ON GELFAND PAIRS ASSOCIATED WITH SOLVABLE LIE GROUPS

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ABSTRACT. Let G be a locally compact group, and let K be a compact subgroup of $\text{Aut}(G)$, the group of automorphisms of G . There is a natural action of K on the convolution algebra $L^1(G)$, and we denote by $L_K^1(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair (K, G) is called a Gelfand pair if $L_K^1(G)$ is commutative. In this paper we consider the case where G is a connected, simply connected solvable Lie group and $K \subseteq \text{Aut}(G)$ is a compact, connected group. We characterize such Gelfand pairs (K, G) , and determine a moduli space for the associated K -spherical functions.

INTRODUCTION

Let G be a locally compact group, and let K be a compact subgroup of $\text{Aut}(G)$, the group of automorphisms of G . There is a natural action of K on the convolution algebra $L^1(G)$, and we denote by $L_K^1(G)$ the subalgebra of those elements in $L^1(G)$ that are invariant under this action. The pair (K, G) is called a Gelfand pair if $L_K^1(G)$ is commutative. A more general and more usual definition of Gelfand pairs assumes that K is a compact subgroup of G . One then defines (K, G) to be a Gelfand pair if the subalgebra of K -bi-invariant elements in $L^1(G)$ is commutative. This is the case, for example, if (G, K) is a Riemannian symmetric pair, as was shown by Gelfand in 1950, [Ge]. In this paper we consider the case where G is a connected, simply connected solvable Lie group and $K \subseteq \text{Aut}(G)$ is a compact, connected group.

For the remainder of the paper, unless otherwise stated, S will denote a connected, simply connected solvable Lie group and N will denote a connected, simply connected nilpotent Lie group, with corresponding Lie algebras $\mathcal{S}$, $\mathcal{N}$, and K will denote a compact, connected subgroup of the appropriate automorphism group.

The classification of Gelfand pairs involving solvable groups presupposes a classification for such pairs involving nilpotent groups, which is the subject we

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first consider. An important reduction is given by

Theorem A. *If (K, N) is a Gelfand pair then N is at most two step.*

The proof is based on the observation that (K, G) is a Gelfand pair if, and only if, products (as sets) of K -orbits in G commute, i.e. for each $x, y \in G$, $(K \cdot x)(K \cdot y) = (K \cdot y)(K \cdot x)$.

The criterion that we generally use to determine if (K, N) is a Gelfand pair is contained in a theorem due to Carcano, [Ca], which we now recall. Let $\pi \in \hat{N}$, and denote by K_π the set of all elements $k \in K$ such that $\pi_k \simeq \pi$ where π_k is the element of $\hat{N}$ defined by $\pi_k(x) = \pi(k \cdot x)$ for all $x \in N$. Then there is a projective representation W_π of K_π on H_π , the representation space of π . W_π is called the *intertwining representation* for π . If σ is the cocycle of W_π there is a decomposition

$$W_\pi = \sum_{T \in \hat{K}_\pi^a} c(T, W_\pi) T,$$

where $c(T, W_\pi)$ denotes the multiplicity of T in W_π . Carcano's theorem states that (K, N) is a Gelfand pair if $c(T, W_\pi) \leq 1$ for all π in a set of full Plancherel measure, and that, conversely, if (K, N) is a Gelfand pair then $c(T, W_\pi) \leq 1$ for every $\pi \in \hat{N}$.

Since the representations of 2-step nilpotent groups factor through tensor products of representations of Heisenberg $\times$ abelian groups, the classification of Gelfand pairs (K, N) reduces to classification of Gelfand pairs (K, H_n) , where H_n is the $2n + 1$ -dimensional Heisenberg group. We realize H_n as $\mathbf{C}^n \times \mathbf{R}$ with multiplication given by $(z, t)(z', t') = (z + z', t + t' + 2\Im z\bar{z}')$. If $K \subseteq \text{Aut}(H_n)$, then, after conjugating by an element of $\text{Aut}(H_n)$ if necessary, we may assume that $K \subseteq U(n)$, the group of $n \times n$ unitary matrices acting on $\mathbf{C}^n$ in the usual fashion. Given such a K , we denote by $K_{\mathbf{C}}$ its complexification, which may be considered as a subgroup of $Gl(n, \mathbf{C})$. We denote by $\mathbf{C}[\mathbf{C}^n]$ the polynomial ring over $\mathbf{C}^n$. There is a natural action of $K_{\mathbf{C}}$ on $\mathbf{C}[\mathbf{C}^n]$.

Theorem B. *Suppose that K acts irreducibly on $\mathbf{C}^n$. (K, H_n) is a Gelfand pair if, and only if, $K_{\mathbf{C}}$ acts without multiplicity on $\mathbf{C}[\mathbf{C}^n]$.*

Victor Kac, [Ka], has given a complete list of all such groups $K_{\mathbf{C}}$ acting without multiplicity on $\mathbf{C}[\mathbf{C}^n]$. If the action of K on $\mathbf{C}^n$ is not irreducible, consider the irreducible decomposition $\mathbf{C}^n = \sum_{j=1}^p V_j$, and let K_j denote the subgroup of $U(V_j)$ given by the (irreducible) action of K on V_j . The subset of H_n given by $V_j \times \mathbf{R}$ is isomorphic to H_{m_j} , where $m_j = \dim(V_j)$. For $n_1, \dots, n_p \in \mathbf{Z}^+$ let $\mathbf{P}^{n_1, \dots, n_p} = \bigotimes_{j=1}^p \mathbf{P}_{j, n_j}$, where $\mathbf{P}_{j, n_j}$ is a K_j -irreducible subspace of $\mathbf{C}[V_j]$.

Theorem C. *(K, N) is a Gelfand pair if, and only if, the subrepresentations of K on the various $\mathbf{P}^{n_1, \dots, n_p}$ are all distinct.*

We next consider the free, two-step nilpotent Lie group on n -generators, $F(n)$. We identify its Lie algebra $\mathcal{F}(n)$ with $\mathbf{R}^n \oplus \Sigma_n$, where $\mathbf{R}^n$ is viewed as $1 \times n$ real matrices, Σ_n is the set of $n \times n$ skew symmetric matrices, and the bracket is defined by $[(u, U), (v, V)] = (0, u^t v - v^t u)$. The automorphism group of $\mathcal{F}(n)$ is identified with $Gl(n, \mathbf{R}) \times \text{Hom}(\mathbf{R}^n, \Sigma_n)$ with the action of (A, ν) on (u, U) given by $(A, \nu) \cdot (u, U) = (uA, A^t U A + \nu(u))$. Thus, $O(n)$, the group of $n \times n$ orthogonal matrices is a maximal compact subgroup of $\text{Aut}(\mathcal{F}(n))$. We denote by $SO(n)$ the subgroup of matrices of determinant one.

Theorem D. *Let K be a closed (not necessarily connected) subgroup of $SO(n)$. $(K, F(n))$ is a Gelfand pair if, and only if $K = SO(n)$.*

Suppose now that a two-step N is given with $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{N}$. (If this condition is not satisfied, then N has an abelian direct product factor that does not play a role in the current considerations.) Given a compact, connected $K \subseteq \text{Aut}(N)$, we fix a K -invariant inner product, $\langle \cdot, \cdot \rangle$, on $\mathcal{N}$, and denote by $\mathcal{N}_1$, the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$. Let $X_1, \dots, X_n$ be an orthonormal basis for $\mathcal{N}_1$. Define the homomorphism $\lambda: \mathcal{F}(n) \rightarrow \mathcal{N}$ by setting $\lambda(e_i) = X_i$ (where $e_1, \dots, e_n$ is the standard basis for $\mathbf{R}^n$), and $\lambda(E_{i,j}) = [X_i, X_j]$, (where $E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathcal{F}(n)$). Let $\mathcal{K}$ denote the kernel of λ ($\subseteq \Sigma_n$). Note that $\lambda: \mathbf{R}^n \rightarrow \mathcal{N}_1$ is an isometry (where $\mathcal{F}(n)$ is equipped with the (standard) inner product $\langle (u, U), (v, V) \rangle = uv^t + \frac{1}{2} \text{tr}(UV^t)$). Given $k \in K$, we define $\tilde{k} \in \text{Aut}(\mathcal{F}(n))$ by $\tilde{k}(e_i) = \lambda^{-1}(k \cdot (\lambda(e_i)))$ and $\tilde{k}(E_{i,j}) = [\tilde{k} \cdot e_i, \tilde{k} \cdot e_j]$, and set $\tilde{K} = \{\tilde{k} | k \in K\}$. Then $\tilde{K} \subseteq O(n)$, and one has that K is maximal compact if, and only if, $\tilde{K} = O_{\mathcal{N}}(n) := \{A \in O(n) | A \cdot \mathcal{K} (= A^t \mathcal{K} A) = \mathcal{K}\}$.

Let $\mathcal{Z}$ denote the orthogonal complement in Σ_n of $\mathcal{K}$, and set $\mathcal{N}_{\mathcal{Z}} = \mathbf{R}^n \oplus \mathcal{Z}$ with Lie bracket defined by $[(u, U), (v, V)]_{\mathcal{Z}} = P_{\mathcal{Z}}(u^t v - v^t u)$, where $P_{\mathcal{Z}}$ is the orthogonal projection of Σ_n onto $\mathcal{Z}$. Then $\mathcal{N}_{\mathcal{Z}} \simeq \mathcal{N}$ and $\tilde{K} \subseteq \text{Aut}(\mathcal{N}_{\mathcal{Z}})$.

For nonzero $B \in \mathcal{Z}$, let $\mathcal{N}_B$ denote the subset of $\mathcal{N}_{\mathcal{Z}}$ given by $\mathbf{R}^n B \oplus \mathbf{R}B$, i.e. the range of B in $\mathbf{R}^n$ plus the line through B , and define a Lie bracket similar to the above by following the bracket in $\mathcal{F}(n)$ with the orthogonal projection onto $\mathbf{R}B$. The quotient Lie algebra $\mathcal{N}_{\mathcal{Z}}/\mathcal{Z}_0$, where $\mathcal{Z}_0$ is the orthogonal complement in $\mathcal{Z}$ of $\mathbf{R}B$ is isomorphic to the direct sum of ideals $\mathcal{N}_B$ and $(\mathbf{R}^n B)^\perp$, the latter being commutative. Let H_B denote the simply connected Lie group corresponding to $\mathcal{N}_B$, and given $b \in (\mathbf{R}^n B)^\perp$, let $\tilde{K}_{(b,B)} = \{\tilde{k} \in \tilde{K} | \tilde{k} \cdot (b, B) = (b, B)\}$.

Theorem E. *(K, N) is a Gelfand pair if $(\tilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for all (b, B) in a set of full Plancherel measure, and conversely, if (K, N) is a Gelfand pair, then $(\tilde{K}_{(b,B)}, H_B)$ is a Gelfand pair for all $B \in \mathcal{Z}$, $b \in (\mathbf{R}^n B)^\perp$.*

We demonstrate the use of Theorem E in two examples. In the first, let N be the group whose Lie algebra has a basis X, Y_1, Y_2, Z_1, Z_2 , and with all non-trivial commutators determined by $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$. We show that there is no compact subgroup $K \subseteq \text{Aut}(N)$ for which (K, N) is a Gelfand pair.

In the second example, we give a short proof of a theorem due to H. Leptin [Le] which states that if K is the n -dimensional torus (and N is a two-step group with $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, the center of $\mathcal{N}$) then (K, N) is a Gelfand pair if, and only if, N is the quotient of the direct product of n -copies of H_1 , with K lifting to a $U(1)$ action on each factor H_1 .

We turn now to solvable groups. The essential new ingredient is another theorem due to H. Leptin, which was privately communicated to the authors. Since a proof has not appeared in the literature, we include his proof here.

Theorem (Leptin). *Let $\mathcal{S}$ be a solvable Lie algebra with nilradical $\mathcal{N}$. Let K be a compact, connected subgroup of $\text{Aut}(\mathcal{S})$, and let $\mathcal{S}_0 = \{X \in \mathcal{S} \mid k \cdot X = X, \forall k \in K\}$. Then $\mathcal{S} = \mathcal{S}_0 + \mathcal{N}$.*

For $X \in \mathcal{S}$, let i_X denote the inner-automorphism of S determined by $\exp X$, and denote by $\text{rad}(S)$ the simply connected nilpotent Lie group whose Lie algebra is the nilradical of $\mathcal{S}$. Using Leptin's theorem we can prove

Theorem F. *(K, S) is a Gelfand pair if, and only if, $(K, \text{rad}(S))$ is a Gelfand pair, and for each $X \in \mathcal{S}_0$, $y \in S$ there is a $k \in K$ such that $i_X(y) = k \cdot y$.*

Finally, we consider the K -spherical functions associated to a Gelfand pair (K, S) . Recall that a K -spherical function ϕ is a continuous, complex valued function defined on S satisfying $\phi(e) = 1$ and $\int_K \phi(xk \cdot y) dk = \phi(x)\phi(y)$ for each $x, y \in S$. It is well known that integration against a K -spherical function, ϕ , defines a complex homomorphism on $L_K^1(S)$, that this homomorphism is continuous if ϕ is bounded, and that each continuous homomorphism of $L_K^1(S)$ is obtained in this manner. We denote by $\Delta(K, S)$ the set of continuous homomorphisms on $L_K^1(S)$. It follows from Theorem F, that if (K, S) is a Gelfand pair then S has polynomial growth, [Je], and hence that $L^1(S)$ is a symmetric Banach $*$ -algebra, [Lu]. From this one can show that the bounded K -spherical functions are positive definite, in sharp contrast to the case when (G, K) is a Riemannian symmetric pair (cf. [He]).

We first consider Gelfand pairs (K, N) . One shows that if $\pi \in \hat{N}$ and $\pi' = \pi_k$, then the intertwining representations W_π and $W_{\pi'}$ have the same irreducible subspaces.

