ENDOMORPHISM RINGS OF FORMAL A_0 -MODULES

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ABSTRACT. Let A_0 be the valuation ring of a finite extension K_0 of $\mathbb{Q}_p$ and $A \supset A_0$ be a complete discrete valuation ring with the perfect residue field. We consider the endomorphism rings of n -dimensional formal A_0 -modules Γ over A of finite A_0 -height with reduction absolutely simple up to isogeny. Especially we prove commutativity of $\text{End}_{A, A_0}(\Gamma)$. Given an arbitrary finite unramified extension K_1 of K_0 , a variety of examples (different dimensions and different A_0 -heights) is constructed whose absolute endomorphism rings are isomorphic to the valuation ring of K_1 .

Let K_0 be a finite extension of $\mathbb{Q}_p$ and A_0 the valuation ring of K_0 ; let $K \supset K_0$ be a complete discrete valuation field with the perfect residue field k of characteristic $p > 0$; let A be the valuation ring of K .

It is known that the fraction field of the endomorphism ring of a one-dimensional formal group of height h over A is a finite extension of $\mathbb{Q}_p$ of degree dividing h (cf. Lubin [7]).

In Theorem 1 and Proposition 2, we prove a higher-dimensional analogue of the above fact: if an n -dimensional formal A_0 -module Γ over A satisfies an assumption that the reduction $\Gamma_k = \Gamma \otimes_A k$ of Γ is an absolutely simple formal A_0 -module up to isogeny and of finite A_0 -height $h \geq n$ (h is relatively prime to n), then we prove that the fraction field Λ of the endomorphism ring of Γ over A as a formal A_0 -module is a finite extension field of K_0 of degree dividing h such that $e(\Lambda/K_0)$ divides $e(K/K_0)$ and that $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.

In the corollary of Theorem 2, we give examples: for any positive integers h and n with $h \geq n + 1$ ($h \geq 1$ if $n = 1$) and for any positive divisor g of h , there exists an n -dimensional formal A_0 -module over A_0 of A_0 -height h whose absolute A_0 -endomorphism ring is the valuation ring of the unramified extension of K_0 of degree g (cf. Cox [1] and Yamasaki [11]).

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1. NOTATIONS

In this paper, a field means a commutative field and we use the following notations.

p = a prime number.

Z_p = the ring of p -adic integers.

Q_p = the field of p -adic numbers.

K_0 = a finite extension of Q_p with the residue field k_0 .

A_0 = the valuation ring of K_0 .

K ($\supset K_0$) = a complete discrete valuation field with the perfect residue field k of characteristic $p > 0$.

π = a prime element of K .

A = the valuation ring of K .

$\bar{k}$ = an algebraic closure of k .

$[M : C]$ = the dimension of a vector space M over a field C .

For an extension E/E' of complete discrete valuation fields,

$e(E/E')$ = the ramification index of E over E' .

$f(E/E')$ = the residue degree of E over E' .

$M_t(T)$ = the total matrix ring of degree t over a ring T .

(a, b) = the (positive) greatest common divisor of integers a and b .

2. ENDOMORPHISM RINGS

In this paper, a formal group means a formal group law.

An n -dimensional formal A_0 -module over a commutative A_0 -algebra S is an n -dimensional commutative formal group over S such that there is an endomorphism $[a]$ of a formal group over S for each $a \in A_0$ whose Jacobian matrix is aI_n (I_n = the unit matrix of degree n) and that $a \rightarrow [a]$ is a ring homomorphism. If a formal A_0 -module is of height H as a formal group, we define the A_0 -height of Γ as the number $H/[K_0 : Q_p]$ (cf. [2], [3, III. 4.3, 5.5] and [5, V. 29.7.2]). A formal A_0 -homomorphism over S between formal A_0 -modules is a homomorphism over S of the formal groups which commutes with $[a]$ for all $a \in A_0$. We write $\text{End}_{S, A_0}(\Psi)$ the formal A_0 -endomorphism ring of a formal A_0 -module Ψ over S . An isogeny over S between formal A_0 -modules is a formal A_0 -homomorphism that, as a homomorphism of formal groups, is an isogeny.

