A NEW 3-DIMENSIONAL SHRINKING CRITERION

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ABSTRACT. We introduce a new shrinking criterion for cell-like upper semicontinuous decompositions G of topological 3-manifolds, such that the embedding dimension (in the sense of Štan'ko) of the nondegeneracy set of G is at most one. As an immediate application, we prove a recognition theorem for 3-manifolds based on a new disjoint disks property.

1. INTRODUCTION

In 1979 M. Starbird developed two important shrinking criteria for 0-dimensional cell-like upper semicontinuous decompositions G of Euclidean 3-space E^3 , called DDPI and DDPII (stands for "the disjoint disks property") [St]: G is said to have the DDPI if for all disjoint tame disks $D_1, D_2 \subset E^3$ so that $\partial D_1 \cup \partial D_2$ misses the nondegenerate elements of G , and for every open set $V \subset E^3$ which contains all the elements of G intersecting both D_1 and D_2 , there are (1) a homeomorphism $g: E^3 \rightarrow E^3$ such that $g|_{E^3 - V} = \text{id}$ and (2) disks $D'_1, D'_2 \subset E^3$ obtained from $g(D_1)$ and $g(D_2)$, respectively, by replacement of subdisks so that each replacement subdisk used in getting from $g(D_i)$ to D'_i , $i = 1, 2$, lies in V and so that no element of G intersects both D'_1 and D'_2 . If one can always assume that already $g(D_i) = D'_i$, $i = 1, 2$, then G is said to have the DDPII. Starbird proved in [St] that such decompositions are always shrinkable. If one replaces E^3 by an arbitrary topological 3-manifold M , then Starbird's result can be generalized as follows: the quotient space M/G of the decomposition G is a 3-manifold if and only if G has DDPI [Re2].

In the present paper we propose a new shrinking criterion called the *resolution disjoint disks property* (RDDP): a cell-like upper semicontinuous decomposition G of a topological 3-manifold M is said to have the RDDP if for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every collection of k pairwise disjoint, tame embeddings $f_i: B^2 \rightarrow M$, there exist maps $g_i: B^2 \rightarrow X = M/G$ satisfying (i) $\rho(g_i, \pi f_i) < \varepsilon$;

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and (ii) for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \emptyset$. (One could define instead the RDDP as a property of the resolution $\pi: M \rightarrow X$ of the generalized 3-manifold X .) Our disjoint disks property applies to all those decompositions G whose nondegeneracy set N_G has embedding dimension (in the sense of M. A. Štan'ko [Št, Ed1]) at most one:

1.1. Shrinking criterion. *Let G be a cell-like, upper semicontinuous decomposition of a 3-manifold M such that $\dim N_G \leq 1$. Then G is shrinkable if and only if G has the RDDP.*

In the second part of this paper we apply this shrinking criterion to obtain another 3-dimensional recognition theorem. Recall that the recognition problem for topological n -manifolds asks for a list of simple geometric properties which a space X (usually assumed to be an ENR $\mathbb{Z}$ -homology n -manifold) should possess in order to be a genuine n -manifold [Ca]. (For a review of the history of this problem see the survey [Re1].) We introduce a new general position property for generalized 3-manifolds, called the *light map separation property* (LMSP): a metric space (X, ρ) is said to have the LMSP if for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every map $f: B \rightarrow X$ of a collection of k standard 2-cells $B = \coprod_{i=1}^k B_i^2$ into X such that: (i) $N_f \subset \text{Int } B$, where $N_f = \{y \in B \mid f^{-1}(f(y)) \neq y\}$; (ii) $\dim N_f \leq 0$; and (iii) $\dim Z_f \leq 0$, where $Z_f = \{x \in X \mid x \in f(B_i^2) \cap f(B_j^2) \text{ for some } i \neq j\}$; there exists a map $F: B \rightarrow X$ such that (1) $\rho(F, f) < \varepsilon$; (2) $F|_{\partial B} = f|_{\partial B}$; and (3) for every $i \neq j$, $F(B_i^2) \cap F(B_j^2) = \emptyset$. We first establish the (nontrivial) fact that every 3-manifold has the LMSP and then show that sometimes the LMSP can be applied to detect nonsingular spaces:

1.2. Recognition theorem. *A (metric) space X is a topological 3-manifold if and only if (i) X is the image under a proper cell-like map $f: M^3 \rightarrow X$, where M^3 is a 3-manifold and $\dim f(N_f) \leq 0$; and (ii) X has the LMSP.*

