IDENTITIES ON QUADRATIC GAUSS SUMS

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ABSTRACT. Given a local field F , each multiplicative character θ of the split algebra $F \times F$ or of a separable quadratic extension of F has an associated generalized Gauss sum γ_θ^F . It is a complex valued function on the character group of $F^\times \times F$, meromorphic in the first variable. We define a pairing between such Gauss sums and study its properties when F is a nonarchimedean local field. This has important applications to the representation theory of $GL(2, F)$ and correspondences [GL3].

INTRODUCTION

The multiplicative group $F^\times$ of a local field F is a split extension of the value group $|F^\times|$ by the compact group of the units. Hence, the group $\mathcal{A}(F^\times)$ of continuous homomorphisms of $F^\times$ in $\mathbb{C}^\times$ is a one-dimensional complex Lie group, with connected component of identity the image of $\mathbb{C}$ under the map

$$s \mapsto (t \mapsto |t|^s).$$

We have written $|t|$ for the normalized absolute value of t .

The group $F^\times$ acts on functions on F by translations:

$$t: f \mapsto f^t, \quad f^t(x) = f(tx), \quad t \in F^\times.$$

This gives an action of $F^\times$ on the space $\mathcal{S}(F)$ of Schwartz-Bruhat functions on F , hence also an action on the space $\mathcal{S}'(F)$ of tempered distributions on F :

$$\langle t.D | f^t \rangle = \langle D | f \rangle.$$

For each χ in $\mathcal{A}(F^\times)$, the space of tempered distributions of type χ under the action of $F^\times$ is one-dimensional (e.g. [W3]). The choice of a nontrivial additive unitary character ψ of F defines an identification of F with its Pontrjagin dual by $(u, v) \mapsto \psi(uv)$, hence a self-dual Haar measure $d_\psi u$ on F , and a Fourier transform

$$\hat{f}(v) = \int_F f(u) \psi(uv) d_\psi u, \quad f \in \mathcal{S}(F).$$

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The Fourier transform exchanges distributions of type χ with distributions of type $t \mapsto |t|\chi(t)^{-1}$.

Fix an additive Haar measure du on F and let d^*u be the measure $|u|^{-1/2}du$ on F . Then, for each $\chi \in \mathcal{A}(F^\times)$, we have a measure χd^* on $F^\times$, which is holomorphic in χ . It is well known [W3] that χd^* extends to a unique meromorphic distribution Δ_χ of type $|\chi|^{1/2}$ on F , which has simple poles. As $\hat{\Delta}_\chi$ is a multiple of $\Delta_{\chi^{-1}}$, denote their ratio by $\gamma^F(\chi, \psi)$, called the gamma factor attached to χ and ψ :

$$\hat{\Delta}_\chi = \gamma^F(\chi, \psi) \Delta_{\chi^{-1}}.$$

It is a meromorphic function of χ , satisfying the complement formula

$$\gamma^F(\chi, \psi) \gamma^F(\chi^{-1}, \psi^{-1}) = 1,$$

and also

$$\begin{aligned} \gamma^F(\chi, \psi^t) &= \chi(t)^{-1} \gamma^F(\chi, \psi), \quad t \in F^\times, \\ \gamma^F(1, \psi) &= 1. \end{aligned}$$

In case F is a nonarchimedean local field, the gamma factor $\gamma^F(\chi, \psi)$ is a Gauss sum; more precisely, it is an analytic continuation of the following integral taken in principal value:

$$\int_F \chi(u) \psi(u) d_\psi^* u.$$

Let K be a quadratic étale algebra over F , that is, K is either an F -algebra isomorphic (in two ways) to $F \times F$, or a separable quadratic field extension of F . Then the norm group $N_{K/F}(K^\times)$ has index 1 or 2 according as K splits or not over F . Denote by $\eta_{K/F}$ the character of $F^\times$ with kernel $N_{K/F}(K^\times)$. For a character θ of $K^\times$, a character χ of $F^\times$, and ψ as above, we define the quadratic Gauss sum with $T_{K/F}$ the trace form from K to F :

$$\gamma_\theta^F(\chi, \psi) = \lambda_{K/F}(\psi) \gamma^K(\theta \chi \circ N_{K/F}, \psi \circ T_{K/F}),$$

where

$$\lambda_{K/F}(\psi) = \gamma^F(\eta_{K/F}, \psi).$$

Note that in case K is split over F , the group $\mathcal{A}(K^\times)$ is isomorphic to the product of two copies of $\mathcal{A}(F^\times)$ so that we may identify θ with a couple $\{\mu, \nu\}$ of characters of $F^\times$; in this case, we have $\lambda_{K/F}(\psi) = 1$, and

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi \mu, \psi) \gamma^F(\chi \nu, \psi).$$

It is known that these quadratic Gauss sums have the following properties:

(a) if the additive character ψ is changed to ψ^t , $t \in F^\times$, then

$$\gamma_\theta^F(\chi, \psi^t) = \chi(t)^{-2} \omega(t)^{-1} \gamma_\theta^F(\chi, \psi), \quad \omega(t) = \eta_{K/F}(t) \theta(t);$$

(b) it satisfies the Davenport-Hasse identity, namely, when $\theta = \mu \circ N_{K/F}$ for some character μ of $F^\times$, one has

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi\mu, \psi) \gamma^F(\chi\mu\eta_{K/F}, \psi);$$

(c) for F nonarchimedean and χ of conductor large enough

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi, \psi) \gamma^F(\chi\omega, \psi),$$

where ω is as in (a); this is the deep twist property.

In this article, we shall study, for F nonarchimedean, a pairing between two such quadratic Gauss sums γ_θ^F and $\gamma_{\theta'}^F$ relative to two quadratic étale algebras K and K' . It is defined for the two meromorphic functions

$$\chi \mapsto \gamma_\theta^F(\chi, \psi) \quad \text{and} \quad \chi \mapsto \gamma_{\theta'}^F(\chi^{-1}, \psi)$$

having no common pole as the finite part of a contour integral over $\mathcal{A}(F^\times)$ enclosing the poles of $\gamma_{\theta'}^F(\chi^{-1}, \psi)$ but no poles of $\chi \mapsto \gamma_\theta^F(\chi, \psi)$:

$$\langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = \oint_{\mathcal{A}(F^\times)} \gamma_\theta^F(\chi, \psi) \gamma_{\theta'}^F(\chi^{-1}, \psi) d\chi.$$

Our purpose is to derive further properties of the quadratic Gauss sums from this pairing, with the goal of reestablishing Langland's correspondences on representations of degree two semisimple algebras over F . Thus, some of the results would become immediate consequences if one were to grant these correspondences.

Our main result (Theorem 1) expresses the value of this pairing in terms of a gamma factor coming from the étale F -algebra $B = K \otimes_F K'$ and the character $\theta \times \theta'$ of $B^\times$ defined by

$$(\theta \times \theta')(z) = \theta \circ N_{B/K}(z) \theta' \circ N_{B/K'}(z).$$

In Theorem 2, we show that for K' split the formula reduces to a formula which has appeared already in [L, GL1, GL2], called the multiplicative formula for γ_θ^F . When K and K' are not isomorphic and when the product of the restriction to $F^\times$ of θ and θ' is the character $t \mapsto |t|^{-1}$, the value of the pairing $\langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi$ can be simplified (§3.3). In particular, it depends only on the fields K and K' . This fact is used in [GL3] to characterize the degree two monomial representations of the local Weil group W_F over F . Each character θ of $K^\times$ determines a two-dimensional representation $\text{Ind}_K^F \theta$ of W_F . We prove in §3.4 that the quadratic Gauss sums γ_θ^F parametrize the isomorphism classes of these representations $\text{Ind}_K^F \theta$.

1. PREPARATION

1.1. We introduce some notations for any nonarchimedean field. In general, the field will be indicated by a subscript, but it will be deleted for the base field

F . The ring of integers of F is $\mathcal{O} = \mathcal{O}_F$, its group of units is $\mathcal{O}^\times = \mathcal{O}_F^\times$, its maximal ideal is $\mathcal{P} = \mathcal{P}_F$. As $\mathcal{O}$ and $\mathcal{O}^\times$ are open compact subgroups of F , $F^\times$, respectively, we choose Haar measures $du = d_F u$ on F , $d^\times t = d_F^\times t$ on $F^\times$, respectively, giving to them the volume 1. Then $d^\times t = L_F |t|^{-1} dt$, where $L_F = (1 - q^{-1})^{-1}$ is the value at the character $t \mapsto |t|$ of the L -function of F , and $q = q_F$ is the module of F .

On the group $\hat{F}$ of characters of F , there is an absolute value $||$ defined on $\psi \in \hat{F}$ as the smallest number c in the value group of F such that $\psi(u) = 1$ for $c|u| \leq 1$. Then $|\psi^t| = |\psi||t|$ for t in F . The self-dual Haar measure on F associated to the bicharacter $\psi(uv)$, for ψ nontrivial in $\hat{F}$, is $d_\psi u = |\psi|^{1/2} du$.

We define a modification $\Gamma^F = \Gamma$ of the gamma factor γ^F by taking the finite part of the following integral:

$$\Gamma(\chi, \psi) = \int_F \chi(t) \psi(t) d^\times t.$$

It is given in terms of γ^F by

$$\Gamma(\chi, \psi) = L_F |\psi|^{-1/2} \gamma^F(\chi q^{1/2}, \psi)$$

and satisfies the following complement formula:

$$\Gamma(\chi, \psi) \Gamma(\chi^{-1} q^{-1}, \psi) = L_F^2 |\psi|^{-1} \chi(-1).$$

Here, we have used the convention which identifies a nonzero complex number Z with the character $t \mapsto Z^{\text{ord } t}$ of $F^\times$, where $\text{ord } t = -\log_q |t|$.

For a character χ of $F^\times$, we write $a(\chi)$ for its conductor, and $A(\chi) = q^{a(\chi)}$; so, $A(\chi)$ is the smallest number $c \geq 1$ such that, for t a unit, we have $\chi(t) = 1$ when $c|t - 1| \leq 1$, i.e. $\text{ord}(t - 1) \geq a(\chi)$ means $|t - 1| A(\chi) \leq 1$. We define also $A'(\chi)$ to be $A(\chi)$ if χ ramifies and to be q otherwise. We denote by $|\chi|$ the character $t \mapsto |\chi(t)|$, so that it coincides with $|Z|$ when χ is given by the nonzero complex number Z .