Theorem G. *Let (K, N) be a Gelfand pair. Then ϕ is a bounded K -spherical function if, and only if, there is a $\pi \in \hat{N}$ and a $\xi \in V_\alpha \subseteq \mathbf{H}_\pi$, $\|\xi\| = 1$, such that for each $x \in N$,*

$$\phi(x) = \phi_{\pi, \xi}(x) := \int_K \langle \pi(k \cdot x)\xi, \xi \rangle dk,$$

where V_α is an irreducible subspace for the intertwining representation W_π . Furthermore, bounded K -spherical functions $\phi_{\pi, \xi} = \phi_{\pi', \xi'}$ if, and only if, $\pi' = \pi_k$ for some $k \in K$ and ξ, ξ' belong to the same V_α .

Theorem G states that there is a 1-1 correspondence between $\Delta(K, N)$ and the fibered product $\hat{N}/K \times_\pi \sigma(W_\pi, \mathbf{H}_\pi)$, where $\hat{N}/K$ denotes the K -orbits in $\hat{N}$, and $\sigma(W_\pi, \mathbf{H}_\pi)$ denotes the irreducible components of W_π in $\mathbf{H}_\pi$.

Suppose now that (K, S) is a Gelfand pair. Let $X_1, \dots, X_p$ be a basis for a complement of $\mathcal{N}$, the nilradical of $\mathcal{S}$, in $\mathcal{S}_0$. For each $y \in S$, there exist unique $n(y) \in N (= \exp(\mathcal{N}))$ and $\mathbf{t}(y) \in \mathbf{R}^p$ such that $y = n(y)\Pi_i \exp(t_i(y)X_i)$.

Theorem H. ϕ is a bounded K -spherical function on S if, and only if, $\phi|_N$ is a bounded K -spherical function on N and there exists $\mathbf{a} \in \mathbf{R}^p$ such that $\phi(y) = \phi(n(y))e^{i\langle \mathbf{a}, \mathbf{t}(y) \rangle}$. Thus,

$$\Delta(K, S) = \Delta(K, N) \times \mathbf{R}^p.$$

Remarks. A number of authors, in addition to those already mentioned, have considered Gelfand pairs of the form (K, N) , and the associated K -spherical functions. In [HR] it is shown that the usual action of a maximal torus in $U(n)$ on H_n provides an example of a Gelfand pair, and the K -spherical functions are expressed in terms of Laguerre polynomials. The paper [KR] exhibits examples (K, N) , where N is an irreducible group of Heisenberg type and K is either $\text{Spin}(n)$ or a maximal connected compact subgroup of $\text{Aut}(n)$. In [Ca], examples are presented where N arises as the Šilov boundary of a Siegel domain of type II and $K = SU(p) \times U(q)$. The generalized Laguerre polynomials introduced in [Hz] are shown in [Di] to be associated to certain Gelfand pairs $(U(n), H_n)$.

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PRELIMINARIES

Consider a unimodular group G with $K \subseteq G$ a compact subgroup. We denote the L^1 -functions that are invariant under both the left and right actions of K on G by $L^1(G//K)$. These form a subalgebra of the group algebra $L^1(G)$ with respect to the convolution product

$$(1.1) \quad f * g(x) = \int_G f(y)g(y^{-1}x) dy = \int_G f(xy^{-1})g(y) dy.$$

According to the traditional definition, one says that $K \subseteq G$ is a Gelfand pair if $L^1(G//K)$ is commutative.

Suppose now that K is a compact group acting on G by automorphisms via some homomorphism $\phi: K \rightarrow \text{Aut}(G)$. One can form the semidirect product $K \ltimes G$, with group law

$$(1.2) \quad (k_1, x_1)(k_2, x_2) = (k_1 k_2, x_1 k_1 \cdot x_2),$$

where we write $k \cdot x$ for $\phi(k)(x)$. Right K -invariance of a function $f: K \ltimes G \rightarrow \mathbb{C}$ means that $f(k, x)$ depends only on x . Accordingly, if one defines $f_G: G \rightarrow \mathbb{C}$ by $f_G(x) = f(e, x)$, then one obtains a bijection $L^1(K \ltimes G/K) \simeq L_K^1(G)$ given by $f \mapsto f_G$. Here $L_K^1(G)$ denotes the K -invariant functions on G , i.e. those $f \in L^1(G)$ such that $f(k \cdot x) = f(x)$ for all $x \in G$ and $k \in K$. One verifies easily that this map respects the convolution product and we see that $K \subseteq K \ltimes G$ is a Gelfand pair if, and only if, the convolution algebra $L_K^1(G)$ is commutative. Thus, the definition given in the introduction agrees with the more standard one.

Note that if (K_1, G) is a Gelfand pair and $K_1 \subseteq K_2$, then (K_2, G) is also a Gelfand pair. Also note that we can assume that K acts faithfully on G since we can always replace K by $K/\ker(\phi)$. In this way we can regard K as a compact subgroup of $\text{Aut}(G)$. It is a useful fact that the Gelfand pair property depends only on the conjugacy class of K in $\text{Aut}(G)$.

Lemma 1.3. *Let K, L be compact groups acting on G which are conjugate inside $\text{Aut}(G)$. Then (K, G) is a Gelfand pair if, and only if, (L, G) is a Gelfand pair.*

Proof. For $f \in L^1(G)$, define $f^L \in L_L^1(G)$ by

$$(1.4) \quad f^L(x) = \int_L f(l \cdot x) dl.$$

The map $f \mapsto f^L$ is onto $L_L^1(G)$. Suppose that $L = uKu^{-1}$ for some $u \in \text{Aut}(G)$. Then

$$\begin{aligned} f^L(x) &= \int_K f((uku^{-1}) \cdot x) dk \\ &= \int_K (f \circ u)(k \cdot (u^{-1}(x))) dk \\ &= (f \circ u)^K(u^{-1}(x)). \end{aligned}$$

It follows that $f^L(u(x)) = (f \circ u)^K(x)$ and that $L_L^1(G) \rightarrow L_K^1(G): f \mapsto f \circ u := \Phi(f)$ is a vector space isomorphism.

Let dx denote Haar measure on G . Then $u^*(dx) = \Delta(u) dx$ for some nonzero real number $\Delta(u)$. We will show that $\Phi(f) * \Phi(g) = \Delta(u)\Phi(f * g)$. It

follows that $f * g = g * f \Leftrightarrow \Phi(f) * \Phi(g) = \Phi(g) * \Phi(f)$. We compute

$$\begin{aligned}
 (\Phi(f) * \Phi(g))(x) &= \int_G \Phi(f)(y) \Phi(g)(y^{-1}x) dy \\
 &= \int_G (f \circ u)(y) (g \circ u)(y^{-1}x) dy \\
 &= \int_G f(u(y)) g(u(y^{-1}u(x))) dy \\
 &= \int_G f(y) g(y^{-1}u(x)) u^*(dy) \\
 &= \Delta(u) \int_G f(y) g(y^{-1}u(x)) dy \\
 &= \Delta(u)(f * g)(u(x)) \\
 &= \Delta(u)\Phi(f * g)(x). \quad \square
 \end{aligned}$$

Suppose now that G is a Lie group. For $D \in \mathcal{E}'(G)$, the space of compactly supported distributions, define the K -average D^K by

$$(1.5) \quad \langle D^K, f \rangle = \langle D, f^K \rangle,$$

for each $f \in C_c^\infty(G)$, where f^K is defined by (1.4). The space of K -invariant, compactly supported distributions is

$$(1.6) \quad \mathcal{E}'_K(G) = \{D \in \mathcal{E}'(G) \mid D^K = D\} = \{D^K \mid D \in \mathcal{E}'(G)\}.$$

If δ_x is the delta function at $x \in G$ then $\delta_x^K \in \mathcal{E}'_K(G)$ has compact support $K \cdot x$. One has

$$(1.7) \quad \langle \delta_x^K, f \rangle = \int_K f(k \cdot x) dk.$$

Lemma 1.8. *The K -invariant test functions are dense in $\mathcal{E}'_K(G)$.*

Proof. Merely note that if $\{u_n\} \subseteq \mathcal{E}(G)$, and $u_n \rightarrow D \in \mathcal{E}'(G)$, then $u_n^K \rightarrow D^K = D$, for each $D \in \mathcal{E}'_K(G)$. $\square$

The convolution of distributions $D_1, D_2 \in \mathcal{E}'(G)$ is defined by

$$(1.9) \quad \langle D_1 * D_2, f \rangle = \langle D_1(x), \langle D_2, l_{x^{-1}}f \rangle \rangle,$$

where $l_x f(y) = f(x^{-1}y)$. In particular, one has

$$(1.10) \quad \langle \delta_x^K * \delta_y^K, f \rangle = \int_K \int_K f((k_1 \cdot x)(k_2 \cdot y)) dk_1 dk_2.$$

Lemma 1.11. *If (K, G) is a Gelfand pair then convolution in $\mathcal{E}'_K(G)$ is commutative.*

Proof. This follows immediately from commutativity of $L_K^1(G)$ and Lemma 1.8. $\square$

Theorem 1.12. (K, G) is a Gelfand pair if, and only if, for all $x, y \in G$, $xy \in (K \cdot y)(K \cdot x)$.

Proof. Suppose that $xy \notin (K \cdot y)(K \cdot x)$. We will show that $\delta_x^K * \delta_y^K \neq \delta_y^K * \delta_x^K$, so (K, G) fails to be a Gelfand pair by Lemma 1.11. Indeed, one can find a non-negative test function $f: G \rightarrow \mathbf{R}$ with $f(xy) = 1$ and $f((K \cdot y)(K \cdot x)) = \{0\}$ by compactness of $(K \cdot y)(K \cdot x)$. But then (1.10) shows that $\langle \delta_x^K * \delta_y^K, f \rangle$ is positive, whereas $\langle \delta_y^K * \delta_x^K, f \rangle = 0$.

Conversely, suppose $xy \in (K \cdot y)(K \cdot x)$ for all $x, y \in G$, and let $f, g \in L_K^1(G)$. Then

$$f * g(x) = \int_G f(xy)g(y^{-1})dy = \int_G f((k_3 \cdot y)x)g(y^{-1})dy,$$

where $xy = (k_1 \cdot y)(k_2 \cdot x) = k_2((k_3 \cdot y)x)$. Note that k_1, k_2 , and k_3 depend on the integration variable y . Using K -invariance of f we write

$$\begin{aligned} f * g(x) &= \int_G \int_K f(k \cdot ((k_3 \cdot y)x))g(y^{-1})dk dy \\ &= \int_G \int_K f((k \cdot y)(k k_3^{-1} \cdot x))g(y^{-1})dk dy \end{aligned}$$

via $k \mapsto k k_3^{-1}$

$$= \int_K \int_G f(y(k k_3^{-1} \cdot x))g(k^{-1} \cdot y^{-1})dy dk$$

via $y \mapsto k^{-1} \cdot y$

$$= \int_G \int_K f(y(k k_3^{-1} \cdot x))g(y^{-1})dk dx$$

using K -invariance

$$= \int_G \int_K f(y(k \cdot x))g(y^{-1})dk dy$$

via $k \mapsto k k_3$

$$= \int_K g * f(k \cdot x)dk$$

changing the order of integration

$$= g * f(x)$$

using K -invariance. $\square$

It is not difficult to check that the condition in Theorem 1.12 is equivalent to the more symmetrical condition that $(K \cdot x)(K \cdot y) = (K \cdot y)(K \cdot x)$.

THREE-STEP GROUPS

We now begin our consideration of Gelfand pairs that involve nilpotent groups. Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$. Recall the descending central series for $\mathcal{N}$,

$$(2.1) \quad \mathcal{N} = \mathcal{N}^{(1)} \supset \mathcal{N}^{(2)} \supset \dots \supset \mathcal{N}^{(n)} \supset \mathcal{N}^{(n+1)} = \{0\},$$

where $\mathcal{N}^{(k)} = [\mathcal{N}, \mathcal{N}^{(k-1)}]$ for $k > 1$. We say that N is an n -step group if $\mathcal{N}^{(n)} \neq \{0\}$.

Fix any inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{N}$, and let $\mathcal{N}_k$ denote the orthogonal complement to $\mathcal{N}^{(k+1)}$ inside $\mathcal{N}^{(k)}$ for $1 \leq k \leq n-1$. Also, set $\mathcal{N}_n = \mathcal{N}^{(n)}$ so that

$$(2.2) \quad \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_n \quad \text{and} \quad \mathcal{N}^{(k)} = \mathcal{N}_k \oplus \cdots \oplus \mathcal{N}_n$$

for $1 \leq k \leq n$.

Lemma 2.3. *Let N be an n -step group with $n \geq 3$. Then*

$$[\mathcal{N}_1, \mathcal{N}^{(n-1)}] \neq \{0\}.$$

Proof. Suppose $[\mathcal{N}_1, \mathcal{N}^{(n-1)}] = \{0\}$, and choose any n elements $X_1, X_1, \dots, X_{n-1}, Y \in \mathcal{N}$. Then $W = [X_1, [X_2, [\cdots [X_{n-2}, X_{n-1}] \cdots]]]$ is an element of $\mathcal{N}^{(n-1)}$, and writing $Y = U + V$ where $U \in \mathcal{N}_1$, $V \in \mathcal{N}^{(2)}$, we see that

$$[Y, W] = [U, W] + [V, W] = [V, W] = 0$$

since $[\mathcal{N}_1, \mathcal{N}^{(n-1)}] = 0$ and any n -fold bracket of terms in $\mathcal{N}^{(2)}$ must vanish. However, this shows that $\mathcal{N}$ cannot be n -step since all n -fold brackets in $\mathcal{N}$ are zero. $\square$

The main result of this section is

Theorem 2.4. *If N is an n -step group with $n \geq 3$ then there are no Gelfand pairs (K, N) .*

Proof. Since K is compact, there is a K -invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{N}$. Indeed, such an inner product can be obtained by averaging an arbitrary one with respect to the K -action. Form the decomposition (2.2) using this inner product and choose any $X \in \mathcal{N}_1$, $Y \in \mathcal{N}_{n-1}$ with $[X, Y] \neq 0$. This is possible by Lemma 2.3, and the observations that $\mathcal{N}^{(n-1)} = \mathcal{N}_{n-1} \oplus \mathcal{N}_n$ and $\mathcal{N}_n$ is contained in the center.

