INTERPRETATIONS OF EUCLIDEAN GEOMETRY

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ABSTRACT. Following Tarski, we view n -dimensional Euclidean geometry as a first-order theory E_n with an infinite set of axioms about the relations of *betweenness* (among points on a line) and *equidistance* (among pairs of points). We show that for $k < n$, E_n does not admit a k -dimensional interpretation in the theory RCF of real closed fields, and we deduce that E_n cannot be interpreted r -dimensionally in E_s , when $r \cdot s < n$.

1. INTRODUCTION

We know since Descartes that points of the Euclidean n -dimensional space can be identified with n -tuples of real numbers. Moreover, each statement of n -dimensional geometry that involves a variable x representing a point has its counterpart in algebra that is a statement involving n variables $x_1, \dots, x_n$ (the “coordinates” of x). In such a situation, we say (to be made precise later in Definition 2.2) that we have an *n -dimensional interpretation* of the language of geometry in the language of algebra.

In the sequel E_n will be the first-order theory with equality that has been proposed by Alfred Tarski [T2] for describing the n -dimensional Euclidean geometry. We shall denote by $L_{\beta\delta}$ the language of E_n ; this is the first-order language whose vocabulary consists of three predicate symbols: $=$ (binary), β (ternary), and δ (quaternary). If $\mathcal{E}_n$ is the Euclidean n -dimensional space, i.e., the usual model of E_n whose universe is $\mathbf{R}^n$, and $a, b, c, d \in \mathbf{R}^n$, then the meaning of the relations β and δ is as follows:

$$\begin{aligned}\beta(a, b, c) & \text{ iff } b \text{ lies between } a \text{ and } c \text{ (possibly } b = a \text{ or } b = c), \\ \delta(a, b, c, d) & \text{ iff } a \text{ is as distant from } b \text{ as } c \text{ is from } d.\end{aligned}$$

As Tarski has shown, his axiomatization of E_n yields a complete theory. Thus, the theorems of E_n are just those sentences in $L_{\beta\delta}$ that are true in $\mathcal{E}_n$.

The preceding Cartesian n -dimensional interpretation of the language $L_{\beta\delta}$ in the language of algebra (see the end of §2, where e stands for equality) is also an *interpretation of theories* in the sense that the algebraic counterparts of *theorems* of E_n are *theorems* of RCF, i.e., of the theory of real closed fields,

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of which the field of real numbers is one. The language of RCF is the language L_{OF} of ordered fields, i.e., the f.o. language whose vocabulary consists of the symbols: $+$, $-$, $\cdot$ (binary functions), $<$, $=$ (binary predicates), and 0 , 1 (constants). Thus, we have the Cartesian n -dimensional interpretation of $L_{\beta\delta}$ in L_{OF} , which is an interpretation of the theory E_n in the theory RCF.

Mycielski [M] has conjectured that an interpretation of E_n in RCF cannot be achieved more economically than in the preceding Cartesian way, that is, using k -tuples of variables for some $k < n$ instead of n -tuples. In 1980, Boffa [B] proved that this is so for the case when $n = 2$ and $k = 1$. The general conjecture is confirmed by the main result of this paper:

Theorem 1.1. *For every $n \geq 2$, there exists a theorem φ_n of E_n such that for every $k < n$, the image of φ_n under any k -dimensional interpretation of the language $L_{\beta\delta}$ in the language L_{OF} is not a theorem of RCF.*

Corollary 1.1. *If $r \cdot s < n$, there does not exist an r -dimensional interpretation of the theory E_n in the theory E_s .*

Indeed, if such an interpretation were to exist, one could superimpose it with the Cartesian s -dimensional interpretation of E_s in RCF, thus obtaining an rs -dimensional interpretation of E_n in RCF, contrary to Theorem 1.1.

Our following proof combines the basic idea of Boffa in [B, p. 123] with a result of Łojasiewicz [L] on triangulation of semialgebraic sets (proved also in [C, VdD1, and H]).

In 1984, Szczerba announced in [Sz, p. 677] that he had found the affirmative answer to Mycielski's conjecture; however, his work is not known to us and does not seem to have been published.

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2. INTERPRETATIONS

Given a formula α , we say that a variable x_j is *substitutable* for a variable x_i in α if, by replacing x_i with x_j at any of its free occurrences, we get a free occurrence of x_j , or, in other words, if no free occurrence of x_i is within the scope of the quantifier $\exists x_j$.