For $|\chi| < q^{1/2}$ the gamma factor $\gamma^F(\chi, \psi)$ is given by the convergent integral

$$\gamma^F(\chi, \psi) = \int_{|t| \leq A'(\chi) |\psi|^{-1}} \chi(t) \psi(t) d_\psi^* t,$$

with $d_\psi^* t = |t|^{-1/2} d_\psi t$. When χ ramifies, only the shell $|t| = A(\chi) |\psi|^{-1}$ contributes; moreover, there is χ_ψ in this shell such that, for any character μ of $F^\times$ satisfying $A(\mu)^2 \leq A(\chi)$,

$$\gamma^F(\chi \mu, \psi) = \mu(\chi_\psi) \gamma^F(\chi, \psi).$$

From this, it follows that for $K = F \times F$ and θ a character of $K^\times$ given by the characters μ and ν of $F^\times$, we have the relation

$$\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi, \psi) \gamma^F(\chi \omega, \psi) \quad \text{if } A(\mu)^2 \text{ and } A(\nu)^2 \leq A(\chi), \quad \omega = \mu \nu.$$

1.2. For K an étale F -algebra, that is, a product of the extensions E of F , its ring of integers $\mathcal{O}_K$ is the product of the $\mathcal{O}_E$'s, and the group $\mathcal{O}_K^\times$ of units is the product of the $\mathcal{O}_E^\times$'s. Let $T = T_{K/F}$ be the trace form from K to F . Then the pairing $(x, y) \mapsto T(xy)$ from $K \times K$ to F is nondegenerate. By composition with $F \rightarrow F/\mathcal{O}_F$, we get an orthogonality relation between the $\mathcal{O}_K$ -submodules of K . The index of $\mathcal{O}_K$ in its orthogonal is called the discriminant $D_{K/F}$: it is the product of the $D_{E/F}$'s. The self-dual Haar measure $d_{K,\psi}$ on K associated to the bicharacter $\psi \circ T(xy)$ is $|\psi \circ T|^{1/2} d_K$, with $|\psi \circ T|$ the product of the $|\psi \circ T_{E/F}|$'s; hence $d_{K,\psi}$ is the product of the $d_{E,\psi}$'s. From Corollary 3 to Proposition 4 of Chapter VIII-1 in [W2], we have $|\psi \circ T| = |\psi|^{[K:F]} D_{K/F}^{-1}$. Let $N = N_{K/F}$ be the norm map on K : it is the product of the norm maps $N_{E/F}$. We introduce the two numbers

$$L_K = \prod_E L_E = \prod_E (1 - q_E^{-1})^{-1} \quad \text{and} \quad L_{K/F} = L_K / L_F^{[K:F]}.$$

Note that L_K is also the integral over $\mathcal{O}_K$ of the function $|Nx|$ for the measure $d_K^\times$. We denote by $|x|_K$ the absolute value $\text{Max}_E |N_{E/F} x_E|$ on K , $x = (x_E)$.

1.3. A quadratic étale F -algebra K has a conjugation $\bar{}$; its norm $N = N_{K/F}$ and trace $T = T_{K/F}$ are also given by $Nx = x\bar{x}$, $Tx = x + \bar{x}$. For each nontrivial additive character ψ of F , the quadratic character $\psi \circ N$ is nondegenerate and defines a fourth roots of unity $\lambda(\psi \circ N)$ by the functional equation [W1, G]:

$$\int_K \hat{f}(y) \psi \circ N(y) dy = \lambda(\psi \circ N) \int_K f(x) \psi^{-1} \circ N(x) dx$$

where f lies in the Schwartz-Bruhat space $\mathcal{S}(K)$ of compactly supported locally constant functions on K , and its Fourier transform is

$$\hat{f}(y) = \int_K f(x) \psi \circ T(x\bar{y}) d_{K,\psi} x.$$

With f the characteristic function of a sufficiently small ball around 0, we get

$$\lambda(\psi \circ N) = \int_{|x|_K \leq R} \psi \circ N(x) d_{K,\psi} x, \quad \text{for } R \text{ large enough.}$$

Moreover, decomposing F with respect to the norm group of K , we get

$$\lambda(\psi \circ N) = \gamma^F(\eta_{K/F}, \psi),$$

with the notations as in the introduction. Note that $\lambda(\psi \circ N) = 1$ for K split.

1.4. If K is a quadratic étale F -algebra and θ is a character of $K^\times$, we define a two-dimensional representation of the Weil group W_F of F as follows. If K a field, its Weil group W_K appears as the kernel of the composition of the class field theory map $W_F \rightarrow F^\times$ with $\eta_{K/F}$; it has index two in W_F and θ gives a one-dimensional representation of W_K from the map $W_K \rightarrow K^\times$, hence by

induction a two-dimensional representation $\text{Ind}_K^F \theta$ of W_F . If K is split, then θ is given by two characters μ and ν of $F^\times$; in this case $\text{Ind}_K^F \theta$ is the sum of the two one-dimensional representations of W_F defined by μ and ν .

Lemma. *Assume that K is a separable quadratic extension of F . Given a character θ of $K^\times$ and two characters α and β of $F^\times$, the following conditions are equivalent:*

- (i) $\theta = \alpha \circ N_{K/F} = \beta \circ N_{K/F}$, $\beta = \alpha \eta_{K/F}$,
- (ii) $\text{Ind}_K^F \theta = \alpha \oplus \beta$,
- (iii) $\gamma_\theta^F(\chi, \psi) = \gamma^F(\chi \alpha, \psi) \gamma^F(\chi \beta, \psi)$ for any character χ of $F^\times$.

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i) comes from the fact that $\text{Ind}_K^F \theta$ is reducible if and only if the character θ is fixed under the conjugation of K over F . (i) $\Rightarrow$ (iii) is the Davenport-Hasse identity. Finally (iii) $\Rightarrow$ (i) uses the fact that θ is not regular since γ_θ^F has poles, so $\theta = \alpha \circ N_{K/F}$ and $\gamma_\theta^F(\chi, \psi)$ is $\gamma^F(\chi \alpha, \psi) \gamma^F(\chi \alpha \eta_{K/F}, \psi)$ with poles at $\alpha^{-1} q^{-1/2}$ and $\alpha^{-1} \eta_{K/F} q^{-1/2}$, which are also $\alpha^{-1} q^{-1/2}$ and $\beta^{-1} q^{-1/2}$: this gives the implication.

1.5. Lemma. *Let K be a quadratic F -algebra, with norm N , trace T , discriminant D , absolute value $|\cdot|_K$. Fix a nontrivial additive character ψ of F . If the element R of $|K^\times|_K$ satisfies $R|\psi| \geq D$, then, the Fourier transform with respect to $\psi \circ T$ of the function g on K defined by*

$$g(x) = \psi(-Nx) \text{ for } |x - 1|_K \leq R \text{ and } 0 \text{ otherwise}$$

is equal to $\lambda_{K/F}(\psi)^{-1} \bar{g}$.

Proof. We write the inequality $R|\psi| \geq D$ as $R^{-1}|\psi \circ T|^{-1}$; this gives $\psi(Nx) = 1$ for $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$; the orthogonal of this subgroup with respect to the bicharacter $\psi \circ T(x\bar{y})$ is $|y|_K \leq R$. From the reduction theorem in [G], the λ -factor of the quadratic group $(K, \psi \circ N)$ is also the λ -factor of the factor group of the ideal $|x|_K \leq R$ modulo the ideal $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$ for the quadratic character $\psi \circ N$, hence

$$\lambda_{K/F}(\psi) \int_{|x|_K \leq R} f(x) \psi \circ N(x)^{-1} dx = \int_{|x|_K \leq R} \hat{f}(y) \psi \circ N(y) dy,$$

for all f in $\mathcal{S}(K)$. We write now the value at $1+z$ of the Fourier transform of the given function g : $\hat{g}(1+z) = \psi(1+Tz) \int_{|x|_K \leq R} \psi(-Nx + Txz) d_{K, \psi} x$. We apply then the above formula to the function $f(x) = \psi(Txz)$ for $|x|_K \leq R$ and 0 elsewhere:

$$\hat{g}(1+z) = \psi(1+Tz) \lambda_{K/F}(\psi)^{-1} \int_{\substack{|x-z|_K \leq R^{-1}|\psi \circ T|^{-1} \\ |x|_K \leq R}} \psi(Nx) d_{K, \psi} x \int_{|y|_K \leq R} d_{K, \psi} y.$$

By integrating first on the ball $|x|_K \leq R^{-1}|\psi \circ T|^{-1}$, the first integral is seen to be 0 unless $|z|_K \leq R$; in this case, we get

$$\hat{g}(1+z) = \psi(1+Tz) \lambda_{K/F}(\psi)^{-1} \psi(Nz) = \lambda_{K/F}(\psi)^{-1} \psi \circ N(1+z),$$

which proves the lemma.

1.6. Biquadratic étale F -algebras. A biquadratic étale F -algebra B is a four-dimensional étale F -algebra containing at least two quadratic sub- F -algebras K and K' . Then, the map $(x, y) \mapsto xy$ gives an isomorphism from $K \otimes_F K'$ onto B . The conjugation of K (resp. K') with respect to F extends to an F -involution on B with K' (resp. K) as fixed points. These two involutions commute, and their composition has fixed points a third quadratic subalgebra K'' , and K, K', K'' are all the three quadratic sub- F -algebras in B . When B is a field, it is a biquadratic extension of F , and K, K', K'' are the three quadratic extensions of F contained in B . When B contains exactly one split quadratic F -algebra, it is a direct product of two isomorphic quadratic separable extensions of F : we see B as $K \times K$ with the two subfields $\{(x, x) | x \in K\}$, $\{(x, \bar{x}) | x \in K\}$ and the split algebra $F \times F$ embedded naturally; in this case, the three involutions are respectively $(x, y) \mapsto (y, x)$, $(\bar{y}, \bar{x})$, $(\bar{x}, \bar{y})$, we have isomorphisms from $K \otimes_F K$ onto B and from $K \otimes_F (F \times F)$ onto B given by $x \otimes y \mapsto (xy, x\bar{y})$ and $x \otimes (u, v) \mapsto (xu, xv)$ respectively. Finally, when B contains more than one split quadratic F -algebra, then it is completely split, and isomorphic to F^4 ; we see then K, K', K'' as fixed points of the involutions $(t, u, v, w) \mapsto (v, w, t, u)$, (w, v, u, t) , (u, t, w, v) respectively.

In each case, we have $\eta_{K/F} \eta_{K'/F} \eta_{K''/F} = 1$. This implies that the product $\lambda_{K/F}(\psi) \lambda_{K'/F}(\psi) \lambda_{K''/F}(\psi)$ is independent of the character ψ ; we denote it by $\lambda_{B/F}$. If θ is a character of $B^\times$, we define for χ and ψ as above

$$\gamma_\theta^F(\chi, \psi) = \lambda_{B/F} \gamma_\theta^B(\theta \chi \circ N_{B/F}, \psi \circ T_{B/F}).$$

For the number $L_{B/F}$ defined in 1.2, we have

$$L_{B/F} = L_{K/F} L_{K'/F} L_{K''/F},$$

expressing the inductivity property of the L -function. The discriminant $D_{B/F}$ satisfies a similar relation ([S, Chapter VI.2] with the Artin representation, and [W2, Corollary 2 of Theorem 5, Chapter XII.4] with the Herbrand distribution):

$$D_{B/F} = D_{K/F} D_{K'/F} D_{K''/F}.$$

Also, we have the relations

$$\eta_{B/K} = \eta_{K'/F} \circ N_{K/F}.$$

In particular, the elements of F and those of $K_1 = \text{Ker } N_{K/F}$ are norms from elements of B . We remark also that the conjugation of B over K' when restricted to K induces the conjugation of K over F . Finally, the product $\lambda_{B/F} \eta_{K/F}(-1)$ is the factor $\lambda_{B/K}(\psi \circ T_{K/F})$ for any ψ as above. We write it simply $\lambda_{B/K}$.