Edwards' celebrated shrinking theorem [Ed2] characterizes those cell-like maps $f: M^n \rightarrow X$ from an n -manifold ($n \geq 5$) to a finite-dimensional space that can be approximated by homeomorphisms, in terms of a disjoint disks property having the important quality of being measured solely in X . Striving to cast our results in the same vein, here we attempt to treat disjoint disks properties of X alone, unlike Starbird's, which pertain in a fundamental way to the domain and the explicit decomposition there. We are most successful with the LMSP, which certainly pertains just to X , while the RDDP is more of a hybrid, because the decomposition map is used to identify those singular disks that can be mapwise separated in X . Nevertheless, the RDDP has the useful feature, the significance of which is demonstrated in Edwards' argument, of being preserved under the operation of taking limits in the space of all (cell-like) maps $M \rightarrow X$. We should candidly acknowledge the negative side: that both the RDDP and the LMSP entail unpleasant complications by employing

an arbitrary finite number of domain 2-cells, unlike the properties of [Ca, Ed2, St], which involve merely a pair of domain 2-cells.

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2. PRELIMINARIES

We shall be working in the category of locally compact, metrizable spaces and continuous maps throughout the paper. Manifolds will be assumed to have no boundary unless specified. A space X is *cell-like* if there exists an n -manifold N and an embedding $f: X \rightarrow N$ such that $f(X)$ is *cellular* in N , i.e., $f(X) = \bigcap_{i=1}^{\infty} B_i^n$, where $\{B_i^n\}_{i \geq 1}$ is a properly nested decreasing sequence of n -cells in N . A map defined on an ANR X is *cell like* if its point-inverses are cell-like sets. A closed map is *proper* if its point inverses are compact. The *nondegeneracy set* of a map $f: X \rightarrow Y$ is the set $N_f = N(f) = \{x \in X \mid f^{-1}(f(x)) \neq x\}$. If for a subset $A \subset Y$, $N_f \cap f^{-1}(A) = \emptyset$, then we say that the map is *one-to-one* over A . A subset $Z \subset X$ is *locally simply coconnected* (1-LCC) if for every $x \in X$ and every neighborhood $U \subset X$ there is a neighborhood $V \subset U$ of x such that $\pi_1(V - Z) \rightarrow \pi_1(U - Z)$ is trivial.

Let G be a decomposition of a space X into compact subsets and let $\pi: X \rightarrow X/G$ be the corresponding quotient map, H_G the collection of all nondegenerate elements of G , and N_G their union (i.e., $N_G = N(\pi)$). A decomposition G is *upper semicontinuous* if π is a closed map. An upper semicontinuous decomposition G of a separable metrizable space X is *k-dimensional* if $\dim \pi(N_G) = k$, $k \in \mathbb{N}$.

A compactum $K \subset M^m$ in a PL m -manifold M has *embedding dimension* $\leq n$, written as $\text{dem } K \leq n$, if for every closed subpolyhedron $L \subset M$ with $\dim L \leq m - n - 1$, there exist arbitrarily small ambient PL isotopies of M with support arbitrarily close to $K \cap L$, which move L off K [Ed1, Št].

A compact, contractible 3-manifold with boundary C is a *fake 3-cell* if C is not a topological 3-cell. A topological space X satisfies *Kneser finiteness* if no compact subset of X contains more than a finite number of pairwise disjoint fake 3-cells.

A space X is a *generalized n -manifold* ($n \in \mathbb{N}$) if (i) X is a euclidean neighborhood retract (ENR), i.e., for some integer m , X embeds in E^m as a retract of an open subset of E^m ; and (ii) X is a homology n -manifold, i.e., for every $x \in X$, $H_*(X, X - x; \mathbb{Z}) \cong H_*(E^n, E^n - 0; \mathbb{Z})$. In dimension ≥ 3 X may fail to be locally euclidean at some (or perhaps all) points. We call such exceptions *singularities* of X and they form the *singular set* of X , $S(X) = \{x \in X \mid x \text{ has no neighborhood in } X \text{ homeomorphic to } E^n\}$. Note that $S(X)$ is always closed and if $S(X) \neq X$ then $M(X) = X - S(X)$ is an open n -manifold. A *resolution* of an n -dimensional ANR X is a pair (M, f)

consisting of a topological n -manifold M and a proper, surjective cell-like map $f: M \rightarrow X$. Consequently, if X has a resolution then X is a generalized n -manifold [La1]. A resolution (M, f) of X can always be assumed to be *conservative*, i.e., the map f is one-to-one over $M(X)$. (See [BrLa] for $n = 3$, [Qu] for $n = 4$, and [Si] or [Ed2] for $n \geq 5$.)