Let Γ be an n -dimensional formal A_0 -module over A of finite A_0 -height. Let $\Gamma_k = \Gamma \otimes_A k$ be the formal A_0 -module over k obtained by reducing the coefficients of Γ modulo the maximal ideal of A .

We put $\Lambda = Q_p \otimes_{Z_p} \text{End}_{A, A_0}(\Gamma)$. Λ is a K_0 -algebra. As Γ is of finite A_0 -height, we identify Λ with its image in $Q_p \otimes_{Z_p} \text{End}_{k, A_0}(\Gamma_k)$ through reduction (cf. [5, IV.21.8.19]).

Let $\text{END}_{*, A_0}(\Gamma)$, the absolute A_0 -endomorphism ring of Γ , be the union of $\text{End}_{B, A_0}(\Gamma \otimes_A B)$ where B runs over all the valuation rings of finite extensions of K . ($\Gamma \otimes_A B$ is the scalar extension of Γ to B .)

We assume,

- (*) Γ_k is an n -dimensional formal A_0 -module such that the scalar extension $\Gamma_{\bar{k}} = \Gamma_k \otimes_k \bar{k}$ of Γ_k to $\bar{k}$ is simple as a formal A_0 -module up to isogeny (i.e. Γ_k is absolutely simple up to isogeny) and of finite A_0 -height h ($\geq n$).

Remark 1. (i) $(h, n) = 1$ by [5, V.29.8.3] (cf. [3, III.4, Corollary 2 of Proposition 8]).

(ii) For examples of Γ and Γ_k satisfying (*), see §4 and [9, Proposition 5].

We put $D = Q_p \otimes_{Z_p} \text{End}_{\bar{k}, A_0}(\Gamma_{\bar{k}})$. By assumption (*), D is a division algebra over K_0 . We put $\Omega = \Lambda \otimes_{K_0} K$.

The following simple proof of Proposition 1 is due to the referee.

Proposition 1. *Under our assumption (*), D is a central division algebra over K_0 of dimension h^2 .*

Proof. By [8, II] (or [5, V.28.5.9]), $\Gamma_{\bar{k}}$ is isogeneous to a product $\Gamma_1^{t_1} \times \Gamma_2^{t_2} \times \dots \times \Gamma_m^{t_m}$, where each Γ_i is simple up to isogeny of finite height H_i , Γ_i is not isogeneous to Γ_j for $i \neq j$, and the decomposition is unique up to isogeny. All this is taking place in the category of formal groups, not of formal A_0 -modules. Δ denotes the scalar extension $Q_p \otimes_{Z_p} \text{End}_{\bar{k}, Z_p}(\Gamma_{\bar{k}})$ of the endomorphism ring of $\Gamma_{\bar{k}}$ as a formal group. It follows that $\Delta \cong \bigoplus M_{t_i}(\Delta_i)$, where each Δ_i is a central division algebra over Q_p of dimension H_i^2 . From the definition of Γ as a formal A_0 -module, we see that K_0 injects into Δ and that D is the commutant of K_0 in Δ . Since D is a division algebra, the center of Δ must be a field. Thus Γ is isogeneous to $\Gamma_1^{t_1}$ and $\Delta \cong M_{t_1}(\Delta_1)$. From standard theorems about central division algebras, we see that D is a central division algebra over K_0 (double commutant theorem) and that

$$[D : Q_p][K_0 : Q_p] = [\Delta : Q_p] = t_1^2 H_1^2.$$

Thus we have

$$[D : K_0] = \left(\frac{t_1 H_1}{[K_0 : Q_p]} \right)^2 = h^2,$$

where the last equality follows from the definition of A_0 -height.