2. PRELIMINARY RESULTS

2.1. Some measures. Let K be a degree n étale F -algebra, with its trace form T and its norm form N . The bilinear form $(x, y) \mapsto T(xy)$ on K is

nondegenerate; hence, it defines a self-pairing on the top exterior power $\bigwedge^n K$, and a Haar measure $d_{K/F}$ on K . In other words, we have, by definition of the discriminant $D_{K/F}$,

$$(2.1) \quad \int_{\mathcal{O}_K} d_{K/F} x = D_{K/F}^{-1/2}.$$

The maps $x \mapsto x/Tx$ and $x \mapsto Tx$ give a decomposition of the complement in K of the hyperplane $\text{Ker } T$ as the product of the affine hyperplane K_T consisting of trace 1 elements, by $F^\times$. This in turn yields a decomposition of the differential forms on K , hence a decomposition of the Haar measure $d_{K/F} x$ as $|Tx|^{n-1} d_T(x/Tx) dTx$, where $d_T y$ is a measure on K_T invariant under translations by $\text{Ker } T$. This is also

$$\frac{d_{K/F} x}{|Nx|} = \frac{d_T(x/Tx) dTx}{|N(x/Tx)| |Tx|}.$$

When passed to the normalized Haar measures on $K^\times$ and $F^\times$, this defines a measure $d^\bullet y$ on K_T by

$$d_K^\times x = d^\bullet(x/Tx) d_F^\times Tx.$$

In terms of the measure d_T introduced above and the number L_K in (1.2) this gives

$$d^\bullet y = D_{K/F}^{1/2} \frac{L_K}{L_F} \frac{d_T y}{|Ny|}.$$

When K is a field, this can also be stated as follows. The projective space deduced from K as an n -dimensional space over F is also the factor group $K^\times/F^\times$; the affine hyperplane K_T of K imbeds in $K^\times/F^\times$ as the open subset image by $x \mapsto x/Tx$ of the complement of the hyperplane $\text{Ker } T$ of K ; the Haar measure on $K^\times/F^\times$ given by the quotient of the normalized Haar measures on $K^\times$ and $F^\times$ induces on this open subset the measure $d^\bullet y$. It is a bounded measure since it gives to K_T the volume of $K^\times/F^\times$ with respect to the quotient measure $d_K^\times x/d_F^\times t$; this volume is computed by integrating the characteristic function of units in $K^\times$, which has volume 1: this gives

$$\int_{K_T} d^\bullet y = \int_{K^\times/F^\times} (d_K^\times x/d_F^\times t) = [|K^\times|_K : |F^\times|_K] = e_{K/F}$$

with $e_{K/F}$ the ramification index of F in K .

If K is a separable quadratic extension of F , with conjugation $x \mapsto \bar{x}$, then the homography $x \mapsto x^{-1} - 1$ of the projective line $K \cup \{\infty\}$ sends $K_T \cup \{\infty\}$ onto K_1 since $N(x^{-1} - 1) - 1 = (1 - Tx)/Nx$. As $y^{-1} - 1 = \bar{y}/y$ for $y \in K_T$, this map coincides on K_T with $x \mapsto \bar{x}/x$, which is a homomorphism from $K^\times$ onto K_1 (by Hilbert Theorem 90) with kernel $F^\times$. Hence it identifies the groups K_1 and $K^\times/F^\times$ in a compatible way with the two maps $K_T \rightarrow K^\times/F^\times$

and $K_T \rightarrow K_1$. This shows that, for f an integrable function on K_T with respect to the measure $d^\bullet y$,

$$\int_{K_T} f(y) d^\bullet y = e_{K/F} \int_{K_1} f((1+w)^{-1}) d^\times w,$$

where $d^\times w$ is the normalized Haar measure on K_1 since the ramification index $e_{K/F}$ is the volume of K_T under $d^\bullet y$. If K is $F \times F$, then K_1 is isomorphic to $F^\times$ by $t \mapsto (t, t^{-1})$, and $x \mapsto x^{-1} - 1$ sends $(K \cup \{\infty\}) \setminus (\{0\} \times F) \cup (F \times \{0\})$ onto $F^\times$, and $(K_T \cup \{\infty\}) \setminus \{(1, 0), (0, 1)\}$ bijectively onto $F^\times$ since $y^{-1} - 1 = \bar{y}/y$ for $y \in K_T$. In this case, for f an integrable function on K_T with respect to $d^\bullet y = dy_1/|y_1(1-y_1)|$ at $y = (y_1, y_2)$, we have

$$\int_{K_T} f(y) d^\bullet y = \int_{F^\times} f((1+t^{-1})^{-1}, (1+t)^{-1}) d^\times t.$$

2.2. A formula for the gamma functions Γ^K .

Lemma. *Let K be a finite separable extension of F , and let K_T be the affine F -hyperplane of K consisting of elements y with trace $Ty = 1$. If $\mu \in \mathcal{A}(K^\times)$ has the module of its restriction to $F^\times$ larger than q^{-1} , then μ is integrable on K_T with respect to the measure $d^\bullet y$ and*

$$\int_{K_T} \mu(y) d^\bullet y = \Gamma^K(\mu, \psi \circ T) / \Gamma(\mu|_{F^\times}, \psi).$$

The proof proceeds by analytic continuation from the case $q_K^{-1/n} < |\mu| < 1$, where n is the degree of K over F ; this later case is straightforward.

2.3. The pairing $\langle \cdot, \cdot \rangle_\psi$. Let $h(\chi, \psi)$ and $h'(\chi, \psi)$ be two rational functions of the characters χ of $F^\times$, depending on the nontrivial additive character ψ of F . We assume they satisfy the following conditions:

(a) there are numbers a, a' and $\omega, \omega' \in \mathcal{A}(F^\times)$ such that, for χ of large enough conductor,

$$\begin{aligned} h(\chi, \psi) &= a \gamma^F(\chi, \psi) \gamma^F(\chi \omega, \psi), \\ h'(\chi, \psi) &= a' \gamma^F(\chi, \psi) \gamma^F(\chi \omega', \psi); \end{aligned}$$

(b) no pole of h is the inverse of a pole of h' .

We define then $\langle h|h' \rangle_\psi \in \mathbb{C} \cup \{\infty\}$ as follows. Observe that for $\mu \in \mathcal{A}(F^\times)$, the function $\chi \mapsto h(\chi \mu, \psi)$ satisfies a) with ω replaced by $\omega \mu^2$; except for a finite number of μ 's, condition b) is satisfied for $h(\chi \mu, \psi)$ and $h'(\chi, \psi)$. We denote by $d\chi$ the 1-form on $\mathcal{A}(F^\times)$ read as $\frac{1}{2\pi i} Z^{-1} dZ$ on each connected component $\chi \mathbb{C}^\times$. Due to property (a), for n large enough, say, $n > N$, the integral

$$\oint_{a(\chi)=n} h(\chi Z, \psi) h'(\chi^{-1}, \psi) d\chi, \quad Z \in \mathbb{C}^\times,$$

taken on simple positive contours around the origin in each component of conductor $a(\chi) = n$, is 0 if $\omega\omega'$ ramifies, and otherwise is

$$\begin{aligned} aa'\omega(-1) \oint_{a(\chi)=n} Z^{-2 \operatorname{ord} \psi - 2n} (\omega\omega')^{-\operatorname{ord} \psi - n} d\chi \\ = aa'(1 - q^{-1})^2 Z^{-2 \operatorname{ord} \psi} (\omega\omega')^{-\operatorname{ord} \psi} (qZ^{-2}(\omega\omega')^{-1})^n; \end{aligned}$$

hence for $|Z^2\omega\omega'| > q$, the series

$$\sum_{n \geq N} \oint_{a(\chi)=n} h(\chi Z, \psi) h'(\chi^{-1}, \psi) d\chi$$

converges absolutely with sum

$$aa'\omega(-1)(1 - q^{-1})^2 (Z^2\omega\omega')^{-\operatorname{ord} \psi} \frac{(Z^2\omega\omega')^{-N-1} q^{N+1}}{1 - qZ^{-2}(\omega\omega')^{-1}}.$$

Then we define the number $\langle h|h' \rangle_\psi$ as the finite part of the integral

$$\oint_{\mathcal{A}(F^\times)} h(\chi, \psi) h'(\chi^{-1}, \psi) d\chi,$$

where a simple positive contour around the origin is taken in each component of $\mathcal{A}(F^\times)$, containing the poles of $\chi \mapsto h'(\chi^{-1}, \psi)$ but not those of $\chi \mapsto h(\chi, \psi)$. This means that $\langle h|h' \rangle_\psi$ is given for N large enough by

$$\begin{aligned} \langle h|h' \rangle_\psi &= \oint_{a(\chi) \leq N} h(\chi, \psi) h'(\chi^{-1}, \psi) d\chi \\ &+ \begin{cases} 0 & \text{if } \omega\omega' \text{ ramifies,} \\ aa'\omega(-1)(1 - q^{-1})^2 \frac{(\omega\omega')^{-N-\operatorname{ord} \psi}}{\omega\omega' - q} q^{N+1} & \text{otherwise.} \end{cases} \end{aligned}$$

It is finite unless $\omega\omega' = q$. In the case $h'(\chi^{-1}, \psi) = bh(\chi, \psi)^{-1}$, we have $aa'\omega(-1) = b$, $\omega' = \omega^{-1}$, so $\omega\omega'$ is not q and

$$\langle h|h' \rangle_\psi = b \oint_{a(\chi) \leq N} d\chi + b(1 - q^{-1})^2 \frac{q^{N+1}}{1 - q};$$

since $\oint_{a(\chi) \leq N} d\chi$ is the measure of units t satisfying $\operatorname{ord}(t - 1) \geq N$, that is,

$$q^N, (1 - q^{-1}) = -(1 - q^{-1})^2 \frac{q^{N+1}}{1 - q},$$

we have shown that $\langle h|h' \rangle_\psi = 0$ in this case.

We shall write

$$\langle h|h' \rangle_\psi = \oint_{\mathcal{A}(F^\times)} h(\chi, \psi) h'(\chi^{-1}, \psi) d\chi.$$

An example of functions h, h' satisfying (a) and (b) are the quadratic Gauss sums γ_θ^F and $\gamma_{\theta'}^F$.

