EXTENDING H^p FUNCTIONS FROM SUBVARIETIES TO REAL ELLIPSOIDS

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ABSTRACT. Let Ω be a domain in C^n which is a somewhat generalized type of the real ellipsoid. Let V be a subvariety in Ω which intersects $\partial\Omega$ transversally. Then there exists an operator $E: H^p(V) \rightarrow H^p(\Omega)$ satisfying $Ef|_V = f$.

1. INTRODUCTION

The H^p extension problem from subvarieties in a strictly pseudoconvex domain was studied by Henkin [10], Adachi [1, 2, 3], and Hatziafratis [9] when $p = \infty$, and then, by Cumenge [6], and Beatrous [4] when $p \in (0, \infty)$. Let D be a real ellipsoid, i.e.,

$$D = \left\{ x + iy \in C^N : \sum_{i=1}^N x_i^{2n_i} + \sum_{i=1}^N y_i^{2m_i} < 1 \right\},$$

where $n_1, \dots, n_N$ and $m_1, \dots, m_N$ are positive integers. Then Diederich-Fornaess-Wiegerinck [7] proved Hölder estimates for solutions of $\bar{\partial}$ problem in D . In their proof, they used the explicit integral formula which involves the support function $\Phi(\zeta, z)$, depending holomorphically on z . On the other hand, Hatziafratis [8] constructed the integral formula for holomorphic functions in a subvariety of a bounded pseudoconvex domain with smooth boundary. His results are the extension of an earlier work of Stout [12]. In the present paper, by applying the integral formula constructed by Hatziafratis, we study the H^p extension from subvarieties in a convex domain Ω , which is a somewhat generalized type of the real ellipsoid D . Finally, we shall adopt the convention of denoting by c any positive constant which does not depend on the relevant parameters in the estimate.

2. PRELIMINARIES AND RESULTS

Let $s_i(x_i)$, $t_i(y_i)$ be real analytic functions on $[0, a]$. We set

$$\phi_i(x_i) = s_i(x_i^2) \quad \text{and} \quad \psi_i(y_i) = t_i(y_i^2).$$

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Suppose that ϕ_i, ψ_i ($i = 1, \dots, N$) satisfy the following conditions:

- (i) $\phi_i''(x_i) \geq 0, \psi_i''(y_i) \geq 0$ for $i = 1, \dots, N$.
- (ii) $\phi_i(a) > 1, \psi_i(a) > 1, \phi_i(0) = \psi_i(0) = 0$ for $i = 1, \dots, N$.
- (iii) $\phi_i''(x_i) > 0$ for $x_i \neq 0, \psi_i''(y_i) > 0$ for $y_i \neq 0$.

We set

$$\rho(z) = \sum_{i=1}^N \phi_i(x_i) + \sum_{i=1}^N \psi_i(y_i) - 1 \quad \text{for } z = (x_1 + iy_1, \dots, x_N + iy_N).$$

Let

$$\Omega = \{z \in C^N : \rho(z) < 0\}.$$

Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ which intersects $\partial\Omega$ transversally. Suppose that $\tilde{V}$ is written in the following form:

$$\tilde{V} = \{z \in \tilde{\Omega} : h_1(z) = \dots = h_m(z) = 0\} \quad (m < N),$$

where $h_1, \dots, h_m$ are holomorphic functions in $\tilde{\Omega}$ which satisfy

$$\partial h_1 \wedge \dots \wedge \partial h_m \wedge \partial \rho \neq 0 \quad \text{on } \tilde{V} \cap \partial\Omega.$$

Let $V = \tilde{V} \cap \Omega$. For any bounded domain G with smooth boundary in a complex variety, we denote by $H^p(G)$ the Hardy class on G . Then we have

Theorem 1. *Let $f \in H^p(V)$ ($1 \leq p < \infty$). Then there exists a function $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$.*

Theorem 2. *Suppose that V has no singular points. Let f be a holomorphic function on V with $\int_V |f|^p d\sigma < \infty$ ($1 \leq p < \infty$). Then there exists a holomorphic function F on Ω satisfying $F = f$ on V , and $\int_\Omega |F|^p d\mu < \infty$, where $d\sigma$ and $d\mu$ are Lebesgue measures on Ω and V , respectively.*

We set

$$h_i(x_i, \xi_i) = \phi_i(x_i) - \phi_i(\xi_i) - \phi_i'(\xi_i)(x_i - \xi_i).$$

Then we have

Lemma 1. $h_i(x_i, \xi_i) > 0$ for $x_i \neq \xi_i$.