Let X be a generalized 3-manifold with 0-dimensional singular set and let $p \in X$. Then p has arbitrarily small neighborhoods $N \subset X$ such that $\partial N \cap S(X) = \emptyset$ and ∂N is a closed orientable surface of some genus $n \geq 0$. If this n can always be $\leq m$, but not also $\leq m - 1$, then we say that the *genus of X at p* is equal to m , $g(X, p) = m$ [La2].

3. PROOF OF THE SHRINKING CRITERION

3.1. Lemma. *Let M be a 3-manifold and G a cell-like, upper semicontinuous decomposition of M such that G has the RDDP and $\text{dem } N_G \leq 1$. Then each $g \in G$ is cellular in M .*

Proof. Since $\text{dem } N_G \leq 1$, each $g \in G$ has a neighborhood in M embeddable in E^3 [Ar, Lemma (5.3)]. Use the RDDP to prove that for every $x \in M/G$, $\{x\}$ is 1-LCC embedded in M/G . Therefore each $g = \pi^{-1}(x) \in G$ satisfies McMillan's Cellularity Criterion (see [Da, Corollary 18.4A]) and is thus cellular in M [Mc1].

3.2. Lemma. *Let G be a cell-like upper semicontinuous decomposition of a topological 3-manifold M with the RDDP and let $\pi: M \rightarrow M/G$ be the corresponding quotient map. Then for every $\varepsilon > 0$ and every finite collection $f_1, \dots, f_k: B^2 \rightarrow M$ of pairwise disjoint, tame embeddings satisfying $\pi f_i(\partial B^2) \cap \pi f_j(B^2) = \emptyset$ whenever $i \neq j$, there exist maps $g_1, \dots, g_k: B^2 \rightarrow M/G$ satisfying:*

- (i) *for every i , $\rho(g_i, \pi f_i) < \varepsilon$;*
- (ii) *for every i , $g_i|_{\partial B^2} = \pi f_i|_{\partial B^2}$; and*
- (iii) *for every $i \neq j$, $g_i(B^2) \cap g_j(B^2) = \emptyset$.*

Proof. This follows in a straightforward fashion by imposing motion controls and using either the fact that M/G is 1-LC or the combination of M/G being an ANR and a controlled version of the Borsuk Homotopy Extension Theorem [Bo].

We shall first prove the 0-dimensional special case of the Shrinking Criterion (1.1).

3.3. Lemma. *Let G be a 0-dimensional cell-like upper semicontinuous decomposition of 3-manifold M such that $\text{dem } N_G \leq 1$. Then G is shrinkable if and only if G has the RDDP.*

Proof. The only if direction is obvious so we prove the other implication. Let $W \subset M$ be an open set containing N_G . It suffices to show that, for any

two disjoint 2-cells B_1 and B_2 locally flatly embedded in M , there exists an embedding $h: B_1 \cup B_2 \rightarrow M$ such that h moves no point outside W , $h(W \cap (B_1 \cup B_2)) \subset W$, and $\pi h(B_1) \cap \pi h(B_2) = \emptyset$, (see Lemma 3.1 and [Št]).

Use the embedding dimension hypothesis to adjust the given 2-cells B_1 and B_2 slightly (with controls in M , not just in $X = M/G$) to achieve $(\partial B_1 \cup \partial B_2) \cap N_G = \emptyset$ and $\dim((B_1 \cup B_2) \cap N_G) \leq 0$.

Set $Z = \pi(B_1) \cap \pi(B_2)$. For every $x \in Z$ choose a neighborhood U_x with $U_x \subset \pi(W)$ and $U_x \cap \pi(\partial B_1 \cup \partial B_2) = \emptyset$. Since $Z \subset \pi(N_G)$, a 0-dimensional set, it is possible to extract a cover $\{V_i\}$ of Z refining $\{U_x\}$ and consisting of mutually exclusive open sets.