Let $\exp$ denote the exponential map from $\mathcal{N}$ to N . We will show that for $x = \exp(X)$, $y = \exp(Y)$ one has $xy \notin (K \cdot y)(K \cdot x)$. Suppose otherwise, and pick $k_1, k_2 \in K$ so that $xy = (k_1 \cdot y)(k_2 \cdot x)$. By the Baker-Campbell-Hausdorff formula one has

$$(2.5) \quad X + Y + \frac{1}{2}[X, Y] = k_2 \cdot X + k_1 \cdot Y + \frac{1}{2}[k_1 \cdot Y, k_2 \cdot X],$$

where $(k, X) \mapsto k \cdot X$ is the derived action of K on $\mathcal{N}$.

Since any automorphism of $\mathcal{N}$ must preserve each $\mathcal{N}^{(k)}$, we have $k_1 \cdot Y \in \mathcal{N}^{(n-1)}$. Thus X and $k_2 \cdot X$ differ by an element $W \in \mathcal{N}^{(n-1)}$, so that $k_2 \cdot X = X + W$. As $\mathcal{N}_1$ and $\mathcal{N}^{(n-1)}$ are orthogonal subspaces in $\mathcal{N}$ and the K -action preserves orthogonality, we see that $W = 0$. That is $k_2 \cdot X = X$, and (2.5) becomes

$$(2.6) \quad Y + \frac{1}{2}[X, Y] = k_1 \cdot Y + \frac{1}{2}[k_1 \cdot Y, X].$$

The same trick now shows that $k_1 \cdot Y = Y$, since the two differ by an element of $\mathcal{N}_n$. Finally, (2.6) becomes $[X, Y] = [Y, X]$, which is impossible since $[X, Y] \neq 0$. $\square$

SOME REPRESENTATION THEORY

This section will serve to introduce some notation and to describe a result due to G. Carcano. Since this result is of primary importance to our analysis, we will include a sketch of the proof.

If π and π' are irreducible unitary representations of N , we write $\pi \simeq \pi'$ to indicate that π and π' are unitarily equivalent. We denote by $\hat{N}$ the equivalence classes of irreducible unitary representations of N . Given $k \in K$ and $\pi \in \hat{N}$ we denote by π_k the representation defined by

$$(3.1) \quad \pi_k(x) = \pi(k \cdot x).$$

The stabilizer of π under this action is

$$(3.2) \quad K_\pi = \{k \in K : \pi_k \simeq \pi\}.$$

We denote by $\mathcal{O}_\pi$ the coadjoint orbit in $\mathcal{N}^*$ corresponding to π according to the Kirillov theory, and note that K_π is also the stabilizer of $\mathcal{O}_\pi$ under the dual action of K on $\mathcal{N}^*$.

For each $k \in K_\pi$, one can choose an intertwining operator $W_\pi(k)$ with $\pi_k(x) = W_\pi(k)\pi(x)W_\pi(k)^{-1}$ for each $x \in N$. The map $k \mapsto W_\pi(k)$ need not be a representation of K_π . Indeed, the $W_\pi(k)$'s are only characterized up to multiplicative constants in the circle $\mathbf{T}$ by the intertwining condition. In fact, there will be a map

$$(3.3) \quad \sigma (= \sigma_\pi): K_\pi \times K_\pi \rightarrow \mathbf{T}$$

for which $W_\pi(k_1 k_2) = \sigma(k_1, k_2)W_\pi(k_1)W_\pi(k_2)$. The map σ can be made measurable and is called the multiplier for the projective representation W_π . We call W_π the *intertwining representation* for the representation π .

Many aspects of representation theory can be extended to projective representations as well (cf. [Ma]). In particular, compactness of K_π implies that W_π decomposes as a direct sum of irreducible (projective) representations. Writing $c(T, W_\pi)$ for the multiplicity of T in W_π , one has

$$(3.4) \quad W_\pi = \sum_{T \in \hat{K}_\pi^\sigma} c(T, W_\pi)T.$$

Here, $\hat{K}_\pi^\sigma$ denotes the set of unitary equivalence classes of projective representations of K_π with multiplier $\sigma (= \sigma_\pi)$. The following theorem is from [Ca].

Theorem 3.5. *If (K, N) is a Gelfand pair, then $c(T, W_\pi) \leq 1$ for all $\pi \in \hat{N}$, and conversely, if $c(T, W_\pi) \leq 1$ for almost all (with respect to Plancherel measure) $\pi \in \hat{N}$ then (K, N) is a Gelfand pair.*

Proof. For completeness we sketch what is essentially Carcano's proof.

Let $\pi \in \hat{N}$ and let W_π be the intertwining representation of K_π with multiplier σ . If $\bar{T}$ is any irreducible projective representation of K_π with multiplier $\bar{\sigma}$, then

$$(3.6) \quad R(k, x) = \bar{T}(k) \otimes \pi(x)W_\pi(k)$$

is an irreducible representation of $K_\pi \ltimes N$ whose restriction to N is a multiple of π , and the induced representation $\text{Ind}_{K_\pi \ltimes N}^{K \ltimes N}(R)$ is irreducible for $K \ltimes N$. By considering all π and $\bar{T}$, one obtains all equivalence classes of irreducible representations of $K \ltimes N$ in this manner (cf. [Ma]).

It is well known that if $K \subset G$ is a Gelfand pair, then for each irreducible representation π of G , the space of K -fixed vectors has dimension $c(1_K, \pi|_K) \in \{0, 1\}$ (cf. [He]). For the representation R given by (3.6), one has

$$\text{Ind}_{K_\pi \ltimes N}^{K \ltimes N}(R)|_K \simeq \text{Ind}_{K_\pi}^K(R|_{K_\pi}) = \text{Ind}_{K_\pi}^K(\bar{T} \otimes W_\pi),$$

and by Frobenius reciprocity for compact groups,

$$c(1_K, \text{Ind}_{K_\pi}^K(\bar{T} \otimes W_\pi)) = c(1_K|_{K_\pi}, \bar{T} \otimes W_\pi) = c(1_{K_\pi}, \bar{T} \otimes W_\pi).$$

This last value can be written as $c(T, W_\pi)$ since 1_{K_π} has multiplicity 1 in $\bar{T} \otimes T$ and multiplicity 0 in $\bar{T} \otimes S$ for S not equivalent to T . This shows the necessity of the condition.

Now suppose $\pi \in \hat{N}$ satisfies the multiplicity condition. Denote the Hilbert space on which it acts by H_π , and form the decomposition

$$(3.7) \quad H_\pi = \sum_{T \in \hat{K}_\pi^\sigma} H_{\pi, T}$$

into K_π -irreducible subspaces. (If T is not a subrepresentation of W_π , then $H_{\pi, T} = \{0\}$.) If $f \in L_K^1(N)$ then one shows that the operator $\pi(f)$ commutes with every $W_\pi(k)$. Since each factor $H_{\pi, T}$ in (3.7) occurs only once, $\pi(f)$ must preserve these factors and thus, acts as a scalar in each by Schur's Lemma. It follows that if $f, g \in L_K^1(N)$ then the operators $\pi(f)$ and $\pi(g)$ commute and hence $\pi(f * g) = \pi(g * f)$. When this equality holds for almost all $\pi \in \hat{N}$, one concludes that $f * g = g * f$ by appealing to the Plancherel Theorem. $\square$

We remark that the result holds more generally for compact actions on separable locally compact groups.

HEISENBERG GROUPS

The $(2n + 1)$ -dimensional Heisenberg group H_n has Lie algebra $\mathcal{H}_n$ with basis $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ and structure equations given by $[X_i, Y_i] = Z$. The group $Sp(n, \mathbf{R})$ of real $2n \times 2n$ symplectic matrices acts on $\text{Span}(X_1, \dots, X_n, Y_1, \dots, Y_n)$ by automorphisms of $\mathcal{H}_n$. It is well known that $U(n) = Sp(n, \mathbf{R}) \cap O(2n) = Sp(n, \mathbf{R}) \cap SO(2n)$ is a maximal compact

connected subgroup of $\text{Aut}(H_n)$ (cf. [Ho]). (The full automorphism group contains inner automorphisms, dilations and an involution that sends Z to $-Z$ in addition to these symplectic automorphisms.) If one models H_n as $\mathbf{C}^n \times \mathbf{R}$, as we generally will, then $U(n)$ becomes the group of $n \times n$ unitary matrices acting on $\mathbf{C}^n$ in the usual fashion.

We recall the representation theory of H_n . A generic set of coadjoint orbits in $\mathcal{H}_n^*$ is parametrized by nonzero $\lambda \in \mathbf{R}$, where the orbit $\mathcal{O}_\lambda$ is the hyperplane in $\mathcal{H}_n^*$ of all functionals taking the value λ at Z . The action of $U(n)$ on $\mathcal{H}_n^*$ preserves each $\mathcal{O}_\lambda$. Hence, if π_λ is the element of $\widehat{H}_n$ corresponding to $\mathcal{O}_\lambda$, then $U(n)$ also preserves the equivalence class of π_λ .

One can realize π_λ in the Fock space

$$(4.1) \quad \mathbf{H}_\lambda(\mathbf{n}) = \left\{ \text{entire } f: \mathbf{C}^n \rightarrow \mathbf{C} \mid \int_{\mathbf{C}^n} e^{-2|\lambda||w|^2} |f(w)|^2 dw < \infty \right\}$$

as

$$(4.2) \quad \pi_\lambda(z, t)f(w) = e^{-i\lambda t + \lambda(2\langle w, z \rangle - |z|^2)} f(w - z)$$

for $\lambda > 0$ and

$$(4.3) \quad \pi_\lambda(z, t)f(w) = e^{-i\lambda t - \lambda(2\langle w, \bar{z} \rangle - |z|^2)} f(w - \bar{z})$$

for $\lambda < 0$. Here $\langle w, z \rangle$ denotes the Hermitian inner product on $\mathbf{C}^n$. We refer the reader to [Ho or Ta] for a discussion of the Fock model.

Define $W_\lambda(k): \mathbf{H}_\lambda(\mathbf{n}) \rightarrow \mathbf{H}_\lambda(\mathbf{n})$ by

$$(4.4) \quad W_\lambda(k)f(z) = f(k^{-1}z).$$

Then $W_\lambda(k)$ intertwines $\pi_\lambda(z, t)$ and $(\pi_\lambda)_k(z, t) = \pi_\lambda(kz, t)$. We verify this for $\lambda > 0$. Indeed,

$$\begin{aligned} W_\lambda(k)(\pi_\lambda(k^{-1}z, t)f)(w) &= \pi_\lambda(k^{-1}z, t)f(k^{-1}w) \\ &= e^{-i\lambda t + \lambda(2\langle k^{-1}w, k^{-1}z \rangle - |k^{-1}z|^2)} f(k^{-1}w - k^{-1}z) \\ &= e^{-i\lambda t + \lambda(2\langle w, z \rangle - |z|^2)} W_\lambda(k)f(w - z) \\ &= (\pi_\lambda(z, t)W_\lambda(k)f)(w), \end{aligned}$$

and hence

$$(4.5) \quad W_\lambda(k)\pi_\lambda(z, t)W_\lambda(k)^{-1} = \pi_\lambda(kz, t)$$

as claimed. That is, $U(n)$ is the stabilizer of the equivalence class of $\pi_\lambda \in \widehat{H}_n$ under the action of $U(n)$ and $W_\lambda: \mathbf{H}_\lambda(\mathbf{n}) \rightarrow \mathbf{H}_\lambda(\mathbf{n})$ is the intertwining representation as in (3.4). (We remark that up to a factor of $\det(k)^{\frac{1}{2}}$, W_λ lifts to the oscillator representation on the double cover $MU(n)$ of $U(n)$ (cf. [Ta]).) It follows that for any compact subgroup $K \subseteq U(n)$, $K_{\pi_\lambda} = K$, and the intertwining representation of K is given by the restriction of W_λ to K .

Given a compact, connected subgroup $K \subseteq U(n)$, we denote its complexification by $K_{\mathbf{C}}$. The action of K on $\mathbf{C}^n$ yields a representation of $K_{\mathbf{C}}$ on

$\mathbb{C}^n$, and one can view $K_{\mathbb{C}}$ as a subgroup of $Gl(n, \mathbb{C})$. (A discussion of the complexification construction can be found in [BtD].)

A finite dimensional representation $\rho: G \rightarrow Gl(V)$ in a complex vector space V is said to be *multiplicity free* if each irreducible G -module occurs at most once in the associated representation on the polynomial ring $\mathbb{C}[V]$ (given by $(x \cdot p)(z) = p(\rho(x^{-1})z)$).

Theorem 4.6. *Let K be a compact, connected subgroup of $U(n)$ acting irreducibly on $\mathbb{C}^n$. The following are equivalent: (i) (K, H_n) is a Gelfand pair. (ii) The representation of $K_{\mathbb{C}}$ on $\mathbb{C}^n$ is multiplicity free. (iii) The representation of $K_{\mathbb{C}}$ on $\mathbb{C}^n$ is equivalent to one of the representations in the following table:*

Multiplicity Free Representations		
Group	Acting On	Subject To
$Sl(n, \mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
$Gl(n, \mathbb{C})$	$\mathbb{C}^n$	$n \geq 1$
$Sp(k, \mathbb{C})$	$\mathbb{C}^n$	$n = 2k$
$\mathbb{C}^* \times Sp(k, \mathbb{C})$	$\mathbb{C}^n$	$n = 2k$
$\mathbb{C}^* \times SO(n, \mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
$Gl(k, \mathbb{C})$	$S^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = k(k+1)/2, k \geq 2$
$Sl(k, \mathbb{C})$	$\Lambda^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = \binom{k}{2}$ and k is odd
$Gl(k, \mathbb{C})$	$\Lambda^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = \binom{k}{2}$
$Sl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$	$n = kl, k \neq l$
$Gl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$	$n = kl$
$Gl(2, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 4k$
$Sl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 6k$
$Gl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 6k$
$Gl(4, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 32$
$Sl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 8k, k > 4$
$Gl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 8k, k > 4$
$\mathbb{C}^* \times Spin(7, \mathbb{C})$	$\mathbb{C}^n$	$n = 8$
$\mathbb{C}^* \times Spin(9, \mathbb{C})$	$\mathbb{C}^n$	$n = 16$
$Spin(10, \mathbb{C})$	$\mathbb{C}^n$	$n = 16$
$\mathbb{C}^* \times Spin(10, \mathbb{C})$	$\mathbb{C}^n$	$n = 16$
$\mathbb{C}^* \times G_2$	$\mathbb{C}^n$	$n = 7$
$\mathbb{C}^* \times E_6$	$\mathbb{C}^n$	$n = 27$

Proof. The complexification $K_{\mathbb{C}}$ of K is connected, reductive, algebraic (cf. [BtD]) and acts irreducibly on $\mathbb{C}^n$. Moreover, the representation of K on $\mathbb{C}^n$ is multiplicity free if, and only if, the complexified representation of $K_{\mathbb{C}}$ on $\mathbb{C}^n$ is multiplicity free. The multiplicity free irreducible linear representations of connected, reductive, algebraic groups have been classified by V. Kac. The table given here is taken from Theorem 3 of [Ka]. This gives the equivalence of (ii) and (iii).