Proof. We consider the case $x_i < 0, \xi_i \geq 0$. The other cases are similar. Then we have

$$\begin{aligned} h_i(x_i, \xi_i) &= \phi_i(-x_i) - \phi_i(\xi_i) - \phi_i'(\xi_i)(x_i - \xi_i) \\ &= -2x_i \phi_i'(\xi_i) + \frac{1}{2} \phi_i''(c_i)(x_i + \xi_i)^2 \quad \text{for some } c_i \neq 0. \end{aligned}$$

Since $\phi_i'(\xi_i) > 0$ for $\xi_i > 0$, we obtain the desired result.

For simplicity, we omit the index i in Lemma 2 and Lemma 3. In some neighborhood of 0, $\phi(x)$ can be written in the following form:

$$\phi(x) = b_k x^{2k} + b_{k+1} x^{2k+2} + \dots \quad (b_k > 0, k \geq 1).$$

Then we have the following lemma.

Lemma 2. *There exist positive constants ε and c such that*

$$h(x, \xi) \geq c[\phi''(\xi)(x - \xi)^2 + (x - \xi)^{2k}] \quad \text{for } |x| < \varepsilon, \quad |\xi| < \varepsilon.$$

Proof. Let $x(t), \xi(t)$ be real analytic functions in a neighborhood of 0 satisfying $x(0) = \xi(0) = 0$. Then

$$\begin{aligned} x(t) &= v_0 t^p + o(t^p), \quad \xi(t) = w_0 t^p + o(t^p), \\ p &\geq 1, \quad v_0^2 + w_0^2 \neq 0. \end{aligned}$$

To prove the Lemma 2, by the curve selection lemma of Bruna-Castillo [5], it is sufficient to show that for $|t|$ sufficiently small,

$$h(x(t), \xi(t)) \geq c[\phi''(\xi(t))(x(t) - \xi(t))^2 + (x(t) - \xi(t))^{2k}].$$

Since $\phi(x) = b_k x^{2k} + b_{k+1} x^{2k+2} + \dots$, it is easily shown that

$$h(x(t), \xi(t)) = b_k [v_0^{2k} - w_0^{2k} - 2kw_0^{2k-1}(v_0 - w_0)]t^{2pk} + o(t^{2pk}).$$

On the other hand we have

$$\begin{aligned} &\phi''(\xi(t))(x(t) - \xi(t))^2 + (x(t) - \xi(t))^{2k} \\ &= [2k(2k-1)b_k w_0^{2k-2}(v_0 - w_0)^2 + (v_0 - w_0)^{2k}]t^{2kp} + o(t^{2pk}). \end{aligned}$$

By Diederich-Fornaess-Wiegerinck [7], there exists a positive constant δ such that

$$\begin{aligned} &v_0^{2k} - w_0^{2k} - 2kw_0^{2k-1}(v_0 - w_0) \\ &> \delta[w_0^{2k-2}(v_0 - w_0)^2 + (v_0 - w_0)^{2k}] \quad \text{for } v_0 \neq w_0. \end{aligned}$$

Therefore, if $v_0 \neq w_0$, we have for $|t|$ sufficiently small,

$$h(x(t), \xi(t)) \geq c[\phi''(\xi(t))(x(t) - \xi(t))^2 + (x(t) - \xi(t))^{2k}].$$

