QUANTIZATION OF CURVATURE OF HARMONIC TWO-SPHERES IN GRASSMANN MANIFOLDS

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ABSTRACT. Various pinching theorems for curvature of minimal two-spheres in Grassmann manifolds have been proved. In particular, we show that when the curvature is large, then the minimal map from S^2 into $G(m, N)$ must be either holomorphic or antiholomorphic. Also, minimal two-spheres of curvature $\kappa \geq 2$ in $G(2, 4)$ have been classified.

0. INTRODUCTION

It is known that starting from a harmonic map f from a Riemann surface into a complex Grassmann manifold, one can define sequences of harmonic maps by using the ∂ -transform and the $\bar{\partial}$ -transform. These sequences are called harmonic sequences [CW]. When f is holomorphic, the sequence generated by f via the ∂ -transform is called a pseudoholomorphic sequence, and the corresponding maps are called pseudoholomorphic maps.

In 1980 A. M. Din and W. J. Zakrzewski [DZ], and then in 1983 J. Eells and J. C. Wood [EC], showed that all harmonic maps from S^2 to the complex projective space CP^n are, in fact, pseudoholomorphic. Recently, F. Burstall, J. Rawnsley and S. Salamon announced in [BRS] that all stable harmonic maps from S^2 into irreducible Hermitian symmetric spaces are either holomorphic or antiholomorphic. It is also known that when the target space is a general complex Grassmann manifold, holomorphic curves do not generate all harmonic maps [CW], [Ra].

On the other hand, M. Rigoli showed in [R] that for a linearly full nonsingular holomorphic curve from S^2 into CP^n , if its curvature $\kappa \geq 4/n$, then $\kappa = 4/n$ is constant. Then J. Bolton et al. [BJRW] generalized this result to pseudoholomorphic maps, and showed that for a linearly full isometric pseudoholomorphic map of position r from S^2 into CP^n , if its curvature $\kappa \geq 4/(2r(n-r) + n)$, then $\kappa = 4/(2r(n-r) + n)$ is constant.

In this paper, we prove various pinching theorems of curvature for harmonic maps and show that for any nonsingular harmonic map from S^2 into a complex Grassmann manifold, if the curvature of its induced metric is large, then

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it is actually a holomorphic or antiholomorphic curve of constant curvature. Precisely, we show that for any harmonic isometric immersion from S^2 to $G(m, N)$, if it is not antiholomorphic (or not holomorphic), and if its curvature $\kappa \geq 4/\text{rank}(\partial f)$ (or $\kappa \geq 4/\text{rank}(\bar{\partial} f)$), then f must be a holomorphic map (or antiholomorphic map) of constant curvature $4/\text{rank}(\partial f)$ (or $4/\text{rank}(\bar{\partial} f)$). When f generates an orthogonal harmonic sequence and if its curvature $\kappa \geq \kappa(f)$, a number associated to the sequence generated by f , then $\kappa = \kappa(f)$ is constant, and f is a holomorphic map which essentially generates a Frenet pseudoholomorphic sequence. We also show that for a harmonic map f in a Frenet harmonic sequence of position j generated by a harmonic map g , if its curvature $\kappa \geq \kappa(g, j)$, a number associated to the sequence and j , then again $\kappa = \kappa(g, j)$ is constant and the directrix g is holomorphic. In particular, f is a pseudoholomorphic map of constant curvature $\kappa(g, j)$. Notice that when the directrix is holomorphic and the target space is CP^n , we recover the pinching theorem of J. Bolton, et al.

Some relations between the pinching conditions of curvature and the ranks of the ∂ -transform and the $\bar{\partial}$ -transform are discussed in this paper. As examples of using the technique developed in this paper, we prove some pinching theorems for harmonic two-spheres in $G(2, 4)$ and $G(2, 6)$. As a by-product, we show that for a minimal two-sphere in $G(2, 4)$, if its curvature $\kappa \geq 2$, then κ is either 2 or 4, and all these maps are explicitly classified up to $u(4)$ -congruences.

The main tool used is the method of moving frames. Extensively using this method, we are able to develop a system of partial differential equations associated to a family of invariants. The study of these invariants plays a key role in proving the main results of this paper.

Definitions and basic formulas are given in §1. General pinching theorems are proved in §2. In §3, we give a detail discussion of harmonic two-spheres in $G(2, 4)$ and $G(2, 6)$.

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1. HARMONIC MAPS, HARMONIC SEQUENCES AND PSEUDOHOLOMORPHIC MAPS

The complex Grassmann manifold $G(m, N)$ is the set of all m -dimensional complex linear subspaces of C^N through the origin, which can be realized as the homogeneous space $U(N)/(U(m) \times U(N - m))$, where

$$\begin{aligned} \pi: U(N) &\rightarrow G(m, N) \\ g &\rightarrow [g \cdot 0] \end{aligned}$$

is a principal $U(m) \times U(N - m)$ -bundle, 0 is the $N \times m$ matrix

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix},$$

and I_m is the $m \times m$ identity matrix.

Let $W = (W_{AB}) = g^{-1} dg$ be the Maurer-Cartan form of $U(N)$, where $1 \leq A, B \leq N$, and let e be a local section of $\pi: U(N) \rightarrow G(m, N)$. Set

$$(1.1) \quad \Psi_{AB} = e^*(W_{AB})$$

and

$$(1.2) \quad ds_{G(m, N)}^2 = \sum_{\substack{1 \leq i \leq m \\ m < \alpha \leq N}} \Psi_{i\alpha} \bar{\Psi}_{i\alpha}.$$

Then $ds_{G(m, N)}^2$ is a $U(N)$ -invariant Hermitian metric on $G(m, N)$. When $m = 1$ and $N = n + 1$, this is just the Fubini-Study metric on CP^n of constant holomorphic curvature 4.

Suppose M is a Riemann surface with a Riemannian metric ds_M^2 . Then locally

$$(1.3) \quad ds_M^2 = \theta \bar{\theta}$$

and

$$(1.4) \quad d\theta = iw \wedge \theta,$$

where θ is a local unitary coframe of bidegree $(1, 0)$ which is determined up to a $U(1)$ -valued function, and w is the real-valued Levi-Civita connection form associated to the coframe θ . Also,

$$(1.5) \quad dw = \frac{i}{2} \kappa \theta \wedge \bar{\theta},$$

where κ is the Gaussian curvature of ds_M^2 .

Let f be a smooth map from M to $G(m, N)$. Choose a local unitary frame $e = (e_1, \dots, e_N)$ along f such that $e_1, \dots, e_m$ span f . Then for $1 \leq A \leq N$

$$(1.6) \quad de_A = \sum_{B=1}^n e_B w_{BA},$$

where $w_{AB} = f^*(\Psi_{AB}) = e^*(W_{AB})$ are the entries of the pull back of the Maurer-Cartan form of $U(N)$ via e .

The Maurer-Cartan structure equations give

$$(1.7) \quad dw_{AB} = - \sum_{c=1}^N w_{AC} \wedge w_{CB}$$

and

$$(1.8) \quad w_{AB} + \overline{w_{BA}} = 0.$$

For $1 \leq j \leq m$ and $m + 1 \leq \alpha \leq N$, set

$$(1.9) \quad w_{\alpha j} = a_{\alpha j} \theta + b_{\alpha j} \bar{\theta}.$$

Then f is an isometric immersion if and only if

$$(1.10) \quad \sum_{\substack{1 \leq j \leq m \\ m < \alpha \leq N}} a_{\alpha j} b_{\alpha j} = 0$$

and

$$(1.11) \quad \sum_{\substack{1 \leq j \leq m \\ m < \alpha \leq N}} (a_{\alpha j} \overline{a_{\alpha j}} + b_{\alpha j} \overline{b_{\alpha j}}) = 1.$$

The harmonicity condition for f is

$$(1.12) \quad da_{\alpha j} + ia_{\alpha j} w + \sum_{\beta=m+1}^n a_{\beta j} w_{\alpha\beta} - \sum_{t=1}^m a_{\alpha t} w_{tj} \equiv 0 \pmod{\theta},$$

which is equivalent to

$$(1.13) \quad db_{\alpha j} - ib_{\alpha j} w + \sum_{\beta=m+1}^n b_{\beta j} w_{\alpha\beta} - \sum_{t=1}^m b_{\alpha t} w_{tj} \equiv 0 \pmod{\bar{\theta}}.$$

In terms of matrix notation, we can rewrite these equations as

$$(1.10') \quad \text{tr}(A^t \bar{B}) = 0,$$

$$(1.11') \quad |A|^2 + |B|^2 = 1,$$

$$(1.12') \quad dA = -iwA + A\Phi_{11} - \Phi_{22}A \pmod{\theta}$$

and

$$(1.13') \quad dB = iwB + B\Phi_{11} - \Phi_{22}B \pmod{\bar{\theta}},$$

where

$$A = \begin{pmatrix} a_{m+1,1} & \cdots & a_{m+1,m} \\ \vdots & & \vdots \\ a_{N,1} & \cdots & a_{N,m} \end{pmatrix}, \quad B = \begin{pmatrix} b_{m+1,1} & \cdots & b_{m+1,m} \\ \vdots & & \vdots \\ b_{N,1} & \cdots & b_{N,m} \end{pmatrix},$$

$$\Phi_{11} = \begin{pmatrix} w_{1,1} & \cdots & w_{1,m} \\ \vdots & & \vdots \\ w_{m,1} & \cdots & w_{m,m} \end{pmatrix} \quad \text{and} \quad \Phi_{22} = \begin{pmatrix} w_{m+1,m+1} & \cdots & w_{m+1,N} \\ \vdots & & \vdots \\ w_{N,m+1} & \cdots & w_{N,N} \end{pmatrix}.$$

Here we define the norm $|x|$ of a matrix x by $|x|^2 = \text{tr}(x^t \bar{x})$ in a standard way.