Find collections $D_1, \dots, D_k$ (resp., $E_1, \dots, E_n$) of pairwise disjoint 2-cells in B_1 (resp., B_2) whose interiors cover $\pi^{-1}(Z) \cap B_1$ (resp., $\pi^{-1}(Z) \cap B_2$), whose boundaries miss N_G , and whose images under π are contained in some element of the cover $\{V_i\}$. Note that for $i = 1, \dots, k$, $\pi(\partial D_i)$ misses all the other singular disks $\pi(D_j)$. Hence, it is possible to extract a subdisk D_i^* of $\text{Int } D_i$ whose boundary again misses N_G , whose interior contains $D_i \cap \pi^{-1}(Z)$, and which is large enough that $\pi(D_i - D_i^*)$ misses the other sets $\pi(D_j)$ (it necessarily misses $\bigcup \pi(E_j)$). Let E_i^* denote a subdisk of E_i with similar properties. Choose additional disks $D_{k+1}, \dots, D_K$ and $E_{n+1}, \dots, E_N$ in B_1 and B_2 disjoint from the others and subject to the same size controls, such that

$$\bigcup_{i=1}^k \pi^{-1} \pi(D_i^*) \subset \bigcup_{j=1}^K \text{Int } D_j$$

and similarly for the disks E_i^* and E_j . Moreover, for $j \in \{k+1, \dots, K\}$ require D_j to be contained in every $\pi^{-1}(V_i)$ such that some $D_s \subset \pi^{-1}(V_i)$, where $s \in \{1, \dots, k\}$ and $\pi(D_s) \cap \pi(D_j) \neq \emptyset$, and require the same of the additional disks E_j in B_2 . Define

$$P_1 = B_1 - \bigcup_{j=1}^K \text{Int } D_j \quad \text{and} \quad P_2 = B_2 - \bigcup_{j=1}^N \text{Int } E_j.$$

Choose a positive number δ less than each of the following:

$$\begin{aligned} & \rho(\pi(P_1), \pi(D_i^*)); \\ & \rho(\pi(P_2), \pi(E_i^*)); \\ & \rho(\pi(D_i), X - V_s), \quad \text{where } V_s \text{ contains } \pi(D_i); \\ & \rho(\pi(E_j), X - V_t), \quad \text{where } V_t \text{ contains } \pi(E_j); \\ & \rho\left(\pi\left(B_1 - \bigcup \text{Int } D_i^*\right), \pi(B_2)\right); \text{ and} \\ & \rho\left(\pi(B_1), \pi\left(B_2 - \bigcup \text{Int } E_i^*\right)\right). \end{aligned}$$

Apply Lemma 3.2 to approximate $\pi|(\bigcup D_i^*) \cup (\bigcup E_i^*)$ by a map f agreeing with π on the various boundaries, with f δ -close to the restricted π , and with

the images under f of these disks pairwise disjoint. Use the cell-likeness of π to approximately lift f to a map F of the same domain back into M , with F the identity on $\partial((\bigcup D_i^*) \cup (\bigcup E_i^*))$. There is no loss of generality in then assuming f was obtained with $\pi F = f$. Extend F to $F: B_1 \cup B_2 \rightarrow M$ via the inclusion elsewhere.

Now the idea is to invoke Dehn's Lemma for replacing F on each of the disks D_i (resp., E_i) by an embedding with the sort of properties allowing global reconstitution of B_1 (resp., B_2). The size controls above ensure that $\pi F(B_1) \cap \pi F(B_2) = \emptyset$ and that no two of the disks $f(D_i)$ (nor of the disks $f(E_i)$) intersect. According to Dehn's Lemma, there are tame disks $d_1, \dots, d_k$ and $e_1, \dots, e_n$ in $M - (P_1 \cup F(B_2)), M - (P_2 \cup F(B_1))$, respectively, with $\partial d_i = \partial D_i$ ($\partial e_i = \partial E_i$), with pairwise disjoint images under π , and with each image in the same element of $\{V_j\}$ as $\pi F(D_i)$ (or $\pi F(E_i)$). Now do disk trading, adjusting d_i and e_i , and remove all intersections of the resulting disks d'_i (resp., e'_i) with $D_{k+1}, \dots, D_K$ (resp., $E_{n+1}, \dots, E_N$). Then

$$B' = \left(B_1 - \bigcup_{i=1}^k D_i \right) \cup \bigcup_{i=1}^k d'_i \quad \text{and} \quad B'' = \left(B_2 - \bigcup_{i=1}^n E_i \right) \cup \bigcup_{i=1}^n e'_i$$

are 2-cells in M whose images under π are disjoint.