In case $v_0 = w_0$, we take ξ as a new parameter. Then we have

$$x(\xi) = \xi + \lambda \xi^\alpha + o(|\xi|^\alpha),$$

where α is a rational number greater than 1. We write $x(\xi)$ in the following form $x(\xi) = \xi + \xi^\alpha w(\xi)$. Then we have

$$\begin{aligned} h(x(\xi), \xi) &= \sum_{j=k}^{\infty} b_j \{(\xi + \xi^\alpha w(\xi))^{2j} - \xi^{2j} - 2j\xi^{2j-1}\xi^\alpha w(\xi)\} \\ &= \sum_{j=k}^{\infty} b_j \sum_{i=2}^{2j} \binom{2j}{i} \xi^{2j-i+\alpha i} w(\xi)^i \\ &= b_k \binom{2k}{2} \lambda^2 \xi^{2k-2+2\alpha} + o(|\xi|^{2k-2+2\alpha}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\phi''(\xi)(x(\xi) - \xi)^2 + (x(\xi) - \xi)^{2k} \\ &= 2k(2k-1)b_k \lambda^2 \xi^{2k-2+2\alpha} + o(|\xi|^{2k-2+2\alpha}) + \lambda \xi^{2k\alpha} + o(|\xi|^{2k\alpha}) \\ &= O(|\xi|^{2k-2+2\alpha}). \end{aligned}$$

Therefore we have

$$h(x(\xi), \xi) \geq c[\phi''(\xi)(x(\xi) - \xi)^2 + (x(\xi) - \xi)^{2k}],$$

for $|\xi|$ sufficiently small. This completes the proof of Lemma 2.

Lemma 3. Let $\phi(x), h(x, \xi)$ be as in Lemma 2. Then we have

$$h(x, \xi) \geq c[\phi''(\xi)(x - \xi)^2 + (x - \xi)^{2k}]$$

for $|x| \leq a, |\xi| \leq a$.

Proof. From Lemma 2, there exists an $\varepsilon > 0$ such that

$$h(x, \xi) \geq c[\phi''(\xi)(x - \xi)^2 + (x - \xi)^{2k}] \quad \text{for } |x| \leq \varepsilon, |\xi| \leq \varepsilon.$$

If $|\xi| \geq \varepsilon, |x - \xi|$ small, then $|x| \geq \varepsilon/2$. Suppose that $x < 0, \xi > 0$. Then we obtain

$$h(x, \xi) > -2x\phi'(\xi) > c.$$

The other cases are proved similarly. This completes the proof of Lemma 3.

Thus we have proved that

$$(1) \quad \begin{aligned} & \phi_i(x_i) - \phi_i(\xi_i) - \phi'_i(\xi_i)(x_i - \xi_i) \\ & \geq c[\phi''_i(\xi_i)(x_i - \xi_i)^2 + (x_i - \xi_i)^{2n_i}], \end{aligned}$$

$$(2) \quad \begin{aligned} & \psi_i(y_i) - \psi_i(\eta_i) - \psi'_i(\eta_i)(y_i - \eta_i) \\ & \geq c[\psi''_i(\eta_i)(y_i - \eta_i)^2 + (y_i - \eta_i)^{2m_i}] \end{aligned}$$

for $\xi_i \in [0, a], \eta_i \in [0, a], i = 1, \dots, N$. Without loss of generality, we may assume that $m_i \leq n_i$ for $i = 1, \dots, N$. We set, for $\zeta_j = \xi_j + i\eta_j$, $\rho_j(\zeta_j) = \phi_j(\xi_j) + \psi_j(\eta_j)$ and

$$\begin{aligned} F_j(\zeta_j, z_j) &= -2 \frac{\partial \rho}{\partial \zeta_j}(\zeta_j)(z_j - \zeta_j) \\ &+ \gamma[(\psi''_j(\eta_j) - \phi''_j(\xi_j))(z_j - \zeta_j)^2 + (z_j - \zeta_j)^{2m_j}], \end{aligned}$$

where γ will be determined later. Using (1), (2), we obtain, for $z_j = x_j + iy_j$,

$$\begin{aligned} & \operatorname{Re}(-\rho_j(\zeta_j) + \rho_j(z_j) + F_j(\zeta_j, z_j)) \\ &= -\phi_j(\xi_j) - \psi_j(\eta_j) + \phi_j(x_j) + \psi_j(y_j) \\ & \quad - [\phi'_j(\xi_j)(x_j - \xi_j) + \psi'_j(\eta_j)(y_j - \eta_j)] \\ & \quad + \gamma(\psi''_j(\eta_j) - \phi''_j(\xi_j))\{(x_j - \xi_j)^2 - (y_j - \eta_j)^2\} + \gamma \operatorname{Re}[(z_j - \zeta_j)^{2m_j}] \\ & \geq \phi''_j(\xi_j)\{(c - \gamma)(x_j - \xi_j)^2 + \gamma(y_j - \eta_j)^2\} + \gamma \operatorname{Re}[(z_j - \zeta_j)^{2m_j}] \\ & \quad + \psi''_j(\eta_j)\{(c - \gamma)(y_j - \eta_j)^2 + \gamma(x_j - \xi_j)^2\} + c(y_j - \eta_j)^{2m_j}. \end{aligned}$$

