *If q is a quasinilpotent derivation on V, then $\lim_{n\to\infty} e^{-nd}qe^{nd} = 0$ (in operator norm topology).*

Proof. Let q be given by $q(f) = df * \mu$ $(f \in V)$. Since $\mu(\{0\}) = 0$ [6, Remark 1], given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mu|([0,\delta)) < \varepsilon$. Let μ_1 and μ_2 be defined by $\mu_1(E) = \mu(E \cap [0,\delta))$ and $\mu_2(E) = \mu(E \cap [\delta, 1))$ for every Borel subset E of $[0,1)$ and define derivations q_1 and q_2 by $q_1(f) = df * \mu_1$ and $q_2(f) = df * \mu_2$ $(f \in V)$. Then

$$\|e^{-nd}qe^{nd}\| \leq \|e^{-nd}q_1e^{nd}\| + \|e^{-nd}q_2e^{nd}\|, \qquad n = 1, 2, \ldots. \tag{1}$$

Since for every $x \in [0,1)$,

$$\|e^{-nd}q_1e^{nd}(\delta_x)\| = \|x\delta_x * e^{-nd}\mu_1\| \leq \|e^{-nd}\mu_1\| < \varepsilon, \tag{2}$$

we have, by Proposition 1,

$$\|e^{-nd}q_1e^{nd}\| < \varepsilon. \tag{3}$$

Also for $x \in [0,1)$

$$\|e^{-nd}q_2e^{nd}(\delta_x)\| = x\int_{[0,1-x)} e^{-ny}\,d|\mu_2|(y) \leq e^{-n\delta}x\int_{[0,1-x)} d|\mu_2|(y). \tag{4}$$

Thus, by Proposition 1,

$$\|e^{-nd}q_2e^{nd}\| \leq e^{-n\delta}\sup\left\{x\int_{[0,1-x)} d|\mu_2|(y) : x \in [0,1)\right\} = e^{-n\delta}\|q_2\|, \tag{5}$$

where the last equality follows from the formula for the norm of a derivation [6, Theorem 2].

From (1), (3) and (5) we get $\lim_{n\to\infty} e^{-nd}qe^{nd} = 0$, and the proof is complete.

Lemma 5. *Suppose θ is a principal automorphism of V. Then there exists a quasinilpotent derivation q, such that*

$$\theta = \text{SO-}\lim(e^{-d}e^{q})^{n}e^{nd}.$$

Proof. The proof of Lemma 2 showed that for every principal automorphism θ there exists a quasinilpotent derivation q on V such that for every positive integer n,

$$\theta e^{-nd}\theta^{-1}e^{nd} = (e^{-d}e^{q})^{n}e^{nd}. \tag{1}$$

Hence by Lemma 3

$$\theta(f) = \lim_{n\to\infty}\theta(e^{-nd}\theta^{-1}e^{nd})(f) = \lim(e^{-d}e^{q})^{n}e^{nd}(f), \tag{2}$$

and the proof is complete.

The following result relates the convergence of an infinite product of operators to the absolute convergence of a related infinite series. The proof is almost the same as the proof for infinite products of complex numbers. However, we have been unable to find a reference for the result in the required form, so we include a proof, for completeness.

Proposition 2. *Suppose (T_n) is a sequence of bounded operators on a Banach space X and I is the identity operator on X. If the series $\sum\|T_n\|$ is convergent, then the sequence $P_n = (I+T_1)\cdots(I+T_n)$, $n = 1, 2, \ldots$, is convergent in the norm topology of the bounded linear operators on X.*

Proof. For $n = 1, 2, \ldots$, we have

$$\|P_n\| \le (1+\|T_1\|)(1+\|T_2\|)\cdots(1+\|T_n\|) \le e^{\|T_1\|+\cdots+\|T_n\|}.$$

Thus $(\|P_n\|)$ is bounded, by an upper bound M say. If $m > n$, then

$$\begin{aligned}\|P_m - P_n\| &= \|(I+T_1)\cdots(I+T_m) - (I+T_1)\cdots(I+T_n)\| \\ &\le \|(I+T_1)\cdots(I+T_n)\|\,\|(I+T_{n+1})\cdots(I+T_m) - 1\| \\ &\le M[(1+\|T_{n+1}\|)\cdots(1+\|T_m\|) - 1],\end{aligned}$$

by the norm inequality for the Banach algebras. Since $\sum\|T_n\| < \infty$ the infinite product $\prod_{n=1}(1+\|T_n\|)$ is convergent. Therefore $[(1+\|T_{n+1}\|)\cdots(1+\|T_m\|) - 1] \to 0$, as m and $n \to \infty$. Thus (P_n) is a Cauchy sequence and thus convergent.