The desired homeomorphism $h: B_1 \cup B_2 \rightarrow B' \cup B'' \subset M$ is one sending D_i onto d'_i (E_i onto e'_i) and reducing to the identity elsewhere.

3.4. Lemma. *Let G be a cellular upper semicontinuous decomposition of a topological 3-manifold M such that G has the RDDP and $\text{dem } N_G \leq 1$. Let $A \subset M/G$ be a closed subset and denote by G_A the decomposition induced over A , i.e., $G_A = \{\pi^{-1}(a) \mid a \in A\} \cup \{\{x\} \mid x \in M - \pi^{-1}(A)\}$, where $\pi = M \rightarrow M/G$ is the quotient map. Then G_A is also upper semicontinuous, cellular, and has the RDDP.*

The proof is a routine lifting argument which exploits the induced cell-like map $p: M/G_A \rightarrow M/G$. One can work with disks $D_1, \dots, D_k$ in M for which $N_G \cap (\bigcup D_i)$ is 0-dimensional and obtain motion control in M/G_A by only lifting images of those 2-simplexes σ in some small mesh triangulation for which $\sigma \cap N_{G_A} \neq \emptyset$.

3.5. Lemma. *Let G be a cell-like decomposition of a 3-manifold M such that G has the RDDP, and let $\{h_i: M \rightarrow M \mid i = 1, 2, \dots\}$ be a sequence of homeomorphisms of M onto itself such that $\pi h_i: M \rightarrow M/G$ converges uniformly to a map $p: M \rightarrow M/G$. Then the decomposition $G_p = \{p^{-1}(x) \mid x \in M/G\}$ induced by p has the RDDP.*

Proof. Consider any collection of k pairwise disjoint, tame embeddings $f_i: B^2 \rightarrow M$. Given $\varepsilon > 0$, choose j sufficiently large that $\rho(\pi h_j, p) < \varepsilon/2$. Applying the RDDP to the embeddings $h_j f_i$ ($i = 1, \dots, k$), one can find maps

$g_i: B^2 \rightarrow M/G$ having pairwise disjoint images and satisfying $\rho(g_i, \pi h_j f_i) < \varepsilon/2$. Clearly then $\rho(g_i, p f_i) < \varepsilon$.

Proof of (1.1). The only if direction is obvious so we prove the other implication. By [KoWa], $\dim Y = 3$ where $Y = M/G$. For classical reasons (see [Wa]), $\pi(N_G)$ is 1-dimensional. Hence, Y contains a 2-dimensional F_σ -set F such that $\dim(Y - F) = \dim(F \cap \pi(N_G)) = 0$. Express F as the union of compacta $A_i \subset Y$, $i \in \mathbb{N}$.

By construction and Lemmas 3.3 and 3.4, the decompositions G_i induced over A_i are shrinkable. As in [Ed2] (see [Da, Chapter 24]), $\pi: M \rightarrow Y$ can be approximated by a proper cell-like map $p: M \rightarrow Y$ such that p is one-to-one over F and $\text{dem } N_p \leq 1$ (p arises as the limit of maps p_i , where p_i is one-to-one over A_i ; given a sequence of triangulations T_j of M with mesh $T_j \rightarrow 0$ as $j \rightarrow \infty$, and $T_j^{(1)} \cap N_G = \emptyset$ (where $T_j^{(1)}$ is the 1-skeleton of T_j), one can choose p_i to be one-to-one over $p(T_j^{(1)})$, $1 \leq j \leq i$, and can impose controls so p is one-to-one over both $F = \bigcup_{i=1}^\infty A_i$ and $\bigcup_{i=1}^\infty p(T_i^{(1)})$. With Lemma 3.5 certifying that the 0-dimensional decomposition G_p has the RDDP, another application of Lemma 3.3 shows that p can be approximated by homeomorphisms. Thus, the same is true of π , or, equivalently, G is shrinkable [Da, Chapter 5].

