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$\bar{\partial}_b$-EQUATIONS ON CERTAIN UNBOUNDED WEAKLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We found an explicit closed formula for the relative fundamental solution of $\bar{\partial}_b$ on the surface $H_k = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = |z_1|^{2k}\}$. We then make estimates of the relative fundamental solution in terms of the nonisotropic metric associated with the surface. The estimates lead us to the regularity results. We also study the problem of finding weights ω so that $\bar{\partial}_b$ as an operator from L^2_ω to L^2 has a closed range. We find the best possible weight among radial weights.

1. INTRODUCTION

Let Ω be a weakly pseudoconvex domain in $\mathbb{C}^2$ with a smooth boundary $\partial\Omega$, and let r be its defining function, i.e., $\Omega = \{z \in \mathbb{C}^2 | r(z) < 0\}$ and $dr(z) \neq 0$ when $r(z) = 0$. Then we can define a holomorphic vector field which is tangential to $\partial\Omega$ by

$$L = \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}. \tag{1.1}$$

Ω is said to have a finite type if a finite number of commutators of $\operatorname{Re}(L)$ and $\operatorname{Im}(L)$ together with $\operatorname{Re}(L)$ and $\operatorname{Im}(L)$ spans the tangent space to $\partial\Omega$ at every point [NSW1]. The tangential Cauchy Riemann operator, denoted by $\bar{\partial}_b$, can be abstractly defined as a restriction of the Cauchy Riemann operator $\bar{\partial}$ [FoK]. In this paper, we adopt the following analytic definition because of its relation to L. In terms of local coordinates, $\partial\Omega$ can be represented as

$$\operatorname{Im} z_2 = \phi(z_1, \operatorname{Re} z_2)$$

for some smooth function ϕ. Then, $\bar{\partial}_b$ is locally defined by

$$\bar{\partial}_b u = (\bar{L}u)d\bar{z}_1 \tag{1.2}$$

where L is as defined in (1.1) with $r(z) = \phi(z_1, \operatorname{Re} z_2) - \operatorname{Im} z_2$. We denote by $L^2(\partial\Omega)$ the square integrable functions on $\partial\Omega$ and by $H^2(\partial\Omega)$ the functions in $L^2(\partial\Omega)$ annihilated by $\bar{L}$. Then, the orthogonal projection from $L^2(\partial\Omega)$ to $H^2(\partial\Omega)$ is called the Szegö projection.

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What we are mainly concerned with in this paper is solving

$$\bar{\partial}_b u = g$$

for a (0,1) form g on $\partial\Omega$, or equivalently,

$$\overline{L}u = f \tag{1.3}$$

for $f \in L^2$, so that

$$u \perp H^2(\partial\Omega) \tag{1.4}$$

when Ω is a certain unbounded domain in $\mathbb{C}^2$.

When Ω is a bounded pseudoconvex domain of finite type in $\mathbb{C}^n$, many results have been obtained. First of all, the existence of a smooth solution for a smooth data was proved by J.-P. Rosay and the existence and regularity in L^2 and Sobolev spaces were achieved by H. Boas, J. J. Kohn, and M. Shaw [BS, K2, R, S]. Quite recently, an optimal Hölder estimate of the solution when $n = 2$ was established by C. Fefferman and J. J. Kohn, and independently by M. Christ under the assumption that $\bar{\partial}_b$ has a closed range [C, FK]. In particular, a microlocal analysis of a certain subelliptic operator was used in both works, and M. Christ made pointwise estimates of the relative fundamental solutions in terms of nonisotropic pseudometric defined by A. Nagel, E. Stein, and S. Wainger [NSW1].

If Ω is not bounded, the situation is quite different. In particular, the range of $\bar{\partial}_b$ is not closed in L^2 and the solution operator is not bounded in L^2.

The domain we will be working on is $\Omega = \{(z_1, z_2)\colon \operatorname{Im} z_2 > |z_1|^{2k}\}$ where k is a positive integer. It is easy to see that Ω is a pseudoconvex domain of finite type. We denote the boundary of Ω by H_k. On H_k the tangential Cauchy Riemann operator $\bar{\partial}_b$ is globally defined by $\bar{\partial}_b u = (\overline{L}u)d\bar{z}_1$ where $L = \frac{\partial}{\partial z_1} + ik\bar{z}_1|z_1|^{2(k-1)}\frac{\partial}{\partial z_2}$. If we make a change of variables $z_1 \mapsto z$ and $\operatorname{Re} z_2 \mapsto t$, then

$$L = \frac{\partial}{\partial z} + ik\bar{z}|z|^{2(k-1)}\frac{\partial}{\partial t}. \tag{1.5}$$

If $k = 1$, then L is the well known Lewy operator on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$.

This paper mainly consists of three parts: finding the relative fundamental solution, its estimates, and regularities of the solution. And in the process of doing these, we introduce a weight ω which makes the range of $\bar{\partial}_b$ closed in the weighted L^2-space L^2_ω.

H_1 after coordinate changes is a group (called the Heisenberg group) with the group structure $(z, t)\cdot(w, s) = (z + w, t + s + 2\operatorname{Im}(z\bar{w}))$. See [N and FS] for the properties of the Heisenberg group. In [GKS], Greiner, Kohn, and Stein found the relative fundamental solution for the boundary Kohn Laplacian $\overline{L}L$. From this one can easily see the relative fundamental solution of L:

$$K(z, t) = \frac{1}{\pi^2}\frac{z}{|z|^4 + t^2}.$$

In this paper, we generalize this result and obtain an explicit formula for the relative fundamental solution of L for arbitrary k. We find in Theorem 3.1 that the relative fundamental solution K is given by

$$K((z,t),(w,s)) = \frac{1}{4\pi^2 B}\left[\frac{z}{A^{1/k} - z\bar{w}} + \frac{w}{\overline{A}^{1/k} - \bar{z}w}\right] \tag{1.6}$$

where $A = \frac{1}{2}[|z|^{2k} + |w|^{2k} - i(t-s)]$ and $B = \frac{1}{2}[|z|^{2k} - |w|^{2k} + i(t-s)]$ (We work with L instead of $\overline{L}$ because of its relation to the Szegö projection.) We can see from (1.6) that K is homogeneous of degree $-2k-1$ with respect to the dilation $\delta(z,t) = (\delta z, \delta^{2k} t)$ on H_k. More precisely,

$$K(\delta(z,t),\delta(w,s)) = \delta^{-2k-1}K((z,t),(w,s)).$$

By considering the homogeneity of the relative fundamental solution K, it is evident that we can not expect the L^2-boundedness of the solution operator without a weight. Roughly speaking, the solution operator increases the homogeneity by 1 and hence causes trouble at ∞. Therefore, we introduce a weight to deal with this difficulty at ∞. The weight we introduce is

$$\omega(z,t) = (1 + d((0,0),(z,t)))^{-2} \tag{1.7}$$

where

$$d((z,t),(w,s)) = \left|\left(\frac{|z|^{2k} + |w|^{2k}}{2} + i\frac{(s-t)}{2}\right)^{1/k} - z\bar{w}\right|^{1/2} \tag{1.8}$$

which is introduced by K. Diaz in [D] to estimate the the Szegö kernel on H_k. The method employed to find the relative fundamental solution is similar to those in [GS and N].

The main theorem in this paper is as follows.

Main Theorem. *Let ω be the weight in* (1.7), *and let $f \in L^2$ satisfy $Pf = 0$ where P is the Szegö projection. Then, there exists a unique solution $u \in L^2_\omega$ which satisfies* (1.3) *and* (1.4), *and the solution operator $f \mapsto u$ is bounded from L^2 to L^2_ω. Furthermore, if $f \in L^\infty$ with a compact support, then $u \in \mathrm{Lip}(\frac{1}{2k})$.*

This theorem immediately implies that L as an operator from L^2_ω to L^2 has a closed range. It is also proved that the exponent -2 is the best possible one among the radial weights.

The proof of the theorem relies on the pointwise estimates of the relative fundamental solution. As we can see from (1.6), the relative fundamental solution has a nonisotropic nature. Therefore, in order to obtain an optimal estimate, a nonisotropic metric should be introduced which is related to the geometry of the surface H_k. In this paper, we use the nonisotropic pseudodistance (1.8). In [D], K. Diaz showed that this pseudodistance is essentially equivalent to that in [NSW1,2]. If K is the relative fundamental solution of L on H_k, we establish the following basic estimates

$$|K(x,y)| \lesssim \frac{\delta}{|B(x,\delta)|} \tag{1.9}$$

and

$$|DK(x,y)| \lesssim \frac{1}{|B(x,\delta)|} \tag{1.10}$$

where $\delta = d(x,y)$, $B(x,\delta) = \{y \in H_k : d(x,y) < \delta\}$, and D is either L or $\overline{L}$ with respect to x or y. Quite recently, M. Christ obtained these estimates on general compact 3-dimensional CR manifolds as mentioned earlier. Here and throughout this paper, $A \lesssim (\gtrsim) B$ means that there is a universal constant C such that $A \le (\ge) CB$ and $A \approx B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

The method of finding the relative fundamental solution of L also enables us to find the Szegö kernel on H_k. But, the Szegö kernels on H_k and on much more general domains were studied in several papers [C, D, GS, NRSW].

This work is a part of my dissertation under professor Alexander Nagel at the University of Wisconsin-Madison. I would like to take this opportunity to express my thanks to him.

2. Relative fundamental solution on H_ϕ

The object of this section is finding the formula for the relative fundamental solution of L on H_k. Since our method works for all radial functions, we work with the following space. Let ϕ be a convex polynomial such that $\phi(0) = \phi'(0) = 0$. And let $H_\phi = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = \phi(|z_1|)\}$. If $\phi(r) = r^{2k}$, then $H_\phi = H_k$. Once we find the formula for the relative fundamental solution of L on H_ϕ, we will apply $\phi(r) = r^{2k}$ to obtain an explicit and closed formula on H_k in the next section. The method to find the formula is as follows: we first decompose the operator L into a sum of ordinary differential equations and after solving each ordinary differential equation in terms of an integral operator in a proper L^2-space we sum them up.