Lemma 6. *Suppose θ is a principal automorphism of V. Let q be the quasinilpotent derivation related to θ as in Lemma 5, and let μ be the locally finite measure representing the derivation q by $q(f) = df * \mu$ $(f \in V)$. Then for each $0 < \delta < 1$, we have*

$$\int_{(0,\delta)} \frac{1}{x}\, d|\mu|(x) < \infty. \tag{†}$$

Proof. First we show that there is a j_0 such that one has

$$(1)\qquad \sum_{j=j_0}^{\infty} \|e^{-jd} q e^{jd}(\delta_x)\| < 2\|\theta\|\,\|\theta^{-1}\|$$

for every $x \in [0,1)$. By Lemma 4, $\lim_{j\to\infty} e^{-jd} q e^{jd} = 0$ (in norm) so there exists j_0 such that for $j \geq j_0$, $\|e^{-jd} q e^{jd}\| < 1/4$. We have

$$(2)\qquad \|(e^{-jd} e^{q} e^{jd} - I)(\delta_x)\| = \left\| e^{-jd} q e^{jd}(\delta_x) + \frac{e^{-jd} q^2 e^{jd}}{2!}(\delta_x) + \cdots + \frac{e^{-jd} q^k e^{jd}}{k!}(\delta_x) + \cdots \right\|.$$

For $j \geq j_0$ we then have

$$(3)\qquad \left\| \frac{e^{-jd} q^2 e^{jd}}{2!}(\delta_x) + \cdots + \frac{e^{-jd} q^k e^{jd}}{k!}(\delta_x) + \cdots \right\|$$
$$= \left\| \frac{e^{-jd} q e^{jd}}{2!}(e^{-jd} q e^{jd})(\delta_x) + \cdots + \frac{(e^{-jd} q e^{jd})^{k-1}(e^{-jd} q e^{jd})}{k!}(\delta_x) + \cdots \right\|$$
$$\leq \|e^{-jd} q e^{jd}(\delta_x)\| \left(\frac{1}{2!}\left(\frac{1}{4}\right) + \frac{1}{3!}\left(\frac{1}{4}\right)^2 + \cdots + \frac{1}{k!}\left(\frac{1}{4}\right)^{k-1} + \cdots \right)$$
$$< \frac{1}{2}\|e^{-jd} q e^{jd}(\delta_x)\|.$$

From (3) and

$$(4)\qquad \|e^{-jd} q e^{jd}(\delta_x)\| \leq \left\| e^{-jd} q e^{jd}(\delta_x) + \cdots + e^{-jd}\frac{q^k}{k!} e^{jd}(\delta_x) + \cdots \right\| + \left\| e^{-jd}\frac{q^2}{2!} e^{jd}(\delta_x) + \cdots + e^{-jd}\frac{q^k}{k!} e^{jd}(\delta_x) + \cdots \right\|$$

it follows,

$$(5)\qquad \|e^{-jd} q e^{jd}(\delta_x)\| < 2\left\| e^{-jd} q e^{jd}(\delta_x) + \cdots + e^{-jd}\frac{q^k}{k!} e^{jd}(\delta_x) + \cdots \right\| = 2\|(e^{-jd} e^{q} e^{jd} - I)(\delta_x)\|,$$

for $j \geq j_0$.

Thus to prove (1) it suffices to prove that

$$(6)\qquad \sum_{j=1}^{\infty} \|(e^{-jd} e^{q} e^{jd} - I)(\delta_x)\| < \infty.$$

To prove (6), from $\theta e^{-d}\theta^{-1} = e^{-d} e^{q}$ we get

$$(7)\qquad e^{-jd} e^{q} e^{jd} - I = e^{-(j-1)d}\theta e^{(j-1)d} e^{-jd}\theta^{-1} e^{jd} - I.$$