Let L be the tangent space (or the Lie algebra) of the scalar extension $\Gamma \otimes_A K$ of Γ to K . Then L is an n -dimensional vector space over K , a faithful Λ -module and a bimodule over Λ and K . L is thus a nontrivial module over Ω (cf. [5, II.14.2]).

Theorem 1 is a higher-dimensional analogue of [7, Theorem 2.3.2] (or [5, IV.23.2.6]).

Theorem 1. *Under our assumption $(*)$, Λ is a finite extension field of K_0 .*

Proof. By Proposition 1, the K_0 -subalgebra Λ of D is a division algebra over K_0 of dimension dividing h^2 . Let Z be the center of Λ . Then we have $[\Lambda : Z] = h'^2$ with h' dividing h . $Z \otimes_{K_0} K$ is a finite direct sum of finite extensions K_i ($\supset Z$) of K . Hence every minimal left ideal of $\Lambda \otimes_Z K_i$ has dimension over K divisible by h' , and so does every minimal ideal of $\Omega \cong \Lambda \otimes_Z (Z \otimes_{K_0} K)$. Ω is a semisimple K -algebra. Since L is a nontrivial Ω -module of dimension n over K , h' divides n .

On the other hand, $(h, n) = 1$ and h' divides h . Thus $h' = 1$ and Λ is a field.

Remark 2. If Γ_k is absolutely simple up to isogeny and of A_0 -height ∞ , then we have $\dim \Gamma = \dim \Gamma_k = \dim \Gamma_{\bar{k}} = 1$ (cf. [5, V.29.8.3]) and so $\text{End}_{A, A_0}(\Gamma)$ is commutative.

Proposition 2. *Under our assumption $(*)$, $[\Lambda : K_0]$ divides h . Furthermore $e(\Lambda/K_0)$ divides $e(K/K_0)$ and $f(\Lambda/K_0)$ divides $f(K/K_0)$ if $f(K/K_0)$ is finite.*

Proof. By Theorem 1, Λ ($\supset K_0$) is a subfield of D . Therefore, by Proposition 1, $[\Lambda : K_0]$ divides h .

Let F be a minimal ideal of Ω . By Theorem 1, F is a composite of K and Λ over K_0 . We have

$$e(F/\Lambda)e(\Lambda/K_0) = e(F/K)e(K/K_0).$$

Then $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $e(F/K)$ and so $[F : K]$. Therefore $e(\Lambda/K_0)/(e(\Lambda/K_0), e(K/K_0))$ divides $n = [L : K]$, since Ω is semisimple and L is a nontrivial Ω -module.

On the other hand, $e(\Lambda/K_0)$ divides $[\Lambda : K_0]$ and so h . Hence, by $(h, n) = 1$, $e(\Lambda/K_0)$ divides $e(K/K_0)$.

For the residue degrees, the same argument holds if $f(K/K_0)$ is finite.

Corollary (of Theorem 1 and Proposition 2). *Under our assumption $(*)$, the absolute A_0 -endomorphism ring of Γ is commutative. Its fraction field is an extension of K_0 of degree dividing h and has the ramification index dividing $e(K/K_0)$.*

Proof. Let K^* be the composite of K and the fraction field of the Witt vector ring over $\bar{k}$. Let A^* be the valuation ring of K^* . We remark $e(K^*/K_0) = e(K/K_0)$. By [10, Theorem 3.2] (or [5, IV.23.2.2]) and [5, IV.21.1.4, Remarks (ii)], $\text{END}_{*, A_0}(\Gamma)$ is contained in $\text{End}_{A^*, A_0}(\Gamma \otimes_A A^*)$. Hence our result follows from Theorem 1 and Proposition 2.