The holomorphic tangential vector field L on H_ϕ is given by

$$L = \frac{\partial}{\partial z} + i\frac{\partial \phi(|z|)}{\partial z}\frac{\partial}{\partial t} \tag{2.1}$$

after the change of variables $\operatorname{Re} z_2 \mapsto t$ and $z_1 \mapsto z$. For this L, solving the equation (1.3) and (1.4) is equivalent to solving the following problem:

Problem. Let P be the orthogonal projection to the kernel of $\overline{L} = -L^*$ in $L^2(H_\phi)$, i.e., the Szegö projection. Solve

$$Lu = g - Pg \tag{2.2}$$

so that u is orthogonal to the kernel of L.

In terms of the polar coordinates $z = re^{i\theta}$, L can be expressed as

$$L = \frac{e^{-i\theta}}{2}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta} + i\phi'(r)\frac{\partial}{\partial t}\right). \tag{2.3}$$

And every function $g \in C_0^\infty(\mathbf{R}^3)$ has a Fourier series decomposition

$$g(r,\theta,t) = \sum_{n=-\infty}^{\infty} g_n(r,t)e^{in\theta} \tag{2.4}$$

where

$$g_n(r,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(r,\psi,t)e^{-in\psi}\,d\psi .$$

Therefore one can see that

$$\begin{aligned} Lg(r,\theta,t) &= \frac{e^{-i\theta}}{2}\sum_{n=-\infty}^{\infty}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\theta}+i\phi'(r)\frac{\partial}{\partial t}\right)(g_n(r,t)e^{in\theta}) \\ &= \frac{e^{-i\theta}}{2}\sum_{n=-\infty}^{\infty}\left(\frac{\partial}{\partial r}+\frac{n}{r}+i\phi'(r)\frac{\partial}{\partial t}\right)g_n(r,t)e^{in\theta} . \end{aligned}$$

Define a partial Fourier transform $\mathfrak{F}$ with respect to the t variable by

$$\mathfrak{F}v(r,\theta,\tau)=\int_{t\in\mathbf{R}} v(r,\theta,t)e^{-2\pi it\tau}\,dt .$$

It then follows that

$$\begin{aligned} \mathfrak{F}Lg(r,\theta,\tau) &= \frac{e^{-i\theta}}{2}\sum_{n=-\infty}^{\infty}\left(\frac{\partial}{\partial r}+\frac{n}{r}-\phi'(r)2\pi\tau\right)\mathfrak{F}g_n(r,\tau)e^{in\theta} \\ &= \frac{e^{-i\theta}}{2}\sum_{n=-\infty}^{\infty}\left(M_{\omega_n^{-1}}\frac{\partial}{\partial r}M_{\omega_n}\mathfrak{F}g_n\right)(r,\tau)e^{in\theta} \end{aligned}$$

where M_{ω_n} is the multiplication operator $M_{\omega_n}f=\omega_n f$ where $\omega_n(r,\tau)=r^n e^{-2\pi\tau\phi(r)}$. If $g\in C_0^\infty(\mathbf{R}^3)$, then each g_n also has a bounded support in $\mathbf{R}^+\times\mathbf{R}$ and hence the infinite sum is well defined. These can be summarized in one formula: If g is of the form (2.4), then

$$Lg(r,\theta,t)=\frac{e^{-i\theta}}{2}\sum_{n=-\infty}^{\infty}\left(\mathfrak{F}^{-1}M_{\omega_n^{-1}}\frac{\partial}{\partial r}M_{\omega_n}\mathfrak{F}g_n\right)(r,\tau)e^{in\theta}, \tag{2.5}$$

where

$$\omega_n(r,\tau)=r^n e^{-2\pi\tau\phi(r)} . \tag{2.6}$$

With the help of the formula (2.5), one can reduce the original problem to the following subproblem which is easier to solve. We note that, for fixed n and τ, M_{ω_n} is an isometry from $L^2(\mathbf{R}^+,dr)$ to $L^2(\mathbf{R}^+,\omega_n^{-2}r\,dr)$.

Subproblem. Fix n and τ. $\frac{\partial}{\partial r}$ is to be considered as an operator on $L^2(\mathbf{R}^+,\omega_n^{-2}r\,dr)$. Let $P_{n,\tau}$ be the orthogonal projection to the kernel of $(\frac{\partial}{\partial r})^*$, the adjoint of $\frac{\partial}{\partial r}$ in $L^2(\mathbf{R}^+,\omega_n^{-2}r\,dr)$. Then, solve

$$\frac{\partial u}{\partial r}=h-P_{n,\tau}h \tag{2.7}$$

so that u is orthogonal to the kernel of $\frac{\partial}{\partial r}$ in $L^2(\mathbf{R}^+,\omega_n^{-2}r\,dr)$.

This reduction is possible because of the following proposition.

Proposition 2.1. *Let $g \in C_0^\infty(\mathbb{R}^3)$ be of the form* (2.4). *Suppose that, for each fixed n and τ, $u_{n,\tau}$ is a solution of*

$$\frac{\partial}{\partial r} u_{n,\tau} = M_{\omega_n} \mathfrak{F} g_{n-1} - P_{n,\tau}(M_{\omega_n} \mathfrak{F} g_{n-1})$$

so that $u_{n,\tau} \perp \operatorname{Ker}(\frac{\partial}{\partial r})$ in $L^2(\mathbb{R}^+, \omega_n^{-2} r\, dr)$. Then, u defined by

$$u(r,\theta,t) = 2 \sum_{n=-\infty}^{\infty} (\mathfrak{F}^{-1} M_{\omega_n^{-1}} u_n)(r,t) e^{in\theta}, \tag{2.8}$$

where $u_n(r,\tau) = u_{n,\tau}(r)$, belongs to $L^2(\mathbb{R}^3)$ and satisfies (2.2) *as well as the orthogonality requirement, namely, $u \perp \operatorname{Ker}(L)$.*

Proof. To begin with, we note that $u \in L^2(\mathbb{R}^3)$ as long as $u_{n,\tau} \in L^2(\mathbb{R}^+, \omega_n^{-2} r\, dr)$. If $f(r,\theta,t) = \sum_{n=-\infty}^{\infty} f_n(r,t) e^{in\theta}$ is in $\operatorname{Ker}(L)$, then

$$Lf(r,\theta,t) = \frac{e^{-i\theta}}{2} \sum_{n=-\infty}^{\infty} \left(\mathfrak{F}^{-1} M_{\omega_n^{-1}} \frac{\partial}{\partial r} M_{\omega_n} \mathfrak{F} f_n \right)(r,t) e^{in\theta} = 0.$$

Therefore, $\mathfrak{F}^{-1} M_{\omega_n^{-1}} \frac{\partial}{\partial r} M_{\omega_n} \mathfrak{F} f_n(r,t) \equiv 0$ and hence $\frac{\partial}{\partial r} M_{\omega_n} \mathfrak{F} f_n(r,t) \equiv 0$ for each n. It then follows that $M_{\omega_n} \mathfrak{F} f_n(\cdot,\tau) \in \operatorname{Ker}(\frac{\partial}{\partial r})$ in $L^2(\mathbb{R}^+, \omega_n^{-2} r\, dr)$ for each n and τ. But

$$\begin{aligned}
(u,f)_{L^2} &= 2 \sum_{n=-\infty}^{\infty} \int_{r>0} \int_t (\mathfrak{F}^{-1} M_{\omega_n^{-1}} u_n)(r,t) \overline{f_n(r,t)} r\, dr\, dt \\
&= 2 \sum_{n=-\infty}^{\infty} \int_{r>0} \int_\tau u_n(r,\tau) \overline{M_{\omega_n} \mathfrak{F} f_n(r,\tau)} \omega_n^{-2}(r,\tau) r\, dr\, d\tau = 0
\end{aligned}$$

since $u_{n,\tau} \perp \operatorname{Ker}(\frac{\partial}{\partial r})$. Therefore, $u \perp \operatorname{Ker}(L)$.

On the other hand,

$$\begin{aligned}
Lu(r,\theta,t) &= e^{-i\theta} \sum_{n=-\infty}^{\infty} \left(\mathfrak{F}^{-1} M_{\omega_n^{-1}} \frac{\partial}{\partial r} u_n \right)(r,t) e^{in\theta} \\
&= e^{-i\theta} \sum_{n=-\infty}^{\infty} (\mathfrak{F}^{-1} M_{\omega_n^{-1}} M_{\omega_n} \mathfrak{F} g_{n-1})(r,t) e^{in\theta} \\
&\qquad - e^{-i\theta} \sum_{n=-\infty}^{\infty} (\mathfrak{F}^{-1} M_{\omega_n^{-1}} P_{n,\tau} M_{\omega_n} \mathfrak{F} g_{n-1})(r,t) e^{in\theta} \\
&= g(r,\theta,t) - Pg(r,\theta,t).
\end{aligned}$$

The last equality comes from the following lemma. □

Lemma 2.2. *Let P and $P_{n,\tau}$ be the orthogonal projections as above. If $g \in L^2(\mathbb{R}^3)$ has a Fourier series decomposition $g(r,\theta,t) = \sum_{n=-\infty}^{\infty} g_n(r,t) e^{in\theta}$, then*

$$Pg(r,\theta,t) = \sum_{n=-\infty}^{\infty} (\mathfrak{F}^{-1} M_{\omega_n^{-1}} P_{n,\tau} M_{\omega_n} \mathfrak{F} g_{n-1})(r,t) e^{i(n-1)\theta}. \tag{2.9}$$

In particular, the series in the right-hand side converges in $L^2(\mathbf{R}^3)$.

Proof. L^2 convergence of the series is trivial. We first observe the following facts:

(1) $\omega_n^{-1}\omega_{n+1}^{-1} = r^{-2n-1}e^{-4\pi\tau\phi(r)} = \omega_{n+1}^{-2}r$ and hence $M_{\omega_n^{-1}}M_{\omega_{n+1}^{-1}} = M_{\omega_{n+1}^{-2}r}$.

(2) $(\frac{\partial}{\partial r})^* = M_{\omega_n^2 r^{-1}}\frac{\partial}{\partial r}M_{\omega_n^{-2}r}$ in $L^2(\mathbf{R}^+, \omega_n^{-2}r\,dr)$.

(3) $\frac{\partial}{\partial r}M_{\omega_n^{-2}r}P_{n,\tau} = M_{\omega_n^{-2}r}(\frac{\partial}{\partial r})^*P_{n,\tau} = 0$ for all n.