3.6. Corollary. *Let X be a generalized 3-manifold with a resolution $\pi: M \rightarrow X$ such that $\text{dem } N_\pi \leq 1$. Then X is a topological 3-manifold if and only if the decomposition $G = \{\pi^{-1}(x) \mid x \in X\}$ of M has the RDDP.*

4. PROOF OF THE RECOGNITION THEOREM

4.1. Lemma. *Let G be a 0-dimensional cell-like upper semicontinuous decomposition of a 3-manifold M such that the quotient space M/G has the LMSP and each $g \in G$ has a neighborhood in M embeddable in E^3 . Then G has the RDDP.*

Proof. By [DaRo], we may assume that $\text{dem } N_G \leq 1$. Given any finite collection of pairwise disjoint, tame embeddings $f_i: B^2 \rightarrow M$ we can adjust them slightly, in M , so that $\dim(N_G \cap (\bigcup_{i=1}^k f_i(B^2))) \leq 0$. Then the map $f: B \rightarrow M/G$ given by $f = \coprod_{i=1}^k \pi f_i$, where $B = \coprod_{i=1}^k B_i^2$, defines the kind of map to which the LMSP applies, leading to a map $F: B \rightarrow M/G$ which shows G has the RDDP.

4.2. Lemma. *Every 3-manifold M has the LMSP.*

Proof. Consider a map $f: B \rightarrow M$ satisfying the hypotheses of LMSP and $\varepsilon > 0$. Using the hypothesis that $Z(f)$ is 0-dimensional, we successively determine compact 3-manifolds with boundary R, Q , and P satisfying:

- (1) each component of R has diameter less than ε ;
- (2) $\text{Int } R \supset Q \supset \text{Int } Q \supset P \supset \text{Int } P \supset Z(f)$;

- (3) $\pi_1(P) \rightarrow \pi_1(Q)$ is trivial;
 (4) B has a triangulation $\mathcal{T}$ with 1-skeleton

$$\Gamma \subset B - [N(f) \cup f^{-1}(P)];$$

- (5) For each 2-simplex $\sigma \in \mathcal{T}$, $f(\sigma) \cap P \neq \emptyset$ implies $f(\sigma) \subset \text{Int } Q$, and $f(\sigma) \cap Q \neq \emptyset$ implies $f(\sigma) \subset \text{Int } R$.

The correct procedure is first to select R , Q , then to find $\mathcal{T}$ with $\Gamma \cap N(f) = \emptyset$, with $f(\sigma) \cap Z(f) \neq \emptyset$ implying $f(\sigma) \subset \text{Int } Q$ and with $f(\sigma) \cap Q \neq \emptyset$ implying $f(\sigma) \subset \text{Int } R$, and finally to identify P subject to (2)–(5).

We will verify the LMSP by adjusting f to a new map $F: B \rightarrow M$ such that F agrees with f on $\Gamma \cup [B - f^{-1}(R)]$, any two distinct points $F(x), f(x)$ belong to a component of R , and the images under F of the various disks B_i^2 comprising B are pairwise disjoint. In the course of these map adjustments we will also modify P , without changing Q or R , always maintaining (1)–(5) above. In particular, F will coincide with f on all 2-simplexes σ for which $f(\sigma) \cap Q = \emptyset$.

To get started, use the Simplicial Approximation Theorem and general position to make the map f PL on $f^{-1}(\text{Int } Q) - \Gamma$, without changing f on Γ , in order to achieve the following:

- (6) f is transverse to the 2-manifold ∂P .

Consider now the finite collection $\mathcal{C} = \{J \mid J \text{ is a simple closed curve from } f^{-1}(\partial P)\}$. Let $c(\partial P)$ be the complexity of ∂P defined by McMillan [Mc2],

$$c(\partial P) = \sum_{p \geq 0} (p+1)^2 g(p),$$

where $g(p)$ denotes the number of components of ∂P of genus p .

We show how to reduce $c(\partial P)$ to a minimum in a finite number of cut-and-paste operations, after which we obtain the map F by carefully trading singular disks in a modified $f(B)$ for others near ∂P .

For $L \in \mathcal{C}$ name the disk B_i^2 such that $L \subset B_i^2$, and let E_L denote the subdisk of B_i^2 bounded by L . Assume L is an innermost curve with respect to B_i^2 . There are three cases to consider.

Case I. $f(L) \not\subset *$ on ∂P and $f(E_L) \subset P$. Then apply the Loop Theorem to find an embedded disk $H \subset P - f(\Gamma)$ such that $H \cap \partial P = \partial H$ and $\partial H \cap f(\bigcup_{j \neq i} B_j^2) = \emptyset$. Thicken H to a 3-cell $C = H \times I$ in $\text{Int } P$ for which $H = H \times \{1/2\}$ and $(\partial H) \times I \subset \partial P - f(\bigcup_{j \neq i} B_j^2)$. Redefine f on those 2-simplexes σ of $\bigcup_{j \neq i} B_j^2$ whose images meet C to eliminate such intersections, starting with innermost curves in the domain, so $f(\sigma) \subset Q - C$ and all new images lie in $P - C$. Make a compression of ∂P along H , forming a new P' in $P - \text{Int } C$. This operation maintains conditions (1)–(5), and the redefinition of f ensures that the new singular set satisfies $Z(f) \subset \text{Int } P'$.