For a harmonic map f from M into $G(m, N)$, using the ∂ -transform and the $\bar{\partial}$ -transform S. S. Chern and J. G. Wolfson defined new harmonic maps, $\partial f: M \rightarrow G(m_1, N)$, if f is not antiholomorphic, and $\bar{\partial} f: M \rightarrow G(m_{-1}, N)$, if f is not holomorphic. These maps satisfy

$$(1.14) \quad \partial f(x) = \text{span} \left\{ \sum_{\alpha=m+1}^N a_{\alpha j}(x) e_{\alpha}(x) : 1 \leq j \leq m \right\}$$

and

$$(1.15) \quad \bar{\partial}f(x) = \text{span} \left\{ \sum_{\alpha=m+1}^N b_{\alpha j}(x) e_{\alpha}(x) : 1 \leq j \leq m \right\}$$

for $x \in M$, except for at most isolated points. Here m_1 and m_{-1} are positive integers, called the rank of ∂f and the rank of $\bar{\partial}f$, respectively. Successively using these transforms they obtained two sequences of harmonic maps

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \rightarrow \dots$$

and

$$f \equiv f_0 \xrightarrow{\bar{\partial}} f_{-1} \xrightarrow{\bar{\partial}} f_{-2} \rightarrow \dots,$$

which are called the harmonic sequences, where for $j \geq 1$, $f_j \equiv \partial f_{j-1} : M \rightarrow G(m_j, N)$, and for $j \leq -1$, $f_j \equiv \bar{\partial} f_{j+1} : M \rightarrow G(m_j, N)$. We say that f_j and f_i are orthogonal, if for $x \in M$, $f_j(x) \perp f_i(x)$ as linear subspaces of C^N with respect to the standard Hermitian inner product. If $f_j \perp f_i$ for $i \neq j$ in a sequence, we say this sequence is orthogonal. In addition, if m_j 's are equal, we have

$$f_0 \xleftrightarrow{\partial} f_1 \xleftrightarrow{\partial} f_2 \xleftrightarrow{\partial} f_3 \dots$$

which is called a Frenet harmonic sequence. It is a finite sequence, and f_0 is called the directrix of this sequence.

When f is holomorphic, it will generate a harmonic sequence,

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \rightarrow \dots$$

which is orthogonal and therefore is finite. This sequence is called a pseudoholomorphic sequence, and each map f_j is called a pseudoholomorphic map generated by f with position j .

A locally defined smooth function $V : M \supset U \rightarrow C^L$ is of analytic type, if either v is identically zero on U or for any local complex coordinate z centered about x , the local representation of v has the form $v(z) = z^r g(z)$ where $g(z)$ is a C^L -valued smooth function such that $g(0) \neq 0$.

The following proposition will be repeatedly used in this paper, whose proof can be found in [Z].

Proposition 1.1. *Let U be an open subset of M , C be a $k \times k$ -matrix-valued smooth function defined on U , and θ be a unitary coframe field defined on U . Suppose that C satisfies*

$$(1.16) \quad dC = C\eta + \phi\eta - C\Psi \text{ mod } \theta,$$

where η is an imaginary valued 1-form and ϕ and Ψ are $k \times k$ -matrix-valued 1-forms with $(\text{tr } \phi - \text{tr } \Psi)$ imaginary valued. Then $\det C$ is a function of analytic type, and

$$(1.17) \quad (\Delta \log |\det C|) \theta \wedge \bar{\theta} = 2kd\eta + 2d(\text{tr } \phi) - 2d(\text{tr } \Psi)$$

on $U \setminus Z$, where Z is the set of zeros of $\det C$ on U .

Remark. When $k = 1$ and $\phi = \Psi = 0$, this is the formula given in [EGT].

2. QUANTIZATION OF CURVATURE OF HARMONIC TWO-SPHERES IN $G(m, N)$

Let f be a harmonic map from a Riemann surface M into $G(m, N)$. Take a local unitary frame $e = (e_1, \dots, e_N)$ along f such that $e_1, \dots, e_m$ span f . Then

$$e^*(W) = \phi = \begin{bmatrix} \phi_{11} & -{}^t\bar{A}\bar{\theta} & -{}^t\bar{B}\bar{\theta} \\ A\theta + B\bar{\theta} & & \phi_{22} \end{bmatrix},$$

where $A, B \in C^\infty(M((N-m) \times m))$, $\phi_{11} \in C^\infty(u(m) \otimes T^*M)$ and $\phi_{22} \in C^\infty(u(N-m) \otimes T^*M)$. Here we denote $C^\infty(M(m_1 \times m_2))$ as the set of all locally defined $m_1 \times m_2$ -matrix-valued smooth functions and $C^\infty(u(k) \otimes T^*M)$ as the set of all locally defined $u(k)$ -valued smooth 1-forms, where $u(k)$ is the Lie algebra of $U(k)$.

The frame e is determined up to $U(m) \times U(n-m)$ -transformations. Changing such a local frame, $\tilde{e} = e \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, where $K_1 \in C^\infty(U(m))$ and $K_2 \in C^\infty(U(N-m))$, we shall have

$$\tilde{e}^*(W) = \tilde{\phi} = \begin{bmatrix} \tilde{\phi}_{11} & -{}^t\tilde{A}\tilde{\theta} - {}^t\tilde{B}\tilde{\theta} \\ \tilde{A}\tilde{\theta} + \tilde{B}\tilde{\theta} & \tilde{\phi}_{22} \end{bmatrix}$$

where $\tilde{A} = K_2^{-1}AK_1$ and $\tilde{B} = K_2^{-1}BK_1$. So,

$$(2.1) \quad {}^t\tilde{A}\tilde{B} = \bar{K}_1^{-1}{}^tA\bar{B}\bar{K}_1.$$

Setting $C = {}^tA\bar{B}$ and $\tilde{C} = {}^t\tilde{A}\tilde{B}$, we see that $\det C = \det \tilde{C}$ and $\text{tr } C = \text{tr } \tilde{C}$. Notice that changing a local unitary coframe θ into $\tilde{\theta}$, $\det C$ becomes $u_1 \det C$ and $\text{tr } C$ becomes $u_2 \text{tr } C$, where u_1 and u_2 are smooth functions of absolute value 1. Thus $|\det C|$ and $|\text{tr } C|$ are globally defined nonnegative invariants on M , which are continuous.

Proposition 2.1. *$\det C$ and $\text{tr } C$ are of analytic type. Furthermore, away from zeros of $|\det C|$ and $|\text{tr } C|$, they satisfy*

$$(2.2) \quad \Delta \log |\det C| = 2m\kappa$$

and

$$(2.3) \quad \Delta \log |\text{tr } C| = 2\kappa,$$

where κ is the Gaussian curvature of the metric ds_M^2 .

Proof. Taking the exterior derivative of C and using (1.12) and (1.13) we obtain

$$(2.4) \quad dC = -2iwC + {}^t\phi_{11}C - C{}^t\phi_{11} \mod \theta.$$

Applying Proposition 1.1 to (2.4),

$$(2.5) \quad \Delta \log |\det C| \theta \wedge \bar{\theta} = -4mi dw.$$

(2.2) then follows from using (1.5) to (2.5).

For (2.3), we take the trace on both sides of (2.4). Then

$$(2.6) \quad d(\operatorname{tr} C) = -2iw(\operatorname{tr} C) \pmod{\theta},$$

and the conclusion again is obtained by using (1.5) and Proposition 1.1 for the case of $k = 1$. Q.E.D.