The equivalence of (i) and (ii) is an immediate consequence of Theorem 3.5 once one observes that for each $\lambda \neq 0$, W_{λ} is the completion of the associated representation of K on $\mathbb{C}[\mathbb{C}^n]$. $\square$

Remarks. Some comments are in order regarding the table. $\mathbb{C}^*$ denotes the nonzero complex numbers, S^2 the symmetric 2-tensors and Λ^2 the alternating 2-tensors. The group $\mathbb{C}^* \times Sp(k, \mathbb{C})$ acts on $\mathbb{C}^{2k}$ via $(\lambda, A) \cdot v = \lambda v A$. We can view $\mathbb{C}^* \times Sp(k, \mathbb{C})$ as the group of $n \times n$ complex matrices that transform the standard symplectic structure on $\mathbb{C}^n$ into a scalar multiple of itself. There are similar interpretations for the other groups $\mathbb{C}^* \times G$. $\text{Spin}(n, \mathbb{C}) = \text{Spin}(n, \mathbb{R})_{\mathbb{C}}$ is a double cover of $SO(n, \mathbb{C})$ and acts by the complexified half-spin representation. $\text{Spin}(7, \mathbb{C})$ and $\text{Spin}(9, \mathbb{C})$ are simply connected and $\pi_1(\text{Spin}(10, \mathbb{C})) = \mathbb{Z}_2$.

Suppose now that the action of K on $\mathbb{C}^n$ is reducible, and let

$$(4.7) \quad \mathbb{C}^n = \sum_{j=1}^p V_j$$

be a decomposition of $\mathbb{C}^n$ into K -irreducible (not necessarily complex) subspaces. If (K, H_n) is a Gelfand pair, then the V_{α}' s are orthogonal with respect to the skew-symmetric form on $\mathbb{C}^n$ given by $\Lambda: (z, w) \mapsto \Im \langle z, w \rangle$. Indeed, if $z_i \in V_{\alpha_i}$ for $i = 1, 2$ then by Theorem 1.12 there exist $k_1, k_2 \in K$ such that $(z_1, 0)(z_2, 0) = (k_2 \cdot z_2, 0)(k_1 \cdot z_1, 0)$. It follows that

$$(4.8) \quad \sum_i z_i = \sum_i k_i \cdot z_i$$

and that

$$(4.9) \quad \Lambda(z_1, z_2) = \Lambda(k_2 \cdot z_2, k_1 \cdot z_1).$$

Since the V_{α} 's are orthogonal with respect to the usual Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ and are K -invariant, one concludes from (4.8) that $k_i \cdot z_i = z_i$, for $i = 1, 2$, and hence from (4.9) that $\Lambda(z_1, z_2) = 0$. It now follows that the V_{α} 's have complex structure, i.e. $iV_{\alpha} = V_{\alpha}$. Suppose not, and let $z \in V_{\alpha}$ such that $iz \notin V_{\alpha}$. Then $iz = \sum_{\beta} z_{\beta}$, and $z_{\beta} \neq 0$ for some $\beta \neq \alpha$. Thus,

$$|z|^2 = -\Lambda(z, iz) = \sum_{\beta} -\Lambda(z, z_{\beta}) = -\Lambda(z, \bar{z}_{\alpha}) < |z|^2.$$

Finally, since the V_{α} 's are invariant under multiplication by i , the skew-symmetric form Λ is nondegenerate on each V_{α} . Therefore, if $m_j = \dim(V_j)$,

$H_{m_j} \simeq V_j \times \mathbf{R}$. (This isomorphism is made explicit in the proof of Theorem 5.12.)

Let K_j denote the subgroup of $U(V_j)$, the group of unitary transformations on V_j obtained by the restriction of K to V_j , and let

$$(4.10) \quad \mathbf{C}[V_j] = \sum_{n=0}^{\infty} \mathbf{P}_{j,n}$$

be the decomposition of the polynomial ring over V_j into K_j -irreducible subspaces, with the convention that $\mathbf{P}_{j,0} = \{0\}$. For each p -tuple $(n_1, \dots, n_p) \in (\mathbf{Z}^+)^p$, let $\mathbf{P}^{n_1, \dots, n_p} = \mathbf{P}_{1,n_1} \otimes \dots \otimes \mathbf{P}_{p,n_p}$. If $W_{\lambda,j}$ denotes the intertwining representation associated to the pair (K_j, H_{m_j}) as above, then for each $k \in K$, the restriction of W_{λ} to $\mathbf{P}^{n_1, \dots, n_p}$ is given by $W_{\lambda,1} \otimes \dots \otimes W_{\lambda,p}$. Thus, if (K, H_n) is a Gelfand pair, Theorem 4.6 implies that (K_j, H_{m_j}) is a Gelfand pair for each $j = 1, \dots, p$. But it also implies the stronger condition that the subrepresentations of K on $\mathbf{P}^{n_1, \dots, n_p}$, as $(n_1, \dots, n_p)$ ranges over $(\mathbf{Z}^+)^p$, are distinct. This establishes the necessity of the condition in the following theorem. The sufficiency is an immediate consequence of Theorem 4.6 and the observation that

$$\mathbf{C}[\mathbf{C}^n] = \sum_{(n_1, \dots, n_p) \in (\mathbf{Z}^+)^p} \mathbf{P}^{n_1, \dots, n_p}.$$

Theorem 4.11. *(K, H_n) is a Gelfand pair if, and only if, the subrepresentations of W_{λ} on $\mathbf{P}^{n_1, \dots, n_p}$ are distinct as $(n_1, \dots, n_p)$ ranges over $(\mathbf{Z}^+)^p$.*

We consider two examples. For the first, let K be the subgroup of matrices of determinant one in $U(2) \times U(1) \subseteq U(3)$, i.e. $K = \{(A, \overline{\det(A)}) | A \in U(2)\}$. The decomposition of $\mathbf{C}^3$ corresponding to (4.7) is $\mathbf{C}^3 = \mathbf{C}^2 \oplus \mathbf{C}$, in the obvious sense, and corresponding to (4.10) one has that $\mathbf{C}[\mathbf{C}^2] = \sum_{n=1}^{\infty} \mathbf{P}_{1,n}$, where $\mathbf{P}_{1,n}$ is the space of homogeneous polynomials in z_1, z_2 of degree n , and $\mathbf{C}[\mathbf{C}] = \sum_{n=1}^{\infty} \mathbf{P}_{2,n}$, where $\mathbf{P}_{2,n} = \mathbf{C}z_3^n$. The intertwining representation of K on $\mathbf{P}^{n_1, n_2}$ is equivalent to the representation $A \mapsto (\det(A))^{n_2} W_{\lambda}(A)$ of $U(2)$ on $\mathbf{P}_{1,n_1}$. These representations are clearly irreducible and inequivalent for distinct (n_1, n_2) . Thus (K, H_3) is a Gelfand pair.

For the second example, let K be the subgroup of $U(1) \times U(1)$ consisting of all matrices of determinant one. In this case, both (K_1, H_1) and (K_2, H_1) are Gelfand pairs, and in fact, the subrepresentations of the intertwining representations of K_1 and K_2 on $\mathbf{C}[\mathbf{C}]$ are distinct (corresponding to $\mathbf{Z}^+$ for K_1 , and $\mathbf{Z}^-$ for K_2). However, the intertwining representation on $\mathbf{P}^{n,n}$ is the identity for each n , and thus (K, H_2) is not a Gelfand pair.

We conclude this section with an immediate corollary to Theorem 4.11.

Corollary 4.12. *Let K_j be a compact subgroup of $U(n_j)$ for $1 \leq j \leq p$, $K = \prod K_j$, and let $n = \sum n_j$. Then (K, H_n) is a Gelfand pair if, and only if (K_j, H_{n_j}) is a Gelfand pair for $1 \leq j \leq p$.*

FREE GROUPS

In this section we turn our attention to the free, two-step nilpotent Lie group on n -generators, $F(n)$. We realize its Lie algebra, $\mathcal{F}(n)$, as $\mathbf{R}^n \oplus \Sigma_n$, where $\mathbf{R}^n$ is viewed as $1 \times n$ real matrices, Σ_n is the space of real $n \times n$ skew symmetric matrices, and the Lie bracket is given by

$$(5.1) \quad [(u, U), (v, V)] = (0, u^t v - v^t u).$$

The group law is thus

$$(5.2) \quad (u, U)(v, V) = (u + v, U + V + \frac{1}{2}(u^t v - v^t u)).$$

Lemma 5.3. *There is a bijection between $\text{Aut}(F(n)) \simeq \text{Aut}(\mathcal{F}(n))$ and the set $Gl(n, \mathbf{R}) \times \text{Hom}(\mathbf{R}^n, \Sigma_n)$.*

Proof. The exponential map establishes the isomorphism

$$\text{Aut}(F(n)) \simeq \text{Aut}(\mathcal{F}(n)).$$

For $(A, \nu) \in Gl(n, \mathbf{R}) \times \text{Hom}(\mathbf{R}^n, \Sigma_n)$, define $\phi_{(A, \nu)}: \mathcal{F}(n) \rightarrow \mathcal{F}(n)$ by

$$(5.4) \quad \phi_{(A, \nu)}(u, U) = (uA, A^t U A + \nu(u)).$$

It is easy to check that $\phi_{(A, \nu)}$ is a Lie algebra automorphism. On the other hand, if $\phi: \mathcal{F}(n) \rightarrow \mathcal{F}(n)$ is any given automorphism, then $\phi = \phi_{(A, \nu)}$, where A and ν are the composites

$$\mathbf{R}^n \hookrightarrow \mathcal{F}(n) \xrightarrow{\phi} \mathcal{F}(n) \rightarrow \mathbf{R}^n$$

and

$$\mathbf{R}^n \hookrightarrow \mathcal{F}(n) \xrightarrow{\phi} \mathcal{F}(n) \rightarrow \Sigma_n$$

respectively. $\square$

Note that the correspondence in Lemma 5.3 becomes a group isomorphism if the set $Gl(n, \mathbf{R}) \times \text{Hom}(\mathbf{R}^n, \Sigma_n)$ is given the group structure

$$(5.4) \quad (A, \nu)(B, \mu) = (AB, A \cdot \mu + \nu B),$$

with $Gl(n, \mathbf{R})$ acting on Σ_n by $A \cdot V = A^t V A$. In particular, we see that a maximal compact subgroup of $\text{Aut}(F(n))$ can be identified with $O(n)$, the group of real orthogonal matrices. This acts on $\mathcal{F}(n)$ by

$$(5.5) \quad A \cdot (u, U) = (uA, A \cdot U) = (uA, A^t U A),$$

and preserves the inner product

$$(5.6) \quad \langle (u, U), (v, V) \rangle = uv^t + \frac{1}{2} \text{tr}(UV^t).$$

Suppose that $\mathcal{Z}$ is a subspace of Σ_n . We define a Lie algebra $\mathcal{N}_{\mathcal{Z}} := \mathbf{R}^n \times \mathcal{Z}$ with bracket

$$(5.7) \quad [(u, U), (v, V)]_{\mathcal{Z}} = (0, P_{\mathcal{Z}}(u^t v - v^t u)),$$

where $P_{\mathcal{Z}}$ is the orthogonal projection of Σ_n onto $\mathcal{Z}$.