Case II. $f(L) \neq *$ on ∂P and $f(E_L) \subset \text{Int } Q - \text{Int } P$. Consequently, $f(E_L) \cap Z(f) = \emptyset$. Again use the Loop Theorem to obtain an embedding disk $H \subset \text{Int } Q - f(\Gamma)$ such that $H \cap P = H \cap \partial P = \partial P$ and $\partial H \cap f(\bigcup_{j \neq i} B_j^2) = \emptyset$. Thicken H to a 3-cell $C = H \times I$, as before, with $C \subset \text{Int } Q$ disjoint from $f(\Gamma) \cup f(\bigcup B_j^2)$ and with $C \cap \partial P = (\partial H) \times I$. Redefine f on those 2-simplexes σ of B_i^2 for which $f(\sigma) \cap P = \emptyset$ but $f(\sigma) \cap C \neq \emptyset$, in particular, on those where $f(\sigma) \not\subset Q$, so the new images lie in $\text{Int } R - P \cup [\text{Int } C \cup f(\bigcup_{j \neq i} B_j^2)]$. Make a compression of ∂P along H , forming a new P' in $P \cup \text{Int } C$. This operation also maintains conditions (1)–(5), and here the redefinition of f is indispensable for obtaining (5).

Remark. According to [Mc2], $c(\partial P') < c(\partial P)$ in both Case I and Case II.

Case III. $f(L) \simeq *$ on ∂P . Consider the universal cover $p: \mathbf{H}^2 \rightarrow S$ where $S \subset \partial P$ is the component of ∂P containing $f(L)$. Then the loop $f(L)$ lifts into $\mathbf{H}^2$ as a collection of loops. Take one such lift $\gamma \subset p^{-1}f(L)$. It separates $\mathbf{H}^2$ into finitely many components $K_1 \dots, K_{r+1}$ where only the closure of K_{r+1} is noncompact.

Subcase III.a. For every $1 \leq t \leq r$, $f(\bigcup_{j \neq i} B_j^2) \cap p(K_t) = \emptyset$. Then we can cut $f(E_L)$ off from ∂P near $p(\bigcup_{t=1}^r K_t)$, eliminating the simple closed curve L from the collection $f^{-1}(\partial P)$, without introducing new singularities or new intersections with P . In light of the next subcase it is worth emphasizing that here L need not be innermost in B .

Subcase III.b. For some $1 \leq t \leq r$, $f^{-1}(f(\bigcup_{j \neq i} B_j^2) \cap p(K_t))$ contains a simple closed curve L' . Then L' has a special lift γ' to K_t , implying that L' falls under Case III and all but one of the components of $H^2 - \gamma'$ lie in K_t . Eventually we obtain a curve L' (not necessarily innermost with respect to B) for which Subcase III.a applies.

Finally, when $c(\partial P)$ is minimal, all curves $L \in \mathcal{C}$ must fall under Case III, which shows how they can all be eliminated via a new map $F: B \rightarrow M$ with $Z(F) \subset \text{Int } P$ and $F(B) \cup P = \emptyset$. The subsequent images of the various disks B_i^2 are pairwise disjoint, as required.

Proof of (1.2). The forward implication follows immediately from Lemma 4.2. We concentrate on the reverse implication, where by [KoWa] X is 3-dimensional and thus by [La1] it is a generalized 3-manifold. Let $G = \{f^{-1}(x) \mid x \in X\}$ be the associated cell-like upper semicontinuous decomposition of M . Since $\dim \pi(N_G) = \dim f(N_f) \leq 0$, G is 0-dimensional.

Let $C_0 = \bigcup \{g \in G \mid g \text{ has no neighborhood in } M \text{ embeddable in } E^3\}$. Then by [ReLa2] the set $f(C_0)$ is locally finite in X . Let G_0 denote the (cell-like) decomposition of M consisting of the components of C_0 and the

singletons from $M - C_0$. Consider $M_1 = M/G_0$ and the associated decomposition $G_1 = \pi_0(G) = \{\pi_0(g) \mid g \in G\}$ of M_1 , where $\pi_0: M \rightarrow M_1$ is the quotient map. Clearly M_1 is a generalized 3-manifold and $S(M_1) \subset \pi_0(C_0)$.