2.4. We give an example of the pairing involving the beta function. In general, for μ, ν characters of $K^\times$, and ψ a nontrivial additive character of K , we define the beta function B^K of K by

$$(2.4.1) \quad B^K(\mu, \nu) = \Gamma^K(\mu, \psi) \Gamma^K(\nu, \psi) / \Gamma^K(\mu\nu, \psi),$$

which is independent of the choice of ψ . We have used the traditional notation B^K .

Proposition. *Let K be a separable quadratic extension of F , and $\theta, \theta' \in \mathcal{A}(K^\times)$. Assume $|\theta\theta'| > q_K^{-1/2}$, and $\theta\theta' \neq 1$ if θ and θ' are liftings of characters of $F^\times$. Then*

$$(2.4.2) \quad \int_{\mathcal{A}(F^\times)} B^K(\theta\chi \circ N, \theta'\chi^{-1} \circ N) d\chi = \int_{K_T} (\theta\bar{\theta}')(y) d^\bullet y,$$

where the left-hand side is defined from the pairing in §2.3, and $d^\bullet y$ has been defined in §2.1.

Proof. The condition $|\theta\theta'| > q_K^{-1/2}$ assures the convergence of both integrals. As they are analytic in this domain, we prove the identity under the conditions $|\theta| < 1$, $|\theta'| < 1$, $|\theta\theta'| > q_K^{-1/2}$. Let m be a positive integer and choose a positive number R_m satisfying

$$\Gamma^K(\theta\chi \circ N, \psi \circ T) = \int_{|Nx| \leq R_m} \theta(x) \chi(Nx) \psi(Tx) d^\times x,$$

and

$$\Gamma^K(\theta'\chi^{-1} \circ N, \psi \circ T) = \int_{|Ny| \leq R_m} \theta'(y) \chi^{-1}(Ny) \psi(Ty) d^\times y$$

for all characters $\chi \in \mathcal{A}(F^\times)$ with conductor $a(\chi) \leq m$. The orthogonal in $F^\times$ of this subgroup of $\mathcal{A}(F^\times)$ is $1 + \mathcal{P}^m$, where $\mathcal{P}$ is the valuation ideal of F . So $\Gamma^K(\theta\theta', \psi \circ T)$ times the left-hand side of (2.4.2) is the limit as m tends to infinity of

$$\oint_{a(\chi) \leq m} \left(\int_{|Nx| \leq R_m, |Ny| \leq R_m} \theta(x) \theta'(y) \psi(Tx + Ty) \chi(Nx/Ny) d^\times x d^\times y \right) d\chi,$$

which is equal to

$$\begin{aligned} & |\mathcal{O}^\times / (1 + \mathcal{P}^m)| \int_{|Nx| \leq R_m, |Ny| \leq R_m, N(y/x) \in 1 + \mathcal{P}^m} \theta(x) \theta'(y) \psi(Tx + Ty) d^\times x d^\times y \\ &= |\mathcal{O}^\times / (1 + \mathcal{P}^m)| \int_{Nw \in 1 + \mathcal{P}^m, |Nx| \leq R_m |N(1+w)|} (\theta\theta')(x) \psi(Tx) \theta'(w) \\ & \quad \times (\theta\theta')(1+w)^{-1} d^\times x d^\times w, \end{aligned}$$

by the change of variables $(x, y) \mapsto (x(1+w)^{-1}, xw(1+w)^{-1})$. For m large enough, the subgroup $1 + \mathcal{P}^m$ is the image by N of some subgroup $1 + \mathcal{P}_K^{m'}$ such that $w \mapsto \theta'(w)(\theta\theta')(1+w)^{-1}$ on $N^{-1}(1 + \mathcal{P}^m) = K_1(1 + \mathcal{P}_K^{m'})$ is constant

$\text{mod}(1 + \mathcal{P}_K^{m'})$; then, because $|\mathcal{O}^\times / (1 + \mathcal{P}^m)| \int_{1 + \mathcal{P}^m} d^\times t = 1$, our expression is, with the normalized Haar measure $d^\times w$ on K_1 ,

$$e \int_{x \in K^\times, w \in K_1, |Nx| \leq R_m |N(1+w)|} (\theta\theta')(x) \psi(Tx) \theta'(w) (\theta\theta')(1+w)^{-1} d^\times x d^\times w.$$

We choose now a positive number r for which

$$\Gamma^K(\theta\theta', \psi \circ T) = \int_{|Nx| \leq r} (\theta\theta')(x) \psi(Tx) d^\times x.$$

Then, for $|N(1+w)| > r/R_m$, the ball $|Nx| \leq R_m |N(1+w)|$ contains the ball $|Nx| \leq r$. We write our expression as

$$\begin{aligned} e \Gamma^K(\theta\theta', \psi \circ T) & \int_{w \in K_1, |N(1+w)| > r/R_m} \theta'(w) (\theta\theta')(1+w)^{-1} d^\times w \\ & + e \int_{\substack{x \in K^\times, w \in K_1 \\ |Nx| \leq R_m |N(1+w)|, |N(1+w)| \leq r/R_m}} (\theta\theta')(x) \psi(Tx) \theta'(w) \\ & \quad \times (\theta\theta')(1+w)^{-1} d^\times x d^\times w. \end{aligned}$$

By Cayley transform $y = (1+w)^{-1}$, $\theta'(w) (\theta\theta')(1+w)$ becomes $(\theta\bar{\theta}')(y) = \theta(y) \theta'(\bar{y})$, and the first term is

$$\Gamma^K(\theta\theta', \psi \circ T) \int_{K_T, |Ny| < R_m/r} (\theta\bar{\theta}')(y) d^\bullet y.$$

The assumption $|\theta\theta'| > q_K^{-1/2}$ implies that this integral has a limit when m , hence R_m , goes to infinity, by Lemma 2.2. We prove now that the second term goes to 0. Let σ be the real number such that $|\theta\theta'(x)| = |Nx|^\sigma$. By assumption, $\sigma < 1/2$. The second term now reads

$$\int_{\substack{x \in K^\times, y \in K_T \\ |N(xy)| \leq R_m, |Ny| \geq R_m/r}} (\theta\theta')(x) (\theta\bar{\theta}')(y) \psi(Tx) d^\times x d^\bullet y$$

and, in absolute value, is dominated by

$$\begin{aligned} & \int_{\substack{x \in K^\times, y \in K_T \\ |N(xy)| \leq R_m, |Ny| \geq R_m/r}} |N(xy)|^\sigma d^\times x d^\bullet y \\ & = \int_{x \in K^\times, |Nx| \leq R_m} |Nx|^\sigma d^\times x \int_{y \in K_T, |Ny| \geq R_m/r} d^\bullet y. \end{aligned}$$

In the right-hand side, the first integral is $O(R_m^\sigma)$, the second is $O(R_m^{-1/2})$ as seen in the proof of lemma. So, the second term is $O(R_m^{\sigma-1/2})$, and goes to 0 when m goes to infinity. This achieves the proof of the proposition.

2.5. Given two quadratic étale F -algebras K and K' , and $B = K \otimes_F K'$, we denote by B_* the subgroup of $K^\times \times K'^\times$ consisting of elements (x, x') such

that $N_{K/F}x = N_{K'/F}x'$. It is a closed subgroup of $K^\times \times K'^\times$ and we have a homomorphism from $B^\times$ to B_* given by

$$(2.5.1) \quad x \mapsto (N_{B/K}x, N_{B/K'}x)$$

with kernel $K_1'' = \text{Ker } N_{K''/F}$, where K'' denotes the third quadratic sub- F -algebra in B . As $\mathcal{O}_K^\times \times \mathcal{O}_{K'}^\times$ is the maximal compact subgroup of $K^\times \times K'^\times$, its intersection $\mathcal{O}_{B_*}^\times$ with B_* is the maximal compact subgroup of B_* , and it is open in B_* . The Haar measure $d_{B_*}^\times$ gives to $\mathcal{O}_{B_*}^\times$ the volume 1. The group B_* appears also as the orthogonal in $K^\times \times K'^\times$ of the group $\mathcal{A}(F^\times)$ embedded in $\mathcal{A}(K^\times \times K'^\times)$ by $\chi \mapsto (\chi \circ N_{K/F}, \chi^{-1} \circ N_{K'/F})$. By Poisson summation formula [W1], this implies that there is a number $c_1 > 0$ such that, for h in the Schwartz-Bruhat space $\mathcal{S}(K^\times \times K'^\times)$ one has

$$(2.5.2) \quad \int_{\mathcal{A}(F^\times)} \left(\int_{K^\times \times K'^\times} h(x, x') \chi(N_{K/F}x / N_{K'/F}x') d_K^\times x d_{K'}^\times x' \right) d\chi \\ = c_1 \int_{B_*} h(x, x') d_{B_*}^\times(x, x').$$

Lemma. Assume that K and K' are not isomorphic. Then

- (a) the image of $B^\times$ in B_* by (2.5.1) is an index 2 subgroup;
- (b) the restriction of (2.5.1) to $\mathcal{O}_B^\times$ has image in $\mathcal{O}_{B_*}^\times$ a subgroup of index $e_{K''/F}$ if K or K' splits over F , and of index 2 otherwise;
- (c) $c_1 = 1$ if K or K' splits over F , otherwise $c_1 = f_{K''/F}$, the modular degree of K'' over F .

Proof. (1) Assume first $K' = F \times F$. Then B_* is the set of $(w, (u, v)) \in K^\times \times (F \times F)^\times$ such that $w\bar{w} = uv$. The algebra B is $K \times K$ and (2.5.1) is $(x, y) \mapsto (xy, (x\bar{x}, y\bar{y}))$. We check now that the image of $B^\times$ is the kernel of the map $(w, (u, v)) \mapsto \eta_{K/F}(u)$ on B_* : if $(w, (u, v))$ lies in B_* and u is a norm $x\bar{x}$ from $K^\times$, so is $v = w\bar{w}/u$. Put $y = wx^{-1}$, then $v = y\bar{y}$ and $w = xy$. Thus $(w, (u, v)) = (xy, (x\bar{x}, y\bar{y}))$ lies in the image of $B^\times$. This gives an isomorphism between the cokernel of the map (2.5.1) and the cokernel of $\eta_{K/F}$, and proves (a) in our case. For (b), we observe that the inverse image of $\mathcal{O}_{B_*}^\times$ in $B^\times$ is the group $\mathcal{O}_B^\times$ of units of B , and that the cokernel of (2.5.1) restricted to $\mathcal{O}_B^\times$ is isomorphic to $\mathcal{O}_F^\times / N_{K/F}\mathcal{O}_K^\times$, which has order equal to the ramification index $e_{K/F}$ of K over F . For (c), we apply (2.5.2) to the characteristic function of $\mathcal{O}_K^\times \times \mathcal{O}_{K'}^\times$: the integral

$$\int_{\mathcal{O}_K^\times \times \mathcal{O}_F^\times \times \mathcal{O}_F^\times} \chi(w\bar{w}) \chi^{-1}(uv) d_K^\times w d_F^\times u d_F^\times v$$

is 0 unless χ is unramified, and then the left-hand side of (2.5.2) is 1, as is the right-hand side when $c_1 = 1$. This proves the lemma when K or K' splits.