Following this proposition, we have

Corollary 2.1 (Vanishing Theorem). *If M is a closed Riemann surface, then either the genus g of M is positive or both $|\det c|$ and $|\operatorname{tr} c|$ vanish identically. In particular, if the Gaussian curvature $\kappa \geq 0$, then either $\kappa = 0$ or both $|\det c|$ and $|\operatorname{tr} c|$ are identically zero.*

Proof. Use the Gauss-Bonnet formula and Stokes' Theorem on (2.2) and (2.3). Q.E.D.

Remarks. 1. Corollary 2.1 is known [CW].

2. When $M = S^2$, then $|\operatorname{tr} C| = 0$, which means that harmonic maps from S^2 into $G(m, N)$ are weakly conformal (i.e., (1.10) holds).

We say that the ∂ -transform (resp. the $\bar{\partial}$ -transform) is degenerate, if $\operatorname{rank}(\partial f) < \operatorname{rank}(f)$ (resp. $\operatorname{rank}(\bar{\partial} f) < \operatorname{rank}(f)$).

Corollary 2.2. *For a harmonic map from S^2 into $G(m, 2m)$, one of the ∂ -transform and the $\bar{\partial}$ -transform must be degenerate.*

Proof. Observe that in this case $\det C = \det A \cdot \det \bar{B}$, and generically $\operatorname{rank}(A) = \operatorname{rank}(\partial f)$ and $\operatorname{rank}(B) = \operatorname{rank}(\bar{\partial} f)$. Q.E.D.

Now suppose that $M = S^2$ is the Riemann sphere and f is a harmonic isometric immersion so that κ is the Gaussian curvature of the induced metric $f^*ds_{G(m, N)}^2$. The following theorem says that if the curvature is large enough, then f is forced to be holomorphic or antiholomorphic.

Theorem 2.1. *If f is not antiholomorphic, and if $\kappa \geq 4/\operatorname{rank}(\partial f)$, then f is a holomorphic map of constant curvature $4/\operatorname{rank}(\partial f)$. Similarly, if f is not holomorphic, and if $\kappa \geq 4/\operatorname{rank}(\bar{\partial} f)$, then f is an antiholomorphic map of constant curvature $4/\operatorname{rank}(\bar{\partial} f)$.*

Proof. Suppose f is not antiholomorphic. Choose a local unitary frame $e = (e_1, \dots, e_N)$ along f so that $e_1, \dots, e_m$ span f and $e_{m+1}, \dots, e_{m+k_0}$ span ∂f , where $k_0 = \operatorname{rank}(\partial f)$. Furthermore, we can require that $e_{k_0+1}, \dots, e_m$ span the kernel of the ∂ -transform. The pull back of the Maurer-Cartan form of $U(N)$ by e is then

$$\phi = \begin{bmatrix} \Omega_{11} & \Omega_{12} & -{}^t\bar{A}_{11}\bar{\theta} - {}^t\bar{B}_{11}\theta & -{}^t\bar{B}_{21}\theta \\ \Omega_{21} & \Omega_{22} & -{}^t\bar{B}_{12}\theta & -{}^t\bar{B}_{22}\theta \\ A_{11}\theta + B_{11}\bar{\theta} & B_{12}\bar{\theta} & \Omega_{33} & \Omega_{34} \\ B_{21}\bar{\theta} & B_{22}\bar{\theta} & \Omega_{43} & \Omega_{44} \end{bmatrix},$$

where $A_{11}, B_{11} \in C^\infty(u(k_0 \times k_0))$, $B_{21} \in C^\infty(M((N - m - k_0) \times k_0))$, $B_{12} \in C^\infty(M(k_0 \times (m - k_0)))$, $B_{22} \in C^\infty(M(m - k_0) \times (m - k_0))$, $\Omega_{11}, \Omega_{33} \in$

$C^\infty(u(k_0) \otimes T^*S^2)$, $\Omega_{21} \in C^\infty(M((m - k_0) \times k_0) \otimes T^*S^2)$, $\Omega_{12} = -\bar{\tau}\Omega_{21}$, $\Omega_{22} \in C^\infty(u(m - k_0) \otimes T^*S^2)$, $\Omega_{43} \in C^\infty(M((N - m - k_0) \times k_0) \otimes T^*S^2)$, $\Omega_{34} = -\bar{\tau}\Omega_{43}$ and $\Omega_{44} \in C^\infty(u(N - m - k_0) \otimes T^*S^2)$. From (1.11'), we have

$$(2.7) \quad |A_{11}|^2 + |B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2 + |B_{22}|^2 = 1.$$

It can be easily checked that $|\det A_{11}|$ is a well-defined invariant on S^2 .

Harmonicity condition (1.12') of f gives

$$(2.8) \quad d \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} = -iw \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \\ - \begin{bmatrix} \Omega_{33} & \Omega_{34} \\ \Omega_{43} & \Omega_{44} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mod \theta,$$

from which it follows that

$$(2.9) \quad dA_{11} = -iwA_{11} + A_{11}\Omega_{11} - \Omega_{33}A_{11} \mod \theta,$$

$$(2.10) \quad \Omega_{43}A_{11} = 0 \mod \theta$$

and

$$A_{11}\Omega_{12} = 0 \mod \theta.$$

(2.10) shows that Ω_{12} and Ω_{43} are matrices of bidegree (1,0) forms. So

$$(2.11) \quad \Omega_{12} = X_{12}\theta$$

and

$$(2.12) \quad \Omega_{43} = X_{43}\theta,$$

where X_{12} and X_{43} are matrix-valued smooth functions.

Applying Proposition 1.1 to (2.9) and using (1.5), we get

$$(2.13) \quad \Delta \log |\det A_{11}| \theta \wedge \bar{\theta} = k_0 \kappa \theta \wedge \bar{\theta} + 2d(\operatorname{tr} \Omega_{11} - \operatorname{tr} \Omega_{33}).$$

By (1.7), (2.7), (2.10) and (2.11),

$$(2.14) \quad \Delta \log |\det A_{11}| \\ = k_0 \kappa + 2(|X_{12}|^2 + |X_{43}|^2 - 2|A_{11}|^2 + 2|B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2) \\ \geq k_0 \kappa - 4.$$

Since $\kappa \geq 4/k_0$,

$$(2.15) \quad \Delta \log |\det A_{11}| \geq 0.$$

Notice that (2.15) holds on S^2 except for at most finitely many points, and that $|\det A_{11}|$ assumes its positive maximum value at some point where (2.15) is valid. By the maximum principle of subharmonic functions, it therefore must be constant. Then by (2.7) and (2.14) again, $\kappa = 4/k_0$ and $B_{11} = B_{12} = B_{21} = B_{22} = 0$, which shows that f is a holomorphic map of constant curvature $4/k_0$.

A similar argument will work for proving the second claim of the statement. Q.E.D.

Recall that a holomorphic map or an antiholomorphic map from the Riemann sphere into a complex Grassmann manifold is always stable. Thus we have

Corollary 2.3. *If $\text{rank}(\partial f) \neq 0$ (resp. $\text{rank}(\bar{\partial} f) \neq 0$), and if $\kappa \geq 4/\text{rank}(\partial f)$ (resp. $K \geq 4/\text{rank}(\bar{\partial} f)$), then f is stable.*

Remark. One can construct a holomorphic map from S^2 into $G(m, N)$ with $\text{rank}(\partial f) = k_0$ and constant curvature $\kappa = 4/k_0$ in terms of homogeneous coordinates in the following way,

$$f \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \left[\begin{array}{cccccccc} Z_0 & & & & & & & \\ Z_1 & & & & & & & \\ & Z_0 & & & & & & \\ & Z_1 & & & & & & \\ & & \ddots & & & & & \\ & & & Z_0 & & & & \\ & & & Z_1 & & & & \\ & & & & 0 & & 1 & \\ & & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & 1 \\ & & & & & & & 0 \end{array} \right] \left\{ \begin{array}{l} 0 \\ \vdots \\ 0 \end{array} \right\} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 2k_0 \\ \\ m - k_0 \\ N - m - k_0 \end{array}$$

When $k_0 > 0$, a small perturbation of this map will produce a holomorphic curve of nonconstant curvature κ with $\text{rank}(\partial f) = k_0$ and arbitrarily small positive value of $|\kappa - 4/k_0|$.