Hence,

$$\begin{aligned}(8)\quad &\|(e^{-jd}e^{q}e^{jd}-I)(\delta_x)\|\\ &=\|e^{-(j-1)d}\theta e^{(j-1)d}(e^{-jd}\theta^{-1}e^{jd}-e^{-(j-1)d}\theta^{-1}e^{(j-1)d})(\delta_x)\|\\ &\le \|e^{-(j-1)d}\theta e^{(j-1)d}\|\,\|(e^{-jd}\theta^{-1}e^{jd}-e^{-(j-1)d}\theta^{-1}e^{(j-1)d})(\delta_x)\|\\ &\le \|\theta\|\,\|(e^{-jd}\theta^{-1}e^{jd}-e^{-(j-1)d}\theta^{-1}e^{(j-1)d})(\delta_x)\|,\end{aligned}$$

by Lemma 1(b). Now since θ^{-1} is a principal automorphism (by Lemma 2) we have

$$\theta^{-1}(\delta_x)=\delta_x+\nu_x,\ \text{with } \alpha(\nu_x)\ge x,\ \text{and } \nu_x(\{x\})=0,\ \text{for every } x\in[0,1).$$

Thus

$$(9)\quad (e^{-jd}\theta^{-1}e^{jd}-e^{-(j-1)d}\theta^{-1}e^{(j-1)d})(\delta_x)=e^{jx}e^{-jd}\nu_x-e^{(j-1)x}e^{-(j-1)d}\nu_x.$$

Since for every Borel subset E of $[0,1)$

$$(10)\quad [e^{jx}e^{-jd}\nu_x-e^{-(j-1)x}e^{-(j-1)d}\nu_x](E)=\int_E[e^{-j(y-x)}-e^{-(j-1)(y-x)}]\,d\nu_x(y),$$

we have

$$\begin{aligned}(11)\quad &\|e^{jx}e^{-jd}\nu_x-e^{-(j-1)x}e^{-(j-1)d}\nu_x\|\\ &=\int_{(x,1)}[e^{-(j-1)(y-x)}-e^{-j(y-x)}]\,d|\nu_x|(y).\end{aligned}$$

From (8), (9) and (11) it follows

$$\begin{aligned}&\sum_{j=j_0}^{\infty}\|(e^{-jd}e^{q}e^{jd}-I)(\delta_x)\|\\ &\le\|\theta\|\sum_{j=j_0}^{\infty}\int_{(x,1)}[e^{-(j-1)(y-x)}-e^{-j(y-x)}]\,d|\nu_x|(y)\\ (12)\quad &=\|\theta\|\int_{(x,1)}\sum_{j=j_0}^{\infty}[e^{-(j-1)(y-x)}-e^{-j(y-x)}]\,d|\nu_x|(y)\\ &=\|\theta\|\int_{(x,1)}e^{-(j_0-1)(y-x)}\,d|\nu_x|(y)\\ &\le\|\theta\|\int_{(x,1)}d|\nu_x|(y)\le\|\theta\|\,\|\theta^{-1}(\delta_x)\|\le\|\theta\|\,\|\theta^{-1}\|.\end{aligned}$$

Therefore (1) holds.

Now to obtain the growth condition (†) we note that

$$\begin{aligned}(13)\quad \|(e^{-jd}qe^{jd})(\delta_x)\|&=e^{jx}\|e^{-jd}q(\delta_x)\|=xe^{jx}\int_{(x,1)}e^{-jy}\,d|\delta_x*\mu|(y)\\ &=xe^{jx}\int_{(x,1)}e^{-jy}\,d(\delta_x*|\mu|(y))=x\int_{(0,1-x)}e^{-jy}\,d|\mu|(y).\end{aligned}$$

From (12) and (13) and 5, for $x = 1-\delta$ we get

$$\sum_{j=j_0}^{\infty} \int_{(0,\delta)} e^{-jy}\, d|\mu|(y) < \frac{2}{1-\delta}\|\theta\|\,\|\theta^{-1}\|, \tag{14}$$

or equivalently,

$$\int_{(0,\delta)} \frac{e^{-j_0 y}}{1-e^{-y}}\, d|\mu|(y) < \frac{2}{1-\delta}\|\theta\|\,\|\theta^{-1}\|.$$

Since near 0, $e^{-j_0 y}/(1-e^{-y}) \sim 1/y$, we get the growth condition (†) for the measure μ.