(4) Since $L^* = -\overline{L} = -\frac{e^{i\theta}}{2}(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta} - i\phi'(r)\frac{\partial}{\partial t})$, we have

$$L^*g(r,\theta,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\mathfrak{F}^{-1}M_{\omega_n}\frac{\partial}{\partial r}M_{\omega_n^{-1}}\mathfrak{F}g_n\right)(r,t)e^{i(n+1)\theta}.$$

Let us set the right-hand side of (2.9) to be Qg. It then follows from above facts that

$$L^*Qg(r,\theta,t) = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\mathfrak{F}^{-1}M_{\omega_n}\frac{\partial}{\partial r}M_{\omega_{n+1}^{-2}r}P_{n+1,\tau}M_{\omega_{n+1}}\mathfrak{F}g_n\right)(r,t)e^{i(n+1)\theta} = 0.$$

Therefore, $Qg \in \operatorname{Ker}(L^*)$.

On the other hand, if $g \in \operatorname{Ker}(L^*)$, then $\frac{\partial}{\partial r}M_{\omega_{n-1}^{-1}}\mathfrak{F}g_{n-1}(\cdot,\tau) = 0$ for each fixed n and τ. Therefore, by (2),

$$(\tfrac{\partial}{\partial r})^*M_{\omega_n^2 r^{-1}}M_{\omega_{n-1}^{-1}}\mathfrak{F}g_{n-1} = (\tfrac{\partial}{\partial r})^*M_{\omega_n}\mathfrak{F}g_{n-1} = 0$$

and hence $P_{n,\tau}M_{\omega_n}\mathfrak{F}g_{n-1} = M_{\omega_n}\mathfrak{F}g_{n-1}$. So, $Qg = g$.

It is easy to show that $QQ = Q$. Now, we shall show that $Q^* = Q$, which implies that Q is the orthogonal projection onto $\operatorname{Ker}(L^*)$, and hence $Q = P$. Let f and g be in $L^2(\mathbf{R}^3)$. Then,

$$\begin{aligned}
(Q^*g, f) &= (g, Qf) \\
&= \sum_{n=-\infty}^{\infty}\int_{r>0}\int_t g_n(r,t)\overline{(\mathfrak{F}^{-1}M_{\omega_{n+1}^{-1}}P_{n+1,\tau}M_{\omega_{n+1}}\mathfrak{F}f_n(r,t))}r\,dr\,dt \\
&= \sum_{n=-\infty}^{\infty}\int_{r>0}\int_\tau (M_{\omega_{n+1}}\mathfrak{F}g_n)(r,\tau)\overline{(P_{n+1,\tau}M_{\omega_{n+1}}\mathfrak{F}f_n)(r,\tau)}\omega_{n+1}^{-2}r\,dr\,d\tau \\
&= \sum_{n=-\infty}^{\infty}\int_{r>0}\int_\tau (\mathfrak{F}^{-1}M_{\omega_{n+1}^{-1}}P_{n+1,\tau}M_{\omega_{n+1}}\mathfrak{F}g_n)(r,t)\overline{f_n(r,t)}r\,dr\,dt \\
&= (Qg, f).
\end{aligned}$$

The fourth equality is true because $P_{n+1,\tau}$ is an orthogonal projection in $L^2(\mathbf{R}^+, \omega_n^{-2}r\,dr)$. Therefore, $Q^* = Q$ and the proof is completed. □

Let, for each fixed n and τ, $Q_{n,\tau}$ be the orthogonal projection to $\operatorname{Ker}(\frac{\partial}{\partial r})$ in $L^2(\mathbf{R}^+, \omega_n^{-2}r\,dr)$. We shall seek a solution of the subproblem (2.7) in the form of an integration against some kernel function. Therefore, first of all, we write $P_{n,\tau}$ and $Q_{n,\tau}$ as integral operators.

Lemma 2.3. *Let $P_{n,\tau}$ be the orthogonal projection to the kernel of $(\frac{\partial}{\partial r})^*$ in $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. Then $P_{n,\tau}$ can be represented as*

$$P_{n,\tau}h(r) = \int_0^\infty h(y) G_{n,\tau}(r, y) y\,dy \tag{2.10}$$

for any $h \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ where

$$G_{n,\tau}(r, y) = \begin{cases} \left(ry \int_0^\infty \omega_n^2 r^{-1}\,dr \right)^{-1} \omega_n^2(r, \tau) & \text{if } \tau > 0 \text{ and } n \geq 1 \\ 0 & \text{otherwise}. \end{cases} \tag{2.11}$$

Proof. Since $(\frac{\partial}{\partial r})^* = M_{\omega_n^2 r^{-1}} \frac{\partial}{\partial r} M_{\omega_n^{-2} r}$ in $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$, $(\frac{\partial}{\partial r})^* h = 0$ if and only if $\omega_n^{-2} r h(r)$ is constant, i.e., $h(r) = c\omega_n^2 r^{-1}$ for some constant c. But, if $c \neq 0$, then $c\omega_n^2 r^{-1} \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ if and only if $\tau > 0$ and $n \geq 1$. Therefore,

$$\begin{aligned} P_{n,\tau}h(r) &= \begin{cases} \dfrac{(\omega_n^2 r^{-1}, h)_n}{(\omega_n^2 r^{-1}, \omega_n^2 r^{-1})_n} \omega_n^2 r^{-1} & \text{if } \tau > 0 \text{ and } n \geq 1, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \left(\int_0^\infty \omega_n^2 r^{-1}\,dr \right)^{-1} \int_0^\infty h(r)\,dr\, \omega_n^2(r, \tau) r^{-1} & \text{if } \tau > 0 \text{ and } n \geq 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $(\cdot, \cdot)_n$ is the inner product in $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. So, the lemma follows. □

Lemma 2.4. *Let $Q_{n,\tau}$ be the orthogonal projection to* $\operatorname{Ker}(\frac{\partial}{\partial r}) \subset L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. *Then, $Q_{n,\tau}$ can be represented as*

$$Q_{n,\tau}h(r) = \int_0^\infty h(y) K_{n,\tau}(y) y\,dy \tag{2.12}$$

for any $h \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. Here,

$$K_{n,\tau}(y) = \begin{cases} \left(\int_0^\infty \omega_n^{-2}(x, \tau) x\,dx \right)^{-1} \omega_n^{-2}(y, \tau) & \text{if } \tau < 0 \text{ and } n \leq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.13}$$

In particular, $Q_{n,\tau}h$ is independent of r as one can expect.

Proof. Since $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ contains nonzero constants if and only if $\tau < 0$ and $n \leq 0$, $k \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ belongs to $\operatorname{Ker}(\frac{\partial}{\partial r})$ if and only if

$$k(r) = \begin{cases} \text{constant} & \text{if } \tau < 0 \text{ and } n \leq 0, \\ 0 & \text{otherwise}. \end{cases}$$

The rest goes in the same way as the proof of Lemma 2.3. □

Being equipped with the integral formulas of $P_{n,\tau}$ and $Q_{n,\tau}$, one can solve the subproblem in terms of an integral operator and hence the problem by the formula (2.8). We recall that $\omega_n(r,\tau) = r^n e^{-2\pi\tau\phi(r)}$. Considering the singularities of functions in $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ near 0, we define an integral operator $A_{n,\tau}$ by

$$A_{n,\tau}h(r) = \begin{cases} \int_0^r h(y)\,dy & \text{if } n \geq 1, \\ -\int_r^\infty h(y)\,dy & \text{if } n \leq 0. \end{cases} \tag{2.14}$$

for $h \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$ with a bounded support. We note that $A_{n,\tau}h$ is well defined and $\frac{\partial}{\partial r}(A_{n,\tau}h)(r) = h(r)$ for both cases.

Lemma 2.5. *If $h \in C^\infty(\mathbf{R}^+)$ has a bounded support on $\mathbf{R}^+$, then for each n and τ, $A_{n,\tau}(h - P_{n,\tau}h) \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$.*

Proof. Let $\tau > 0$ and $n \geq 1$. Then, since $y\int_0^\infty G_{n,\tau}(r,y)\,dr = 1$, it follows that

$$\int_0^\infty (h - P_{n,\tau}h)\,dr = \int_0^\infty h(r)\,dr - \int_0^\infty \int_0^\infty h(y) G_{n,\tau}(r,y) y\,dy\,dr = 0. \tag{2.15}$$

Choose $R > 0$ so that $\operatorname{supp}(h) \subset [0, R]$. Then, because of (2.15),

$$A_{n,\tau}(h - P_{n,\tau}h)(r) = \int_0^r (h - P_{n,\tau}h)(y)\,dy = -\int_r^\infty (h - P_{n,\tau}h)(y)\,dy,$$

and hence if $r > R$, then

$$A_{n,\tau}(h - P_{n,\tau}h)(r) = \int_r^\infty (P_{n,\tau}h)(y)\,dy.$$

Therefore,

$$\begin{aligned} &\int_0^\infty |A_{n,\tau}(h - P_{n,\tau}h)(r)|^2 \omega_n^{-2}(r,\tau) r\,dr \\ &\leq \int_0^R \left| \int_0^r A_{n,\tau}(h - P_{n,\tau}h)(r) \right|^2 \omega_n^{-2}(r,\tau) r\,dr \\ &\quad + \int_R^\infty \left| \int_r^\infty (P_{n,\tau}h)(x)dx \right|^2 \omega_n^{-2}(r,\tau) r\,dr \\ &= I_1 + I_2 \end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned} I_1 &\leq \int_0^R \| h - P_{n,\tau}h \|_{L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)} \int_0^r \omega_n^2(y,\tau) y^{-1}\,dy\ \omega_n^{-2}(r,\tau) r\,dr \\ &\leq C \int_0^R \int_0^r e^{-4\pi\tau\phi(y)} y^{2n-1}\,dy\ e^{4\pi\tau\phi(r)} r^{-2n+1}\,dr \\ &\leq C \int_0^R \int_0^r y^{2n-1}\,dy\ r^{-2n+1}\,dr \leq C. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_2 &= \int_R^\infty \left| \int_r^\infty \int_0^\infty g(x) G_{n,\tau}(y,x) x\,dx\,dy \right|^2 \omega_n^{-2}(r,\tau) r\,dr \\ &= \int_R^\infty \left| \int_0^\infty g(x)\,dx \right|^2 \left[\int_r^\infty \omega_n^2(y,\tau) y^{-1}\,dy \right]^2 \omega_n^{-2}(r,\tau) r\,dr \\ &\le C \int_R^\infty \left[\int_r^\infty e^{-4\pi\tau\phi(y)} y^{2n-1}\,dy \right]^2 e^{4\pi\tau\phi(r)} r^{-2n+1}\,dr \le C \end{aligned}$$

since $\tau > 0$. Therefore, $A_{n,\tau}(h - P_{n,\tau}h) \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$.