3. A LEMMA

Let $K' (\subset K)$ be the composite of K_0 and the fraction field (in K) of the Witt vector ring over k . Then K is a totally ramified finite extension of K' and

$e(K'/K_0) = 1$. Let A' be the valuation ring of K' . Let τ' be the Frobenius of K' over K_0 (i.e. the K_0 -automorphism of K' satisfying $a^{\tau'} \equiv a^{p^{f(K_0/Q_p)}} \pmod{\text{the maximal ideal of } A'}$ for all $a \in A'$).

In §§3 and 4, we assume that there exists an extension τ of τ' to an automorphism of K .

We write R the A -module of formal power series

$$x = a_h t^h + a_{h+1} t^{h+1} + \cdots + a_n t^n + \cdots$$

in an indeterminant t , with coefficients $a_n \in A$, where the exponent h is arbitrary. The A -module R is made into a ring by the multiplication law

$$(at^i)(bt^j) = (ab^{\tau^i})t^{i+j} \quad \text{for all } a, b \in A.$$

We also write $A_\tau[[t]]$ the subring of R with only terms of nonnegative exponents (cf. Hilbert-Witt ring and localized Hilbert-Witt ring in [3, III.4.1]).

Lemma (a generalization of the claim in [3, III.5, Proof of Theorem 3]). *Let $x = \sum a_i t^i \in A_\tau[[t]]$ with $a_i \in A$ be such that $v(a_i) > 0$ for all $0 \leq i \leq s-1$ and $v(a_s) = 0$, where v is the normalized discrete valuation of K with $v(\pi) = 1$. Suppose that $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to the left $A_\tau[[t]]$ -ideal $A_\tau[[t]]x$ generated by x . Then we have $u = 0$, i.e. $b_i = 0$ for all $0 \leq i \leq s-1$.*

Proof. We remark that v is invariant under τ . We take $\sum c_i t^i \in A_\tau[[t]]$ such that

$$\sum b_h t^h = \left(\sum c_i t^i \right) \left(\sum a_j t^j \right) = \sum \sum c_i a_j^{\tau^i} t^{i+j}.$$

Then we have, for all integers $h \geq 0$,

$$0 = c_{s+h} a_0^{\tau^{s+h}} + c_{s+h-1} a_1^{\tau^{s+h-1}} + \cdots + c_0 a_{s+h}.$$

Hence we have, for all integers $h \geq 1$,

$$\begin{aligned} v(c_h) &= v(c_h) + v(a_s^{\tau^h}) = v(c_h a_s^{\tau^h}) \\ &= v(-c_{s+h} a_0^{\tau^{s+h}} - c_{s+h-1} a_1^{\tau^{s+h-1}} - \cdots - c_{h+1} a_{s-1}^{\tau^{h+1}} - c_{h-1} a_{s+1}^{\tau^{h-1}} - \cdots - c_0 a_{s+h}) \\ &\geq \text{Min}\{v(c_{s+h}) + v(a_0), v(c_{s+h-1}) + v(a_1), \dots, v(c_{h+1}) + v(a_{s-1}), \\ &\quad v(c_{h-1}) + v(a_{s+1}), \dots, v(c_0) + v(a_{s+h})\} \\ &\geq \text{Min}\{v(c_0), v(c_1), \dots, v(c_{h-1}), v(c_{h+1}) + 1, \dots, v(c_{s+h}) + 1\} \end{aligned}$$

and, for $h = 0$,

$$v(c_0) \geq \text{Min}\{v(c_1), v(c_2), \dots, v(c_s)\} + 1.$$

Therefore if $v(c_{h'}) \geq q$ for all $0 \leq h' \leq h$ and $v(c_{h''}) \geq q-1$ for all $h'' \geq h+1$, then $v(c_{h+1}) \geq q$. Also if $v(c_h) \geq q$ for all integers $h \geq 0$, then $v(c_0) \geq q+1$.

Using induction on h and q , we have $v(c_h) \geq q$ for all integers $h \geq 0$ and $q \geq 1$. Hence we have $c_h = 0$ for all integers $h \geq 0$ and therefore $u = 0$.