For example, for any $\varepsilon > 0$ one can define $f_\varepsilon: S^2 \rightarrow G(m, N)$ by

$$f_\varepsilon \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \left[\begin{array}{cccccccc} (1 + \varepsilon)Z_0 & & & & & & & \\ & Z_1 & & & & & & \\ & & Z_0 & & & & & \\ & & Z_1 & & & & & \\ & & & \ddots & & & & \\ & & & & Z_0 & & & \\ & & & & Z_1 & & & \\ & & & & & 0 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & & \ddots \\ & & & & & & & 1 \\ & & & & & & & 0 \end{array} \right] \left\{ \begin{array}{l} 0 \\ \vdots \\ 0 \end{array} \right\} \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} 2k_0 \\ \\ m - k_0 \\ N - m - k_0 \end{array}$$

It is holomorphic. In terms of the complex coordinate Z on S^2 , the induced metric is $ds^2 = h(Z)|dz|^2$, where

$$h(Z) = \frac{k_0 - 1}{(1 + |Z|^2)^2} + \frac{(1 + \varepsilon)^2}{(|Z|^2 + (1 + \varepsilon)^2)^2}.$$

Its curvature is

$$\begin{aligned} \kappa = \frac{4}{h^3(Z)} & \left[\frac{(k_0 - 1)^2}{1 + |Z|^2} + \frac{(1 + \varepsilon)^6}{(|Z|^2 + (1 + \varepsilon)^2)^6} \right. \\ & + \frac{(k_0 - 1)(1 + \varepsilon)^4}{(1 + |Z|^2)^2(|Z|^2 + (1 + \varepsilon)^2)^4} \\ & + \frac{(k_0 - 1)(1 + \varepsilon)^2}{(1 + |Z|^2)^4(|Z|^2 + (1 + \varepsilon)^2)^2} \\ & \left. - \frac{2(k_0 - 1)(1 + \varepsilon)^2|Z|^2(2\varepsilon + \varepsilon)^2}{(1 + |Z|^2)^4(|Z|^2 + (1 + \varepsilon)^2)^4} \right], \end{aligned}$$

which is not constant. Since $\lim_{\varepsilon \rightarrow 0} f_\varepsilon = f$, one certainly can make $|\kappa - 4/k_0|$ as small as desired.

Now, if the curvature κ is not that big, the following theorem describes some relations between the pinching conditions of the curvature κ and the ranks of the ∂ -transform and the $\bar{\partial}$ -transform.

Theorem 2.2. *Let f be a harmonic isometric immersion from S^2 into $G(m, N)$, where $N \geq 2m$. Let $k_1 = \text{rank}(\partial f)$, $k_2 = \text{rank}(\bar{\partial} f)$ and κ be the Gaussian curvature. We have the following.*

(i) *If $0 < k_1 + k_2 \leq m$ and $\kappa \geq 4/(k_1 + k_2)$, then $\kappa = 4/(k_1 + k_2)$ is constant, and $f = [f_1 \oplus f_2 \oplus V]: S^2 \rightarrow G(m, n)$, where $f_1 \perp f_2 \perp V$ such that $f_1: S^2 \rightarrow G(k_1, N)$ is holomorphic, $f_2: S^2 \rightarrow G(k_2, N)$ is antiholomorphic and V is a constant complex vector subspace of C^N of dimension $m - k_1 - k_2$.*

(ii) *If $0 < k_1 < m$, $0 < k_2 < m$ and $k_1 + k_2 > m$, then $\min \kappa < 4/(k_1 + k_2)$.*

(iii) *If $k_1 = m$ and $0 < k_2 < m$ (or $0 < k_1 < m$ and $k_2 = m$) when $N > 2m$, then $\min \kappa < 8/(2k_1 + 3k_2)$; (or $\min \kappa < 8/(2k_2 + 3k_1)$); when $N = 2m$, then $\min \kappa < 4/(k_1 + 2k_2)$ (or $\min \kappa < 4/(k_2 + 2k_1)$).*

(iv) *If $k_1 = k_2 = m$ and $\kappa \geq 1/m$, then $\kappa = 1/m$ is constant and $\partial f \perp \bar{\partial} f$. Furthermore, f is a pseudoholomorphic map in the following sequence,*

$$f_{-1} \xrightarrow{\partial} f \equiv f_0 \xrightarrow{\bar{\partial}} f_1,$$

where $f_{-1}: S^2 \rightarrow G(m, N)$ is holomorphic and $f_1: S^2 \rightarrow G(m, N)$ is antiholomorphic.

Remarks. 1. The hypothesis that $N \geq 2m$ is not crucial, since for any harmonic map f , its orthogonal complement, denoted by $f^\perp$, is also a harmonic map having the same induced metric as that of f . Moreover, $\text{rank}(\partial f)$

$= \text{rank}(\bar{\partial}(f^\perp))$ and $\text{rank}(\bar{\partial}f) = \text{rank}(\partial(f^\perp))$. So we can replace f by $f^\perp$ if it is necessary.

2. (iii) says that when $N > 2m$, if $\kappa \geq 8/(2m + 3\text{rank}(\bar{\partial}f))$ (or $\kappa \geq 8/(2m + 3\text{rank}(\partial f))$), then the ∂ -transform (or $\bar{\partial}$ -transform) must be degenerate; when $N = 2m$, if $\kappa \geq 4/(m + 2\text{rank}(\bar{\partial}f))$ (or $\kappa \geq 4/(m + 2\text{rank}(\partial f))$), then the ∂ -transform (or $\bar{\partial}$ -transform) must be degenerate.

Proof. Choosing a unitary frame e as in the proof of Theorem 2.1 and adopting the notations there we have

$$(2.16) \quad \Delta \log |\det A_{11}| = k_1 \kappa + 2(|X_{12}|^2 + |X_{43}|^2 - 2|A_{11}|^2 + 2|B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2).$$

Change such a unitary frame e into $\tilde{e}$ such that $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_N)$, where $\tilde{e}_1, \dots, \tilde{e}_m$ span f , $\tilde{e}_{m+1}, \dots, \tilde{e}_{m+k_2}$ span $\bar{\partial}f$ and $\tilde{e}_{k_2+1}, \dots, \tilde{e}_m$ span the kernel of the $\bar{\partial}$ -transform. If we denote the corresponding new quantities by adding tildes, then a similar argument leads to

$$(2.17) \quad \Delta \log |\det \tilde{B}_{11}| = k_2 \kappa + 2(|\tilde{X}_{12}|^2 + 2|\tilde{A}_{11}|^2 + |\tilde{A}_{21}|^2 - 2|\tilde{B}_{11}|^2 + |\tilde{A}_{12}|^2 + |\tilde{X}_{43}|^2)$$

Since $\tilde{e} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} e$, where $K_1 \in C^\infty(U(m))$ and $K_2 \in C^\infty(U(N-m))$, we have $\tilde{A} = K_2^{-1} A K_1$ and $\tilde{B} = K_2^{-1} B K_1$, and therefore $|\tilde{A}|^2 = |A|^2$ and $|\tilde{B}|^2 = |B|^2$.

If $0 < k_1 + k_2 \leq m$ and $\kappa \geq 4/(k_1 + k_2)$, then combining (2.16) and (2.17), together with (1.11'), we shall have

(2.18)

$$\begin{aligned} \Delta \log (|\det A_{11}| |\det \tilde{B}_{11}|) &= (k_1 + k_2) \kappa + 2(|X_{12}|^2 + |\tilde{X}_{12}|^2 + |X_{43}|^2 + |\tilde{X}_{43}|^2 - 2|A_{11}|^2 - 2|\tilde{B}_{11}|^2 \\ &\quad + 2|B_{11}|^2 + |B_{12}|^2 + |B_{21}|^2 + 2|\tilde{A}_{11}|^2 + |\tilde{A}_{12}|^2 + |\tilde{A}_{21}|^2) \\ &\geq (k_1 + k_2) \kappa - 4(|A_{11}|^2 + |\tilde{B}_{11}|^2) \\ &= (k_1 + k_2) \kappa - 4 \geq 0. \end{aligned}$$

Using the same argument as in the proof of Theorem 2.1, we see that the globally defined invariant $|\det A_{11}| |\det \tilde{B}_{11}|$ is constant and $\kappa = 4/(k_1 + k_2)$ is constant. Moreover, (2.18) implies

$$(2.19) \quad |X_{12}| = |X_{43}| = |B_{11}| = |B_{12}| = |B_{21}| = 0,$$

and

$$|\tilde{X}_{12}| = |\tilde{X}_{43}| = |\tilde{A}_{11}| = |\tilde{A}_{12}| = |\tilde{A}_{21}| = 0$$

which mean that $\partial f \perp \bar{\partial} f$.