If $n \le 0$, then $P_{n,\tau}h = 0$ and

$$A_{n,\tau}h(r) = -\int_r^\infty h(y)\,dy.$$

Therefore, $A_{n,\tau}(h - P_{n,\tau}h)$ has a bounded support and hence it belongs to $L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$.

If $n \ge 1$ and $\tau < 0$, then $P_{n,\tau}h = 0$ and $A_{n,\tau}h =$ constant for all $r > R$. So one can easily see that $\int_0^\infty |A_{n,\tau}h(r)|^2 \omega_n^{-2}(r,\tau) r\,dr < \infty$ since $\tau < 0$. This completes the proof. □

For each n and τ, we define an operator $T_{n,\tau}$ by

$$T_{n,\tau}h(r) = A_{n,\tau}(h - P_{n,\tau}h)(r) - Q_{n,\tau}A_{n,\tau}(h - P_{n,\tau}h)(r) \tag{2.16}$$

for a $C^\infty(\mathbf{R}^+)$ function h with a bounded support. Then, by Lemma 2.5, $T_{n,\tau}h \in L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. Moreover, $\frac{\partial}{\partial r} T_{n,\tau}h = h - P_{n,\tau}h$ and $T_{n,\tau}h \perp \operatorname{Ker}(\frac{\partial}{\partial r}) \subset L^2(\mathbf{R}^+, \omega_n^{-2} r\,dr)$. We then define an operator T, accordingly to the formula (2.8), by

$$Tg(r,\theta,t) = 2 \sum_{n=-\infty}^{\infty} (\mathfrak{F}^{-1} M_{\omega_n^{-1}} T_{n,\tau} M_{\omega_n} \mathfrak{F} g_{n-1})(r,t) e^{in\theta} \tag{2.17}$$

for each $g(r,\theta,t) = \sum_{n=-\infty}^{\infty} g_n(r,t) e^{in\theta} \in C_0^\infty(\mathbf{R}^3)$. Then, by Proposition 2.1, one can see that $LTg = g - Pg$ and $Tg \perp \operatorname{Ker}(L)$. Moreover, by Lemma 2.5 and (2.17), it is easy to see that $Tg \in L^2(\mathbf{R}^3)$. In short, we obtain the following theorem.

Theorem 2.6. *Let f be in $C_0^\infty(\mathbf{R}^3)$ and satisfy the condition $Pf = 0$. Then, there exists a unique solution $u \in L^2$ such that $Lu = f$ and $u \perp \operatorname{Ker}(L)$. Moreover, the solution formula is given by* (2.16) *and* (2.17).

We now express T as an integral operator against a singular kernel. That singular kernel is going to be our relative fundamental solution. If $n \ge 1$, then $Q_{n,\tau} = 0$ and hence

$$T_{n,\tau}h(r) = A_{n,\tau}(h - P_{n,\tau}h) = \int_0^\infty h(y) B_{n,\tau}(r,y)\,dy \tag{2.18}$$

where

$$(2.19)\qquad B_{n,\tau}(r,y)=\begin{cases}1-y\displaystyle\int_0^r G_{n,\tau}(x,y)\,dx & \text{if } 0\le y\le r,\\ -y\displaystyle\int_0^r G_{n,\tau}(x,y)\,dx & \text{if } r<y\end{cases}$$

with $G_{n,\tau}$ given in (2.11).

On the other hand, if $n\le 0$, then $P_{n,\tau}=0$ and hence

$$(2.20)\qquad T_{n,\tau}h(r)=A_{n,\tau}h(r)-Q_{n,\tau}A_{n,\tau}h(r)=\int_0^\infty h(y)B_{n,\tau}(r,y)\,dy$$

where

$$(2.21)\qquad B_{n,\tau}(r,y)=\begin{cases}-1+\displaystyle\int_0^y K_{n,\tau}(x)x\,dx & \text{if } y>r,\\ \displaystyle\int_0^y K_{n,\tau}(x)x\,dx & \text{if } 0<y<r,\end{cases}$$

with $K_{n,\tau}$ given in (2.13). Now, it follows from the formula (2.17) that

$$\begin{aligned}Tg(r,\theta,t)&=2\sum_{n=-\infty}^{\infty}(\mathfrak{F}^{-1}M_{\omega_n^{-1}}T_{n,\tau}M_{\omega_n}\mathfrak{F}g_{n-1})(r,t)e^{in\theta}\\ &=2\sum_{n=-\infty}^{\infty}\left[\int_{y>0}\int_s g_{n-1}(y,s)\int_\tau e^{2\pi i\tau(t-s)}\omega_n^{-1}(r,\tau)\right.\\ &\qquad\qquad\left.\times\,\omega_n(y,\tau)B_{n,\tau}(r,y)\,d\tau\,dy\,ds\right]e^{in\theta}\\ &=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\int_{y>0}\int_s e^{-i(n-1)\psi}g(y,\psi,s)\right.\\ &\qquad\left.\times\left(\int_\tau e^{2\pi i\tau(t-s)}\omega_n^{-1}(t,\tau)\omega_n(y,\tau)B_{n,\tau}(r,y)\,d\tau\right)d\psi\,dy\,ds\right]e^{in\theta}\end{aligned}$$

if $g(y,\psi,s)=\sum_{n=-\infty}^{\infty}g_n(y,s)e^{in\psi}$. Therefore, we have

$$(2.22)\qquad Tg(r,\theta,t)=\int_{-\pi}^{\pi}\int_{y>0}\int_s g(y,\psi,s)K((r,\theta,t),(y,\psi,s))y\,dy\,d\psi\,ds$$

where

$$\begin{aligned}&K((r,\theta,t),(y,\psi,s))\\ &\quad=\frac{1}{\pi y}\sum_{n=-\infty}^{\infty}e^{in(\theta-\psi)+i\psi}\int_\tau e^{2\pi i\tau(t-s)}\omega_n^{-1}(r,\tau)\omega_n(y,\tau)B_{n,\tau}(r,y)\,d\tau.\end{aligned}$$

Recall that $\omega_n(r,\tau)=r^ne^{-2\pi\tau\phi(r)}$. If we put $z=re^{i\theta}$ and $w=ye^{i\psi}$, then

$$\begin{aligned}(2.23)\quad &K((z,t),(w,s))\\ &\quad=\frac{1}{\pi\bar w}\sum_{n=-\infty}^{\infty}\left(\frac{\bar w}{\bar z}\right)^n\int_\tau e^{2\pi i\tau(t-s)}e^{2\pi\tau(\phi(|z|)-\phi(|w|))}B_{n,\tau}(|z|,|w|)\,d\tau\end{aligned}$$

Here, $B_{n,\tau}$ is the kernel given in (2.19) and (2.21). In conclusion, we establish the following theorem.

Theorem 2.7. *Let $K((z,t),(w,s))$ be as in (2.23), and let*

$$Tf(z,t) = \int_{w,s} f(w,s)K((z,t),(w,s))\,dV(w,s). \tag{2.24}$$

Then, for any $f \in C_0^\infty(\mathbf{R}^3)$, $Tf \in L^2(\mathbf{R}^3)$ and satisfies $LTf = f - Pf$ and $Tf \perp \operatorname{Ker}(L)$. The integration in (2.24) is defined in the sense of distributions.

By the same computation as we did for Theorem 2.7, the following theorem can be obtained from (2.9).

Theorem 2.8. *The Szegö projection on H_ϕ is given by*

$$S((z,t),(w,s)) = \frac{1}{2\pi}\sum_{n=0}^{\infty}(z\bar{w})^n \int_0^\infty \frac{e^{2\pi i\tau(t-s)-2\pi\tau[\phi(|z|)+\phi(|w|)]}}{\int_0^\infty r^{2n}e^{-4\pi\tau\phi(r)}r\,dr}\,d\tau. \tag{2.25}$$

The integration is defined in the sense of distributions.

3. Relative fundamental solution on H_k

In this section, we deduce an explicit closed formula for the relative fundamental solution of L on H_k, i.e., the case when $\phi(r) = r^{2k}$, from the formula (2.23) in previous section. Recall that

$$K((z,t),(w,s)) = \frac{1}{\pi\bar{w}}\sum_{n=-\infty}^{\infty}\left(\frac{\bar{w}}{\bar{z}}\right)^n \int_\tau e^{2\pi i\tau(t-s)}e^{2\pi\tau(\phi(|z|)-\phi(|w|))}B_{n,\tau}(|z|,|w|)\,d\tau.$$

We may assume that $0 \le |w| \le |z|$ since the computations for $0 \le |z| \le |w|$ case go parallel. Throughout this paper we use the following notations for convenience:

$$A = \tfrac{1}{2}[(|z|^{2k} + |w|^{2k}) - i(t-s)] \tag{3.1}$$

and

$$B = \tfrac{1}{2}[(|z|^{2k} - |w|^{2k}) + i(t-s)]. \tag{3.2}$$

Put

$$I = \frac{1}{\pi\bar{w}}\sum_{n=1}^{\infty}\left(\frac{\bar{w}}{\bar{z}}\right)^n \int e^{4\pi\tau B}B_{n,\tau}(|z|,|w|)\,d\tau$$

and put

$$J = \frac{1}{\pi\bar{w}}\sum_{n=-\infty}^{0}\left(\frac{\bar{w}}{\bar{z}}\right)^n \int e^{4\pi\tau B}B_{n,\tau}(|z|,|w|)\,d\tau$$

where $B_{n,\tau}$ is as in (2.19) and (2.21). Then, the relative fundamental solution K is equal to $I + J$.