We now describe the coadjoint orbits in $\mathcal{F}(n)^*$ and $\mathcal{N}_{\mathcal{Z}}^*$. First, using the inner product (5.6) we identify $\mathcal{F}(n)^*$ with $\mathcal{F}(n)$ and $\mathcal{N}_{\mathcal{Z}}^*$ with $\mathcal{N}_{\mathcal{Z}}$. This gives an inclusion $\mathcal{N}_{\mathcal{Z}}^* \hookrightarrow \mathcal{F}(n)^*$ dual to the projection $P_{\mathcal{Z}}$. For $B \in \Sigma_n$, define a map

$$(5.8) \quad J_B: \mathbf{R}^n \rightarrow \mathbf{R}^n$$

by $\langle J_B(u), v \rangle = \langle B, u^t v - v^t u \rangle$. Similarly, if $B \in \mathcal{Z}$ define a map

$$(5.9) \quad J_B^{\mathcal{Z}}: \mathbf{R}^n \rightarrow \mathbf{R}^n$$

by $\langle J_B^{\mathcal{Z}}(u), v \rangle = \langle B, [(u, 0), (v, 0)]_{\mathcal{Z}} \rangle$. In fact, though, for $B \in \mathcal{Z}$, $J_B = J_B^{\mathcal{Z}}$ since

$$\begin{aligned} \langle J_B^{\mathcal{Z}}(u), v \rangle &= \langle B, P_{\mathcal{Z}}[(u, 0), (v, 0)] \rangle \\ &= \langle B, [(u, 0), (v, 0)] \rangle = \langle J_B(u), v \rangle. \end{aligned}$$

Accordingly, we denote both maps by J_B . One computes

$$\begin{aligned} \langle J_B(u), v \rangle &= \langle B, [(u, 0), (v, 0)] \rangle \\ &= \frac{1}{2} \operatorname{tr}(B(u^t v - v^t u)^t) = \langle uB, v \rangle \end{aligned}$$

to conclude that

$$(5.10) \quad J_B(u) = uB.$$

The coadjoint orbit through $(b, B) \in \mathcal{F}(n)^*$ ($\cong \mathcal{F}(n)$) is

$$\mathcal{O}_{(b, B)} = \operatorname{Ad}^*(F(n))(b, B).$$

For $(u, U), (v, V) \in \mathcal{F}(n)$ one has

$$\begin{aligned} \langle \operatorname{Ad}^* \exp(u, U)(b, B), (v, V) \rangle &= \langle (b, B), (v, V) + [(u, U), (v, V)] \rangle \\ &= bv^t + \frac{1}{2} \operatorname{tr}(BV^t) + \frac{1}{2} \operatorname{tr}(B(u^t v - v^t u)^t) \\ &= \langle (b, B), (v, V) \rangle + \langle J_B(u), v \rangle \\ &= \langle (b + J_B(u), B), (v, V) \rangle. \end{aligned}$$

Thus,

$$(5.11) \quad \mathcal{O}_{(b, B)} = (b, B) + (\operatorname{Image}(J_B), 0) = (b + \mathbf{R}^n B, B).$$

The same reasoning shows that when $B \in \mathcal{Z}$ the orbit $\mathcal{O}_{(b, B)}^{\mathcal{Z}}$ through $(b, B) \in \mathcal{N}_{\mathcal{Z}}^*$ is also given by $(b + \mathbf{R}^n B, B)$, i.e. the inclusion $\mathcal{N}_{\mathcal{Z}}^* \hookrightarrow \mathcal{F}(n)^*$ maps $\mathcal{O}_{(b, B)}^{\mathcal{Z}}$ diffeomorphically to $\mathcal{O}_{(b, B)}$. Accordingly, we denote both of these orbits by $\mathcal{O}_{(b, B)}$, and will write $\mathcal{O}_B$ for $\mathcal{O}_{(0, B)}$.

For n even, the orbits $\mathcal{O}_B := \mathcal{O}_{(0, B)} = \mathbf{R}^n \times \{B\}$ with B nondegenerate provide a generic set of orbits in $\mathcal{F}(n)^*$, while for n odd, the orbits $\mathcal{O}_{(b, B)}$ with $b \in \mathbf{R}^n$ and B of rank $(n-1)$ form a generic set. (Note that these orbits are not distinct since $\mathcal{O}_{(b_1, B)} = \mathcal{O}_{(b_2, B)}$, provided $b_1 - b_2 \in \mathbf{R}^n B$.)

Theorem 5.12. $(SO(n), F(n))$ is a Gelfand pair for all $n \geq 2$.

Proof. The proof is an application of Theorem 3.5. Since the generic orbits in $\mathcal{F}(n)^*$ depend on the parity of n , we consider the cases separately.

Suppose first that $n = 2k$ and let $B \in \Sigma_n$ be nondegenerate. We may also assume that B has distinct eigenvalues which we denote $\pm i\lambda_1, \dots, \pm i\lambda_k$, with $\lambda_j > 0$. The orbits $\mathcal{O}_B = \mathbf{R}^n \times \{B\}$ for such B form a generic set in $\mathcal{F}(n)^*$.

Let $\mathcal{H}_B$ denote the Lie algebra defined in (5.7) with $\mathcal{Z} = \mathbf{R}B$. B is central in $\mathcal{H}_B$ and for $u, v \in \mathbf{R}^n$ one has

$$(5.13) \quad [(u, 0)(v, 0)] = \langle J_B(u), v \rangle B = \omega_B(u, v)B,$$

where $\omega_B(u, v) = uBv^t$ is the skew symmetric bilinear form on $\mathbf{R}^n$ with matrix B . Nondegeneracy of B implies that $\mathcal{H}_B$ is isomorphic to the Heisenberg algebra $\mathcal{H}_k$. We can make this isomorphism explicit by changing the basis on $\mathbf{R}^n$. Suppose B has eigenvectors $\alpha_1, \dots, \alpha_k$ in $\mathbf{C}^k$ corresponding to the eigenvalues $i\lambda_1, \dots, i\lambda_k$. Writing $\alpha_j = v_j + iu_j$, one has $u_j B = \lambda_j v_j$ and $v_j B = -\lambda_j u_j$. The matrix of B in the basis $\{u_1, v_1, \dots, u_k, v_k\}$ is

$$(5.14) \quad B = \begin{pmatrix} \lambda_1 J & 0 & \dots & 0 \\ 0 & \lambda_2 J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k J \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

By scaling the α_j 's we can ensure that $\{u_1, v_1, \dots, u_k, v_k\}$ is an orthonormal basis. Writing $X'_j = (u_j, 0)$, $Y'_j = (v_j, 0)$, and $Z = (0, B)$ in $\mathcal{H}_B$ we obtain a basis in which the Lie bracket in (5.7) becomes $[X'_j, Y'_j] = \lambda_j Z$ with other brackets vanishing. Replacing X'_j by $X_j = (1/\sqrt{\lambda_j})X'_j$, and Y'_j by $Y_j = (1/\sqrt{\lambda_j})Y'_j$ one obtains a basis $\{X_1, Y_1, \dots, X_k, Y_k, Z\}$ for $\mathcal{H}_B$ in which the nonzero brackets are determined by $[X_j, Y_j] = Z$.

Let $Sp(\omega_B) = \{A \in Gl(n, \mathbf{R}) | ABA^t = B\}$. This is the group of linear transformations preserving the symplectic form ω_B . The stabilizer of $\mathcal{O}_B$ under the action of $SO(n)$ is

$$(5.15) \quad K_B = SO(n) \cap Sp(\omega_B) = \{A \in SO(n) | AB = BA\}.$$

K_B also acts on $\mathcal{H}_B$ and stabilizes $\mathcal{O}_B$ regarded as an orbit in $\mathcal{H}_B^*$. In view of (5.14), K_B acts on $\mathcal{H}_B$ as $U(1)^k$ on $\text{Span}(X_1, Y_1, \dots, X_k, Y_k)$. Here each factor $U(1) = SO(2) \cap Sp(1, \mathbf{R}) = \{A \in SO(2) | AJ = JA\}$ acts on $\text{Span}(X_j, Y_j)$ in the usual fashion. The representations of $H_B = \exp(\mathcal{H}_B)$ and $F(n)$ given by $\mathcal{O}_B$ coincide under the orthogonal projection $\mathcal{F}(n) \rightarrow \mathcal{H}_B$ and hence have the same intertwining representations. In view of Corollary 4.12, this must satisfy the conditions of Theorem 3.5, and we conclude that $(SO(n), F(n))$ is a Gelfand pair.

Now consider the case $n = 2k + 1$. Let $b \in \mathbf{R}^n$ and let $B \in \Sigma_n$ have rank $n - 1 = 2k$ and distinct eigenvalues $0, \pm i\lambda_1, \dots, \pm i\lambda_k$ with $\lambda_j > 0$. We obtain a generic set of orbits $\mathcal{O}_{(b, B)}$ in $\mathcal{F}(n)^*$ from such pairs (b, B) .

Let $\mathcal{N}_B$ be defined as in (5.7) with $\mathcal{Z} = \mathbf{R}B$, and let X be any nonzero vector in $\ker(B)$. From (5.10) one concludes that the center of $\mathcal{N}_B$ is given by $\text{Span}(B, X)$ and that $\mathcal{N}_B = \mathcal{H}_B \times \mathbf{R}$ (as Lie algebras) where $\mathcal{H}_B = \mathcal{N}_B / \mathbf{R}X \simeq \mathcal{H}_k$.

In view of (5.5), the stabilizer of $\mathcal{O}_{(b, B)}$ under the action of $SO(n)$ is given by

$$(5.16) \quad \begin{aligned} K_{(b, B)} &= \{A \in SO(n) \mid bA = b \text{ and } AB = BA\} \\ &= \{A \in SO(2k) \mid AB = BA\}, \end{aligned}$$

where we are regarding $SO(2k)$ as the stabilizer of $b \in \mathbf{R}^n$ under the action of $SO(n)$.

$\mathcal{O}_{(b, B)}$ can be viewed as an orbit in $\mathcal{N}_B$ and also as an orbit in $\mathcal{H}_B$. The action of $K_{(b, B)}$ on $\mathcal{N}_B$ descends to $\mathcal{H}_B$ since each $A \in K_{(b, B)}$ preserves $\ker(B)$. Just as in the case where n is even, one shows that this corresponds to the action of $U(1)^k$ on $\mathcal{H}_k$ and completes the proof using Corollary 4.12 and Theorem 3.5. $\square$

Theorem 5.17. *If K is a proper, closed (not necessarily connected) subgroup of $SO(n)$ then $(K, F(n))$ is not a Gelfand pair.*

Proof. As in the proof of Theorem 5.12, one must consider separately the cases n even and n odd. Here we present the argument for the case $n = 2k$. We assume at first that K is connected. The stabilizer of a generic orbit $\mathcal{O}_B$ can be viewed as a compact subgroup A_B of $K_B \simeq U(1)^k$ (see equation (5.15)). We regard A_B as acting on a Heisenberg group H_k and conclude that if $(K, F(n))$ is a Gelfand pair then so is (A_B, H_k) , as in the proof of Theorem 5.12.

For a suitable choice of B , A_B is a proper subgroup of K_B . Indeed, let $C \in SO(n) \setminus K$ and let $\mathbf{T}$ be a maximal torus in $SO(n)$ that contains C . Choose a basis for $\mathbf{C}^k \simeq \mathbf{R}^n$ which transforms $\mathbf{T}$ into the usual $U(1)^k$ and let B be given in this basis by

$$(5.18) \quad \begin{pmatrix} J & 0 & \dots & 0 \\ 0 & 2J & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & kJ \end{pmatrix}.$$

One has $K_B = \mathbf{T}$ so that $A_B = K \cap K_B$ is a proper subgroup of K_B .

$A = A_B$ is a proper connected subgroup of $U(1)^k$ and hence is a torus. One can decompose $\mathbf{C}^k$ into a sum of weight spaces for the action of A ,

$$(5.19) \quad \mathbf{C}^k = \sum_{\alpha \in P} V_\alpha.$$

Here $\alpha \in \mathcal{A}^*$, where $\mathcal{A}$ is the Lie algebra of A ,

$$(5.20) \quad V_\alpha = \{v \in \mathbf{C}^k \mid \exp(X) \cdot v = e^{2\pi i \alpha(X)} v \text{ for all } X \in \mathcal{A}\},$$

and P denotes the set of weights: $P = \{\alpha \in \mathcal{A}^* \mid V_\alpha \neq \{0\}\}$. Each $\alpha \in P$ is an integral form, that is $\alpha(L) \subseteq \mathbb{Z}$, where $L = \ker(\exp: \mathcal{A} \rightarrow A)$. There is a corresponding decomposition of the polynomial functions on $\mathbf{C}^{\mathbf{k}}$:

$$(5.21) \quad \mathbf{C}[\mathbf{C}^{\mathbf{k}}] = \bigotimes \mathbf{C}[V_\alpha].$$

The A -action on $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ preserves each $\mathbf{C}[V_\alpha]$ and acts via the character

$$(5.22) \quad \chi_\alpha(\exp(X)) = e^{2\pi i \alpha(X)}.$$

There are two cases to consider:

- (i) Some weight space V_α has $\dim_{\mathbf{C}}(V_\alpha) > 1$.
- (ii) $\dim_{\mathbf{C}}(V_\alpha) = 1$ for all $\alpha \in P$.

Suppose (i). Any decomposition $V_\alpha = U \oplus W$ into nontrivial subspaces U and W will be preserved by the A -action. Moreover, A will act on the invariant subspaces $\mathbf{C}[U]$ and $\mathbf{C}[W]$ of $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ via the character χ_α . This shows that the action of A on $\mathbf{C}^{\mathbf{k}}$ is not multiplicity free and hence that $(K, F(n))$ is not a Gelfand pair.

Next assume that $\dim_{\mathbf{C}}(V_\alpha) = 1$ for all $\alpha \in P$. In this case, P consists of k weights $\{\alpha_1, \dots, \alpha_k\}$ and we obtain a basis $\{v_1, \dots, v_k\}$ of $\mathbf{C}^{\mathbf{k}}$ by choosing $v_j \in V_{\alpha_j}$ with $v_j \neq 0$. Note that any monomial $v_1^{j_1} v_2^{j_2} \dots v_k^{j_k}$ generates an A -invariant subspace in $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$.

As $\dim(\mathcal{A}) < k$, the weights $\alpha_1, \dots, \alpha_k$ must satisfy some nontrivial linear dependence relation:

$$(5.23) \quad c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_k \alpha_k = 0.$$

In fact, one can find an integer solution $(c_1, c_2, \dots, c_k)$ to this equation, since the forms α_j are integral. Suppose $c_1, \dots, c_l$ are nonnegative and that $c_{l+1}, \dots, c_k$ are negative (after rearranging the weights). Consider the monomials

$$(5.24) \quad p = v_1^{c_1} \dots v_l^{c_l} \quad \text{and} \quad q = v_{l+1}^{-c_{l+1}} \dots v_k^{-c_k}.$$

One has

$$\exp(X)p = e^{2\pi i(c_1 \alpha_1 + \dots + c_l \alpha_l)(X)} p \quad \text{and} \quad \exp(X)q = e^{-2\pi i(c_{l+1} \alpha_{l+1} + \dots + c_k \alpha_k)(X)} q$$

for $X \in \mathcal{A}$. One concludes that the A -irreducible subspaces of $\mathbf{C}[\mathbf{C}^{\mathbf{k}}]$ spanned by p and q are equivalent. As in case (i), the action of A on $\mathbf{C}^{\mathbf{k}}$ is not multiplicity free and $(K, F(n))$ fails to be a Gelfand pair.