(2) Assume now that K and K' are not isomorphic and nonsplit. Let $(u, v) \in B_*$; then the equality $N_{K/F}u = N_{K'/F}v$ shows that this element of $F^\times$ lies in $\text{Im } N_{K/F} \cap \text{Im } N_{K'/F} = \text{Im } N_{B/F}$. Let $z \in B^\times$ with $N_{B/F}z$ equal to this common norm. Its image by (2.5.1) differs from (u, v) by some element of $K_1 \times K'_1$, the product of $\text{Ker } N_{K/F}$ by $\text{Ker } N_{K'/F}$. As the inverse image of $K_1 \times K'_1$ in $B^\times$ by (2.5.1) is $B_1 = \text{Ker } N_{B/F}$, we have shown that the orbits of $B^\times$ acting on B_* through (2.5.1) are the same as the orbits of B_1 acting on $K_1 \times K'_1$. As any element of K_1 lies in $\text{Im } N_{B/K}$, so is in $N_{B/K}B_1$, the number of these orbits is $[K'_1 : N_{B/K'}B_K]$ where $B_K = \text{Ker } N_{B/K}$. We prove now that $N_{B/K'}B_K$ has index 2 in K'_1 . For that, we use the long exact sequence for the cohomology of the group $\text{Gal } B/K''$ acting on the exact sequence

$$1 \rightarrow K^\times \rightarrow B^\times \rightarrow B_K \rightarrow 1$$

of subgroups of $B^\times$ by its Galois action composed with inversion:

$$\begin{aligned} 1 \rightarrow K_1 \rightarrow B_{K''} \rightarrow K'_1 \rightarrow F^\times / N_{K/F}K^\times \xrightarrow{\alpha} K''^\times / N_{B/K''}B^\times \\ \rightarrow K''_1 / N_{B/K''}K^\times \rightarrow 0 \rightarrow \dots \end{aligned}$$

As any element of $F^\times$ is a norm from B to K'' , the arrow α is 0. The image of $B_{K''}$ in K'_1 is $N_{B/K'}B_K$, and the factor group is isomorphic to $F^\times / N_{K/F}K^\times$, which has order 2. This proves the claim, which gives part (a) of the lemma and also part (b) since B_1 consists of units. For (c), we use again the characteristic function of $\mathcal{O}_K^\times \times \mathcal{O}_{K'}^\times$; this shows that c_1 is the index in $\mathcal{O}_F^\times$ of the subgroup generated by $N_{K/F}\mathcal{O}_F^\times$ and $N_{K'/F}\mathcal{O}_{K'}^\times$. As $N_{K/F}\mathcal{O}_K^\times = \mathcal{O}_F^\times$ if K is unramified over F , and $[\mathcal{O}_F^\times : N_{K/F}\mathcal{O}_K^\times] = 2$ otherwise, we see that $c_1 = 1$ unless $N_{K/F}\mathcal{O}_K^\times = N_{K'/F}\mathcal{O}_{K'}^\times$; in this case, the units of F are either in $N_{K/F}K^\times \cap N_{K'/F}K'^\times$ or in the complement of $N_{K/F}K^\times \cup N_{K'/F}K'^\times$, that is, are all in $N_{K''/F}K''^\times$ since $F^\times$ is the union $\text{Im } N_{K/F} \cup \text{Im } N_{K'/F} \cup \text{Im } N_{K''/F}$; but then, K'' is unramified, so $f_{K''/F} = 2$. This gives the proof of (c).

3. MAIN RESULTS

3.1. Theorem 1. *Let K and K' be two quadratic étale F -algebras, and let B be their composite algebra $K \otimes_F K'$. Let θ, θ' be characters of $K^\times$ and $K'^\times$ respectively. Denote by $\omega\eta_{K/F}, \omega'\eta_{K'/F}$ the restrictions of θ, θ' to $F^\times$, and by $\theta \times \theta'$ the character $(\theta \circ N_{B/K})(\theta' \circ N_{B/K'})$ of $B^\times$. Then we have*

$$(3.1.1) \quad \langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = \gamma_{\theta \times \theta'}^F(q^{-1/2}, \psi) / \Gamma(\omega\omega'q^{-2}, \psi).$$

Proof. (a) If K and K' are both split, we see them in $B = F^4$ as the fixed points of the involutions

$$(t, u, v, w) \mapsto (v, w, t, u)$$

and

$$(t, u, v, w) \mapsto (w, v, u, t)$$

respectively. For $\theta = \mu \otimes \nu$, $\theta' = \mu' \otimes \nu'$, we have

$$\theta \times \theta' = (\mu\mu') \otimes (\nu\nu') \otimes (\mu\nu') \otimes (\nu\mu').$$

In this case, formula (3.1.1) has been proved in [L and GL1].

(b) If K and K' are isomorphic fields, we see them as fixed points in $B = K \times K$ under the involutions $(x, y) \mapsto (y, x)$ and $(x, y) \mapsto (\bar{y}, \bar{x})$ respectively, with $\bar{}$ the conjugation of K over F . Then, $\theta \times \theta'$ is the character $(x, y) \mapsto (\theta\theta')(x)(\theta\bar{\theta}')(y)$, and

$$\gamma_{\theta \times \theta'}^F(q^{-1/2}, \psi) = \lambda_{B/K}(\psi)^2 \gamma^K(\theta\theta' q_K^{-1/2}, \psi \circ T) \gamma^K(\theta\bar{\theta}' q_K^{-1/2}, \psi \circ T)$$

with $T = T_{K/F}$. Multiply both θ and θ' by $q_K^{1/2}$, then $\omega\omega'$ is multiplied by q^2 and (3.1.1) takes the form

$$(3.1.2) \quad \oint_{\mathcal{A}(F^\times)} B^K(\theta\chi \circ N, \theta'\chi^{-1} \circ N) d\chi = \Gamma^K(\theta\bar{\theta}', \psi \circ T) / \Gamma(\omega\omega', \psi),$$

with $N = N_{K/F}$ and B^K the beta function (2.4.1) for K . Formula (3.1.2) is the proposition of §2.4 combined with the lemma of §2.2.

(c) The case of K and K' nonisomorphic remains. As they play the same role in the statement, we assume that K is a field. Then the third quadratic étale F -algebra K'' in B defined by K and K' is also a field, satisfying

$$\eta_{K/F} \eta_{K'/F} \eta_{K''/F} = 1.$$

We choose two coset representatives, say, a_+ and a_- of $N_{K''/F}(K''^\times)$ in $F^\times$. Using the complement formula in §1.1 for Γ , we rewrite (3.1.1) as

$$(3.1.3) \quad \langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = |\psi|(\omega\omega')(-1) L_F^{-2} \Gamma(\omega^{-1} \omega'^{-1} q, \psi^{-1}) \gamma_{\theta \times \theta'}^F(q^{-1/2}, \psi).$$

Since both sides of (3.1.3) are meromorphic functions in θ, θ' , it suffices to prove the identity for $q < |\omega\omega'| < q^2$, and we shall so assume. Our strategy is to express the right-hand side of (3.1.3) as an integral over the subgroup $B_\star$ of $K^\times \times K'^\times$, the orthogonal of the group $\mathcal{A}(F^\times)$ embedded in $\mathcal{A}(K^\times \times K'^\times)$ by $\chi \mapsto (\chi \circ N_{K/F}, \chi^{-1} \circ N_{K'/F})$; a suitable form of Poisson summation formula expresses then the right-hand side as a contour integral over $\mathcal{A}(F^\times)$, which is the left-hand side of (3.1.3).

The condition $|\omega\omega'| > q$ implies the integrability near 0 for $d^\times t$ of the character $\omega^{-1} \omega'^{-1} q$. Thus, for R large, one has

$$\Gamma(\omega^{-1} \omega'^{-1} q, \psi^{-1}) = \int_{|t| \leq R} (\omega\omega')(t)^{-1} \psi(-t) |t|^{-1} d^\times t,$$

and

$$\Gamma(\omega^{-1} \omega'^{-1} \eta'' q, \psi^{-1}) = \int_{|t| \leq R} (\omega\omega')(t)^{-1} \eta''(t) \psi(-t) (t)^{-1} d^\times t,$$

with $\eta'' = \eta_{K''/F}$, so

$$\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1}) = \Gamma_+ + \Gamma_-,$$

where $\Gamma_{\pm}$ correspond to the intersection over those t in the ball $|t| \leq R$ which satisfy $t \in a_{\pm}N''(K''^{\times})$, with $N'' = N_{K''/F}$. We express $\Gamma_{\pm}$ as integrals over K'' :

(3.1.4)

$$\begin{aligned} \Gamma_{\pm} &= \Gamma_{\pm, R} \\ &= c^{-1}|a_{\pm}|^{-1} \int_{|N''w| \leq R} (\omega\omega')(a_{\pm}N''w)^{-1} \psi(-a_{\pm}N''w) |N''w|^{-2} d_{K''/F} w, \end{aligned}$$

with c being the measure of $K'' = \text{Ker } N''$ under $d_{K''/F} w/d^{\times} t$.

The condition $|\omega\omega'| < q^2$ assures that the character $\beta = \theta \times \theta'$ of $B^{\times}$ is integrable near 0 with respect to the measure $d_{B, \psi}$; so, for S large, we have

$$\gamma_{\beta}^B(q_B^{-1/2}, \psi \circ T_{B/F}) = \int_{|z|_B \leq S} \beta(z) \psi \circ T_{B/F}(z) d_{B, \psi} z,$$

where $|z|_B = \max(|Nx|, |Ny|)$ if K' is split and z in B correspond to (x, y) in $K \times K$, and $N = N_{K/F}$.

Our first step is to express the product of the two gamma functions in the right-hand side of (3.1.3) as follows:

Lemma.

$$\Gamma(\omega^{-1}\omega'^{-1}q, \psi) \gamma_{\beta}^B(q_B^{-1/2}, \psi \circ T_{B/F}) = |\psi|^{-1} \lambda''(\psi)^{-1} c^{-1} c' \lim_{Q \rightarrow \infty} A_Q$$

where $\lambda''(\psi) = \lambda_{K''/F}(\psi)$ and

$$A_Q = \int_{(x, x') \in B_+, |Nx| \leq Q} (\theta\psi \circ T)(x) (\theta'\psi \circ T')(x') d_{B_+, \psi}^*(x, x'),$$

with $T = T_{K/F}$, $T' = T_{K'/F}$, c' is the index of the image of $\mathcal{O}_B^{\times}$ in $\mathcal{O}_{B_+}^{\times}$ by (2.5.1) and $d_{B_+, \psi}^*(x, x') = |Nx| d_{B_+, \psi}(x, x')$.