Specifying the local frame e further by requiring that $e_{k_1+k_2+1}, \dots, e_m$ span the kernel of the $\bar{\partial}$ -transform and using (2.19), the pull back of the Maurer-Cartan form by e is then of the form

$$\phi = \begin{bmatrix} \phi_1 & 0 & 0 & -{}^t\bar{A}_{11}\bar{\theta} & 0 & 0 \\ 0 & \phi_2 & 0 & 0 & -{}^t\bar{B}_{22}\theta & 0 \\ 0 & 0 & \phi_3 & 0 & 0 & 0 \\ A_{11}\theta & 0 & 0 & \phi_4 & 0 & 0 \\ 0 & B_{22}\bar{\theta} & 0 & 0 & \phi_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_6 \end{bmatrix}$$

where ϕ_i , $i = 1, \dots, 6$, are matrix-valued 1-forms, $A_{11} \in C^\infty(M(k_1 \times k_1))$ and $B_{22} \in C^\infty(M(k_2 \times k_2))$.

Now, we see that $f = [f_1 \oplus f_2 \oplus V]$, where $f_1 \perp f_2 \perp V$, and $f_1 = \text{span}\{e_1, e_2, \dots, e_{k_1}\}: S^2 \rightarrow G(k_1, N)$ is holomorphic,

$$f_2 = \text{span}\{e_{k_1+1}, \dots, e_{k_1+k_2}\}: S^2 \rightarrow G(k_2, N)$$

is antiholomorphic and $V = \text{span}\{e_{k_1+k_2+1}, \dots, e_m\}$ is a constant vector subspace of complex dimension $m - k_1 - k_2$.

Suppose now $0 < k_1 < m$, $0 < k_2 < m$ and $m < k_1 + k_2$. If $\min \kappa \geq 4/(k_1 + k_2)$, then (2.19) would imply $f = [g_1 \oplus g_2]$, where $g_1 \perp g_2$, $g_1: S^2 \rightarrow G(k_1, N)$ is holomorphic and $g_2: S^2 \rightarrow G(m - k_1, N)$ is antiholomorphic, with $\text{rank}(\bar{\partial} g_2) = k_2$. But this means that $m - k_1 \geq k_2$ which contradicts the hypothesis that $k_1 + k_2 > m$.

When $k_1 = m$ and $0 < k_2 < m$ (or $0 < k_1 < m$ and $k_2 = m$), there are two cases.

(a) $N > 2m$. In this case, since $k_1 = m$, $X_{12} = B_{12} = B_{22} = 0$, and $|B|^2 = |B_{11}|^2 + |B_{21}|^2$. Thus (2.16) becomes

$$(2.20) \quad \begin{aligned} \Delta \log |\det A_{11}| &= k_1 \kappa + 2(|X_{43}|^2 - 2|A_{11}|^2 + 2|B_{11}|^2 + |B_{21}|^2) \\ &\geq k_1 \kappa + 2(-2|A|^2 + |B|^2). \end{aligned}$$

Combining it with (2.17), we get

$$(2.21) \quad \begin{aligned} \Delta \log |\det A_{11}|^2 |\det \tilde{B}_{11}|^3 &\geq (2k_1 + 3k_2) \kappa - 8(|A|^2 + |B|^2) \\ &= (2k_1 + 3k_2) \kappa - 8. \end{aligned}$$

If $\min k \geq 8/(2k_1 + 3k_2)$, then it would follow by (2.17) and (2.21) that $|\tilde{A}_{11}| = |\tilde{A}_{12}| = |\tilde{A}_{21}| = |\tilde{X}_{12}| = 0$, which means that $\text{rank}(\tilde{A}) = \text{rank}(\partial f) < m$, contradicting the assumption on $\text{rank} \partial f$.

(b) $N = 2m$. In this case, $X_{12} = X_{34} = B_{12} = B_{21} = B_{22} = 0$ and $|B|^2 = |B_{11}|^2$. So (2.16) is now

$$(2.22) \quad \Delta \log |\det A_{11}| = k_1 \kappa + 4(|B|^2 - |A_{11}|^2),$$

from which, together with (2.17), it follows that

$$(2.23) \quad \Delta \log(|\det A_{11}| |\det \tilde{B}_{11}|^2) \geq (k_1 + 2k_2)\kappa - 4.$$

If $\min \kappa \geq 4/(k_1 + k_2)$, then again $\text{rank}(\partial f) < m$, which is a contradiction.

Finally, suppose $k_1 = k_2 = m$ and $\kappa = 1/m$. (2.16) and (2.17) become

$$(2.24) \quad \begin{aligned} \Delta \log |\det A_{11}| &= k_1 \kappa + 2(|X_{43}|^2 - 2|A_{11}|^2 + |B|^2 + |B_{11}|^2) \\ &\geq m\kappa - 4|A|^2 + 2|B|^2 \end{aligned}$$

and

$$(2.25) \quad \begin{aligned} \Delta \log |\det \tilde{B}_{11}| &= k_2 \kappa + 2(|\tilde{X}_{43}|^2 - 2|\tilde{B}_{11}|^2 + |\tilde{A}|^2 + |\tilde{A}_{11}|^2) \\ &\geq m\kappa - 4|B|^2 + 2|A|^2. \end{aligned}$$

So

$$\Delta \log(|\det A_{11}| |\det \tilde{B}_{11}|) \geq 2m\kappa - 2 \geq 0.$$

Thus $\kappa = 1/m$ is constant, and $X_{43} = B_{11} = \tilde{X}_{43} = \tilde{A}_{11} = 0$. Now it is easy to see that $\partial f \perp \bar{\partial} f$ and

$$f_{-1} \equiv \bar{\partial} f \xrightarrow{\partial} f \equiv f_0 \xrightarrow{\partial} f_1 \equiv \partial f,$$

where f_{-1} is holomorphic and f_1 is antiholomorphic. Q.E.D.

An interesting situation is when the harmonic map f generates an orthogonal harmonic sequence,

$$f \equiv f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \rightarrow \cdots \xrightarrow{\partial} f_n,$$

such that: when $0 \leq j \leq k_0$, then the rank $(f_j) = m_0$; and for $1 \leq t \leq s$, when

$$\sum_{i=0}^{t-1} k_i + 1 \leq j \leq \sum_{i=0}^t k_i,$$

then the rank $(f_j) = m_t$; and $\sum_{i=0}^s k_i = n$.

For such a map, we can prove the following theorem.

Theorem 2.3. *Let f be a harmonic isometric immersion from S^2 into $G(m, N)$. Suppose that f generates an orthogonal harmonic sequence as above, and the Gaussian curvature κ is bounded below by the constant*

$$\kappa(f) = \frac{4 \prod_{j=0}^s (k_j + 1)}{\sum_{r=0}^s \prod_{j=r}^s k_j (k_j + 1) m_r}.$$

Then $k = \kappa(f)$ and f is a holomorphic map which essentially generates a Frenet pseudoholomorphic sequence.

Proof. Suitably choose a smooth unitary frame e so that the pull back of the Maurer-Cartan form via e is

$$\phi = \begin{bmatrix} \phi_0 & -{}^\tau \bar{A}_0 \bar{\theta} & 0 & -{}^\tau \bar{B}_1 \theta & -{}^\tau \bar{B}_2 \theta \\ A_0 \theta & \phi_1 & -{}^\tau \bar{A}_1 \bar{\theta} & & \\ & 0 & A_{j-1} \theta & \phi_j & -{}^\tau \bar{A}_j \bar{\theta} \\ B_1 \bar{\theta} & & & A_{n-1} \theta & \phi_n & -{}^\tau \bar{A}_n \bar{\theta} \\ B_2 \bar{\theta} & & 0 & & A_n \theta & \phi_{n+1} \end{bmatrix}$$

where $B_1 \in C^\infty(M(m_s \times m_0))$, $B_2 \in C^\infty(M((N - \sum_{i=0}^s k_i m_i - m_0) \times m_0))$; for each $r = 0, \dots, s$, when $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, $A_j \in C^\infty(M(m_r \times m_r))$; for $r = 0, \dots, s-1$, $A_{k_0+\dots+k_r} \in C^\infty(M(m_r \times m_{r+1}))$; $A_n \in C^\infty(M((n - \sum_{i=0}^s k_i m_i - m_0) \times m_s))$; for $r = 0, \dots, s$, when $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, $\phi_j \in C^\infty(u(m_r) \otimes T^*S^2)$ and $\phi_{s+1} \in C^\infty(u(N - \sum_{i=0}^s k_i m_i - m_0) \otimes T^*S^2)$. Here we set $k_{-1} = 0$ for notational convenience. Also, we can require