Finally, consider a nonconnected, proper subgroup $K \subseteq SO(n)$. The stabilizer $A' = A'_B$ of a generic orbit $\mathcal{O}_B$ now has the form $A' = A \times F$, where A is a torus with $\dim(A) < k$ and F is a finite abelian group. As before, we decompose $\mathbf{C}^{\mathbf{k}}$ into weight spaces V_α for the action of A . Note that the action of F and A commute so that each V_α is F -invariant. As before, we consider two cases:

(i) Suppose $\dim(V_\alpha) > 1$. Choose two linearly independent vectors $u, v \in V_\alpha$. The actions of A' on the monomials $u^{|F|}$ and $v^{|F|}$ agree and hence the representation of A' on $\mathbf{C}^K$ is not multiplicity free.

(ii) Suppose $\dim(V_\alpha) = 1$ for all α . In this case, the actions of A' on $p^{|F|}$ and $q^{|F|}$ agree, where p and q are given by (5.24). $\square$

TWO-STEP GROUPS

In this section we do not assume that K is a connected group. Suppose now that a two-step N is given with $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{N}$. If this condition is not satisfied, then $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{A}$ where $\mathcal{N}_1$ is a K -invariant, nilpotent Lie algebra with $[\mathcal{N}_1, \mathcal{N}_1]$ spanning the center of $\mathcal{N}_1$, and $\mathcal{A}$ is commutative. Thus, $N = N_1 \times A$ and $L^1(N) = L^1(N_1) \otimes L^1(A)$. It is now easy to show that $L_K^1(N)$ is commutative if, and only if, $L_K^1(N_1)$ is commutative. Thus there is no loss in assuming that $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$.

Given a compact subgroup $K \subseteq \text{Aut}(N)$, we fix a K -invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{N}$, and denote by $\mathcal{N}_1$ the orthogonal complement to $\mathcal{Z}$ in $\mathcal{N}$. Let $X_1, \dots, X_n$ be an orthonormal basis for $\mathcal{N}_1$. Define the homomorphism $\lambda: \mathcal{F}(n) \rightarrow \mathcal{N}$ by setting $\lambda(e_i) = X_i$ (where $e_1, \dots, e_n$ is the standard basis for $\mathbf{R}^n$), and $\lambda(E_{i,j}) = [X_i, X_j]$, (where $E_{i,j} = [(e_i, 0), (e_j, 0)] \in \mathcal{F}(n)$). Let $\mathcal{K}$ denote the kernel of λ ($\subseteq \Sigma_n$). Note that $\lambda: \mathbf{R}^n \rightarrow \mathcal{N}_1$ is an isometry (where $\mathcal{F}(n)$ is equipped with the inner product $\langle (u, U), (v, V) \rangle = (0, uv^t + \frac{1}{2} \text{tr}(UV^t))$). Given $k \in K$, we define $\tilde{k} \in \text{Aut}(\mathcal{F}(n))$ by $\tilde{k}(e_i) = \lambda^{-1}(k \cdot (\lambda(e_i)))$ and $\tilde{k}(E_{i,j}) = [\tilde{k} \cdot e_i, \tilde{k} \cdot e_j]$, and set $\tilde{K} = \{\tilde{k} | k \in K\}$. Note that $\tilde{K} \simeq K$.

Lemma 6.1. *Let K be a compact subgroup of $\text{Aut}(N)$. For any choice of orthonormal basis of $\mathcal{N}_1$, $\tilde{K}$ is a compact subgroup of $O(n)$. If $\tilde{K}, \tilde{K}'$ are constructed using different orthonormal bases of $\mathcal{N}_1$ then $\tilde{K} = A' \tilde{K}' A$ for some $A \in O(n)$. K is a maximal compact subgroup of $\text{Aut}(N)$ if, and only if, $\tilde{K} = O_{\mathcal{K}}(n) := \{A \in O(n) | A \cdot \mathcal{K} := A' \mathcal{K} A = \mathcal{K}\}$.*

Proof. Given $\tilde{k} \in \tilde{K}$, $\tilde{k}(\mathbf{R}^n) \subseteq \mathbf{R}^n$. Thus, there is an $A_k \in Gl(n, \mathbf{R})$ such that $\tilde{k} \cdot (u, U) = (uA_k, A_k \cdot U)$. Since $\lambda: \mathbf{R}^n \rightarrow \mathcal{N}_1$ is an isometry and the inner product on $\mathcal{N}$ is K -invariant, $A_k \in O(n)$. Finally note that $\lambda \tilde{k} = k \lambda$. It follows that $\mathcal{K} = \ker(\lambda)$ is $\tilde{k}$ -invariant, and hence that $\tilde{K} \subseteq O_{\mathcal{K}}(n)$.

Suppose that $A \in O_{\mathcal{K}}(n)$. Define $k_A \in \text{Aut}(N)$ by requiring that $k_A \cdot \lambda((u, U)) = \lambda(A \cdot (u, U))$. It is clear that $A \mapsto k_A: O_{\mathcal{K}}(n) \rightarrow \text{Aut}(N)$ is a 1-1 homomorphism, and hence, since $O(n)$ is a maximal compact subgroup of $Gl(n, \mathbf{R})$, that K is a maximal compact subgroup of $\text{Aut}(N)$ if, and only if, $\tilde{K} = O_{\mathcal{K}}(n)$. $\square$

Let $\mathcal{Z}$ denote the orthogonal complement in Σ_n of $\mathcal{K}$, and let $\mathcal{N}_{\mathcal{Z}} = \mathbf{R}^n \times \mathcal{Z}$ be the Lie algebra defined as in (5.7), i.e. with Lie bracket defined by

$[(u, U), (v, V)]_{\mathcal{Z}} = P_{\mathcal{Z}}(u^t v - v^t u)$, where $P_{\mathcal{Z}}$ is the orthogonal projection of Σ_n onto $\mathcal{Z}$. Let $\tilde{\lambda}: \mathcal{F}(n)/\mathcal{H} \rightarrow \mathcal{N}$ be the canonical isomorphism, define $i: \mathcal{N}_{\mathcal{Z}} \rightarrow \mathcal{F}(n)/\mathcal{H}$ by $i(X) = X + \mathcal{H}$, and let $\tilde{\lambda} = \tilde{\lambda} \circ i$. Then $\tilde{\lambda}$ is a Lie algebra isomorphism. Since $\tilde{K} \subseteq O_{\mathcal{Z}}(n)$, by restriction we may consider $\tilde{K} \subseteq \text{Aut}(N_Z)$, where $N_Z = \exp(\mathcal{N}_{\mathcal{Z}})$. One can easily check that $k \cdot \lambda(X) = \tilde{\lambda}(\tilde{k} \cdot X)$ and thus prove

Lemma 6.2. *(K, N) is a Gelfand pair if, and only if, $(\tilde{K}, N_Z)$ is a Gelfand pair.*

Pick a nonzero $B \in \mathcal{Z}$. Let $\mathcal{N}_B$ denote the Lie algebra defined as in (5.7) with $\mathcal{Z} = \mathbf{R}B$. $\mathcal{N}_B$ is a concrete realization of the quotient Lie algebra $\mathcal{N}_{\mathcal{Z}}/\mathcal{Z}_0$, where $\mathcal{Z}_0$ is the orthogonal complement in $\mathcal{Z}$ of $\mathbf{R}B$. Let $\mathcal{H}_B$ denote the subset of $\mathcal{N}_B$ given by $\mathbf{R}^n B \times \mathbf{R}B$, and define a Lie bracket as in (5.7). Let N_B and H_B denote the corresponding simply connected Lie groups. Since the bilinear form defined on $\mathbf{R}^n$ by B is nondegenerate on its range, one has as in the proof of Theorem 5.12 (see equation (5.13)) that H_B is isomorphic to a Heisenberg group.

Given $b \in (\mathbf{R}^n B)^{\perp}$, the orthogonal complement in $\mathbf{R}^n$ of the range of B , set

$$(6.3) \quad \tilde{K}_{(b, B)} = \{\tilde{k} \in \tilde{K} \mid \tilde{k} \cdot B = B, \text{ and } \tilde{k} \cdot b = b\}.$$

By restriction, we may consider $\tilde{K}_{(b, B)}$ as a subgroup of $\text{Aut}(H_B)$.

Theorem 6.4. *If (K, N) is a Gelfand pair then $(\tilde{K}_{(b, B)}, H_B)$ is a Gelfand pair for all B in $\mathcal{Z}$, and all $b \in (\mathbf{R}^n B)^{\perp}$. Conversely, if $(\tilde{K}_{(b, B)}, H_B)$ is a Gelfand pair for (b, B) in a set of full Plancherel measure, then (K, N) is a Gelfand pair.*

Proof. Recall that we identify Lie algebras and their duals using the selected inner products. Given $B \in \mathcal{Z}$ and $b \in (\mathbf{R}^n B)^{\perp}$ we let $\mathcal{O}_{(b, B)}$ denote the orbit in $\mathcal{N}_Z$ ($\cong \mathcal{N}_Z^*$) through (b, B) . By (5.11), $\mathcal{O}_{(b, B)} = (b + \mathbf{R}^n B, B)$. Thus, $\tilde{K}_{(b, B)}$ is the subgroup of $\tilde{K}$ that preserves the equivalence class of $\pi_{(b, B)}$, the representation of N_Z corresponding to $\mathcal{O}_{(b, B)}$.

As above, let $\mathcal{Z}_0$ be the orthogonal complement in $\mathcal{Z}$ of $\mathbf{R}B$. Then $\mathcal{Z}_0$ is the subset of $\mathcal{Z}$ on which the functional B vanishes. Thus, $\pi_{(b, B)}$ factors through a representation of $N_B = N_Z / \exp(\mathcal{Z}_0)$.

Note that for $u \in \mathbf{R}^n$ and $v \in (\mathbf{R}^n B)^{\perp}$, equation (5.10) implies that

$$\begin{aligned} [(u, 0), (v, 0)]_{\mathbf{R}B} &= P_{\mathbf{R}B}([(u, 0), (v, 0)]) = \langle B, [(u, 0), (v, 0)] \rangle B \\ &= \langle J_B(u), v \rangle B = \langle uB, v \rangle B = 0. \end{aligned}$$

Thus, $\mathcal{N}_B$ is the direct sum of the Heisenberg Lie algebra $\mathcal{H}_B = \mathbf{R}^n B \times \mathbf{R}B$ and the commutative algebra $(\mathbf{R}^n B)^\perp (= (\mathbf{R}^n B)^\perp \times \{0\})$. Writing $N_B = H_B \times (\mathbf{R}^n B)^\perp$, $\pi_{(b,B)}$ factors as $\pi_B \otimes \chi_b$, where π_B is the element of $\hat{H}_B$ corresponding to B and χ_b is the unitary character defined on $(\mathbf{R}^n B)^\perp$ by $\chi_b(v) = e^{2\pi i \langle b, v \rangle}$.

The intertwining representation of $\tilde{K}_{(b,B)}$ fixes the factor χ_b , and thus is multiplicity free if, and only if, the representation of $\tilde{K}_{(b,B)}$ on the space of π_B is multiplicity free. This proves the theorem. $\square$

Remark. If K is a maximal compact, connected subgroup of $\text{Aut}(N)$ then $\tilde{K}_{(b,B)} = O(\mathbf{R}^n B) \times O_b((\mathbf{R}^n B)^\perp)$, where $O_v(V)$ denotes the group of all orthogonal transformations of V that fix $v \in V$. We consider two applications of Theorem 6.4. in the first, let $\mathcal{N}$ be the Lie algebra with basis X, Y_1, Y_2, Z_1, Z_2 , and with all nonzero brackets determined by $[X, Y_j] = Z_j$ for $j = 1, 2$. Let K be a maximal compact subgroup of $\text{Aut}(\mathcal{N})$, and fix a K -invariant inner product on $\mathcal{N}$. Pick an orthonormal basis X_i , $i = 1, 2, 3$, for $\mathcal{Z}^\perp$, and define $\lambda: \mathcal{F}(3) \rightarrow \mathcal{N}$ by requiring that $\lambda(e_i) = X_i$, $i = 1, 2, 3$. Then, $\dim(\mathcal{K} = \ker \lambda) = 1$. Thus, if $\mathcal{Z}$ is the orthogonal complement to $\mathcal{K}$ in Σ_3 , $\dim(\mathcal{Z}) = 2$. Hence, if $B \in \mathcal{Z}$, $B \neq 0$, and $b \in \mathbf{R}^3$, one easily sees that $\tilde{K}_{(b,B)} = \{e\}$. Thus there are no compact subgroups K' of $\text{Aut}(\mathcal{N})$ such that (K', N) is a Gelfand pair.

The next application of Theorem 6.4 will be to offer a short proof of a theorem due to H. Leptin, [Le]. We assume, as always, that $\mathcal{N}$ is the nilpotent Lie algebra of a simply connected group N with $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, the center of $\mathcal{N}$.

Theorem (Leptin). *Suppose that K is the k -torus contained in $\text{Aut}(N)$. Then (K, N) is a Gelfand pair if, and only if, N is the quotient of the direct product of k -copies of the 3-dimensional Heisenberg group H_1 , with K acting trivially on the center of N and lifting to the product of the usual $U(1)$ action on each factor H_1 .*

Proof. Let $\lambda: \mathcal{F}(n) \rightarrow \mathcal{N}$, and $\tilde{K} \subseteq \text{Aut}(F(n))$ be defined as above. Let

$$\mathbf{R}^n = \sum_{i=1}^k V_{\alpha_i}$$

be the decomposition into $\tilde{K}$ -root spaces. First note that if $X_{\alpha_i} \in V_{\alpha_i}$, $i = 1, 2$, and $\alpha_1 \neq \alpha_2$, then $[X_{\alpha_1}, X_{\alpha_2}] = 0$. Indeed, since $(\tilde{K}, N_Z)$ is a Gelfand pair, there exist $k_i \in \tilde{K}$, $i = 1, 2$, such that

$$X_{\alpha_1} + X_{\alpha_2} + \frac{1}{2}[X_{\alpha_1}, X_{\alpha_2}] = k_1 \cdot X_{\alpha_1} + k_2 \cdot X_{\alpha_2} + \frac{1}{2}[k_2 \cdot X_{\alpha_2}, k_1 \cdot X_{\alpha_1}].$$

From the $\tilde{K}$ -invariance of each V_α , one concludes that $k_i \cdot X_{\alpha_i} = X_{\alpha_i}$, and thus that $[X_{\alpha_1}, X_{\alpha_2}] = 0$.