Suppose all the free variables of α (i.e., variables that have at least one free occurrence) are among $x_{i_1}, \dots, x_{i_n}$, where $i_1 < i_2 < \dots < i_n$, and let $x_{j_1}, \dots, x_{j_n}$ be such that x_{j_s} is substitutable for x_{i_s} in α for each $s = 1, \dots, n$. Then the result of replacing each x_{i_s} at every free occurrence by x_{j_s} will be denoted by:

$$\alpha \left[\begin{array}{ccc} x_{i_1} & \cdots & x_{i_n} \\ x_{j_1} & \cdots & x_{j_n} \end{array} \right].$$

If the n -tuple $(x_{i_1}, \dots, x_{i_n})$ coincides with $(x_1, \dots, x_n)$ or with $(x_0, \dots, x_{n-1})$, we shall write the above as $\alpha[x_{j_1}, \dots, x_{j_n}]$.

By a *variant* of α we shall mean any formula obtained from α by a sequence of replacements as follows:

$$\text{replace } \exists x_k \beta \text{ by } \exists x_s \beta \left[\begin{smallmatrix} x_k \\ x_s \end{smallmatrix} \right],$$

where $\exists x_k \beta$ is any subformula of α such that x_s is substitutable for x_k in β . Obviously, such replacements do not affect the free occurrences of variables and they do not change the scopes of quantifiers.

We assume until the end of this section that L is a relational language (i.e., L has no function symbols) and that $x_0, x_1, \dots$ is the list of all variables of L . Let L' be any language that contains the variables $x_0, x_1, \dots$.

Definition 2.1. By a one-dimensional *interpretation* of L in L' , we mean a map $\alpha \mapsto \alpha^0$, which assigns to every formula α of L a formula α^0 of L' so that

- (i) every variable free in α^0 is also free in α ;
- (ii) for any n -place predicate symbol p of L and any $x_{i_1}, \dots, x_{i_n}$, $(p(x_{i_1}, \dots, x_{i_n}))^0$ is $((p(x_1, \dots, x_n))^0)'[x_{i_1}, \dots, x_{i_n}]$, where $(p(x_1, \dots, x_n))^0'$ is a variant of $(p(x_1, \dots, x_n))^0$ in which x_{i_s} is substitutable for x_s for all $s = 1, \dots, n$;
- (iii) if α is of the form $\neg \mu, \mu \vee \eta, \mu \rightarrow \eta, \mu \& \eta, \forall x_i \mu$ or $\exists x_i \mu$, then α^0 is correspondingly $\neg \mu^0, \mu^0 \vee \eta^0, \mu^0 \rightarrow \eta^0, \mu^0 \& \eta^0, \forall x_i \mu^0$ or $\exists x_i \mu^0$.

Obviously, an interpretation $\alpha \mapsto \alpha^0$ is uniquely determined by its restriction to the atomic formulas $p(x_{i_1}, \dots, x_{i_n})$, and thus, up to variants, by its restriction to the formulas $p(x_1, \dots, x_n)$. The images of the latter can be chosen arbitrarily, provided (i) is satisfied.

For example, there is exactly one interpretation of $L_{\beta\delta}$ in L_{OF} such that:

$$\begin{aligned} (\beta(x_1, x_2, x_3))^0 & \text{ is } (x_3 - x_2)(x_2 - x_1) \geq 0, \\ (\delta(x_1, x_2, x_3, x_4))^0 & \text{ is } (x_2 - x_1)^2 = (x_4 - x_3)^2, \text{ and} \\ (x_1 = x_2)^0 & \text{ is } x_1 = x_2. \end{aligned}$$

This is also the obvious interpretation of E_1 in RCF.

In the sequel, we shall treat equality just as any other binary predicate symbol. In particular, it will not necessarily be assumed that $(x_1 = x_2)^0$ is $x_1 = x_2$. To prevent misunderstandings, we shall use henceforth the letter e to denote the binary predicate symbol for equality in any language L . Thus, for example, to define an L -structure $\mathcal{A}$, we shall have to specify also the binary relation on $\mathcal{A}$ corresponding to e . (This will not, in general, be the relation of equality in the structure $\mathcal{A}$.)

Remarks. (1) Interpretations can be superimposed, that is, from an interpretation of L in L' and another from L' to L'' , we get an interpretation of L in L'' .