$$(2.26) \quad A_{k_0+\dots+k_r} = [0, \tilde{A}_{k_0+\dots+k_r}]$$

where $\tilde{A}_{k_0+\dots+k_r} \in C^\infty(M(m_{r+1} \times m_{r+1}))$. Now (1.11') becomes

$$(2.27) \quad |A_0|^2 + |B_1|^2 + |B_2|^2 = 1,$$

and the harmonicity condition (1.12') and the Maurer-Cartan structure equations imply

$$(2.28) \quad dA_j = -iwA_j + A_j\phi_j - \phi_{j+1}A_j \mod \theta,$$

for $j = 0, \dots, n-1$. Thus, for $k_{-1} + \dots + k_{r-1} < j < k_{-1} + \dots + k_r$, where $r = 0, \dots, s$,

$$(2.29) \quad \Delta \log |\det A_j| \theta \wedge \bar{\theta} = 2d(-im_r w + \text{tr } \phi_j - \text{tr } \phi_{j+1}),$$

and so

$$(2.30) \quad \Delta \log |\det A_j| = m_r \kappa + 2(|A_{j-1}|^2 + |A_{j+1}|^2 - 2|A_j|^2),$$

if $j \neq n-1, 0$,

$$\Delta \log |\det A_{n-1}| = m_s \kappa + 2(|A_{n-2}|^2 + |A_n|^2 - 2|A_{n-1}|^2 + |B_1|^2)$$

and

$$\begin{aligned} \Delta \log |\det A_0| &= m_0 \kappa + 2(|A_1|^2 - 2|A_0|^2 + |B_1|^2 + |B_2|^2) \\ &\geq m_0 \kappa + 2(|A_1|^2 - 2|A_0|^2), \end{aligned}$$

if $k_0 \neq 0$.

For $j = k_0 + \dots + k_r$, set

$$\phi_j = \begin{bmatrix} \Omega_{11}^j & \Omega_{12}^j \\ \Omega_{21}^j & \Omega_{22}^j \end{bmatrix},$$

where $\Omega_{11}^j \in C^\infty(\mu(m_r - m_{r+1}) \otimes T^*S^2)$, $\Omega_{22}^j \in C^\infty(\mu(m_{r+1}) \otimes T^*S^2)$ and $\Omega_{21}^j = -{}^t\overline{\Omega_{12}^j} \in C^\infty(M(m_{r+1} \times (m_r - m_{r+1})) \otimes T^*S^2)$. Set also

$$A_{j-1} = \begin{bmatrix} D_1^{j-1} \\ D_2^{j-1} \end{bmatrix},$$

where $D_1^{j-1} \in C^\infty(M((m_r - m_{r+1}) \times m_r))$ and $D_2^{j-1} \in C^\infty(M(m_{r+1} \times m_r))$. Then

$$(2.31) \quad d\tilde{A}_j = -iw\tilde{A}_j + \tilde{A}_j\Omega_{22}^j - \phi_{j+1}\tilde{A}_j \mod \theta$$

and

$$(2.32) \quad \tilde{A}_j \Omega_{21}^j = 0 \pmod{\theta},$$

from which it follows that

$$(2.33) \quad \Omega_{21}^j = X_{21}^j \theta,$$

$$(2.34) \quad \Delta \log |\det \tilde{A}_{k_0+\dots+k_r}| = m_{r+1} \kappa + 2(|D_2^{k_0+\dots+k_r-1}|^2 + |X_2^{k_0+\dots+k_r}|^2 - 2|\tilde{A}_{k_0+\dots+k_r}|^2 + |A_{k_0+\dots+k_r+1}|^2)$$

for $r = 0, \dots, s-1$ and if $k_0 \neq 0$. Whereas if $k_0 = 0$,

$$(2.35) \quad \Delta \log |\det \tilde{A}_0| \geq m_1 \kappa + 2(|X_{21}^0|^2 - 2|\tilde{A}_0|^2 + |A_1|^2).$$

Observe that $|A_{k_0+\dots+k_r}|^2 = |\tilde{A}_{k_0+\dots+k_r}|^2$. For $l = 0, 1, \dots, s$, set

$$(2.36) \quad F_l = \sum_{j=k_{-1}+\dots+k_{l-1}}^{k_{-1}+\dots+k_l-1} (k_{-1} + k_0 + \dots + k_l - j) \Delta \log |\det A_j|,$$

where for notational convenience we set $\det A_{k_0+\dots+k_r} = \det \tilde{A}_{k_0+\dots+k_r}$, even though the first one does not make sense. We set also $D_2^{-1} = 0$. Then F_l , $l = 0, 1, \dots, s$, are well-defined continuous functions on S^2 .

A tedious computation shows that

$$(2.37) \quad \begin{aligned} F_s + \sum_{r=0}^{s-1} \left[\prod_{j=r+1}^s (k_j + 1) \right] F_r \\ \geq \frac{1}{2} \sum_{r=0}^s \prod_{j=r}^s k_r (k_r + 1) m_r \kappa + 2|A_{k_0+\dots+k_s}|^2 \\ - 2 \left[\prod_{j=1}^s (k_j + 1) \right] (k_0 + 1) |A_0|^2 \\ + 2 \left[k_s |D_2^{k_0+\dots+k_{s-1}-1}|^2 + \sum_{r=0}^{s-1} \prod_{j=r+1}^s (k_j + 1) k_r |D_2^{k_0+\dots+k_{r-1}-1}|^2 \right] \\ \geq \frac{1}{2} \sum_{r=0}^s \left[\prod_{j=r}^s k_r (k_j + 1) \right] m_r \kappa - 2 \prod_{j=0}^s (k_j + 1). \end{aligned}$$

Now, if $\kappa \geq k(f)$, then

$$F_s + \sum_{r=0}^{s-1} \left[\prod_{j=r+1}^s (k_j + 1) F_r \right] \geq 0,$$

which means that the function h defined by

$$(2.38) \quad h = \prod_{j=0}^s h_j$$

is a subharmonic function on S^2 except for finitely many points, where for $0 \leq l < s$

$$h_l = \log \left[\prod_{j=k_{-1}+\dots+k_{l-1}}^{k_{-1}+\dots+k_l-1} |\det A_j|^{(k_{-1}+\dots+k_l-j)} \right]^{\prod_{i=l+1}^s (k_i+1)}$$

and

$$h_s = \log \left[\prod_{j=k_{-1}+\dots+k_{s-1}}^{k_{-1}+\dots+k_s-1} |\det A_j|^{(k_{-1}+\dots+k_s-j)} \right].$$

Therefore, h must be constant. It follows from (2.27) and (2.37) that $\kappa = k(f)$.

$$(2.40) \quad \begin{aligned} |B_1| &= |B_2| = 0, \\ |A_n| &= |A_{k_0+\dots+k_s}| = 0 \end{aligned}$$

and

$$(2.41) \quad k_s |D_2^{k_0+\dots+k_{s-1}-1}|^2 + \sum_{r=0}^{s-1} \left[\prod_{j=r+1}^s (k_j+1) k_r |D_2^{k_0+\dots+k_{r-1}-1}|^2 \right] = 0,$$

which imply that f is holomorphic, f_n is antiholomorphic and

$$(2.42) \quad k_s |D_2^{k_0+\dots+k_{r-1}-1}|^2 = 0,$$

for $r = 1, \dots, s$. Thus, either $k_r = 0$ or $D_2^{k_0+\dots+k_{r-1}-1} = 0$. But when $r \geq 2$, $D_2^{k_0+\dots+k_{r-1}-1} = 0$ would imply that $\text{rank}(\partial f_{k_0+\dots+k_{r-1}-1}) = m_r < m_{r-1}$, which is a contradiction. Therefore $k_r = 0$ for $r \geq 2$. Also, $k_1 = 0$, unless $k_0 = 0$. For the latter case by (2.34) and (2.37),

$$(2.43) \quad X_{12} = 0.$$

The pull back of the Maurer-Cartan form is now the following,

$$\phi = \begin{bmatrix} \Omega_{11}^0 & 0 & & & & & & \\ 0 & \Omega_{22}^0 & -{}^\tau \bar{A}_0 \bar{\theta} & & & & & 0 \\ & \tilde{A}_0 \theta & \phi_1 & -{}^\tau \bar{A}_1 \bar{\theta} & & & & \\ & & A_1 \theta & \phi_2 & & & & \\ 0 & & & & A_{n-1} \theta & \phi_n & 0 & \\ & & & & & 0 & \phi_{n+1} & \end{bmatrix}.$$

We see that after a unitary transformation on C^N , f can be decomposed as $f = [\tilde{f} \oplus V_0]: S^2 \rightarrow G(m, N)$, where V_0 is an $(m - m_1)$ -dimensional constant vector subspace of C^N and $\tilde{f}$ is a holomorphic map from S^2 to $G(m_1, N)$, and $\text{rank}(f_j) = m_1$, for $j = 1, \dots, n$. In this sense, f essentially generates a Frenet pseudoholomorphic sequences. Q.E.D.