Theorem. *If θ is an automorphism of V, then there exists a complex number λ, and a quasinilpotent derivation q, such that $\theta = e^{\lambda d} e^{q}$. Thus the group of automorphisms of V is connected in the operator norm topology.*

Proof. It suffices to assume that θ is a principal automorphism. By Lemma 5 we have $\theta = (\mathrm{SO})\text{-}\lim (e^{-d}e^{q})^{n} e^{nd}$, where q is not necessarily the same as the q in the statement of the Theorem. Since for each n, $(e^{-d}d^{q})^{n}e^{nd}$ belongs to the connected component of the identity (in the norm topology) and the connected component of the identity is closed, the proof will be complete if we show that the above limit exists in the operator norm topology. Now if $P_n = (e^{-d}e^{q})^{n}e^{nd}$, $n = 1, 2, \ldots$, then

$$\begin{aligned} P_n &= (e^{-d}e^{q}e^{d})(e^{-2d}e^{q}e^{2d})(e^{-3d}e^{q}e^{3d})\cdots(e^{-nd}e^{q}e^{nd}) \\ &= \exp(e^{-d}qe^{d})\exp(e^{-2d}qe^{2d})\cdots\exp(e^{-nd}qe^{nd}). \end{aligned} \tag{1}$$

For every j let $\exp(e^{-jd}qe^{jd}) = I + T_j$, where

$$T_j = e^{-jd}qe^{jd} + \frac{(e^{-jd}qe^{jd})^2}{2!} + \cdots + \frac{(e^{-jd}qe^{jd})^k}{k!} + \cdots. \tag{2}$$

Since $\lim_{j\to\infty} e^{-jd}qe^{jd} = 0$ (Lemma 4), there exists j_0 such that for $j > j_0$

$$\|T_j\| < 2\|e^{-jd}qe^{jd}\|. \tag{3}$$

For $n > j_0$ we then have

$$P_n = \left(\prod_{j=1}^{j_0} \exp(e^{-jd}qe^{jd})\right)(I + T_{j_0+1})\cdots(I+T_n). \tag{4}$$

Whence by (3) and Proposition 2 it suffices to show that

$$\sum_{j=j_0+1}^{\infty} \|e^{-jd}qe^{jd}\| < \infty. \tag{5}$$

Let q be represented by the locally finite measure μ, $q(f) = df * \mu$. Fix $0 < \delta < 1$. Then by Lemma 6, $\int_{(0,\delta)} \frac{1}{x}\, d|\mu|(x) < \infty$. Let μ_1 and μ_2 be defined by $\mu_1(E) = \mu(E\cap[0,\delta))$, $\mu_2(E) = \mu([\delta,1)\cap E)$, for every Borel subset E of $[0,1)$. Define q_1 and q_2 by $q_1(f) = df * \mu_1$ and $q_2(f) = df * \mu_2$, for every $f \in V$. Then $q = q_1 + q_2$. Thus

$$\text{(6)} \qquad \sum_{j=j_0+1}^{\infty} \|e^{-jd} q e^{jd}\| \le \sum_{j=j_0+1}^{\infty} \|e^{-jd} q_1 e^{jd}\| + \sum_{j=j_0+1}^{\infty} \|e^{-jd} q_2 e^{jd}\|.$$

We have

$$\begin{aligned} \text{(7)} \quad \|e^{-jd} q_2 e^{jd}\| &= \sup\{\|e^{-jd} q_2 e^{jd}(\delta_x)\| : x \in [0,1)\} \\ &= \sup\left\{x \int_{[\delta,1-x)} e^{-jy}\, d|\mu|(y) : x \in [0,1)\right\} \\ &\le e^{-j\delta} \sup\left\{x \int_{[\delta,1-x)} d|\mu|(y) : x \in [0,1)\right\} = e^{-j\delta}\|q_2\|. \end{aligned}$$

Thus

$$\sum_{j=j_0+1}^{\infty} \|e^{-jd} q_2 e^{jd}\| < \infty.$$

An analogous calculation shows that $\|e^{-jd} q_1 e^{jd}\| \le \int_{(0,\delta)} e^{-jy}\, d|\mu|(y)$. However the series

$$\sum_{j=1}^{\infty} \int_{(0,\delta)} e^{-jy}\, d|\mu|(y) = \int_{(0,\delta)} \sum_{j=1}^{\infty} e^{-jy}\, d|\mu|(y),$$

converges since μ satisfies the growth condition $\int_{(0,\delta)} \frac{1}{y}\, d|\mu|(y) < \infty$. Hence the series $\sum_{j=1}^{\infty} \|e^{-jd} q_1 e^{jd}\|$ converges, and the proof is complete.

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