For I, we first observe from (2.19) and (2.21) that

$$B_{n,\tau}(|z|,|w|) = \begin{cases} 1 & \text{if } \tau < 0, \\ |w|\displaystyle\int_{|z|}^\infty G_{n,\tau}(\eta,|w|)\,d\eta & \text{if } \tau > 0. \end{cases}$$

Since $\int_0^\infty \omega_n^2(r,\tau)r^{-1}\,dr = \frac{1}{2k}(4\pi\tau)^{-n/k}\Gamma(n/k)$, it follows that

$$B_{n,\tau}(|z|,|w|) = \begin{cases} 1 & \text{if } \tau < 0, \\ 2k(4\pi\tau)^{n/k}\Gamma(n/k)^{-1}\int_{|z|}^\infty \eta^{2n-1}e^{-4\pi\tau\eta^{2k}}\,d\eta & \text{otherwise}. \end{cases}$$

Therefore,

$$\begin{aligned} I &= \frac{1}{\pi\bar w}\sum_{n=1}^\infty \left(\frac{\bar w}{\bar z}\right)^n \left[\int_0^\infty e^{4\pi\tau B}2k(4\pi\tau)^{n/k}\Gamma(n/k)^{-1}\right. \\ &\qquad\qquad \left.\times \int_{|z|}^\infty \eta^{2n-1}e^{-4\pi\tau\eta^{2k}}\,d\eta\,d\tau + \int_{-\infty}^0 e^{4\pi\tau B}\,d\tau\right] \\ &= \frac{1}{\pi\bar w}\sum_{n=1}^\infty \left(\frac{\bar w}{\bar z}\right)^n (I_n^1 + I_n^2). \end{aligned}$$

Put $\mu_\eta = \eta^{2k} - B$. Then,

$$I_n^1 = \frac{2k}{\Gamma(n/k)}\int_{|z|}^\infty \eta^{2n-1}\int_0^\infty e^{-4\pi\tau\mu_\eta}(4\pi\tau)^{n/k}\,d\tau\,d\eta.$$

And, since $\int_0^\infty e^{-4\pi\tau\mu_\eta}(4\pi\tau)^{n/k}\,d\tau = \frac{1}{4\pi}\mu_\eta^{-n/k-1}\Gamma(n/k+1)$, it follows that

$$I_n^1 = \frac{n}{2\pi}\int_{|z|}^\infty \eta^{2n-1}\mu_\eta^{-n/k-1}\,d\eta.$$

And

$$I_n^2 = \int_{-\infty}^0 e^{4\pi\tau B}\,d\tau = \frac{1}{4\pi B}.$$

Therefore,

$$\begin{aligned} I &= \frac{1}{\pi\bar w}\sum_{n=1}^\infty \left(\frac{\bar w}{\bar z}\right)^n \left(\frac{n}{2\pi}\int_{|z|}^\infty \eta^{2n-1}\mu_\eta^{-n/k-1}d\eta + \frac{1}{4\pi B}\right) \\ &= \frac{1}{2\pi^2\bar w}\int_{|z|}^\infty \sum_{n=1}^\infty n\left(\frac{\bar w\eta^2}{\bar z\mu_\eta^{1/k}}\right)^n \eta^{-1}\mu_\eta^{-1}d\eta + \frac{1}{4\pi^2 B\bar w}\frac{\bar w/\bar z}{1-\bar w/\bar z} \\ &= -\frac{1}{2\pi^2\bar w}\frac{1}{B}\int_{|z|}^\infty \frac{d}{d\eta}\left(1 - \frac{\bar w\eta^2}{\bar z\mu_\eta^{1/k}}\right)^{-1} d\eta + \frac{1}{4\pi^2 B}\frac{1}{\bar z - \bar w} \\ &= \frac{1}{4\pi^2 B}\frac{z}{A^{1/k} - z\bar w}. \end{aligned}$$

On the other hand,

$$\begin{aligned} J &= \frac{1}{\pi\bar w}\sum_{n=-\infty}^0 \left(\frac{\bar w}{\bar z}\right)^n \int e^{4\pi\tau B}B_{n,\tau}(|z|,|w|)\,d\tau \\ &= \frac{1}{\pi\bar w}\sum_{n=0}^\infty \left(\frac{\bar z}{\bar w}\right)^n \int e^{-4\pi\tau B}B_{-n,-\tau}(|z|,|w|)\,d\tau, \end{aligned}$$

here, we made changes of variables $n \mapsto -n$ and $\tau \mapsto -\tau$. If $\tau < 0$ then $B_{n,-\tau} = 0$ for any n. If $\tau > 0$ and $n \geq 0$ then

$$
\begin{aligned}
B_{-n,-\tau}(|z|,|w|) &= \int_0^{|w|} K_{-n,-\tau}(\eta)\eta\, d\eta \\
&= \left(\int_0^\infty \omega_{-n}^{-2}(t, -\tau) r\, dr\right)^{-1} \int_0^{|w|} \omega_{-n}^{-2}(\eta, -\tau)\eta\, d\eta. \\
&= 2k(4\pi\tau)^{(n+1)/k}\Gamma((n+1)/k)^{-1} \int_0^{|w|} r^{2n+1} e^{-4\pi\tau r^{2k}}\, dr,
\end{aligned}
$$

by the same reason as above. Therefore, if we put $\lambda_\eta = B + \eta^{2k}$ then, we can obtain

$$J = \frac{1}{2\pi^2 \bar{w}} \sum_{n=0}^{\infty} (n+1)\left(\frac{\bar{z}}{\bar{w}}\right)^n \int_0^{|w|} \eta^{2n+1} \lambda_\eta^{-(n+1)/k-1}\, d\eta.$$

Note that $\lambda_{|w|} = \overline{A}$ and obtain from the same computation as above that

$$J = \frac{1}{2\pi^2 B} \frac{w}{\overline{A}^{1/k} - \bar{z}w}.$$

Combining all these computations, we finally obtain the following theorem:

Theorem 3.1. *Let*

$$A = \tfrac{1}{2}[|z|^{2k} + |w|^{2k} - i(t-s)]$$

and

$$B = \tfrac{1}{2}[|z|^{2k} - |w|^{2k} + i(t-s)].$$

And let

$$K((z,t),(w,s)) = \frac{1}{4\pi^2 B}\left[\frac{z}{A^{1/k} - z\bar{w}} + \frac{w}{\overline{A}^{1/k} - \bar{z}w}\right]. \tag{3.3}$$

Define an operator T by

$$Tf(z,t) = \int f(w,s) K((z,t),(w,s))\, dV(w,s). \tag{3.4}$$

Then, for any $f \in C_0^\infty(\mathbf{R}^3)$, $Tf \in L^2$ and Tf satisfies $LTf = f - Pf$ and $Tf \perp \operatorname{Ker}(L)$. Here, P is the Szegö projection.

The kernel K in (3.3) is called the relative fundamental solution. By the same computation starting from the identity (2.25), we can also obtain the following theorem.

Theorem 3.2 (Greiner and Stein). *The Szegö kernel on H_k is given by*

$$S((z,t),(w,s)) = \frac{1}{4\pi^2}\left[1 - \frac{z\bar{w}}{A^{1/k}}\right]^{-2} A^{-1/k-1}.$$

Remark 3.3. If $k=1$, $B=\overline{A}-|w|^2=-A+|z|^2$ and hence $z(\overline{A}-\bar{z}w)+w(A-z\bar{w})=B(z-w)$. Therefore,

$$K((z,t),(w,s))=\frac{1}{4\pi^2}\frac{z-w}{|A-z\bar{w}|^2}=\frac{1}{\pi^2}\frac{z-w}{|z-w|^4+|t-s+2\,\mathrm{Im}(z\bar{w})|^2}$$

which is exactly the one in [GKS].

Remark 3.4. The singularity of K may seem strange because B vanishes at the points $|z|=|w|$ and $t=s$. But, at the point $((z,t),(-z,t))$, the singularities are cancelled out if we put the fractions together, namely,

$$K((z,t),(w,s))=\frac{\Lambda((z,t),(w,s))}{|A^{1/k}-z\bar{w}|^2} \tag{3.5}$$

where

$$\Lambda((z,t),(w,s))=\frac{1}{4\pi^2}\left[z\frac{\overline{A}^{1/k}-|w|^2}{\overline{A}-|w|^{2k}}-w\frac{|z|^2-A^{1/k}}{|z|^{2k}-A}\right]. \tag{3.6}$$

4. Pointwise estimates of the relative fundamental solution

In this section, we shall make pointwise estimates of the relative fundamental solution K given by (3.3), in terms of the nonisotropic pseudodistance and the size of ball defined by the pseudodistance. The pseudodistance to be used is the one introduced by K. Diaz. In [D], K. Diaz used the following pseudodistance to estimate the Szegö kernel on H_k:

$$d((z,t),(w,s))=|A^{1/k}-z\bar{w}|^{\frac{1}{2}}. \tag{4.1}$$

We recall that $A=\frac{1}{2}[(|z|^{2k}+|w|^{2k})-i(t-s)]$. She proved

Theorem 4.1 (Diaz). *$(\mathbf{R}^3,d)$ is a space of homogeneous type as defined in* [CW].

The distance function (4.1) is convenient to use since it is in a closed form and the formula itself is very much related to those of the Szegö kernels and the relative fundamental solution. K. Diaz showed that $d((z,t),(w,s))\approx|z-w|+|t-s|^{1/k}$ if z and w are bounded.

Lemma 4.2. (1) $|z-w|\lesssim d((z,t),(w,s))$.

(2) *If $|z|$ is bounded and if $d((z,t),(w,s))=1$, then either $|z-w|\approx 1$ or $|t-s|\approx 1$.*

Proof. Lemma 4.2 follows from Lemma 3.2.1 of [D]. □

Lemma 4.3. *Let $A=\frac{1}{2}[|z|^{2k}+|w|^{2k}-i(t-s)]$. Then*

(1) $|A|\approx|z|^{2k}+d((z,t),(w,s))^{2k}$

(2) $|\overline{A}^{1/k}-|w|^2e^{2\pi ij/k}|\approx|z|^2+d((z,t),(w,s))^2$

(3) $|A^{1/k}-|z|^2e^{2\pi ij/k}|\approx|w|^2+d((z,t),(w,s))^2$, $j=0,1,2,3,\dots,k-1$.

Proof. For convenience, we assume that $k\geq 2$. By the homogeneity of A with respect to the dilation $\delta(z,t)=(\delta z,\delta^{2k}t)$, we may assume that

$$d((z,t),(w,s))=1.$$

We note that if $d((z,t),(w,s))=1$ and z is bounded, then either $|z-w|\approx 1$ or $|t-s|\approx 1$ by Lemma 4.2.