Remark. Given a harmonic map $f: S^2 \rightarrow G(m, N)$, one can define an associated vector bundle of f over S^2 in the following way:

$$\delta_f = \{(x, \nu) \in S^2 \times C^N | \nu \in f(x)\}.$$

If f generates an orthogonal harmonic sequence then the trivial complex vector bundle

$$\pi: S^2 \times C^N \rightarrow S^2$$

splits into orthogonal complex vector subbundles,

$$S^2 \times C^N = \delta_f \oplus \delta_{f_1} \oplus \cdots \oplus \delta_{f_n} \oplus \delta,$$

which are, in fact, holomorphic subbundles. Here δ is the orthogonal complement of $\delta_f \oplus \cdots \oplus \delta_{f_n}$. Theorem 2.3 says that if the curvature is large, then the ranks of δ_{f_j} , $j = 1, \dots, n$, are equal, f is holomorphic and δ is a trivial subbundle.

We say that a map g is in a Frenet harmonic sequence (i.e. constant ranks and orthogonal)

$$f \equiv f_0 \xrightarrow{\vartheta} f_1 \xrightarrow{\vartheta} f_2 \rightarrow \cdots \rightarrow f_n$$

of positive j , if $g = f_j$ for some $j = 0, \dots, n$, where f is harmonic and $\text{rank}(f_j) = m$. For such a map, we show the following pinching theorem.

Theorem 2.4. *Let $g = f_j: S^2 \rightarrow G(m, N)$ be a harmonic isometric immersion, which is in the above Frenet harmonic sequence, and let κ_j be the Gaussian curvature. Suppose that $\kappa_j \geq 4/m[2j(n-j) + n]$, then $\kappa_j = 4/m[2j(n-j) + n]$ is constant, and the directrix f is holomorphic.*

Proof. We use the invariants developed in the proof of Theorem 2.3. These invariants now satisfy

$$(2.44) \quad \Delta \log |\det A_0| = m\kappa_j + 2(|A_1|^2 - 2|A_0|^2 + |B_1|^2 + |B_2|^2),$$

$$\Delta \log |\det A_\tau| = m\kappa_j + 2(|A_{\tau-1}|^2 + |A_{\tau+1}|^2 - 2|A_\tau|^2)$$

for $0 < \tau < n-1$, and

$$\Delta \log |\det A_{n-1}| = m\kappa_j + 2(|A_{n-2}|^2 + |A_n|^2 - 2|A_{n-1}|^2 + |B_1|^2).$$

(1.11') becomes

$$(2.45) \quad |A_{j-1}|^2 + |A_j|^2 = 1, \quad \text{if } j \neq 0, n-1,$$

$$|A_0|^2 + |B_1|^2 + |B_2|^2 = 1, \quad \text{if } j = 0,$$

and

$$|A_{n-1}|^2 + |A_n|^2 + |B_1|^2 = 1, \quad \text{if } j = n-1.$$

The case when $j = 0$ has been treated in Theorem 2.3 by setting $m_r = m$ for $0 \leq r \leq s$, $k_0 = n$ and $k_1 = k_2 = \cdots = 0$. When $j > 0$, set

$$(2.46) \quad h_j = \left[\prod_{i=0}^{j-1} |\det A_i|^{(2j+1)(n-i-1)+2i} \right] \cdot \left[\prod_{i=j}^{n-1} |\det A_i|^{(2j+1)(n-i)} \right].$$

Then

$$(2.47) \quad \begin{aligned} \Delta \log h_j &= \frac{n+1}{2} [(2j(n-j) + n)m\kappa_j - 4(|A_j|^2 + |A_{j-1}|^2)] \\ &\quad + 2(2j+1)(n-1)(|B_1|^2 + |B_2|^2) + 2(2j+1)(|A_n|^2 + |B_1|^2) \\ &\geq \frac{n+1}{2} [(2j(n-j) + n)m\kappa_j - 4]. \end{aligned}$$

The assumption on κ_j implies

$$\Delta \log h_j \geq 0,$$

which means that h_j is constant, $\kappa_j = \frac{4}{m[2j(n-j)+n]}$, and $A_n = B_1 = B_2 = 0$. Therefore the directrix f is holomorphic. Q.E.D.

We say that a map f from M into CP^n is linearly full, if $f(M)$ is not contained in some linear subspace $CP^{n'} \subset CP^n$, where $n' < n$. The following corollary was first proved in [BJRW].

Corollary 2.4. *For a linearly full pseudoholomorphic map f_j of positions j from S^2 into CP^n , if it is an isometric immersion, and if the curvature $\kappa_j \geq 4/(2j(n-j) + n)$, then $\kappa_j = 4/(2j(n-j) + n)$ is constant.*

3. HARMONIC TWO-SPHERES IN $G(2, 4)$ AND $G(2, 6)$

In this section, we look at harmonic maps from S^2 into $G(2, 4)$ and $G(2, 6)$.

Let f be a harmonic isometric immersion from S^2 into $G(2, 4)$. By Corollary 2.2, we know that at least one of the ∂ -transform and the $\bar{\partial}$ -transform is degenerate. Let us say $\text{rank}(\partial f) \leq 1$. Choosing a suitable local unitary frame $e = (e_1, e_2, e_3, e_4)$ along f as before, the pull back of the Maurer-Cartan form is

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & -\bar{a}_{31}\bar{\theta} - \bar{b}_{31}\theta & -\bar{b}_{41}\theta \\ \phi_{21} & \phi_{22} & -\bar{b}_{32}\theta & -\bar{b}_{42}\theta \\ a_{31}\theta + b_{31}\bar{\theta} & b_{32}\bar{\theta} & \phi_{33} & \phi_{34} \\ b_{41}\bar{\theta} & b_{42}\bar{\theta} & \phi_{43} & \phi_{44} \end{bmatrix},$$

where

$$(3.1) \quad |a_{31}|^2 + |b_{31}|^2 + |b_{32}|^2 + |b_{41}|^2 + |b_{42}|^2 = 1.$$

From Corollary 2.1, the vanishing of $\text{tr} C = \text{tr}' A\bar{B}$ implies

$$(3.2) \quad a_{31}\bar{b}_{31} = 0.$$

Harmonicity conditions (1.12') and (1.13') imply

$$(3.3) \quad da_{31} = a_{31}(-iw + \phi_{11} - \phi_{33}) \mod \theta,$$

$$(3.4) \quad \phi_{12} = a_{12}\theta$$

and

$$(3.5) \quad \phi_{43} = a_{43}\theta.$$

By Proposition 1.1, (3.3) means that a_{31} is a function of analytic type. Thus either it is identically zero which means that f is an antiholomorphic map or away from its zeros it satisfies

$$(3.6) \quad \Delta \log |a_{31}| \theta \wedge \bar{\theta} = 2d(-iw + \phi_{11} - \phi_{33}).$$

Suppose that f is not antiholomorphic. Then $b_{31} = 0$. Harmonicity condition (1.13) also implies

$$(3.7) \quad db_{32} = b_{32}(iw + \phi_{22} - \phi_{33}) \mod \bar{\theta},$$

$$(3.8) \quad db_{41} = b_{41}(iw + \phi_{11} - \phi_{44}) \mod \bar{\theta}$$

and

$$(3.9) \quad db_{42} = b_{42}(iw + \phi_{22} - \phi_{44}) + b_{41}\phi_{12} - b_{32}\phi_{43} \mod \bar{\theta},$$

from which it follows again that b_{32} and b_{41} are functions of analytic type.

If b_{32} is not identically zero, then away from its zeros we get

$$(3.10) \quad \Delta \log |b_{32}| \theta \wedge \bar{\theta} = 2d(-iw + \phi_{33} - \phi_{22}).$$

Combining (3.6) and (3.10) and using the Maurer-Cartan structure equations, we have

$$(3.11) \quad \begin{aligned} \Delta \log |a_{31}b_{32}| &= 2\kappa + 4|a_{12}|^2 - 2|a_{31}|^2 + 2|b_{41}|^2 \\ &\quad - 2|b_{32}|^2 - 2|b_{42}|^2 \\ &\geq 2\kappa - 2. \end{aligned}$$

Notice that the function $|a_{31}b_{32}|$ is a globally defined continuous function. If $\kappa \geq 1$, then we see that $\kappa = 1$ and $a_{12} = b_{41} = 0$. Notice also that the local unitary frame e is uniquely determined up to $U(1) \times U(1) \times U(1) \times U(1)$ -transformations, and thus for $i = 1, 2, 3$ and 4, each e_i defines a map $[e_i]: S^2 \rightarrow CP^3$. In this case since $a_{12} = b_{41} = b_{31} = 0$, $[e_1]$ is holomorphic, $[e_2]$ is antiholomorphic and $f = [e_1 \oplus e_2]$.