Proof of the lemma. Choose S large as above, then

$$\begin{aligned} &c\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1}) \gamma_{\beta}^B(\beta q_B^{-1/2}, \psi \circ T_{B/F}) \\ &= c(\Gamma_+ + \Gamma_-) \gamma_{\beta}^B(\beta q_B^{-1/2}, \psi \circ T_{B/F}) \\ &= \lim_{r \rightarrow 0} c(\Gamma_+ - \Gamma_{+, r|a_+|^{-2}} + \Gamma_- - \Gamma_{-, r|a_-|^{-2}}) \gamma_{\beta}^B(\beta q_B^{-1/2}, \psi \circ T_{B/F}) \\ &= \lim_{r \rightarrow 0} (I(r, S, a_+) + I(r, S, a_-)), \end{aligned}$$

where, for $a = a_+$ or a_- and R large as before, we define

$$\begin{aligned} I(r, S, a) &= |a|^{-1} \int_{|z|_B \leq S, r|a|^{-2} \leq |N''w| \leq R} (\omega\omega')(aN''w)^{-1} \psi(-aN''w) \beta(z) \\ &\quad \times \psi \circ T_{B/F}(z) |N''w|^{-2} d_{K''/F} w d_{B, \psi} z. \end{aligned}$$

Note that for $w \in K''^\times$, we have

$$\begin{aligned}\beta(w) &= \theta(N_{B/K}w)\theta'(N_{B/K'}w) = \theta(N''w)\theta'(N''w) = (\omega\omega'\eta\eta')(N''w) \\ &= (\omega\omega')(N''w),\end{aligned}$$

with $\eta = \eta_{K/F}$, $\eta' = \eta_{K'/F}$ so that $\eta'' = \eta\eta'$. Therefore, $I(r, S, a)$ can be simplified as

$$\begin{aligned}&|a|^{-1}(\omega\omega')(a)^{-1} \int_{|z|_B \leq S, r|a|^{-2} \leq |N''w| \leq R} \beta(w^{-1}z)\psi(-aN''w) \\ &\quad \times \psi \circ T_{B/F}(z)|N''w|^{-2}d_{K''/F}wd_{B,\psi}z \\ &= |a|^{-1}(\omega\omega')(a)^{-1} \int_{|z|_B \leq Sr^{-1}|a|^2, r|a|^{-2} \leq |N''w| \leq R} \beta(z)\psi(-aN''w) \\ &\quad \times \psi \circ T_{B/F}(wz)d_{K''/F}wd_{B,\psi}z\end{aligned}$$

by the change of variable $z \mapsto wz$. For a given S , the assumption $q < |\omega\omega'| < q^2$ implies that the integral

$$\int_{|z|_B \leq Sr^{-1}, |N''w| < r} |\beta(z)|d_{K''/F}wd_{B,\psi}z$$

is majorized by a constant multiple of $r \int_{|z|_B \leq Sr^{-1}} |\beta(z)|d_Bz$, which is $O(r^\alpha)$ with $\alpha = |\beta|^{(\log q_B)^{-1}} > 0$. This shows that the difference

$$I(r, S, a) - I(0, Sr^{-1}|a|^2, a)$$

tends to zero as r does. Hence we have

$$c\Gamma(\omega^{-1}\omega'^{-1}q, \psi^{-1})\gamma^B(\beta q_B^{-1/2}, \psi \circ T_{B/F}) = \lim_{Q \rightarrow \infty} (I(Q|a_+|^2, a_+) + I(Q|a_-|^2, a_-))$$

where for R and Q large $I(Q, a) = I(0, Q, a)$, that is,

$$\begin{aligned}I(Q, a) &= |a|^{-1}(\omega\omega')(a)^{-1} \int_{|z|_B \leq Q, |N''w| \leq R} \beta(z)\psi(-aN''w) \\ &\quad \times \psi \circ T_{B/F}(wz)d_{K''/F}wd_{B,\psi}z.\end{aligned}$$

For fixed a and for R large enough, the integral against w has been computed in the lemma of §1.5. It is 0 unless $|N'' \circ T_{B/K''}z| \leq R$; take R large enough, the condition $|z|_B \leq Q$, or equivalently $|N_{B/K''}z|_{K''} \leq R$, implies $|T_{B/K''}z|_{K''} \leq R$, and this integral against w is, for $|z|_B \leq Q$, equal to

$$|\psi|^{-1}|a|^{-1}\eta''(a)\lambda''(\psi)^{-1}\psi(a^{-1}N'' \circ T_{B/K''}z).$$

Noting

$$(\omega\omega')(a)^{-1} = \theta(a)^{-1}\theta'(a)^{-1}\eta(a)\eta'(a) = \theta(a)^{-1}\theta'(a)^{-1}\eta''(a)$$

and

$$N'' \circ T_{B/K''} = T \circ N_{B/K} + T' \circ N_{B/K'},$$

we arrive at

$$I(Q|a|^2, a) = |\psi|^{-1} \lambda''(\psi)^{-1} |a|^{-2} \int_{|z|_B \leq Q|a|^2} (\theta\psi \circ T)(a^{-1} N_{B/K} z) \\ \times (\theta\psi \circ T')(a^{-1} N_{B/K'} z) d_{B/\psi'} z.$$

From the lemma in §2.5, the image of $B^\times$ in $K^\times \times K'^\times$ by the map $(N_{B/K}, N_{B/K'})$ is a subgroup B_*^+ of index 2 of B_* with coset representatives (a_+^{-1}, a_+^{-1}) and (a_-^{-1}, a_-^{-1}) . We express $I(Q|a|^2, a)$ as an integral over B_*^+ :

$$I(Q|a|^2, a) = |\psi|^{-1} \lambda''(\psi)^{-1} c' \int_{(x, x') \in B_*^+, |Nx| \leq Q|a|^2} |a|^{-2} (\theta\psi \circ T)(a^{-1} x) \\ \times (\theta'\psi \circ T')(a^{-1} x') d_{B_*, \psi}^*(x, x') \\ = |\psi|^{-1} \lambda''(\psi)^{-1} c' \int_{(x, x') \in a^{-1} B_*^+, |Nx| \leq Q} (\theta\psi \circ T)(x) (\theta'\psi \circ T')(x') d_{B_*, \psi}^*(x, x')$$

by the change of variables $(x, x') \mapsto (ax, ax')$. Here c' is the index of the maximal compact subgroup of B_*^+ in the maximal compact subgroup of B_* . Adding $I(Q|a_+|^2, a_+)$ and $I(Q|a_-|^2, a_-)$ leads to

$$|\psi|^{-1} \lambda''(\psi)^{-1} c' \int_{(x, x') \in B_*, |Nx| \leq Q} (\theta\psi \circ T)(x) (\theta'\psi \circ T')(x') d_{B_*, \psi}^*(x, x') \\ = |\psi|^{-1} \lambda''(\psi)^{-1} A_Q.$$

This proves the lemma.

Recall that $\lambda_{B/F} = \lambda(\psi) \lambda'(\psi) \lambda''(\psi)$, where $\lambda(\psi) = \lambda_{K/F}(\psi)$, $\lambda'(\psi) = \lambda_{K'/F}(\psi)$, $\lambda''(\psi) = \lambda_{K''/F}(\psi)$, as seen in §1. The lemma gives the right-hand side of (3.1.3) as

$$(3.1.5) \quad L_F^{-2} \lambda(\psi) \lambda'(\psi) c^{-1} c' \lim_{Q \rightarrow \infty} \int_{(x, x') \in B_*, |Nx| \leq Q} (\theta\psi \circ T)(x) (\theta'\psi \circ T')(x') d_{B_*, \psi}^*(x, x').$$

On the other hand, for $|\omega\omega'| < q^2$, the left-hand side of (3.1.3) is given by a convergent integral

$$\lambda(\psi) \lambda'(\psi) \oint_{\mathcal{A}(F^\times)} \gamma^K(\theta\chi \circ N, \psi \circ T) \gamma^{K'}(\theta'\chi^{-1} \circ N', \psi \circ T') d\chi.$$

We assume now $|\omega|$ and $|\omega'| < q$, which is compatible with the preceding assumption $q < |\omega\omega'| < q^2$. For χ unitary with $a(\chi) \leq M$, the two gamma terms $\gamma^K(\theta\chi \circ N, \psi \circ T)$, $\gamma^{K'}(\theta'\chi^{-1} \circ N', \psi \circ T')$ above are given by integrals on sufficiently large compact subsets of K and K' , depending on M , and independent of θ, θ' running through given compact subsets of $\mathcal{A}(K^\times)$, $\mathcal{A}(K'^\times)$:

$$\gamma^K(\theta\chi \circ N, \psi \circ T) = \int_{|Nx| \leq Q(M)} (\theta\psi \circ T)(x) \chi \circ N(x) d_{K, \psi}^* x,$$

$$\gamma^{K'}(\theta' \chi^{-1} \circ N', \psi \circ T) = \int_{|x'|_{K'} \leq Q(M)} (\theta' \psi \circ T')(x') \chi^{-1} \circ N'(x') d_{K', \psi}^* x'.$$

Using (3.1.5), we see that identity (3.1.3) is equivalent to

$$(3.1.6) \quad \begin{cases} \lim_{M \rightarrow \infty} \chi_{a(\chi) \leq M} \left(\int_{|Nx| \leq Q(M), |x'|_{K'} \leq Q(M)} f(x, x') x(Nx/N'x') d_{K, \psi}^* x d_{K', \psi}^* x' \right) d\chi \\ = c^{-1} c' L_F^{-2} \lim_{Q \rightarrow \infty} \int_{(x, x') \in B_*, |Nx| \leq Q} f(x, x') d_{B_*, \psi}^*(x, x'), \end{cases}$$

where $f(x, x') = (\theta \psi \circ T)(x)(\theta' \psi \circ T')(x')$ for $x \in K^\times$, $x' \in K'^\times$. Recall that the measures $d_{K, \psi}^*$, $d_{K', \psi}^*$, $d_{B_*, \psi}^*$ are given by

$$\begin{aligned} d_{K, \psi}^* x &= |\psi| D^{-1/2} L_K^{-1} |Nx|^{1/2} d_K^\times x, & D &= D_{K/F}, \\ d_{K', \psi}^* x' &= |\psi| D'^{-1/2} L_{K'}^{-1} |N'x'|^{1/2} d_{K'}^\times x', & D' &= D_{K'/F}, \\ d_{B_*, \psi}^*(x, x') &= |\psi|^2 D_{B/F}^{-1/2} L_B^{-1} |Nx|^{1/2} |N'x'|^{1/2} d_{B_*}^\times(x, x'), \end{aligned}$$

hence, by §1.6, we rewrite (3.1.6) as

$$(3.1.7) \quad \begin{cases} \lim_{Q \rightarrow \infty} \chi_{a(\chi) \leq M} \left(\int_{|Nx| \leq Q(M), |x'|_{K'} \leq Q(M)} h(x, x') \chi(Nx/N'x') d_K^\times x d_{K'}^\times x' \right) d\chi \\ = c' e''^{-1} \lim_{Q \rightarrow \infty} \int_{(x, x') \in B_*, |Nx| \leq Q} h(x, x') d_{B_*}^\times(x, x'), \end{cases}$$

where $e'' = e_{K''/F}$, and h is now the function on $K^\times \times K'^\times$ given by

$$h(x, x') = (\theta \psi \circ T)(x)(\theta' \psi \circ T')(x') |Nx|^{1/2} |N'x'|^{1/2}.$$