It is easy to see that $|A|\lesssim |z|^{2k}+1$. If $|z|$ is large or if $|z|$ is small and $|t-s|\approx 1$, then $|A|\gtrsim |z|^{2k}+1$. If $|z|$ is small and $|z-w|\approx 1$, then $|z|^{2k}+|w|^{2k}\gtrsim 1$ and hence $|A|\geq \frac{1}{2}(|z|^{2k}+|w|^{2k})\gtrsim 1\geq |z|^{2k}+1$, and this completes the proof of (1).

For (2), we first suppose that $|A|^{1/k}>|w|^2$. If we put $A=|A|e^{i\theta}$, then $|\theta|<\frac{\pi}{2}$ and it follows from (1) that

$$|\overline{A}^{1/k}-|w|^2e^{2\pi ij/k}|\geq |A|^{1/k}\cos(\theta/k)\left|1-\frac{|w|^2\cos(2\pi j/k)}{|A|^{1/k}\cos(\theta/k)}\right|\gtrsim |A|^{1/k}\gtrsim |z|^2+1.$$

for $j=1,2,\dots,k-1$. If $|A|^{1/k}\leq |w|^2$ and either $0\leq\frac{2\pi j}{k}<\frac{3\pi}{4}$ or $\frac{5\pi}{4}<\frac{2\pi j}{k}\leq 2\pi$, then

$$\begin{aligned}||A|^{1/k}\sin(\theta/k)-|w|^2\sin(2\pi j/k)| &\geq |w|^2(|\sin(2\pi j/k)|-|\sin(\theta/k)|)\\ &\gtrsim |w|^2\geq |A|^{1/k}\gtrsim |z|^2+1.\end{aligned}$$

If $|A|^{1/k}\leq |w|^2$ and $\frac{3\pi}{4}\leq\frac{2\pi j}{k}\leq\frac{5\pi}{4}$, it follows that

$$\begin{aligned}||A|^{1/k}\cos(\theta/k)-|w|^2\cos(2\pi j/k)| &= |A|^{1/k}\cos(\theta/k)+|w|^2|\cos(2\pi j/k)|\\ &\gtrsim |w|^2\gtrsim |z|^2+1\end{aligned}$$

since $\cos(\frac{2\pi j}{k})<0$. Therefore, in both cases, we have

$$\begin{aligned}|\overline{A}^{1/k}-|w|^2e^{2\pi ij/k}| &\geq ||A|^{1/k}\sin(\theta/k)-|w|^2\sin(2\pi j/k)|\\ &\quad+||A|^{1/k}\cos(\theta/k)-|w|^2\cos(2\pi j/k)|\\ &\gtrsim |z|^2+1.\end{aligned}$$

If $j=0$, then

$$\begin{aligned}|\overline{A}^{1/k}-|w|^2| &\leq |\overline{A}^{1/k}-\bar{z}w|+|\bar{z}w-|w|^2|\\ &= d((z,t),(w,s))+|w||z-w|\lesssim |w|^2+1\approx |z|^2+1.\end{aligned}$$

The opposite inequality is trivial. (3) can be shown in the same way. □

Lemma 4.4. *Let A be the quantity given in Lemma 4.3, then*

$$\overline{L}_{z,t}(\overline{A}^{1/k}-\bar{z}w)\lesssim d((z,t),(w,s)) \tag{4.2}$$

where the subscript z,t means that $\overline{L}$ acts on (z,t) variables.

Proof. Again, we may assume that $d((z,t),(w,s))=1$. We first see that

$$\begin{aligned}\overline{L}_{z,t}(\overline{A}^{1/k}-\bar{z}w) &= 1/k\overline{A}^{1/k-1}\overline{L}_{z,t}\overline{A}-w=\overline{A}^{1/k-1}|z|^{2(k-1)}z-w\\ &= (\overline{A}^{-1/k}|z|^2)^{k-1}(z-w)+w((|z|^2\overline{A}^{-1/k})^{k-1}-1).\end{aligned}$$

Since $|z|^2|A|^{-1/k} < C$ for some constant C and since $|z-w| \lesssim 1$ provided that $d((z,t),(w,s)) = 1$,

$$\begin{aligned} |(|z|^2\overline{A}^{-1/k})^{k-1} - 1| &\lesssim ||z|^2\overline{A}^{-1/k} - 1| = |\overline{A}^{-1/k}(|z|^2 - \overline{A}^{1/k})| \\ &= |\overline{A}^{-1/k}(|z|^2 - \bar z w) + \overline{A}^{-1/k}(\bar z w - \overline{A}^{1/k})| \\ &\lesssim |A|^{-1/k}(|z|+1), \end{aligned}$$

and hence

$$\begin{aligned} |\overline{L}_{z,t}(\overline{A}^{1/k} - \bar z w)| &\lesssim |z-w| + |A|^{-1/k}(|zw| + |w|) \\ &\approx |z-w| + \frac{|zw|+|w|}{|z|^2+1} \lesssim 1 \end{aligned}$$

since $d((z,t),(w,s)) = 1$. This completes the proof. □

Lemma 4.5. *Let*

$$\Lambda((z,t),(w,s)) = \frac{1}{4\pi^2}\left[z\frac{\overline{A}^{1/k} - |w|^2}{\overline{A} - |w|^{2k}} - w\frac{|z|^2 - A^{1/k}}{|z|^{2k} - A}\right].$$

Then

$$|\Lambda((z,t),(w,s))| \lesssim \frac{\delta}{|z|^{2k-2} + \delta^{2k-2}} \tag{4.3}$$

and

$$|\overline{L}_{z,t}\Lambda((z,t),(w,s))| \lesssim \frac{1}{|z|^{2k-2} + \delta^{2k-2}} \tag{4.4}$$

where $\delta = d((z,t),(w,s))$.

Proof. We may assume that $\delta = 1$ by the homogeneity. One can easily observe that if $d((z,t),(w,s)) = 1$, then $|z|^2 + 1 \approx |w|^2 + 1$ since $|z-w| \lesssim 1$. We will use this fact without mentioning. Since

$$a^k - b^k = (a-b)\prod_{j=1}^{k-1}(a - be^{2\pi ij/k}),$$

we can write Λ as

$$\begin{aligned} \Lambda((z,t),(w,s)) &= (z-w)\frac{\overline{A}^{1/k} - |w|^2}{\overline{A} - |w|^{2k}} + w\left[\frac{\overline{A}^{1/k} - |w|^2}{\overline{A} - |w|^{2k}} - \frac{A^{1/k} - |z|^2}{A - |z|^{2k}}\right] \\ &= (z-w)P_1 + wP_2 \end{aligned}$$

where

$$P_1 = \prod_{j=1}^{k-1}(\overline{A}^{1/k} - |w|^2 e^{2\pi ij/k})^{-1}$$

and

$$P_2 = \frac{\prod_{j=1}^{k-1}(A^{1/k} - |z|^2 e^{2\pi ij/k}) - \prod_{j=1}^{k-1}(\overline{A}^{1/k} - |w|^2 e^{2\pi ij/k})}{\prod_{j=1}^{k-1}(\overline{A}^{1/k} - |w|^2 e^{2\pi ij/k})(A^{1/k} - |z|^2 e^{2\pi ij/k})}.$$

We can easily see from Lemma 4.3(2) that

$$|P_1| \lesssim (|z|^{2k-2}+1)^{-1}.$$

On the other hand,

$$\begin{aligned}
&\left|\prod_{j=1}^{k-1}(A^{1/k}-|z|^2e^{2\pi ij/k})-\prod_{j=1}^{k-1}(\overline{A}^{1/k}-|w|^2e^{2\pi ij/k})\right| \\
&\quad=\left|\prod_{j=1}^{k-1}(A^{1/k}-|z|^2e^{2\pi ij/k})-\prod_{j=1}^{k-1}(A^{1/k}-|w|^2e^{2\pi ij/k})\right| \\
&\qquad+\left|\prod_{j=1}^{k-1}(A^{1/k}-|w|^2e^{2\pi ij/k})-\prod_{j=1}^{k-1}(\overline{A}^{1/k}-|w|^2e^{2\pi ij/k})\right| \\
&\quad=E_1+E_2
\end{aligned}$$

If we multiply the products out, it then follows from Lemma 4.3(a) that

$$\begin{aligned}
E_1 &\lesssim \sum_{j=1}^{k-1}||z|^{2j}-|w|^{2j}||A|^{(k-1-j)/k} \\
&\lesssim \sum_{j=1}^{k-1}||z|-|w||(|z|^{2j-1}+|w|^{2j-1})|A|^{(k-1-j)/k} \\
&\lesssim \sum_{j=1}^{k-1}||z|-|w||(|z|^{2j-1}+1)(|z|^{2(k-1-j)}+1) \\
&\lesssim |z-w|(|z|^{2k-3}+1).
\end{aligned}$$

And

$$\begin{aligned}
E_2 &\lesssim \sum_{j=1}^{k-1}|A^{j/k}-\overline{A}^{j/k}||w|^{2(k-1-j)} \\
&\lesssim |A^{1/k}-\overline{A}^{1/k}|\sum_{j=1}^{k-1}|A|^{(j-1)/k}|w|^{2(k-1-j)} \\
&\lesssim |A^{1/k}-\overline{A}^{1/k}|(|z|^{2k-4}+1).
\end{aligned}$$

But, since $|A^{1/k}-\overline{A}^{1/k}| = 2|A|^{1/k}\sin(\theta/k) \lesssim (|z|^2+1)|t-s|$, we have

$$E_2 \lesssim (|z|^{2k-3}+1)|t-s|.$$

Now, Lemma 4.3(2) and (3) together with above inequalities lead us to the estimate

$$|P_2| \lesssim \frac{(|z|^{2k-3}+1)}{(|z|^2+1)^{2(k-1)}} \lesssim (|z|^{2k-1}+1)^{-1}.$$

And, we finally have

$$|\Lambda((z,t),(w,s))| \lesssim |z-w||P_1|+|w||P_2| \lesssim \frac{1}{|z|^{2k-2}+1}.$$

For $\overline{L}_{z,t}\Lambda$, we perform the logarithmic derivative to have

$$\overline{L}_{z,t}\Lambda = \frac{-z}{\prod_{j=1}^{k-1}(\overline{A}^{1/k} - |w|^2 e^{2\pi i j/k})} \sum_{j=1}^{k-1} \frac{|z|^{2(k-1)}\bar{z}A^{1/k-1}}{(\overline{A}^{1/k} - |w|^2 e^{2\pi i j/k})}$$
$$+ \frac{w}{\prod_{j=1}^{k-1}(A^{1/k} - |z|^2 e^{2\pi i j/k})} \sum_{j=1}^{k-1} \frac{z e^{2\pi i j/k}}{(A^{1/k} - |z|^2 e^{2\pi i j/k})}.$$