The other case is $b_{32} = 0$. Suppose b_{41} is not identically zero. Then by (3.8) and (3.6), we get

$$(3.12) \quad \begin{aligned} \Delta \log |b_{41}a_{31}| &= 2\kappa + 4|a_{43}|^2 - 2|b_{42}|^2 - 2|b_{41}|^2 - 2|a_{31}|^2 \\ &\geq 2\kappa - 2. \end{aligned}$$

Again, if $\kappa \geq 1$, then $\kappa = 1$ and $a_{43} = 0$. In this case, we see that $[e_3]$ is an antiholomorphic map, $[e_4]$ is a holomorphic map and $f = [e_3 \oplus e_4]^\perp$.

Finally, if $b_{32} = b_{41} = 0$, but b_{42} is not identically zero, then by (3.9),

$$(3.13) \quad \Delta \log |b_{42}| \theta \wedge \bar{\theta} = 2d(-iw + \phi_{44} - \phi_{22}).$$

Taking the exterior derivatives on both sides of (3.4) and (3.5), and using the structure equations, we get

$$(3.14) \quad da_{12} = a_{12}(-iw - \phi_{11} + \phi_{22}) \mod \theta$$

and

$$(3.15) \quad da_{43} = a_{43}(-iw + \phi_{33} - \phi_{44}) \mod \theta.$$

Repeating the previous argument, we see that both a_{12} and a_{43} are of analytic type and that $|a_{12}|$ and $|a_{43}|$ are globally defined on S^2 .

We claim that if $k \geq 1$, then a_{12} and a_{43} are identically zero. There are three cases to be checked.

Case (i). Both a_{12} and a_{43} are not identically zero. Then from (3.14) and (3.15),

$$(3.16) \quad \Delta \log |a_{12}| \theta \wedge \bar{\theta} = 2d(-iw - \phi_{11} + \phi_{22})$$

and

$$(3.17) \quad \Delta \log |a_{43}| \theta \wedge \bar{\theta} = 2d(-iw + \phi_{33} - \phi_{44}).$$

These, together with (3.6) and (3.13), would give

$$(3.18) \quad \Delta \log |a_{12}a_{31}a_{43}b_{42}| = 4k \geq 4 > 0.$$

But then $|a_{12}a_{31}a_{43}b_{42}|$ must be constant and $k = 0$, which is a contradiction.

Case (ii). $a_{12} = 0$, but a_{43} is not zero. Then (3.17) holds and the combination of (3.6), (3.13) and (3.17) gives

$$(3.19) \quad \Delta \log |a_{31}a_{43}b_{42}| = 3k - 2 > 1,$$

which again would imply $k = \frac{2}{3}$, contradicting the hypothesis on k .

Case (iii). $a_{43} = 0$, but a_{12} is not zero. This case can be treated in exactly the same way as Case (ii), and it cannot happen either.

Now since $a_{12} = a_{43} = 0$, by reading the pull back of the Maurer-Cartan form we see that e_1 defines a holomorphic map $[e_1]: S^2 \rightarrow CP^1 \subset CP^3$, and that e_2 defines an antiholomorphic map $[e_2]: S^2 \rightarrow CP^1 \subset CP^3$. Up to a unitary transformation the map f is then given by $f = h \circ g$ in the following way.

$$(3.20) \quad \begin{array}{c} \xrightarrow{\quad f \quad} \\ S^2 \xrightarrow[g]{} CP^1 \times CP^1 \xrightarrow[h]{} CP^3 \end{array}$$

where $g = [e_1] \times [e_2]: S^2 \rightarrow CP^1 \times CP^1$ and h is the holomorphic embedding given by

$$h([Z_0, Z_1], [Z_2, Z_3]) = \left[\frac{Z_0}{\sqrt{|Z_0|^2 + |Z_1|^2}}, \frac{Z_1}{\sqrt{|Z_0|^2 + |Z_1|^2}}, \frac{Z_2}{\sqrt{|Z_2|^2 + |Z_3|^2}}, \frac{Z_3}{\sqrt{|Z_2|^2 + |Z_3|^2}} \right].$$

In summary, we have shown

Theorem 3.1. *Let f be a harmonic isometric immersion from S^2 into $G(2, 4)$ with $\text{rank}(\partial f) = 1$ and $\text{rank}(\bar{\partial} f) \geq 1$. If the curvature $\kappa \geq 1$, then either $f = [e_1 \oplus e_2]$ or $f^\perp = [e_1 \oplus e_2]$ with $[e_1]: S^2 \rightarrow CP^3$ holomorphic, $[e_2]: S^2 \rightarrow CP^3$ antiholomorphic and $e_1 \perp e_2$. Moreover, if the curvature κ is not constant 1, then up a unitary transformation f is given by (3.20).*

Remark. Suppose now that $\kappa \geq 2$. Then it happens only when $a_{12} = a_{43} = b_{31} = b_{32} = b_{41} = 0$. From (3.6) and (3.13), we set that

$$(3.21) \quad \Delta \log |a_{31} b_{42}| = 2k - 4 \geq 0.$$

Thus $k = 2$ and $|a_{31} b_{42}|$ is constant. But by (3.1),

$$(3.22) \quad |a_{31}|^2 + |b_{42}|^2 = 1,$$

and thus both $|a_{31}|$ and $|b_{42}|$ are constant. We can specify the unitary frame e so that $a_{31} \geq 0$ and $b_{42} \geq 0$ are constant. Using (3.6), it is easy to see that $a_{31} = b_{42} = \frac{1}{2}$. Thus all invariants are constant. In this case up to a $U(4)$ -congruence, f is the map $V_{1, \bar{1}}$ defined by

$$(3.16) \quad V_{1, \bar{1}} \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \begin{bmatrix} Z_0 & 0 \\ Z_1 & 0 \\ 0 & \bar{Z}_0 \\ 0 & \bar{Z}_1 \end{bmatrix}.$$

Notice that the map $V_{1, 0}: S^2 \rightarrow G(2, 4)$, defined by

$$V_{1, 0} \begin{bmatrix} Z_0 \\ Z_1 \end{bmatrix} = \begin{bmatrix} Z_0 & 0 \\ Z_1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

is the only holomorphic map from S^2 to $G(2, 4)$ with constant curvature 4 up to $U(4)$ -congruences. In [CZ], we also classified all holomorphic maps from S^2 to $G(2, 4)$ of curvature 2. In fact, the argument in [CZ] actually shows that for a holomorphic isometric immersion f from S^2 to $G(2, 4)$, if the curvature $\kappa \geq 2$, then either $\kappa = 2$ or $\kappa = 4$. Theorem 3.1, together with the argument in [CZ], shows the following.

Theorem 3.2. *Up to $U(4)$ -congruence, all isometric harmonic maps from S^2 into $G(2, 4)$ of curvature $\kappa \geq 2$ are given by: $V_{1,1}$, $V_{1,0}$, and the holomorphic curves (and their complex conjugates) of curvature 2 given in [CZ].*

Now suppose that f is a harmonic isometric immersion from S^2 into $G(2, 6)$. Then the only nonholomorphic or nonantiholomorphic cases are those when $\text{rank}(\partial f) \geq 1$ and $\text{rank}(\bar{\partial} f) \geq 1$. These cases can be treated by using Theorem 2.2. In fact, we have the following theorem.

Theorem 3.3. *For a harmonic isometric immersion f from S^2 into $G(2, 6)$, IF $\text{rank}(\partial f) = \text{rank}(\bar{\partial} f) = 2$, and if the curvature $\kappa \geq \frac{1}{2}$, then $\kappa = \frac{1}{2}$ is constant and f is a pseudoholomorphic curve of position one; if $\text{rank}(\bar{\partial} f) = 2$ and $\text{rank}(\partial f) = 1$, or if $\text{rank}(\partial f) = 1$ and $\text{rank}(\bar{\partial} f) = 2$, then $\min \kappa < \frac{8}{7}$; finally, if $\text{rank}(\partial f) = \text{rank}(\bar{\partial} f) = 1$, and if $\kappa \geq 2$, then $\kappa = 2$ is constant.*

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