We observe now that for $|\omega| < q$ and $|\omega'| < q$, the function h_M on $K^\times \times K'^\times$ equal to h on $|Nx| \leq Q(M)$, $|x'|_{K'} \leq Q(M)$, and 0 otherwise, lies in $\mathcal{S}(K^\times \times K'^\times)$. We apply Poisson summation formula (2.5.2) to this function h_M :

$$\begin{aligned} \oint_{\mathcal{A}'(F^\times)} \left(\int_{K^\times \times K'^\times} h_M(x, x') \chi(Nx/N'x') d_K^\times x d_{K'}^\times x' \right) d\chi \\ = c_1 \int_{B_*} h_M(x, x') d_{B_*}^\times(x, x'). \end{aligned}$$

On the left-hand side, only the characters χ with $a(\chi) \leq M$ will contribute, so this formula is

$$\begin{aligned} \oint_{a(\chi) \leq M} \left(\int_{|Nx| \leq Q(M), |x'|_{K'} \leq Q(M)} h(x, x') \chi(Nx/N'x') d_K^\times x d_{K'}^\times x' \right) d\chi \\ = c_1 \int_{(x, x') \in B_*, |Nx| \leq Q(M)} h(x, x') d_{B_*}^\times(x, x'). \end{aligned}$$

This clearly proves (3.1.7), and hence completes the proof of the theorem if we show $c_1 = c' e''^{-1}$; but this is proved in the lemma of §2.5.

3.2. The multiplicative formula for γ_θ^F .

Theorem 2. *Let K be a quadratic étale F -algebra, and $\theta \in \mathcal{A}(K^\times)$. If $\alpha, \beta \in \mathcal{A}(F^\times)$ are not poles of $\chi \mapsto \gamma_\theta^F(\chi, \psi)$, then*

$$(3.2.1) \quad \oint_{\mathcal{A}(F^\times)} \gamma_\theta^F(\chi, \psi) \Gamma(\alpha \chi^{-1}, \psi) \Gamma(\beta \chi^{-1}, \psi) d\chi \\ = \Gamma((\alpha \beta \omega)^{-1}, \psi^{-1}) \gamma_\theta^F(\alpha, \psi) \gamma_\theta^F(\beta, \psi)$$

where $\omega = \eta_{K/F} \theta|_{F^\times}$.

Proof. We apply Theorem 1 to $K' = F \times F$, $\theta'(u, v) = \alpha(u)\beta(v)|uv|^{-1/2}$. Then B is $K \times K$ and

$$(\theta \times \theta')(xy) = \theta(xy) \alpha(Nx) \beta(Ny) |N(xy)|^{-1/2},$$

with $N = N_{K/F}$. As $\lambda_{B/F} = \lambda_{K/F}(\psi)^2$, using the complement formula in §1.1 for Γ , the right-hand side of (3.1.1) is written as

$$|\psi| L_F^{-2} \Gamma((\alpha \beta \omega)^{-1}, \psi^{-1}) \gamma_\theta^F(\alpha, \psi) \gamma_\theta^F(\beta, \psi).$$

On the other hand,

$$\gamma_{\theta'}^F(\chi^{-1}, \psi) = \gamma^F(\alpha q^{1/2} \chi^{-1}, \psi) \gamma^F(\beta q^{1/2} \chi^{-1}, \psi) \\ = (|\psi|^{1/2} L_F^{-1})^2 \Gamma(\alpha \chi^{-1}, \psi) \Gamma(\beta \chi^{-1}, \psi).$$

This gives (3.2.1).

3.3. Theorem 3. *Let K and K' be two nonisomorphic étale F -algebras, and let B be their tensor product over F . Let θ and θ' be two multiplicative characters of K and K' respectively. Assume that the product of the restrictions of θ and θ' to $F^\times$ is trivial. Then, for any nontrivial additive character ψ of F , one has*

$$\gamma^B(\theta \circ N_{B/K} \theta' \circ N_{B/K'}, \psi \circ T_{B/F}) = \theta(-1) = \theta'(-1).$$

Proof. Write β for the character $\theta \circ N_{B/K} \theta' \circ N_{B/K'}$ of $B^\times$. For t in $F^\times$, one has

$$\beta(t) = \theta(t^2) \theta'(t^2) = (\theta \theta')(t^2) = 1,$$

hence $|\beta| = 1$. The identity to prove is equivalent to the relation

$$(3.3.1) \quad \int_{B^\times} \beta(z) \hat{f}(z) d_B^* z = \theta(-1) \int_{B^\times} \beta(z)^{-1} f(z) d_B^* z$$

for any f in (B) ; the integrals are both convergent.

Denote by K'' the third quadratic F -algebra in B determined by K and K' . Since K and K' are not isomorphic, K'' is a field. By restriction to K'' , the two automorphism groups of B over K and K' identify them with $\text{Gal } K''/F$. Hence, the norm maps $N_{B/K}$ and $N_{B/K'}$ coincide on K'' with $N_{K''/F}$. As a consequence the restriction of β to the multiplicative group of

K'' is trivial. Denote by $d_{B,K''}^*$ the quotient measure of d_B^* by $d_{K''}$, so, for $f \in \mathcal{S}(B)$,

$$(3.3.2) \quad \int_{B^\times} \beta(z)^{-1} f(z) d_B^* z = \int_{B^\times / K''^\times} \beta(z)^{-1} \left(\int_{K''} f(wz) d_{K''} w \right) d_{B,K''}^* z.$$

We apply now the Poisson formula to the closed subgroup K'' of B and to the function f

$$(3.3.3) \quad \int_{K''} f(wz) d_{K''} w = |z|_B^{-1} \int_{K_B''} \hat{f}(z^{-1} w') d_{K_B''} w',$$

where K_B'' denotes the orthogonal of K'' in B with respect to the self-duality $(z, z') \mapsto \psi \circ T_{B/F}(zz')$, that is, $K_B'' = \text{Ker } T_{B/K''}$, and where $d_{K_B''}$ denotes the Haar measure on K_B'' associated to $d_{K''}$. For w' in K_B'' , one has $N_{B/K} w' = -N_{B/K'} w'$, hence the image of K_B'' under $N_{B/K}$ is contained in $K \cap K' = F$. So if $w' \in K_B''$ is not 0, one has

$$(3.3.4) \quad \beta(w') = (\theta \circ N_{B/K} w') (\theta' \circ N_{B/K'} w') = \theta(-1) = \theta'(-1).$$

Choose a nonzero element, say s , in K_B'' . We rewrite the right-hand side of (3.3.1) using (3.3.2)–(3.3.4) as

$$(3.3.5) \quad \int_{B^\times / K''^\times} \beta(sz^{-1}) \left(\int_{K_B''} \hat{f}(z^{-1} w') d_{K_B''} w' \right) |z|_B^{-1} d_{B,K''}^* z.$$

We observe now that the measure $|z|_B^{-1} d_{B,K''}^* z$ is $d_{B,K''}^* z^{-1}$. We change the variables $w' = sw$, $z \mapsto sz^{-1}$ in (3.3.5) to get

$$\int_{B^\times / K''^\times} \beta(z) \left(\int_{K''} \hat{f}(zw) d_{K''} w \right) d_{B,K''}^* z$$

which is $\int_{B^\times} \beta(z) \hat{f}(z) d_B^* z$, the left-hand side of (3.3.1). This proves the theorem.

Corollary. *With the same assumptions on K and K' , if now the product of the restrictions to $F^\times$ of θ and θ' is the character $q: t \mapsto |t|^{-1}$, then*

$$(3.3.6) \quad \langle \gamma_\theta^F | \gamma_{\theta'}^F \rangle_\psi = \frac{\lambda_{B/K} \omega(-1)}{\Gamma(\eta_{K/F} \eta_{K'/F} q^{-1}, \psi)},$$

where $\omega(-1) = \theta(-1) \eta_{K/F}(-1)$.

Proof. This follows immediately from Theorems 1 and 3, due to the definition of $\lambda_{B/K}$ given in §1.6.

3.4. Theorem 4. *Let K and K' be two quadratic étale F -algebras, and $\theta \in \mathcal{A}(K^\times)$, $\theta' \in \mathcal{A}(K'^\times)$. Then $\gamma_\theta^F = \gamma_{\theta'}^F$ if and only if one of the following holds:*

(1) *for K' isomorphic to K , then θ' corresponds to θ or to $\bar{\theta}$ by such an isomorphism;*

(2) for K' not isomorphic to K , let $B = K \otimes K'$; then θ and θ' have the same lifts to $B^\times$ and different restrictions to $F^\times$.

Proof. If K and K' are split, then $\gamma_\theta^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\nu\chi, \psi)$ if $\theta = \mu \otimes \nu$, and $\{\mu, \nu\}$ is determined by the zeros (or the poles), with multiplicities, of the function of γ_θ^F . This establishes the theorem in this case.

Assume now K is a field. If $\gamma_\theta^F = \gamma_{\theta'}^F$, the deep twist property shows that $\omega = \omega'$, hence $\theta'|_{F^\times} = \eta_{K/F}\eta_{K'/F}\theta|_{F^\times}$.

(a) If K' is not isomorphic to K , then $\eta_{K/F}\eta_{K'/F}$ is nontrivial, since the third quadratic subalgebra of B is a field.

(b) The equality $\gamma_\theta^F = \gamma_{\theta'}^F$ implies $\gamma_{\theta^{-1}}^F(\chi, \psi) = \omega(-1)\gamma_{\theta'}^F(\chi^{-1}, \psi)^{-1}$, so, as seen in §2.3, $\langle \gamma_{\theta^{-1}}^F | \gamma_{\theta'}^F \rangle_\psi = 0$. By Theorem 1, we then have

$$\gamma^B(\theta^{-1} \circ N_{B/K}\theta' \circ N_{B/K'}, q_B^{-1/2}, \psi \circ T_{B/F}) = 0.$$

(c) If K and K' are isomorphic fields, then B appears as $K \times K$ and γ^B as product of two γ^K 's, so

$$\gamma^K(\theta^{-1}\theta'q_K^{-1/2}, \psi \circ T_{K/F})\gamma^K(\theta^{-1}\bar{\theta}'q_K^{-1/2}, \psi \circ T_{K/F}) = 0$$

which means $\theta' = \theta$ or $\bar{\theta}$.