Corollary. *Let x be as in the lemma. Suppose that $a_0 \neq 0$. If $u = b_0 + b_1 t + b_2 t^2 + \cdots + b_{s-1} t^{s-1}$ with $b_i \in A$ belongs to Rx , then $u = 0$.*

4. EXAMPLES

Let τ be as in §3. $\sigma = \tau$ and $q = p^{f(K_0/Q_p)}$ satisfy the assumption (F) in [6, §2]. Let n and m be positive integers ($m \geq 0$ if $n = 1$) and d an integer with $0 \leq d \leq m+n-1$. Let $\Gamma_{n,m,d}$ be the n -dimensional commutative formal group over A obtained by the following special element $u_{n,m,d}$ as was done in [6]. $u_{n,m,d}$ commutes with $\text{diag}(a, a, \dots, a)$ for all $a \in A_0$. Hence $\Gamma_{n,m,d}$ is an n -dimensional formal A_0 -module over A . By [5, IV.21.1.4, Remarks (ii)], $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ coincides with the endomorphism ring $\text{End}_{A,Z_p}(\Gamma_{n,m,d})$ of $\Gamma_{n,m,d}$ as a formal group over A . $u_{1,m,d} = \pi - t^{m+1}(1+t^d)$, and for $n \geq 2$,

$$u_{n,m,d} = \left(\overbrace{\begin{pmatrix} \pi & -t & 0 & \cdots & 0 \\ 0 & \pi & -t & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -t \\ -t^{m+1}(1+t^d) & 0 & 0 & \cdots & \pi \end{pmatrix}}^n \right) \Bigg\}^n$$

We have the following generalization of [11, Theorem 2] for K .

Theorem 2. *Suppose that there exists an extension τ of τ' to an automorphism of K . Then we have*

$$\text{End}_{A,A_0}(\Gamma_{n,m,d}) \cong \{\text{diag}(a^{\tau^{n-1}}, a^{\tau^{n-2}}, \dots, a) | a^{\tau^d} = a^{\tau^{m+n}} = a \in A\}.$$

Therefore $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ is isomorphic to the valuation ring of invariants of $\tau^{(m+n,d)}$ in A .

Proof. For a ring T , T^n denotes the left free T -module of the n -dimensional row vectors over T .

Let $\{\bar{e}_i\}$ ($1 \leq i \leq n$) be the images of the canonical basis $\{e_i\}$ ($1 \leq i \leq n$) of A^n under the composition of the inclusion $A^n \rightarrow R^n$ and the canonical surjection $R^n \rightarrow R^n/R^n u_{n,m,d}$.

First we assume $n \geq 2$. The left R -module $R^n u_{n,m,d}$ is generated by $te_2 - \pi e_1$, $te_3 - \pi e_2$, $\dots$, $te_n - \pi e_{n-1}$ and $\pi e_n - t^{m+1}(1+t^d)e_1$. Then we have the relations $t\bar{e}_2 = \pi\bar{e}_1$, $t^2\bar{e}_3 = \pi^{\tau+1}\bar{e}_1$, $\dots$, $t^{n-1}\bar{e}_n = \pi^{\tau^{n-2}}\pi^{\tau^3} \cdots \pi^{\tau}\pi\bar{e}_1$ and the annihilator of $\bar{e}_1$ is the left R -ideal

$$R(\pi^{\tau^{n-1}}\pi^{\tau^{n-2}} \cdots \pi^{\tau}\pi - t^{m+n}(1+t^d))$$

of R . Especially

$$R^n/R^n u_{n,m,d} \cong R/R(\pi^{\tau^{n-1}} \cdots \pi^\tau \pi - t^{m+n}(1+t^d))$$

is a monogenic left R -module (cf. [3, III.5.5]).