Assertion. $X - f(C_0)$ is a 3-manifold.

Proof. Every $g' \in G'_1 = \{g \in G_1 \mid g \subset M_1 - \pi_0(C_0)\}$ has a neighborhood in $M'_1 = M_1 - \pi_0(C_0)$ embeddable in E^3 . Let $\pi_1: M_1 \rightarrow X$ be the quotient map of the decomposition G_1 , and set $\pi'_1 = \pi_1 \mid M'_1$. So (M'_1, π'_1) is a resolution of $X' = X - f(C_0)$. Since X has the LMSP, so does X' . Hence Lemma 4.1 applies, implying G'_1 has the RDDP, and by Lemma 3.3 G'_1 is shrinkable. This confirms the assertion.

By [BrLa] we can assume that f is one-to-one over X' . Based on LMSP and the existence of f , it is a simple matter to verify that each $f(c) \in f(C_0)$ is 1-LCC embedded in X (see the proof of [ReLa1, Theorem 3.1]). By Theorem 4 of [BrLa], X is a 3-manifold.

By way of application we have another recognition theorem:

4.3. Corollary. *A space X is a 3-manifold if and only if it satisfies the following properties:*

- (i) *each $x \in X$ is 1-LCC embedded in X ;*
- (ii) *X admits a resolution $\pi: M^3 \rightarrow X$ defined on a 3-manifold;*
- (iii) *$S(X)$ is contained in a finite graph Γ (topologically) embedded in X ;*
- (iv) *X has the LMSP.*

Proof. In case X satisfies properties (i)–(iv), we can assume the resolution $\pi: M^3 \rightarrow X$ is one-to-one over $X - \Gamma$. Let

$$E = \{x \in X \mid \pi^{-1}(x) \text{ has no neighborhood that embeds in } E^3\}.$$

For the reasons set forth in the proof of (1.2), E is a discrete subset of X . Select a countable dense subset D of $\Gamma - E$. As in the proof of Lemma 3.1, each $\pi^{-1}(d)$, $d \in D$, is cellular in M ; consequently, we can approximate π by a cell-like map $f: M^3 \rightarrow X$ such that f is one-to-one over $D \cup (X - \Gamma)$. This verifies that X satisfies the conditions of (1.2), which in turn shows X is a 3-manifold.

Virtually the identical argument yields the next result, an improvement to Corollary 4.3.

4.4. Corollary. *A space X is a 3-manifold if and only if it satisfies the following properties:*

- (i) *X admits a resolution $\pi: M^3 \rightarrow X$ defined on a 3-manifold;*
- (ii) *each $s \in S(X)$ has arbitrarily small neighborhoods whose frontiers B_s are such that $\dim[B_s \cap S(X)] \leq 0$ and $B_s \cap S(X)$ is 1-LCC embedded in X ;*
- (iii) *X has the LMSP.*

5. EPILOGUE

We close by spelling out some unresolved issues. The first pertains to potential improvements to Shrinking Theorem (1.1).

5.1. Conjecture. *If $\pi: M^3 \rightarrow X$ is a resolution of X with the RDDP, then X is a 3-manifold.*

The fundamental difficulty occurs in examining decompositions induced over closed subsets.

5.2. Conjecture. *If G is a cell-like decomposition of a 3-manifold M such that G has the RDDP and if A is a subset of M/G , then the decomposition G_A induced over A has the RDDP.*

In our attempts to improve on the Recognition Theorem (1.2), we repeatedly encountered some form of the problem stated below.

5.3. Conjecture. *Every 3-manifold has the $LMSP^*$, where $LMSP^*$ stands for the $LMSP$ without any hypothesis on the set $Z(f)$.*

Only if Conjecture 5.3 is true does the next one make sense.

5.4. Conjecture. *A space X is a 3-manifold if X has the $LMSP^*$ and it admits a resolution $\pi: M^3 \rightarrow X$ defined on a 3-manifold.*

Finally, it seems that a stronger result than 5.3 might be valid. Compare with [An].

5.5. Conjecture. *Let $f: S^2 \rightarrow M$ be a map of a 2-sphere into a 3-manifold such that N_f is 0-dimensional. Then for each $\varepsilon > 0$ there exists an embedding $F_\varepsilon: S^2 \rightarrow M$ such that $\rho(F_\varepsilon, f) < \varepsilon$.*

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