(d) If now B is a field, then $\theta^{-1} \circ N_{B/K}\theta' \circ N_{B/K'} = 1$, and θ and θ' have the same lift to $B^\times$.

(e) If K is a field and K' is $F \times F$, then $\theta' = \mu \otimes \nu$ with $\mu, \nu \in \mathcal{A}(F^\times)$ and B is $K \times K$; then

$$\gamma^K(\mu \circ N_{K/F}\theta^{-1}q_K^{-1/2}, \psi \circ T_{K/F})\gamma^K(\nu \circ N_{K/F}\theta^{-1}q_K^{-1/2}, \psi \circ T_{K/F}) = 0,$$

so θ is either $\mu \circ N_{K/F}$ or $\nu \circ N_{K/F}$. By the Davenport-Hasse theorem we have $\gamma_\theta^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\mu\eta_{K/F}\chi, \psi)$ or $\gamma^F(\nu\chi, \psi)\gamma^F(\nu\eta_{K/F}\chi, \psi)$. As $\gamma_\theta^F(\chi, \psi) = \gamma_{\theta'}^F(\chi, \psi) = \gamma^F(\mu\chi, \psi)\gamma^F(\nu\chi, \psi)$, we have $\mu\nu^{-1} = \eta_{K/F}$ in both cases; so $\theta = \mu \circ N_{K/F} = \nu \circ N_{K/F}$ and $\theta \circ N_{B/K} = (\mu \otimes \nu) \circ N_{B/K'}$, which means that θ and θ' have the same lift to $B^\times$.

We have proved the “necessary” part of the theorem. If K and K' are isomorphic and θ' corresponds to θ or $\bar{\theta}$, then $\lambda_{K/F}(\psi) = \lambda_{K'/F}(\psi)$ and $\gamma_\theta^F = \gamma_{\theta'}^F$ since both $\chi \circ N$ and $\psi \circ T$ are invariant by the conjugation of K over F . Assume now B is a field, and that $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ with $\theta|_{F^\times} \neq \theta'|_{F^\times}$. We use the third quadratic subextension K'' of B , and write $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ on $K''^\times$: this shows that the restrictions to $N_{K''/F}K''^\times$, which is an index two subgroup of $F^\times$, of θ and θ' are equal. Hence $\theta'|_{F^\times} = \eta_{K''/F}\theta|_{F^\times}$, that is, $\omega = \omega'$. We prove that this implies that θ and θ' are regular over F . Indeed, if $\theta = \mu \circ N_{K/F}$ then $\theta \circ N_{B/K} = \mu \circ N_{K'/F} \circ N_{B/K'} = \theta' \circ N_{B/K'}$; hence $\mu \circ N_{K'/F}$ and θ' coincide on the elements of $K'^\times$ which are norms

from $B^\times$, in particular on $F^\times$: this contradicts $\theta|_{F^\times} \neq \theta'|_{F^\times}$. The relation $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ shows that this character of $B^\times$ is fixed by both $\text{Gal } B/K$ and $\text{Gal } B/K'$, hence by all $\text{Gal } B/F$, that is, also by $\text{Gal } B/K''$. This shows that θ and its conjugate $\bar{\theta}$ over F have the same lift to $B^\times$, and θ being regular, this gives $\bar{\theta} = \theta\eta_{B/K}$, and also $\bar{\theta}' = \theta'\eta_{B/K'}$. As $\eta_{B/K}$ is the lift of $\eta_{K''/F}$ to $K^\times$, the relation $\gamma_\theta^F = \gamma_{\bar{\theta}}^F$ shows that $\gamma_\theta^F(\chi, \psi) = \gamma_\theta^F(\chi\eta_{K''/F}, \psi)$, and also for $\gamma_{\theta'}^F$. We apply Theorem 1 to get the relation $\langle \gamma_{\theta''}^F | \gamma_\theta^F \rangle_\psi = \langle \gamma_{\theta''}^F | \gamma_{\theta'}^F \rangle_\psi$ for $\theta'' \in \mathcal{A}(K''^\times)$, with a finite number of exceptions. We take $\theta'' = \chi \circ N_{K''/F}$ and use the Davenport-Hasse identity to get

$$\gamma_\theta^F(\chi, \psi) \gamma_{\theta'}^F(\chi\eta_{K''/F}, \psi) = \gamma_{\theta'}^F(\chi, \psi) \gamma_{\theta'}^F(\chi\eta_{K''/F}, \psi),$$

that is, $\gamma_\theta^F(\chi, \psi)^2 = \gamma_{\theta'}^F(\chi, \psi)^2$. We define a sign $\varepsilon(\chi)$ by

$$\gamma_{\theta'}^F(\chi, \psi) = \varepsilon(\chi) \gamma_\theta^F(\chi, \psi), \quad \chi \in \mathcal{A}(F^\times).$$

This sign is 1 for χ with large conductor, due to the deep twist property and $\omega = \omega'$. As θ and θ' are regular, the rational functions $\gamma_\theta^F(\chi, \psi)$ and $\gamma_{\theta'}^F(\chi, \psi)$ are monomials on $\mathcal{A}(F^\times)$, and having the same squares, the degree is the same, as $\varepsilon(\chi)$ is constant on each component of $\mathcal{A}(F^\times)$. Theorem 1 shows that $\langle \gamma_{\theta'}^F | \gamma_{\theta^{-1}}^F \rangle_\psi = 0$, hence, for M large enough,

$$\begin{aligned} 0 &= \oint_{a(\chi) \leq M} \gamma_{\theta'}^F(\chi, \psi) \gamma_{\theta^{-1}}^F(\chi^{-1}, \psi) d\chi - q^M (1 - q^{-1}) \\ &= \left(\oint_{a(\chi) \leq M} \varepsilon(\chi) d\chi - q^M (1 - q^{-1}) \right) \omega(-1), \end{aligned}$$

that is, $\sum_{a(\chi) \leq M} \varepsilon(\chi) = \sum_{a(\chi) \leq M} 1$, the summations being on the characters of $\mathcal{O}^\times$ with $a(\chi)$ at most M . As $\varepsilon(\chi)$ is a sign, this implies $\varepsilon(\chi) = 1$ for all χ 's, and $\gamma_\theta^F = \gamma_{\theta'}^F$. The theorem is completely proved.

Corollary. *With K , K' , θ , θ' as in the theorem, we have $\gamma_\theta^F = \gamma_{\theta'}^F$ if and only if $\text{Ind}_K^F \theta = \text{Ind}_{K'}^F \theta'$.*

Proof. (a) This is clear if K and K' are isomorphic since $\text{Ind}_K^F \theta$ and $\text{Ind}_{K'}^F \theta'$ are the same if and only if $\theta' = \theta$ or $\bar{\theta}$, which means $\gamma_\theta^F = \gamma_{\theta'}^F$ by Theorem 4.

(b) Assume K is a field and K' is $F \times F$. Write $\theta' = \mu \otimes \nu$, so that $\text{Ind}_{K'}^F \theta'$ is the direct sum of the two one-dimensional representations μ and ν of $F^\times$, abelianized group of W_F . Theorem 4 shows that $\gamma_\theta^F = \gamma_{\theta'}^F$ if and only if $\theta = \mu \circ N_{K/F}$ and $\nu = \mu\eta_{K/F}$; then $\text{Ind}_K^F \theta = \mu \otimes \text{Ind}_K^F 1 = \mu \oplus \mu\eta_{K/F} = \mu \oplus \nu = \text{Ind}_{K'}^F \mu \otimes \nu = \text{Ind}_{K'}^F \theta'$. Conversely, if $\text{Ind}_K^F \theta$ is the sum of the two characters μ and ν , then $\text{Ind}_K^F \theta$ is not irreducible, so $\theta = \chi \circ N_{K/F}$; but then $\text{Ind}_K^F \theta = \chi \oplus \chi\eta_{K/F}$ so that $\chi = \mu$ or ν and $\nu = \mu\eta_{K/F}$, and the theorem says $\gamma_\theta^F = \gamma_{\theta'}^F$.

(c) Assume now $B = K \otimes K'$ is a field. If $\gamma_\theta^F = \gamma_{\theta'}^F$, then, by Theorem 4, we have $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$ and $\theta|_{F^\times} \neq \theta'|_{F^\times}$; during the proof, we have shown that this implies $\bar{\theta} = \theta\eta_{B/K}$ and $\bar{\theta}' = \theta'\eta_{B/K'}$. As the trace of $\text{Ind}_K^F \theta$ is 0 outside W_K and on W_K it factors through its abelianization $K^\times$ where it is given by $(\theta + \bar{\theta})/2$, this is 0 outside W_B and on W_B it factors through its abelianization $B^\times$ where it is given by $\theta \circ N_{B/K}$. Hence $\text{Ind}_K^F \theta$ and $\text{Ind}_{K'}^F \theta'$ have their traces supported on W_B , where they are equal, so the representations $\text{Ind}_K^F \theta$ and $\text{Ind}_{K'}^F \theta'$ are equivalent. Conversely, this equivalence implies that traces and determinants of two representations are the same; for the determinants, this gives $\theta|_{F^\times} \eta_{K/F} = \theta'|_{F^\times} \eta_{K'/F}$, so $\theta|_{F^\times}$ and $\theta'|_{F^\times}$ are different; for the traces, we get 0 outside $W_K \cap W_{K'} = W_B$, that is, $\bar{\theta} = \theta\eta_{B/K}$, $\bar{\theta}' = \theta'\eta_{B/K'}$, and the coincidence on W_B says that $\theta \circ N_{B/K} = \theta' \circ N_{B/K'}$. Theorem 1 then concludes the proof.

REFERENCES

- [D] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular Functions in One Variable. II, Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin-Heidelberg-New York, 1973, pp. 501–597.
- [G] P. Gérardin, *Groupes quadratiques et applications arithmétiques*, Séminaire de Théorie des Nombres, Paris, 1984–85 (C. Golstein, Ed.), Progress in Math., Vol. 63, Birkhäuser, Boston, Mass., 1986, pp. 79–95.
- [GK] P. Gérardin and P. C. Kutzko, *Facteurs locaux pour GL(2)*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 359–384.
- [GL1] P. Gérardin and W.-C. W. Li, *A functional equation for degree two local factors*, Canad. Math. Soc. Bull. **28** (1985), 355–371.
- [GL2] —, *Fourier transforms of representations of quaternions*, J. Reine Angew. Math. **349** (1985), 121–173.
- [GL3] —, *Degree two monomial representations of local Weil groups*, J. Reine Angew. Math. **394** (1989), 1–30.
- [L] W.-C. W. Li, *Barnes' identities and representations of GL_2 . Part II: Nonarchimedean local field case*, J. Reine Angew. Math. **345** (1983), 69–92.
- [S] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1964.
- [W1] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 153–211.
- [W2] —, *Basic number theory*, Springer-Verlag, Berlin and New York, 1967.
- [W3] —, *Fonctions zeta et distributions*, Séminaire Bourbaki, Juin 1966.

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