Next observe that for $\alpha \in \{\alpha_i \mid 1 \leq i \leq k\}$, $\dim(V_\alpha) = 2$. For this note that if $\tilde{K}_\alpha$ is the action of $\tilde{K}$ on $\mathcal{N}_\alpha := V_\alpha \oplus \mathcal{Z}$, considered as a subalgebra of $\mathcal{N}_{\mathcal{Z}}$, then $(\tilde{K}_\alpha, \exp(\mathcal{N}_\alpha))$ is a Gelfand pair. $\dim(V_\alpha) > 1$, since for each nonzero $X \in V_\alpha$ there is a $Y \in V_\alpha$ such that $[X, Y] \neq 0$, and since $\tilde{K}_\alpha$ acts as a subgroup of $\mathbf{T}$ on $\mathcal{N}_\alpha$, one concludes as in the proof of Theorem 5.17 that $\dim(V_\alpha) = 2$, and so $n = 2k$.

Let $\{e_{2i-1}, e_{2i}\}$ be an orthonormal basis for V_{α_i} , and let

$$\Omega = \text{span}\{E_{2i-1, 2i} \mid 1 \leq i \leq k\}.$$

We will show that if $B \in \mathcal{Z}$, the orthogonal complement to $\mathcal{K} := \ker(\lambda)$ in Σ_{2k} , then $B \in \Omega$. Given such a B , let $\mathbf{R}^n B = \sum_{i=1}^l V_i$ be the decomposition corresponding to the standard form of the skew-symmetric B . Since B is nondegenerate on its range, for each nonzero $X \in \mathbf{R}^n B$ there is a $Y_X \in \mathbf{R}^n B$ such that $[X, Y_X] \neq 0$. Since $(\tilde{K}_B, H_B)$ is a Gelfand pair, one concludes as before, that if $X \in V_i$, then $Y_X \in V_i$. It then follows that $V_i = \text{span}\{\tilde{K}_B \cdot X\}$ for any nonzero $X \in V_i$. This amounts to showing that if $\tilde{K}_B \cdot X = X$ for some $X \in V_i$, then $X = 0$. But this is clear, for otherwise, by Theorem 1.12, there exist $k \in \tilde{K}_B$ such that

$$X + Y_X + \frac{1}{2}[X, Y_X] = X + k \cdot Y_X + \frac{1}{2}[k \cdot Y_X, X].$$

This forces the contradiction that $[X, Y_X] = 0$. It now follows that each V_i equals some V_{α_j} , and hence that $B \in \Omega$. Therefore, $\mathcal{K}$ contains the orthogonal complement to Ω in Σ_{2k} , and $F(n)/\exp(\mathcal{K})$ is the quotient of the direct product of k -copies of H_1 . Finally, since $\tilde{K}$ fixes each element of Ω , K acts trivially on the center of N . $\square$

SOLVABLE GROUPS

We now consider a simply connected solvable Lie group S with Lie algebra $\mathcal{S}$. We denote by $\mathcal{N}_{\mathcal{S}}$, or more simply by $\mathcal{N}$, the nilradical of $\mathcal{S}$. Given a compact subgroup $K \subseteq \text{Aut}(\mathcal{S})$, we set

$$\mathcal{S}_0 = \{X \in \mathcal{S} \mid k \cdot X = X, \forall k \in K\}.$$

The following theorem and proof was communicated to the authors by H. Leptin.

Theorem (Leptin). *If K is connected, then $\mathcal{S} = \mathcal{S}_0 + \mathcal{N}$.*

Proof. Let $\mathcal{S}_{\mathbf{C}} = \mathcal{S} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\mathcal{S}$. Then $K \subseteq \text{Aut}(\mathcal{S}_{\mathbf{C}})$, $(\mathcal{S}_0)_{\mathbf{C}} = (\mathcal{S}_{\mathbf{C}})_0$, and $\mathcal{N}_{\mathcal{S}_{\mathbf{C}}} = (\mathcal{N}_{\mathcal{S}})_{\mathbf{C}}$. Thus, we may assume that $\mathcal{S}$ is complex.

Now, if K is abelian and

$$\mathcal{S}_\chi = \{X \in \mathcal{S} \mid k \cdot X = \chi(k)X, \forall k \in K\},$$

then

$$(7.1) \quad \mathcal{S} = \sum_{\chi \in \hat{K}} \mathcal{S}_\chi.$$

If $X \in \mathcal{S}_\chi$, $X \neq 0$, and λ is an eigenvalue of $\text{ad } X$, then there is a nonzero $Y \in \mathcal{S}$ such that $[X, Y] = \lambda Y$. For $k \in K$,

$$k \cdot (\lambda Y) = [k \cdot X, k \cdot Y] = \chi(k)[X, k \cdot Y].$$

Thus, $\overline{\chi(k)}\lambda$ is also an eigenvalue of $\text{ad } X$ for all $k \in K$. But if $\chi \neq \varepsilon$, the identity, $\chi(K) = \mathbf{T}$, and thus, λt is an eigenvalue of $\text{ad } X$ for all $t \in \mathbf{T}$. It follows that $\lambda = 0$, and so $\text{ad } X$ is nilpotent. Therefore, $\mathcal{S}_\chi \subseteq \mathcal{N}$ for all $\chi \neq \varepsilon$, i.e. $\mathcal{S} = \mathcal{S}_0 + \mathcal{N}$.

We turn now to the general case. Let $t \in \mathbf{T} \subseteq K$, and $X \in \mathcal{S}$. Since $\mathcal{S} = \mathcal{S}'_0 + \mathcal{N}$, where $\mathcal{S}'_0 = \{X \in \mathcal{S} \mid t \cdot X = X, \forall t \in \mathbf{T}\}$, by the argument above, $t \cdot X \equiv X \pmod{\mathcal{N}}$. But every element of K is in a torus, and so for all $k \in K$, $k \cdot X \equiv X \pmod{\mathcal{N}}$. It follows that

$$X_0 := \int_K k \cdot X dk \equiv X \pmod{\mathcal{N}}.$$

Since $X_0 \in \mathcal{S}_0$, the theorem is proven. $\square$

Given $X \in \mathcal{S}$, we define $i_X \in \text{Aut}(S)$ by $i_X(y) = \exp(X)y \exp(-X)$. Consider the following condition:

$$(7.2) \quad \text{For each } X \in \mathcal{S}_0, y \in S, \exists k \in K \ni i_X(y) = k \cdot y.$$

Theorem 7.3. *Suppose K is connected. Then (K, S) is a Gelfand pair if, and only if, (K, N) is a Gelfand pair, and condition (7.2) is satisfied.*

Proof. Suppose (K, S) is a Gelfand pair. By Theorem 1.12, for all $x, y \in N$, $xy \in (K \cdot y)(K \cdot x)$, which implies that (K, N) is a Gelfand pair. Furthermore, if $X \in \mathcal{S}_0$ and $y \in S$, then $\exp(X)y \in (K \cdot y)(K \cdot \exp(X)) = (K \cdot y)\exp(X)$. This proves the necessity of the conditions.

Suppose now the converse. Note that $S = \exp(\mathcal{S}_0)N$. Given $X, Y \in \mathcal{S}_0$, and $x, y \in N$ we compute

$$\begin{aligned} (K \cdot \exp(X)x)(K \cdot \exp(Y)y) &= \exp(X)(K \cdot x)\exp(Y)(K \cdot y) \\ &= \exp(X)\exp(Y)(\exp(-Y)(K \cdot x)\exp(Y))(K \cdot y) \\ &= \exp(X)\exp(Y)(K \cdot x)(K \cdot y) \\ &= \exp(X)\exp(Y)(K \cdot y)(K \cdot x) \\ &= (\exp(X)(K \cdot \exp(Y)y)\exp(-X))(K \cdot (\exp(X)x)) \\ &= (K \cdot \exp(Y)y)(K \cdot \exp(X)x). \end{aligned}$$

Theorem 1.12 implies that (K, S) is a Gelfand pair. $\square$

Recall that a connected Lie group G is said to be *type-R* if the eigenvalues of $\text{ad } X$, as a linear operator on $\mathcal{G}$, are pure imaginary. Note that $i_X(\exp(Y)) = \exp(\text{Ad}(\exp(X)) \cdot Y) = \exp(\exp(\text{ad } X) \cdot Y)$. Thus, if (7.2) is satisfied, and $\|\cdot\|$ is a K invariant norm on $\mathcal{S}$, then for all $X \in \mathcal{S}_0$, $\|\exp(\text{ad } X) \cdot Y\| = \|i_X \cdot Y\| = \|Y\|$. This implies that the eigenvalues of $\text{ad } X$ are pure imaginary for all $X \in \mathcal{S}_0$. The same holds true for $X \in \mathcal{N}$, since $\text{ad } X$ is nilpotent as a

linear operator on $\mathcal{S}$. Thus

Corollary 7.4. *If (K, S) is a Gelfand pair, then S is type-R.*

A very simple example of a Gelfand pair (K, S) involving a non-nilpotent group is given by letting $S = \mathbf{R} \ltimes \mathbf{C}$, with $\mathbf{R}$ acting on $\mathbf{C}$ by $t: z \mapsto e^{it}z$, and $K = U(1)$ acting as usual on $\mathbf{C}$.

SPHERICAL FUNCTIONS

In this section we identify a moduli space for the K -spherical functions associated to a Gelfand pair (K, S) . Recall that a K -spherical function associated to such a pair is a continuous, complex-valued function, ϕ , defined on S , satisfying

$$(8.1) \quad \phi(e) = 1 \quad \text{and} \quad \int_K \phi(xk \cdot y) dk = \phi(x)\phi(y)$$

for all $x, y \in S$. It easily follows that a K -spherical function is K -invariant. One also has that integration against a K -spherical function, ϕ , defines a complex-valued homomorphism on $L_K^1(N)$, that this homomorphism is continuous if ϕ is bounded, and that all continuous homomorphisms of $L_K^1(N)$ are given in this manner (cf. [He]). We first consider K -spherical functions associated to a Gelfand pair (K, N) .

Lemma 8.2. *Suppose ϕ is a bounded K -spherical function on N . Then there is a $\pi \in \hat{N}$ and a unit vector $\xi \in \mathbf{H}_\pi$ such that*

$$\phi(x) = \int_K \langle \pi(k \cdot x)\xi, \xi \rangle dk,$$

for each $x \in N$.

Proof. Let $\lambda_\phi: L_K^1(N) \rightarrow \mathbf{C}$ be given by integration against ϕ .

Since $L^1(N)$ is a symmetric Banach $*$ -algebra, [Le2], there is a representation $\bar{\pi}$ of $L^1(N)$ and a one-dimensional subspace $\mathbf{H}_\phi$ of $\mathbf{H}_{\bar{\pi}}$ such that $(\bar{\pi}|_{L_K^1(N)}, \mathbf{H}_\phi)$ is equivalent to $(\lambda_\phi, \mathbf{C})$. As λ_ϕ is irreducible, the extension $\bar{\pi}$ is also irreducible (cf. [Na]). Using approximate identities at each point of N , one can show that $\bar{\pi}$ is the integrated version of some $\pi \in \hat{N}$, with $\mathbf{H}_\pi = \mathbf{H}_{\bar{\pi}}$.

Choose $\xi \in \mathbf{H}_\phi$ with $\|\xi\| = 1$. Then for each $f \in L_K^1(N)$, $\pi(f)\xi = \lambda_\phi(f)\xi$, so that

$$\begin{aligned} \langle \phi, f \rangle &= \lambda_\phi(f) = \langle \pi(f)\xi, \xi \rangle \\ &= \int_N f(x) \langle \pi(x)\xi, \xi \rangle dx \\ &= \int_K \int_N f(k^{-1} \cdot x) \langle \pi(x)\xi, \xi \rangle dx dk \end{aligned}$$

since f is K -invariant

$$= \int_K \int_N f(x) \langle \pi(k \cdot x) \xi, \xi \rangle dx dk.$$

Since ϕ is K -invariant, we change the order of integration and obtain

$$(8.3) \quad \phi(x) = \int_K \langle \pi(k \cdot x) \xi, \xi \rangle dk. \quad \square$$

Notation. We denote the function defined by (8.3) as $\phi_{\pi, \xi}$.

Corollary 8.4. *If ϕ is a bounded K -spherical function on N , then ϕ is positive definite.*

Recall from §3 that for $\pi \in \hat{N}$ we denote by K_π the subgroup of K that preserves the equivalence class of π , and that W_π denotes the intertwining representation of K_π .

Let $\mathbf{H}_\pi = \sum_\alpha V_\alpha$ be the decomposition of $\mathbf{H}_\pi$ into irreducible subspaces invariant under the action of W_π . The assumption that (K, N) is a Gelfand pair implies that as K_π -modules, the V_α 's are inequivalent for different α 's.

Lemma 8.5. *If $\pi' = \pi_{k_0}$, then $K_{\pi'} = k_0^{-1} K_\pi k_0$.*

Proof. If $k' \in K_{\pi'}$, then $\pi'_{k'} \simeq \pi'$. That is, $\pi'_{k'}(x) = W_{\pi'}(k') \pi'(x) W_{\pi'}^*(k')$ for each $x \in N$. Thus

$$\begin{aligned} \pi_{k_0 k' k_0^{-1}}(x) &= \pi_{k_0 k'}(k_0^{-1} \cdot x) = \pi'_{k'}(k_0^{-1} \cdot x) \\ &= W_{\pi'}(k') \pi'(k_0^{-1} \cdot x) W_{\pi'}^*(k') = W_{\pi'}(k') \pi(x) W_{\pi'}^*(k'). \end{aligned}$$

Thus, $\pi_{k_0 k' k_0^{-1}} \simeq \pi$, so $k_0 k' k_0^{-1} \in K_\pi$. $\square$

Note that for $k' \in K_{\pi'}$, the above calculation shows that we could choose $W_{\pi'}$ so that $W_\pi(k_0 k' k_0^{-1}) = W_{\pi'}(k')$.