We suppose $C = (c_{ij}) \in M_n(A)$ be such that

$$f_{n,m,d}^{-1}(C f_{n,m,d}) \in \text{End}_{A,A_0}(\Gamma_{n,m,d}),$$

where $f_{n,m,d}$ is the transformer of $\Gamma_{n,m,d}$. The left $A_\tau[[t]]$ -module $(A_\tau[[t]])^n u_{n,m,d} C$ is contained in $(A_\tau[[t]])^n u_{n,m,d}$ by [6, Theorem 3]. Then C gives an R -endomorphism of $R^n/R^n u_{n,m,d}$ which stabilizes $\sum_{1 \leq i \leq n} A \bar{e}_i$. ($\sum_{1 \leq i \leq n} A \bar{e}_i$ is a bimodule over $\text{End}_{A,A_0}(\Gamma_{n,m,d})$ and A .) Therefore, we have

$$\begin{aligned} t(c_{21} \bar{e}_1 + c_{22} \bar{e}_2 + \cdots + c_{2n} \bar{e}_n) &= \pi(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n), \\ &\vdots \\ t(c_{n1} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) &= \pi(c_{n-11} \bar{e}_1 + \cdots + c_{n-1n} \bar{e}_n), \\ \pi(c_{n1} \bar{e}_1 + c_{n2} \bar{e}_2 + \cdots + c_{nn} \bar{e}_n) &= t^{m+1}(1+t^d)(c_{11} \bar{e}_1 + \cdots + c_{1n} \bar{e}_n). \end{aligned}$$

By representing $\bar{e}_i$'s with $\bar{e}_1$, we have

$$(**) \quad \begin{cases} \{t(c_{21} + c_{22}(t^{-1}\pi) + \cdots + c_{2n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi)) \\ \quad - \pi(c_{11} + \cdots + c_{1n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi))\} \bar{e}_1 = 0, \\ \vdots \\ \{t(c_{n1} + c_{n2}(t^{-1}\pi) + \cdots + c_{nn}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi)) \\ \quad - \pi(c_{n-11} + \cdots + c_{n-1n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi))\} \bar{e}_1 = 0, \end{cases}$$

and

$$(***) \quad \{\pi(c_{n1} + c_{n2}(t^{-1}\pi) + \cdots + c_{nn}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi)) \\ - t^{m+1}(1+t^d)(c_{11} + \cdots + c_{1n}(t^{-(n-1)}\pi^{\tau^{n-2}} \cdots \pi^\tau \pi))\} \bar{e}_1 = 0.$$

We multiply (**) by t^{n-1} from the left.

Since $n \leq m+n-1$, by the corollary of the lemma we have $c_{i1} = 0$ ($2 \leq i \leq n$), $c_{ik} = c_{i+1,k+1}^\tau$ ($1 \leq i, k \leq n-1$), and $c_{in} = 0$ ($1 \leq i \leq n-1$). Hence we have $c_{ij} = 0$ if $i \neq j$ and $c_{ii} = c_{nn}^{\tau^{n-i}}$ for $1 \leq i \leq n-1$ and so

$$C = \text{diag}(c_{nn}^{\tau^{n-1}}, c_{nn}^{\tau^{n-2}}, \dots, c_{nn}).$$

Since the annihilator of $\bar{e}_1$ is $R(\pi^{\tau^{n-1}} \cdots \pi^\tau \pi - t^{m+n}(1+t^d))$, from (***) we have

$$\{c_{nn}^{\tau^{-(m+1)}}(1+t^d) - t^{m+1}(1+t^d)c_{11}\} \bar{e}_1 = 0.$$

Then, by dividing the above equation by t^{m+1} , we have

$$\{c_{nn}^{\tau^{-(m+1)}} + c_{nn}^{\tau^{-(m+1)}} t^d - c_{nn}^{\tau^{n-1}} - c_{nn}^{\tau^{n+d-1}} t^d\} \bar{e}_1 = 0.$$

From $0 \leq d \leq m+n-1$, by the corollary of the lemma we have $c_{nn}^{\tau^{m+n}} = c_{nn}^{\tau^d} = c_{nn}$.