If we choose $\gamma > 0$ small enough, then it holds that (see [7, Proposition 3.5]),

$$c(y_j - \eta_j)^{2m_j} + \gamma \operatorname{Re}[(z_j - \zeta_j)^{2m_j}] \geq c|z_j - \zeta_j|^{2m_j}.$$

Therefore we have for sufficiently small γ ,

$$(3) \quad \begin{aligned} & -\rho_j(\zeta_j) + \rho_j(z_j) + \operatorname{Re} F_j(\zeta_j, z_j) \\ & \geq c[(\phi_j''(\xi_j) + \psi_j''(\eta_j))|z_j - \zeta_j|^2 + |z_j - \zeta_j|^{2m_j}]. \end{aligned}$$

Now we fix γ so that the inequality (3) holds. We set

$$F(\zeta, z) = \sum_{j=1}^N F_j(\zeta_j, z_j).$$

Then we obtain

$$(4) \quad \begin{aligned} & -\rho(\zeta) + \rho(z) + \operatorname{Re} F(\zeta, z) \\ & \geq c \sum_{j=1}^N \{(\phi_j''(\xi_j) + \psi_j''(\eta_j))|z_j - \zeta_j|^2 + |z_j - \zeta_j|^{2m_j}\} \end{aligned}$$

for $(\zeta, z) \in \Omega \times \Omega$. We set, for $\zeta_j = \xi_j + i\eta_j \in \Omega$, $z_j = x_j + iy_j \in \Omega$, $j = 1, \dots, N$,

$$P_j(\zeta_j, z_j) = -2 \frac{\partial \rho_j}{\partial \zeta_j}(\zeta_j) + \gamma(\psi_j''(\eta_j) - \phi_j''(\xi_j))(z_j - \zeta_j) + (z_j - \zeta_j)^{2m_j-1}.$$

Then

$$F(\zeta, z) = \sum_{j=1}^N P_j(\zeta_j, z_j)(z_j - \zeta_j).$$

For $\varepsilon > 0$ sufficiently small, we set

$$V_\varepsilon = \{z \in V: \rho(z) < -\varepsilon\} \quad \text{and} \quad \Omega_\varepsilon = \{z \in \Omega: \rho(z) < -\varepsilon\}.$$

Let f^* be the boundary value of $f \in H^p(V)$ ($1 \leq p < \infty$). Then $f^* \in L^p(\partial V)$. Now we are going to prove the following.

Proposition 1. For $f \in H^p(V)$ ($1 \leq p < \infty$), and $z \in V$, we have the formula

$$f(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z),$$

where

(i) $K(\zeta, z)$ is written as

$$\sum_{\substack{K=(k_1, \dots, k_{N-m-1}) \\ S=(\alpha_1, \dots, \alpha_{N-m})}} \frac{\alpha_{K,S}(\zeta, z) \bigwedge_{j=1}^{N-m-1} \bar{\partial}_\zeta P_{k_j} \bigwedge_{i=1}^{N-m} d\zeta_{\alpha_i}}{F(\zeta, z)^{N-m}}$$

where $k_1, \dots, k_{N-m-1}$ are positive integers between 1 and N which are different from each other.

(ii) $\alpha_{K,S}(\zeta, z)$ are smooth on $\overline{\Omega} \times \overline{\Omega}$, holomorphic in $z \in \Omega$.