It then follows from Lemma 4.3(2) and (3) that

$$|\overline{L}_{z,t}\Lambda| \lesssim \frac{|z|^2(|z|^2\overline{A}^{-1/k})^{k-1}}{(|z|^2+1)^k} + \frac{|z|^2+1}{(|z|^2+1)^k} \lesssim \frac{1}{|z|^{2k-2}+1}$$

since $|z|^2\overline{A}^{-1/k}$ is bounded. This completes the proof of Lemma 4.5. □

Lemma 4.6. *Let K be the relative fundamental solution of L on H_k. Then, off the diagonal, $-L_{z,t}K$ is the Szegö kernel and $\overline{L}_{w,s}K$ is the complex conjugate of the Szegö kernel.*

Proof. It is easy since $L_{z,t}(B) = L_{z,t}(\frac{w}{\overline{A}^{1/k} - \bar{z}w}) = 0$ and $L_{w,s}B = L_{w,s}(\frac{z}{A^{1/k} - z\bar{w}})$ $= 0$. □

Here and throughout this paper, we denote by $B((z,t),\delta)$ the ball of radius δ defined by the distance function (4.1). Proposition 3.4.4 of [D] shows that $|B((z,t),\delta)| \approx \delta^4(|z|^{2k-2} + \delta^{2k-2})$ where $|\cdot|$ is the Lebesgue measure.

Theorem 4.7. *Let K be the relative fundamental solution for L on H_k. Then, K satisfies estimates*

$$|K((z,t),(w,s))| \lesssim \frac{\delta}{|B((z,t),\delta)|} \tag{4.5}$$

and

$$|DK((z,t),(w,s))| \lesssim \frac{1}{|B((z,t),\delta)|} \tag{4.6}$$

where $\delta = d((z,t),(w,s))$ and D is one of $L_{z,t}, \overline{L}_{z,t}, \overline{L}_{w,s}$, and $L_{w,s}$.

Proof. If we use the identity (3.5), (4.5) comes from Lemma 4.5. Lemma 4.6 and Proposition 6.1.1 in [D] imply (4.6) when $D = L_{z,t}$ or $\overline{L}_{w,s}$. If $D = \overline{L}_{z,t}$, (4.6) comes from Lemmas 4.3 and 4.5 since

$$\overline{L}_{z,t}K = \Lambda \overline{L}_{z,t}\left(\frac{1}{|A^{1/k} - z\bar{w}|^2}\right) + \frac{1}{|A^{1/k} - z\bar{w}|^2}\overline{L}_{z,t}\Lambda.$$

$D = L_{w,s}$ case is also easy to show. □

5. Regularity of the solution operator

In this section, we study the regularity properties of the solution operator T defined by the relative fundamental solution (3.3) in terms of spaces such as L^p, weighted L^2, and the Lipschitz class $\mathrm{Lip}(\alpha)$. Since H_k is unbounded and K is homogeneous of degree $-2k-1$, it is evident that we can not expect L^2-boundedness of T without a weight. In this section the points in H_k are denoted by ζ, ξ, etc.

Our first theorem is analogous to a theorem on the Heisenberg group due to Folland and Stein (Theorem 15.3 of [FS]).

Theorem 5.1. *Let K be a homogeneous kernel function of degree $-2k-1$ and C^∞ off the diagonal, i.e.,*

$$K(\delta\zeta, \delta\xi) = \delta^{-2k-1} K(\zeta, \xi) \quad \textit{for any } \delta > 0$$

where ζ and ξ are points in H_k. Assume that K satisfies the estimate (4.5). *Define an operator T by*

$$Tf(\zeta) = \int_{H_k} f(\xi) K(\zeta, \xi)\, dV(\xi),$$

where dV is the Lebesgue measure on $H_k = \mathbf{R}^3$. Then, for any $f \in L^p$,

$$\| Tf \|_q \lesssim \| f \|_p$$

where $q^{-1} = p^{-1} + (2k+1)/(2k+2) - 1$.

Proof. By Lemma 15.3 of [FS], it suffices to show that $K(\zeta, \cdot)$ is in weak-L^r uniformly in ζ and $K(\cdot, \xi)$ is in weak-L^r uniformly in ξ where $r = (2k+1)/(2k+2)$. But,

$$\int_{|K(\zeta,\xi)| \geq \alpha} dV(\xi) = \alpha^{-r} \int_{|K(\zeta,\xi)| \geq 1} dV(\xi) \lesssim \alpha^{-r}$$

where the equality is from the change of variables and the homogeneity of K, and the inequality is due to the following lemma.

The same argument works for $\int_{|K(\zeta,\xi)| \geq \alpha} dV(\zeta)$. □

Lemma 5.2. *Let K be the same kernel as in Theorem* 5.1. *Then,*

$$\int_{|K(\zeta,\xi)| \geq 1} dV(\xi) \leq C \qquad \textit{independently of } \zeta,$$

and

$$\int_{|K(\zeta,\xi)| \geq 1} dV(\zeta) \leq C \qquad \textit{independently of } \xi.$$

Proof. It follows from the estimate (4.5) that

$$\begin{aligned} \int_{|K(\zeta,\xi)| \geq 1} dV(\xi) &\leq \int_{|K(\zeta,\xi)| \geq 1} |K(\zeta, \xi)|\, dV(\xi) \\ &\lesssim \int_{|K(\zeta,\xi)| \geq 1} \frac{d(\zeta, \xi)}{|B(\zeta, d(\zeta, \xi))|}\, dV(\xi). \end{aligned}$$

But, $d(\zeta,\xi) \gtrsim |B(\zeta,d(\zeta,\xi))| \gtrsim d(\zeta,\xi)^4[|z|^{2k-2} + d(\zeta,\xi)^{2k-2}]$ provided that $|K(\zeta,\xi)| \geq 1$. Therefore, $d(\zeta,\xi)^{-3} \gtrsim d(\zeta,\xi)^{2k-2}$ and hence $d(\zeta,\xi) < C$. It then follows that

$$\begin{aligned}\int_{|K(\zeta,\xi)|\geq 1} dV(\xi) &\lesssim \int_{d(\zeta,\xi)\leq C} \frac{d(\zeta,\xi)}{|B(\zeta,d(\zeta,\xi))|} dV(\xi)\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j} \int_{C2^{-j+1}\leq d(\zeta,\xi)<C2^{-j}} \frac{1}{|B(\zeta,d(\zeta,\xi))|} dV(\xi)\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j} \frac{|B(\zeta,C2^{-j})|}{|B(\zeta,C2^{-j+1})|} \lesssim C.\end{aligned}$$

by the doubling property of d. The second estimate can also be done in the same way. □

Theorem 5.3. *Let K and T be as in Theorem* 5.1. *If f is bounded and has a compact support, then Tf belongs to* Lip$(1/2k)$.

Proof. This follows immediately from Theorem 14(b) of [RS]. □

Lemma 5.4. *Let K be a positive kernel function satisfying the assumptions of Theorem* 5.1. *Then, for* $1 < \lambda < 2k+2$,

$$\int (|\xi|+1)^{-\lambda} K(\zeta,\xi)\, dV(\xi) \lesssim (|\zeta|+1)^{-\lambda+1}. \tag{5.1}$$

Proof. Let M be a constant for the pseudo-triangular inequality, namely,

$$d(\zeta,\xi) \leq M(d(\zeta,\theta) + d(\theta,\xi))$$

for any ζ, ξ, and θ.

If $|\zeta| < 1$, we have to prove

$$\int (|\xi|+1)^{-\lambda} K(\zeta,\xi) dV(\xi) \leq C.$$

Let us split the integral into two parts, namely, the integral over $\{\xi | d(\zeta,\xi) \geq 2M\}$ and that over $\{\xi | d(\zeta,\xi) \leq 2M\}$ and call them I_1 and I_2 respectively. Then, by the proof of Lemma 5.2,

$$I_2 \lesssim \int_{d(\zeta,\xi)\leq 2M} K(\zeta,\xi)\, dV(\xi) \leq C.$$

If $d(\zeta,\xi) \geq 2M$ and $|\zeta| < 1$, then $d(\zeta,\xi) \approx |\xi|$ and hence

$$\begin{aligned}I_1 &\lesssim \int_{d(\zeta,\xi)\geq 2M} \frac{1}{d(\zeta,\xi)^{\lambda}} \frac{d(\zeta,\xi)}{|B(\zeta,d(\zeta,\xi))|} dV(\xi)\\ &= \int_{d(\zeta,\xi)\geq 2M} \frac{1}{d(\zeta,\xi)^{\lambda-1}|B(\zeta,d(\zeta,\xi))|} dV(\xi)\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j(\lambda-1)} \int_{C2^{j-1}\leq d(\zeta,\xi)<C2^{j}} \frac{1}{|B(\zeta,d(\zeta,\xi))|} dV(\xi)\\ &\lesssim \sum_{j=0}^{\infty} 2^{-j(\lambda-1)} \frac{|B(\zeta,C2^{j})|}{|B(\zeta,C2^{j-1})|} \leq C\end{aligned}$$

provided that $\lambda > 1$.