Conversely if $C = (c_{ij})$ satisfies the above conditions, then $u_{n,m,d} C = Cu_{n,m,d}$ and so $f_{n,m,d}^{-1}(C f_{n,m,d}) \in \text{End}_{A, A_0}(\Gamma_{n,m,d})$.

Hence $\text{End}_{A, A_0}(\Gamma_{n,m,d})$ is isomorphic to the invariants of $\tau^{(m+n,d)}$ in A .

Finally, for $n=1$, the analogous argument holds since the annihilator of $\bar{e}_1$ is $R(\pi - t^{m+1}(1+t^d))$.

Remark 3. (i) The field consisting of the invariants of $\tau^{(m+n,d)}$ in K has been determined more explicitly in [11, Theorem 3].

(ii) Suppose that $e(K/K_0) = 1$, $(n, m) = 1$, and k is algebraically closed for simplicity. By [3, III.5.2, Proof of Theorem 2], we have

$$R/R(\pi^{\tau^{n-1}} \cdots \pi^{\tau} \pi - t^{m+n}(1+t^d)) \cong R/R(\pi_0^n - t^{m+n}),$$

where π_0 is a prime element of K_0 . Hence $\Gamma_{n,m,d} \otimes_A k$ is absolutely simple up to isogeny (cf. Proof of the corollary below).

The following corollary is a higher-dimensional analogue of [1, Theorem 5.2.2] (or [5, IV.23.2.16]).

Corollary. *For any positive integers n and h with $h \geq n+1$ ($h \geq 1$ if $n=1$) and for any positive divisor g of h , there exists an n -dimensional formal A_0 -module over A_0 of A_0 -height h whose absolute A_0 -endomorphism ring is the valuation ring of the unramified extension of K_0 of degree g .*

Proof. Let $K = K_0$ and $m = h - n$. Put $d = g$ if $g < h$ and $d = 0$ if $g = h$. Let K_0^* be the completion of the maximal unramified extension of K_0 and A_0^* the valuation ring of K_0^* . As in the proof of the corollary in §2, we have

$$\text{END}_{*, A_0}(\Gamma_{n,m,d}) \subset \text{End}_{A_0^*, A_0}(\Gamma_{n,m,d} \otimes_{A_0} A_0^*).$$

We apply Theorem 2 to $\Gamma_{n,m,d} \otimes_{A_0} A_0^*$. Thus $\text{END}_{*, A_0}(\Gamma_{n,m,d})$ is contained in

$$\{\text{diag}(a^{\tau^{n-1}}, a^{\tau^{n-2}}, \dots, a) | a^{\tau^g} = a \in A_0^*\}.$$

The invariants of τ^g in A_0^* coincide with the valuation ring of the unramified extension of degree g over K_0 . Especially, any A_0 -endomorphism of $\Gamma_{n,m,d} \otimes_{A_0} A_0^*$ is defined over the valuation ring of the finite extension of K_0 . Hence the converse inclusion follows.

For the functor M in [2], we have

$$M(\Gamma_{n,m,d} \otimes_{A_0} k_0) \cong A_{0\tau}[[t]]^n / (A_{0\tau}[[t]]^n u_{n,m,d})$$

as in [4, V.2]. Thus we have

$$[k_0 \otimes_{A_0} M(\Gamma_{n,m,d} \otimes_{A_0} k_0) : k_0] = m + n$$

and therefore $\Gamma_{n,m,d}$ is of A_0 -height $m + n = h$.

Remark 4. If $(n, m) = 1$, then $\Gamma_{n,m,d} \otimes_{A_0} k_0$ is absolutely simple up to isogeny as in Remark 3(ii) (cf. [3, III.5.5]).

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