Proof. We follow the proof of Lemma I.1 of Stout [13]. Fix $z \in V$. As $f \in H^p(V)$, f is holomorphic in a neighborhood of $\overline{V}_\varepsilon$ in V . By the assumption

imposed on V , there exists $\varepsilon_1 > 0$ such that (i) $z \in V_{\varepsilon_1}$ (ii) $\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial \rho \neq 0$ on ∂V for $\varepsilon \in [0, \varepsilon_1]$. So by the theorem of Hatziafratis [8], we have

$$f(z) = \int_{\partial V_\varepsilon} f(w) K(w, z) \quad \text{for } \varepsilon \in (0, \varepsilon_1].$$

If we choose ε_1 sufficiently small, there exist a continuous function $\Delta: \partial V \times [0, \varepsilon_1] \rightarrow C$, and a smooth map $\chi: \partial V \times [0, \varepsilon_1] \rightarrow \overline{V}$ such that for fixed ε , $\chi(\cdot, \varepsilon)$ takes ∂V diffeomorphically onto the surface ∂V_ε , $\chi(\cdot, 0)$ the identity, and such that

$$\int_{\partial V_\varepsilon} f(w) K(w, z) = \int_{\partial V} f(\chi(w, \varepsilon)) K(\chi(w, \varepsilon), z) \Delta(w, \varepsilon).$$

Since $\chi(w, \varepsilon) \rightarrow w$, nontangentially as $\varepsilon \rightarrow 0+$, the dominated convergence theorem implies (cf. Stein [11]) that the integral on the right tends, as $\varepsilon \rightarrow 0+$, to $\int_{\partial V} f^*(w) K(w, z)$. This completes the proof of Proposition 1.

We set

$$F(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z) \quad \text{for } z \in \Omega.$$

Then $F(z)$ is a holomorphic function in Ω which satisfies $F(z) = f(z)$ for $z \in V$. Let $\phi(x)$ be the function as in Lemma 2. Then we have

Lemma 4. *Let A be a positive number, close to 0, and q be a positive integer. Let $x \in [-R, R]$, $0 < R < a/2$. We set*

$$\begin{aligned} I_1 &= \iint_{|\tau| < R; |\sigma| < R} \frac{|\phi'''(x - \sigma)| |\tau| d\sigma d\tau}{[A + \phi''(x - \sigma)(\tau^2 + \sigma^2) + (\tau^2 + \sigma^2)^k]^q}, \\ I_2 &= \iint_{|\tau| < R; |\sigma| < R} \frac{|\phi'''(x - \tau)| |\tau| d\tau d\sigma}{[A + \phi''(x - \tau)(\tau^2 + \sigma^2)]^q}, \\ I_3 &= \iint_{|\tau| < R; |\sigma| < R} \frac{\phi''(x - \tau) d\tau d\sigma}{[A + \phi''(x - \tau)(\tau^2 + \sigma^2)]^q}. \end{aligned}$$

Then, for $j = 1, 2, 3$,

$$\begin{aligned} |I_j| &\leq c |\log A| \quad \text{if } q = 1, \\ |I_j| &\leq c A^{1-q} \quad \text{if } q > 1, \end{aligned}$$

where c is independent of x .

Proof. There exists $\varepsilon > 0$ such that, for $|x| < \varepsilon$,

$$\phi(x) = b_k x^{2k} + b_{k+1} x^{2k+2} + \cdots \quad (b_k > 0, k \geq 1).$$

We may assume that $|x - \sigma| \leq \varepsilon' < \varepsilon$. Then

$$\phi''(x - \sigma) \geq c(x - \sigma)^{2k-2}, \quad |\phi'''(x - \sigma)| \leq c|x - \sigma|^{2k-\mu(k)},$$

where $\mu(k) = 3$ when $k \geq 2$,

$$\mu(k) = 1, -1, -3, \dots, \quad \text{when } k = 1.$$

Therefore we have

$$\begin{aligned} I_1 &\leq c \iint_{|\tau| < R; |\sigma| < R} \frac{|x - \sigma|^{2k - \mu(k)} |\tau| d\tau d\sigma}{[A + |x - \sigma|^{2k-2}(\tau^2 + \sigma^2) + (\tau^2 + \sigma^2)^k]^q}, \\ I_2 &\leq c \iint_{|\tau| < R; |\sigma| < R} \frac{|x - \tau|^{2k - \mu(k)} |\tau| d\tau d\sigma}{[A + |x - \tau|^{2k-2}(\tau^2 + \sigma^2)]^q}, \\ I_3 &\leq c \iint_{|\tau| < R; |\sigma| < R} \frac{|x - \tau|^{2k-2} d\tau d\sigma}{[A + |x - \tau|^{2k-2}(\tau^2 + \sigma^2)]^q}. \end{aligned}$$

From the Lemma 4.1 of Diederich-Fornaess-Wiegerinck [7], we obtain the desired results. This completes the proof of Lemma 4.