To prove the estimate (5.1) when $|\zeta| \geq 1$, it is enough, by the homogeneity, to show that

$$\int |\xi|^{-\lambda} K(\zeta,\xi) dV(\xi) \leq C \tag{5.2}$$

independently of ζ such that $|\zeta| = 1$. Let $|\zeta| = 1$ and split the integral in the left-hand side of (5.2) into three parts: an integral over $\{d(\zeta,\xi) \leq 1/(2M)\}$, one over $\{|\xi| < 1, d(\zeta,\xi) > 1/(2M)\}$, and another over $\{|\xi| > 1, d(\zeta,\xi) \geq 1/(2M)\}$. Let us call them J_1, J_2, and J_3. If $d(\zeta,\xi) \leq 1/(2M)$, then $|\xi| \geq C$ for some constant C since $|\zeta| = 1$. Therefore, it follows from the proof of Lemma 5.2 that

$$J_1 \lesssim \int_{d(\zeta,\xi)\leq 1/2M} K(\zeta,\xi)\, dV(\xi) \lesssim C.$$

If $d(\zeta,\xi) \geq 1/(2M)$, then $|B(\zeta, d(\zeta,\xi))| \geq C$ for some constant C and hence, by the estimate (4.5),

$$J_2 \leq \int_{|\xi|<1} |\xi|^{-\lambda}\, dV(\xi) \lesssim C$$

provided that $\lambda < 2k+2$. If $|\xi| > 1$ and $d(\zeta,\xi) \geq 1/(2M)$, then $|\xi| \gtrsim d(\zeta,\xi)$ and therefore,

$$\begin{aligned} J_3 &\lesssim \int_{d(\zeta,\xi)\geq 1/(2M)} \frac{1}{d(\zeta,\xi)^{\lambda-1}|B(\zeta,d(\zeta,\xi))|}\, dV(\xi) \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j(\lambda-1)} \int_{C2^j \leq d(\zeta,\xi) < C2^{j+1}} \frac{1}{|B(\zeta,d(\zeta,\xi))|}\, dV(\xi) \\ &\lesssim \sum_{j=0}^{\infty} 2^{-j(\lambda-1)} \frac{|B(\zeta, C2^{j+1})|}{|B(\zeta, C2^j)|} \leq C \end{aligned}$$

if $\lambda > 1$. This completes the proof.

Now, we are ready to show our main theorem. We shall make use of a theorem due to S. Bloom [B, Theorem 1].

Theorem 5.5. *Let $K(\zeta,\xi)$ be a positive kernel function satisfying the assumptions in Theorem 5.1, and let $Tf(\zeta) = \int K(\zeta,\xi)f(\xi)\, dV(\xi)$. Let $\omega(\zeta) = (|\zeta|+1)^{-2}$ where $|\zeta| = d(0,\zeta)$. Then, T is a bounded operator from L^2 to L^2_ω.*

Proof. By Theorem 1 of [B], it suffices to show that there is a nonnegative function $\alpha \in L^2$ such that $\omega^{1/2}T(\alpha) \leq C\alpha$ and $T^*(\alpha\omega^{1/2}) \leq C\alpha$ where T^* is the adjoint operator of T, i.e., $T^*f(\zeta) = \int K(\xi,\zeta)f(\xi)\, dV(\xi)$. We note that $K(\xi,\zeta)$ also satisfies the assumptions of Theorem 5.1.

Choose α to be $\alpha(\zeta) = (|\zeta|+1)^{-k-\lambda}$ for some λ. It then follows from Lemma 5.4 that if $1-k < \lambda < k+2$, then

$$T(\alpha) = \int (|\xi|+1)^{-k-\lambda} K(\zeta,\xi)\, dV(\xi) \lesssim (|\zeta|+1)^{-k-\lambda+1} = \alpha(\zeta)\omega(\zeta)^{-1/2}.$$

And it also follows that if $-k < \lambda < k+1$, then

$$T^*(\alpha\omega^{1/2}) = \int (|\xi|+1)^{-k-\lambda-1} K(\zeta,\xi)\, dV(\xi) \le \alpha(\zeta).$$

Choose $\lambda = 3/2$. Then, α obviously belongs to L^2 and proof is completed.

Combining Theorems 3.1 and 5.5, one can show the following theorem.

Theorem 5.6. *Let K and T be as defined in Theorem 3.1 and let $\omega(\zeta) = (|\zeta|+1)^{-2}$. If $f \in L^2(\mathbf{R}^3)$ satisfy the solvability condition $Pf = 0$, then there exists a unique $u \in L^2_\omega$ such that $Lu = f$ and $\| u \|_{2,\omega} \le \| f \|_2$.*

Remark 5.7. In the next section, we will demonstrate that the exponent -2 for the weight is the best possible among the radial weights.

6. Counterexamples

One of the differences between H_k and bounded domains is that the tangential holomorphic vector field L on H_k as an operator from L^2 to L^2 does not have a closed range. In this section, we prove this fact by giving a counterexample. Our example also shows that the exponent -2 for the weight in Theorem 5.6 is the best possible one that one can have.

Let us put $\omega_\alpha(\zeta) = (|\zeta|+1)^{-\alpha}$ for any number α. Then $\omega_2 = \omega$ which was introduced in previous section. If $\alpha > 2$, then by Hölder's inequality and Theorem 5.1, it can be shown that if K and T are as defined in Theorem 5.1, then

$$\| Tf \|_{2,\omega_\alpha} \lesssim \left[\int \omega_\alpha(\zeta)^p dV(\zeta) \right]^{1/p} \| f \|_2$$

where $p = k+1$. But, if $\alpha > 2$, then

$$\begin{aligned} \int \omega_\alpha(\zeta)^p \, dV(\zeta) &= \int (1+|\zeta|)^{-\alpha(k+1)} \, dV(\zeta) \\ &\lesssim C + \int_{|\zeta|>1} (1+|\zeta|)^{-\alpha(k+1)} \, dV(\zeta) \\ &\lesssim C + \int_1^\infty s^{-\alpha(k+1)} s^{2k+1} ds \le C. \end{aligned}$$

Therefore, T is a bounded operator from L^2 to $L^2_{\omega_\alpha}$ if $\alpha > 2$. Now, we prove that T is not bounded from L^2 to $L^2_{\omega_\alpha}$ if $\alpha < 2$.

By the closed range theorem, the operator L has a closed range if and only if $\text{Range}(L) = \text{Ker}(L^*_\alpha)^\perp$ where L^*_α is the adjoint of the operator L from $L^2_{\omega_\alpha}$ into L^2. We can easily see that $L^*_\alpha = \omega_\alpha^{-1} L^* = -\omega_\alpha^{-1}\overline{L}$. Therefore, L has a closed range if and only if $\text{Range}(L) = (H^2)^\perp$.

Theorem 6.1. *Let T be the operator defined in Theorem 5.1. Then, the following are equivalent:*

(1) $L\colon L^2_{\omega_\alpha} \to L^2$ *has a closed range and it is* $(H^2)^\perp$,

(2) $T\colon (H^2)^\perp \to L^2_{\omega_\alpha}$ *is bounded.*

Proof. Well known. □

Theorem 6.2. (2) *is false and hence so is* (1) *if* $\alpha < 2$.

Proof. Since the construction of a function in the following relies only on the homogeneity, we may assume that $k = 1$. Let $\alpha < 2$, and choose β so that $2 < \beta < 3 - \alpha/2$. Define a smooth function g by

$$g(\zeta) = \begin{cases} |\zeta|^{-\beta+1} & \text{if } |\zeta| > 1, \\ 0 & \text{if } |\zeta| < 1/2. \end{cases}$$

and let $f = Lg$. Then, f is homogeneous of degree $-\beta$ for $|\zeta| > 1$. Furthermore, $f \in L^2$ and $Pf = PLg = 0$, namely, $f \in (H^2)^\perp$. We claim that $Tf \notin L^2_{\omega_\alpha}$ if $\alpha < 2$.

In fact, we define F to be f for $|\zeta| > 1$ and extend it to all of $\mathbf{R}^3$ by the homogeneity. Define $f_1 = F - f$. Then,

$$\|f_1\|_{4/3}^{4/3} \le \int_{|\zeta|\le 1} |\zeta|^{-4\beta/3}\, dV(\zeta) \lesssim \int_0^1 s^{-4\beta/3+3}\, ds < \infty$$

since $\beta \le 3 - \alpha/2$. It then follows from Theorem 5.1 that $Tf_1 \in L^2 \subset L^2_{\omega_\alpha}$. On the other hand, TF defined by $TF = Tf_1 + Tf$ is not in $L^2_{\omega_\alpha}$ as one can see easily from the homogeneity. Therefore, $Tf \notin L^2_{\omega_\alpha}$. This completes the proof.

REFERENCES

[B] S. Bloom, *Solving weighted norm inequalities using the Rubio de Francia algorithm*, Proc. Amer. Math. Soc. **101** (1987), 306–312.

[BS] H. P. Boas and M. C. Shaw, *Sobolev estimates for the Lewy operator on weakly pseudo convex boundaries*, Math. Ann. **274** (1986), 221–231.

[C] M. Christ, *Regularity properties of the* $\bar\partial_b$ *equation on weakly pseudoconvex CR manifolds of dimension* 3, J. Amer. Math. Soc. **3** (1988), 587–643.

[CW] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Math., vol. 242, Springer-Verlag, 1971.

[D] K. P. Diaz, *The Szegö kernel as a singular integral kernel on a family of pseudoconvex domains*, Trans. Amer. Math. Soc. **304** (1987), 147–170.

[FK] C. Fefferman and J. J. Kohn, *Hölder estimates on domains of complex dimension two and three dimensional CR manifolds*, Adv. in Math. **70** (1988).

[FoK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Stud., no. 75, Princeton Univ. Press., 1972.

[FS] G. B. Folland and E. M. Stein, *Estimates for the* $\bar\partial_b$ *complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[GKS] P. C. Greiner, J. J. Kohn and E. M. Stein, *Necessary and sufficient conditions for the solvability of the Lewy equation*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 3287–3289.

[K] J. J. Kohn, *The range of the tangential Cauchy-Riemann operator*, Duke Math. J. **53** (1986), 525–545.

[N] A. Nagel, *Vector fields and nonisotropic metrics*, Beijing Lectures in Harmonic Analysis, E. M. Stein, Ed., Princeton Univ. Press, 1986.

[NRSW] A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger, *Estimates for the Bergman and Szegö kernels in certain weakly pseudoconvex domains*, Bull. Amer. Math. Soc. **18** (1988), 55–59.

[NSW1] A. Nagel, E. M. Stein and S. Wainger, *Boundary behavior of functions holomorphic in domains of finite type*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 6596–6599.

[NSW2] ——, *Balls and metrics defined by vector fields* I: *basic properties*, Acta Math. **155** (1985), 103–147.

[R] J.-P. Rosay, *Equation de Lewy-resolubilite globale de l'equation* $\bar{\partial}_b u = f$ *sur la frontiere de domain faiblement pseudo-convexes de* C^2 (*on* $\mathbf{C}^n$), Duke Math. J. **49** (1982), 121–128.

[RS] L. Rothschild and E. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[S] M. C. Shaw, L^2*-estimates and existence theorems for the tangential Cauchy Riemann complex*, Invent. Math. **82** (1985), 133–150.

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