Corollary 8.6. *For $\pi' = \pi_{k_0}$, $\mathbf{H}_\pi$ and $\mathbf{H}_{\pi'}$ have the same decomposition into W_π - and $W_{\pi'}$ -irreducible subspaces respectively.*

Theorem 8.7. (i) $\phi_{\pi, \xi}$ is a K -spherical function if, and only if, $\xi \in V_\alpha$ for some α , and $\|\xi\| = 1$. (ii) $\phi_{\pi, \xi} = \phi_{\pi', \eta}$ if, and only if, there is a $k \in K$ such that $\pi' = \pi_k$ and ξ, η belong to the same V_α .

Proof. Let $f \in L_K^1(N)$. Since f is K_π -invariant, $\pi(f)$ commutes with the action of W_π on $\mathbf{H}_\pi$. Since W_π is multiplicity free, $\pi(f)$ preserves each V_α . Now by Schur's lemma, the irreducibility of W_π on V_α implies that $\pi(f)$ acts as a scalar multiple of the identity on each V_α . Note that this scalar is computed by the formula $\langle \pi(f) \xi, \xi \rangle$ for any $\xi \in V_\alpha$ with $\|\xi\| = 1$.

For $\xi \in V_\alpha$ with $\|\xi\| = 1$, $\phi_{\pi, \xi}$ is clearly a continuous function on N . We only need to show that λ_ϕ (with $\phi = \phi_{\pi, \xi}$) is a homomorphism on $L_K^1(N)$.

Note that for $f \in L_K^1(N)$,

$$\begin{aligned}
 \langle \phi_{\pi, \xi}, f \rangle &= \int_N \int_K \langle \pi(k \cdot x) \xi, \xi \rangle f(x) dk dx \\
 (8.8) \qquad &= \int_K \int_N \langle \pi(x) \xi, \xi \rangle f(k^{-1} \cdot x) dx dk \\
 &= \langle \pi(f) \xi, \xi \rangle.
 \end{aligned}$$

Thus, if $f, g \in L_K^1(N)$,

$$\begin{aligned}
 \lambda_\phi(f * g) &= \langle \pi(f * g) \xi, \xi \rangle = \langle \pi(f) \pi(g) \xi, \xi \rangle \\
 &= \langle \pi(g) \xi, \xi \rangle \langle \pi(f) \xi, \xi \rangle = \lambda_\phi(f) \lambda_\phi(g).
 \end{aligned}$$

Conversely, suppose $\xi \in \mathbf{H}_\pi$, $\|\xi\| = 1$. Write $\xi = \sum t_\alpha \xi_\alpha$ with $\xi_\alpha \in V_\alpha$, $\|\xi_\alpha\| = 1$, $t_\alpha \geq 0$, and $\sum t_\alpha^2 = \|\xi\|^2 = 1$. Then

$$\langle \phi_{\pi, \xi}, f \rangle = \langle \pi(f) \xi, \xi \rangle = \sum_{\alpha, \beta} t_\alpha t_\beta \langle \pi(f) \xi_\alpha, \xi_\beta \rangle = \sum_\alpha t_\alpha^2 \langle \pi(f) \xi_\alpha, \xi_\alpha \rangle$$

since $\pi(f)$ preserves the mutually orthogonal V_α 's

$$= \sum_\alpha t_\alpha^2 \langle \phi_{\pi, \xi_\alpha}, f \rangle.$$

Thus, for $\xi = \sum t_\alpha \xi_\alpha$, $t_\alpha \geq 0$, $\phi_{\pi, \xi} = \sum_\alpha t_\alpha^2 \phi_{\pi, \xi_\alpha}$, and $\|\xi\|^2 = 1$ implies that $\sum t_\alpha^2 = 1$. Note that positive definite homomorphisms are extreme points in the Gelfand space of $L_K^1(N)$, so if $\phi_{\pi, \xi}$ is a positive definite K -spherical function, it cannot be a convex sum of positive definite K -spherical functions. Thus $\xi = \xi_\alpha$ for some α .

Now suppose $\pi' = \pi_{k_0}$ and ξ, η belong to $V_\alpha \subseteq \mathbf{H}_\pi$. Then

$$\langle \phi_{\pi, \xi}, f \rangle = \langle \pi(f) \xi, \xi \rangle = \langle \pi(f) \eta, \eta \rangle$$

since $\pi(f)$ is constant on V_α

$$\begin{aligned}
 &= \int_N \int_K \langle \pi(k \cdot x) \eta, \eta \rangle f(x) dk dx \\
 &= \int_N \int_K \langle \pi(k_0 k \cdot x) \eta, \eta \rangle f(x) dk dx \\
 &= \langle \phi_{\pi', \eta}, f \rangle.
 \end{aligned}$$

Thus, $\phi_{\pi, \xi} = \phi_{\pi', \eta}$.

For the converse of (ii), we need to understand $K \widehat{\propto} N$ via the Mackey machine. Let $\pi \in \widehat{N}$, and suppose the intertwining representation W_π of K_π is a σ -representation, as described in §3. Let T be any $\bar{\sigma}$ -representation of K_π . Then $\rho = T \otimes \pi W_\pi$ is an irreducible representation of $K_\pi \propto N$. Let $\tilde{\rho}$ be the representation of $K \propto N$ induced from ρ . Then $\tilde{\rho} \in \widehat{K \propto N}$, and any irreducible representation of $K \propto N$ is obtained in this manner. More precisely,

$\widehat{K \propto N}$ is given by pairs (π, T) , where $\pi \in \widehat{N}$, and $T \in \widehat{K_\pi^\sigma}$. Another pair (π', T') yields an equivalent representation if, and only if, $\pi' \simeq \pi_{k_0}$ for some k_0 and $T' \simeq T \circ i_{k_0}$, where $i_{k_0}: K_{\pi'} \rightarrow K_\pi = k_0 K_{\pi'} k_0^{-1}$.

As a function on $G = K \propto N$, any positive definite K -spherical function is given as follows: Let $\hat{\rho} \in \widehat{G}$. If there is a K -fixed vector $v \in \mathbf{H}_{\hat{\rho}}$ (the space of K -fixed vectors has dimension at most one), then $\phi(x) = \langle \hat{\rho}(x)v, v \rangle$. This yields a 1-1 correspondence between the representations in $\widehat{G}$ with K -fixed vectors and positive definite K -spherical functions on G (cf. [He]).

By Frobenius reciprocity, we see that the dimension of the space of K -fixed vectors in $\mathbf{H}_{\hat{\rho}}$ equals the dimension of the space of K_π -fixed vectors in $\mathbf{H}_{\hat{\rho}}$. Note that $T \otimes W_\pi$ has K_π -fixed vectors if, and only if, $\bar{T}$ is a subrepresentation of W_π , i.e. $\mathbf{H}_T = V_\alpha$ for some W_π -irreducible component of $\mathbf{H}_\pi$, and $T = \bar{W}_\pi|_{V_\alpha}$. Thus there is a 1-1 correspondence between positive definite K -spherical functions and pairs (π, V_α) , where $\pi \in \widehat{N}$ and $V_\alpha \subseteq \mathbf{H}_\pi$ is a W_π -irreducible component. We will see that these K -spherical functions coincide with the formulas in the statement of the theorem. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V_α , and set

$$(8.9) \quad v = \frac{1}{\sqrt{m}} \sum v_i \otimes v_i,$$

regarded as an element of $\mathbf{H}_\rho = V_\alpha \otimes \mathbf{H}_\pi$. For $k \in K_\pi$,

$$\begin{aligned} \rho(k)v &= \frac{1}{\sqrt{m}} \sum_i \bar{W}_\pi(k)v_i \otimes W_\pi(k)v_i \\ &= \frac{1}{\sqrt{m}} \sum_{i,j,k} \bar{a}_{i,j} v_j \otimes a_{i,k} v_k, \end{aligned}$$

where $A = (a_{i,j})$ is the matrix corresponding to $W_\pi(k)|_{V_\alpha}$. But

$$\sum_i \bar{a}_{i,j} a_{i,k} = (A^* A)_{j,k} = \delta_{j,k}.$$

Thus

$$(8.10) \quad \rho(k)v = \frac{1}{\sqrt{m}} \sum_j v_j \otimes v_j,$$

so v is a K_π -fixed vector in $\mathbf{H}_\rho$.

To construct a corresponding K -fixed vector in $\mathbf{H}_{\hat{\rho}}$, define $f: K \propto N \rightarrow V_\alpha \otimes \mathbf{H}_\pi$ by $f(k, n) = (1 \otimes \pi(n))v$. To ensure that $f \in \mathbf{H}_{\hat{\rho}}$, we need $f(hg) = \rho(h)f(g)$, for $h \in K_\pi \propto N$, $g \in K \propto N$. (Actually it is sufficient to take $g = (k, e)$ with $k \in K$.) We have

$$f((k_\pi, n)(k, e)) = f(k_\pi k, n) = (1 \otimes \pi(n))v.$$

On the other hand,

$$\begin{aligned} \rho(k_\pi, n)f(k, e) &= \bar{W}_\pi(k_\pi) \otimes \pi(n)W_\pi(k_\pi)v \\ &= (1 \otimes \pi(n))\rho(k_\pi)v = (1 \otimes \pi(n))v, \end{aligned}$$

as required. Thus $f \in \mathbf{H}_{\tilde{\rho}}$, and for $k \in K$,

$$\tilde{\rho}(k)f(k', n) = f((k', n)(k, e)) = f(k'k, n) = (1 \otimes \pi(n))v = f(k', n),$$

so f is a K -fixed vector.

We check that f is a unit vector.

$$\begin{aligned} \|f\|^2 &= \int_{(K \rtimes N)/(K_\pi \rtimes N)} \|f(k, n)\|^2 dk dn \\ &= \int_{(K \rtimes N)/(K_\pi \rtimes N)} \|(1 \otimes \pi(n))v\|^2 dk dn \\ &= \int_{K/K_\pi} \|v\|^2 dk = 1, \end{aligned}$$

since

$$\|v\|^2 = \frac{1}{m} \sum_{i=1}^m \|v_i \otimes v_i\|^2 = 1.$$

The K -spherical function $\tilde{\phi}$ on G associated with f is given by $\tilde{\phi}(g) = \langle \tilde{\rho}(g)f, f \rangle$. The restriction ϕ of $\tilde{\phi}$ to N is given by

$$\begin{aligned} \phi(n) &= \langle \tilde{\rho}(n)f, f \rangle \\ &= \int_{K/K_\pi} \langle \tilde{\rho}(n)f(k), f(k) \rangle dk \\ &= \int_{K/K_\pi} \langle f((k, e)(e, n)), f(k) \rangle dk \\ &= \int_{K/K_\pi} \langle f(k, k \cdot n), f(k) \rangle dk \\ &= \int_{K/K_\pi} \langle (1 \otimes \pi(k \cdot n))v, v \rangle dk. \end{aligned}$$

For $k \in K$,

$$\begin{aligned} \langle (1 \otimes \pi(k \cdot n))v, v \rangle &= \frac{1}{m} \sum_{i,j} \langle v_j \otimes \pi(k \cdot n)v_j, v_i \otimes v_i \rangle \\ &= \frac{1}{m} \sum_i \langle \pi(k \cdot n)v_i, v_i \rangle \end{aligned}$$

For $k \in K_\pi$,

$$\begin{aligned} \sum_i \langle \pi(k \cdot n)v_i, v_i \rangle &= \sum_i \langle W_\pi(k)\pi(n)W_\pi(k)^{-1}v_i, v_i \rangle \\ &= \sum_i \langle \pi(n)W_\pi(k)^{-1}v_i, W_\pi(k)^{-1}v_i \rangle \\ &= \sum_i \langle \pi(n)v_i, v_i \rangle, \end{aligned}$$

by an easy trace argument. Thus,

$$\begin{aligned}\phi(n) &= \frac{1}{m} \int_K \sum_i \langle \pi(k \cdot n) v_i, v_i \rangle dk \\ &= \frac{1}{m} \sum_i \phi_{\pi, v_i}(n) = \phi_{\pi, m^{-1/2} \sum v_i}(n).\end{aligned}$$

Thus, $\phi = \phi_{\pi, \xi}$, where ξ is any element of V_α (since any unit vector in V_α can be written as $1/\sqrt{m} \sum v_i$ for some orthonormal basis $\{v_1, \dots, v_n\}$). $\square$

Suppose now that (K, S) is a Gelfand pair. Note that if ϕ is a K -spherical function, $X, Y \in \mathcal{S}_0$, and $y \in S$, then by (8.1)

$$\phi(y \exp X \exp Y) = \phi(y) \phi(\exp X) \phi(\exp Y).$$

One also sees from (8.1) that the restriction of ϕ to $N := \exp(\mathcal{N})$, where $\mathcal{N}$ is the nilradical of $\mathcal{S}$, is a K -spherical function. This indicates how one constructs K -spherical functions on S .

Let $X_1, \dots, X_p$ be a basis for a complement of $\mathcal{N}$, the nilradical of $\mathcal{S}$, in $\mathcal{S}_0$. Since S is simply connected, for each $y \in S$, there exist unique $n(y) \in N (= \exp(\mathcal{N}))$ and $\mathbf{t}(y) \in \mathbf{R}^p$ such that $y = n(y) \prod_i \exp(t_i(y) X_i)$. Thus, if ϕ is a bounded K -spherical function on S then

$$\phi(y) = \phi(n(y)) \prod_i \phi(\exp(t_i(y)))$$

for each $y \in S$. Again by (8.1), for any $X \in \mathcal{S}_0$, the mapping $t \mapsto \phi(\exp(tX))$ is a homomorphism of $\mathbf{R}$ into $\mathbf{C}$. Thus, there exist an $\mathbf{a} \in \mathbf{R}^p$ such that $\phi(y) = \phi(n(y)) e^{i\langle \mathbf{a}, \mathbf{t}(y) \rangle}$. Thus one has

Theorem 8.11. *ϕ is a bounded K -spherical function on S if, and only if, there is a bounded K -spherical function ψ on N and an $\mathbf{a} \in \mathbf{R}^p$ such that $\phi(y) = \psi(n(y)) e^{i\langle \mathbf{a}, \mathbf{t}(y) \rangle}$. Thus $\Delta(K, S) = \Delta(K, N) \times \mathbf{R}^p$.*

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