Let B_i ($i = 0, 1, \dots, M$), be balls with centers on ∂V and radius r_0 which form a cover of ∂V . Let $\tilde{B}_i$ be the ball with the same center as B_i and radius $2r_0$. We set $N - m = s$. Since $\partial h_1 \wedge \dots \wedge \partial h_m \wedge \partial \rho \neq 0$ on ∂V , we may assume, for r_0 sufficiently small,

- (i) $\left| \frac{\partial \rho}{\partial \bar{z}_s}(z) \right| \geq c > 0 \quad \text{for } z \in \tilde{B}_0,$
- (ii) $V \cap \tilde{B}_0 = \{z \in \tilde{B}_0 : z_{s+1} = \dots = z_N = 0\}.$

Then

$$L_j = \frac{\partial \rho}{\partial \bar{z}_s}(z) \frac{\partial}{\partial \bar{z}_j} - \frac{\partial \rho}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \bar{z}_s} \quad (j = 1, \dots, s-1)$$

form a base for the $(0, 1)$ tangential vector fields on $\partial V \cap \tilde{B}_0$. By a simple computation, we have

$$|L_j P_i| \leq \delta_{ji} c [\phi_i''(\xi_i) + \psi_i''(\eta_i) + \gamma(|\psi_i'''(\eta_i)| + |\phi_i'''(\xi_i)|) |z_i - \zeta_i|],$$

for $i \neq s$,

$$|L_j P_s| \leq c \left| \frac{\partial \rho}{\partial \bar{z}_j} \right| \leq c [|\phi_j'(\xi_j)| + |\psi_j'(\eta_j)|].$$

3. PROOF OF THEOREM 1

Without loss of generality, it is sufficient to show that

$$(5) \quad \sup_{\varepsilon > 0} \int_{\partial \Omega_\varepsilon \cap B_0} |F(z)|^p dS_\varepsilon(z) < \infty,$$

where $dS_\varepsilon(z)$ is the element of surface area on $\partial \Omega_\varepsilon$. We set

$$\begin{aligned} \tau_j &= \operatorname{Re}(z_j - \zeta_j), \quad \sigma_j = \operatorname{Im}(z_j - \zeta_j), \quad j = 1, \dots, N, \\ \lambda &= \operatorname{Im} F(\zeta, z), \quad \rho = \rho(\zeta) - \rho(z). \end{aligned}$$

By the transversality of V , for $\zeta \in \partial \Omega \cap \tilde{B}_0$ fixed, $\tau_j, \sigma_j, \lambda, \rho$ ($j = 1, \dots, s-1, s+1, \dots, N$) form coordinates of $\bar{\Omega} \cap \tilde{B}_0$ in such a way that $\tau_j, \sigma_j, \lambda$

($j = 1, \dots, s-1$) form coordinates of $\partial V \cap \tilde{B}_0$ for $z \in \overline{\Omega} \cap \tilde{B}_0$ fixed. We set

$$H(z) = \int_{\partial V \cap \tilde{B}_0} f^*(\zeta) K(\zeta, z).$$

Then we have

$$\begin{aligned} & \int_{\partial \Omega_\varepsilon \cap B_0} |H(z)| dS_\varepsilon(z) \\ & \leq \int_{\partial V \cap \tilde{B}_0} |f^*(\zeta)| \left(\int_{\partial \Omega_\varepsilon \cap B_0} |K(\zeta, z)| dS_\varepsilon(z) \right) d\sigma(\zeta) \\ & \leq c \int_{\partial V \cap \tilde{B}_0} |f(\zeta)| d\sigma(\zeta) \\ & \quad \times \int_{\partial \Omega_\varepsilon \cap B_0} \frac{\prod_{j=1}^{s-1} \{ \phi_j''(\xi_j) + \psi_j''(\eta_j) + (|\psi_j'''(\eta_j)| + |\phi_j'''(\xi_j)|) |z_j - \zeta_j| \} dS_\varepsilon(z)}{[|\rho(z)| + \sum_{j=1}^N \{ (\phi_j''(\xi_j) + \psi_j''(\eta_j)) |\zeta_j - z_j|^2 + |\zeta_j - z_j|^{2m_j} \} + |\lambda|^s]}. \end{aligned}$$

Using Lemma 4, we obtain

$$\begin{aligned} & \int_{\partial \Omega_\varepsilon \cap B_0} |H(z)| dS_\varepsilon(z) \\ & \leq c \int_{\partial V \cap \tilde{B}_0} |f^*(\zeta)| d\sigma(\zeta) \int_{\substack{|\tau_s| < R \\ |\sigma_n| < R}} \left| \log \left(\varepsilon + \sum_{j=s+1}^N (\tau_j^{2m_j} + \eta_j^{2m_j}) \right) \right| d\tau_s \cdots d\sigma_n \\ & \leq c \int_{\partial V} |f^*(\zeta)| d\sigma(\zeta). \end{aligned}$$

Thus we have proved the inequality (5) when $p = 1$. In case $p > 1$, we take q such that $1/p + 1/q = 1$. Let $z \in \tilde{B}_0$. Then for $\zeta \in \partial V \cap \tilde{B}_0$, we can write

$$K(\zeta, z) = T(\zeta, z) d\tau_1 \wedge d\sigma_1 \wedge \cdots \wedge d\tau_{s-1} \wedge d\sigma_{s-1} \wedge d\lambda.$$

By applying Hölder's inequality, we have

$$|H(z)|^p \leq c \left(\int_{\partial V \cap \tilde{B}_0} |f(\zeta)|^p |T(\zeta, z)| d\sigma(\zeta) \right) \left(\int_{\partial V \cap \tilde{B}_0} |T(\zeta, z)| d\sigma(\zeta) \right)^{p/q}.$$

By applying the method in case $p = 1$, we can easily prove the inequality (5), which completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We set $B(\zeta, z) = -2\rho(\zeta) + F(\zeta, z)$ and $B_\varepsilon(\zeta, z) = -2\rho(\zeta) + F(\zeta, z) - 2\varepsilon$. Then we obtain

$$\begin{aligned} \operatorname{Re} B(\zeta, z) & \geq -\rho(\zeta) - \rho(z) \\ & \quad + c \sum_{j=1}^N \{ (\phi_j''(\xi_j) + \psi_j''(\eta_j)) |z_j - \zeta_j|^2 + |z_j - \zeta_j|^{2m_j} \}, \end{aligned}$$

for $(\zeta, z) \in \overline{\Omega} \times \overline{\Omega}$. The integral formula in Proposition 1 also holds when we replace $F(\zeta, z)$ by $B_\varepsilon(\zeta, z)$, V by V_ε , and f^* by f . Since V is nonsingular, by applying Stokes' theorem, we have letting $\varepsilon \rightarrow 0$,

$$f(z) = \int_V f(\zeta) \bar{\partial}_\zeta K(\zeta, z) \quad \text{for } z \in V.$$

Each term of $\bar{\partial}_\zeta K(\zeta, z)$ consists of

$$\frac{\bar{\partial}_\zeta \alpha_{K,S}(\zeta, z) \bigwedge_{j=1}^{s-1} \bar{\partial}_\zeta P_{k_j} \bigwedge_{i=1}^s d\zeta_{\alpha_i}}{B(\zeta, z)^s} \quad \text{and} \quad \frac{\alpha_{K,S}(\zeta, z) \bigwedge_{j=1}^s \bar{\partial}_\zeta P_{k_j} \bigwedge_{i=1}^s d\zeta_{\alpha_i}}{B(\zeta, z)^{s+1}},$$

where $\alpha_{K,S}$ are smooth functions. We set

$$F(z) = \int_V f(\zeta) \bar{\partial}_\zeta K(\zeta, z) \quad \text{for } z \in \Omega.$$

Then $F(z)$ is the holomorphic function in Ω which satisfies $F(z) = f(z)$ for $z \in V$. Using the same method as in the proof of Theorem 1, we can obtain the desired result. This completes the proof of Theorem 2.

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