(2) For interpreting L in L' it is irrelevant which variables are in these languages, provided all variables of L are also in L' .

Given a positive integer k and L as just described, let us denote by $L^{(k)}$ the language such that:

- (a) the only variables of $L^{(k)}$ are x_{ij} , $i, j = 0, 1, \dots$,
- (b) there is a one-to-one correspondence $p \mapsto p^{(k)}$ between the predicate symbols p in L and $p^{(k)}$ in $L^{(k)}$ such that if p is n -place then $p^{(k)}$ is kn -place.

We define a mapping $\alpha \mapsto \alpha^{(k)}$, called the k -spread, from formulas of L to formulas of $L^{(k)}$, as follows. Given α , one obtains $\alpha^{(k)}$ by replacing each occurrence of Qx_i in α (where Q is $\exists$ or $\forall$) by $Qx_{i1}Qx_{i2}\cdots Qx_{ik}$, and every occurrence of x_i , other than in Qx_i , by $x_{i1}, x_{i2}, \dots, x_{ik}$.

Suppose that L'' contains the variables x_{ij} ; $i, j = 0, 1, 2, \dots$. Then one has interpretations of $L^{(k)}$ in L'' .

Definition 2.2. By a k -dimensional interpretation of L in L'' , we mean a map $\alpha \mapsto \alpha^*$ from the formulas of L to the formulas of L'' that is the composite of the k -spread $\alpha \mapsto \alpha^{(k)}$ with a one-dimensional interpretation $\beta \mapsto \beta^0$ of $L^{(k)}$ in L'' (i.e., α^* is $(\alpha^{(k)})^0$).

Obviously, the k -spread is the simplest example of a k -dimensional interpretation. To describe the Cartesian k -dimensional interpretation $\alpha \mapsto \alpha^*$ of $L_{\beta d}$ in L_{OF} , assume that L_{OF} has all x_{ij} ; $i, j = 0, 1, \dots$, among its variables. Then $\alpha \mapsto \alpha^*$ is uniquely determined by the following conditions:

- (i₁) $(\beta(x_1, x_2, x_3))^*$ is the conjunction of the $k + k^2$ formulas

$$(x_{3i} - x_{2i})(x_{2i} - x_{1i}) \geq 0; \quad 1 \leq i \leq k,$$

$$(x_{3i} - x_{2i})(x_{2j} - x_{1i}) = (x_{3j} - x_{2j})(x_{2i} - x_{1j}); \quad 1 \leq i, j \leq k,$$
- (i₂) $(\delta(x_1, x_2, x_3, x_4))^*$ is $\sum_{i=1}^k (x_{2i} - x_{1i})^2 = \sum_{i=1}^k (x_{4i} - x_{3i})^2$,
- (i₃) $(e(x_1, x_2))^*$ is $e(x_{11}, x_{21}) \& e(x_{12}, x_{22}) \& \cdots \& e(x_{1k}, x_{2k})$.

3. INTERPRETATIONS WITH PARAMETERS

Let L, L' be first-order languages where L is relational and all variables of L are also in L' . We denote by L'_p the language obtained by adding to L' new variables $p_0, p_1, \dots$, which we shall call *parameter* or *p-variables*. Let $\alpha \mapsto \exists p\alpha$ be the map that assigns to every α in L'_p the existential closure of α with respect to all the p -variables. We shall call this map the *existential p-closure*.

Definition 3.1. A *one-dimensional interpretation with parameters* of L in L' is a map $\alpha \mapsto \alpha^{(0)}$ from formulas of L to formulas of L' obtained by combining a one-dimensional interpretation $\alpha \mapsto \alpha^0$ of L in L'_p with the existential p -closure and next taking a variant that has no p -variables. (Thus $\alpha^{(0)}$ is a variant without p -variables of $\exists p\alpha^0$.)

Definition 3.2 [M, M-P-S]. Suppose L has only the variables $x_0, x_1, \dots$ and L'' contains all the variables $x_{ij}; i, j = 0, 1, \dots$. Then, by a *k-dimensional interpretation with parameters* of L in L'' , we mean the superposition of the k -spread of L in $L^{(k)}$ with a one-dimensional interpretation with parameters of $L^{(k)}$ in L'' .

Our main result (Theorem 1.1) is valid for interpretations *with parameters*. However, it will suffice to prove this theorem for interpretations *without parameters*, because of the following known fact. (See a remark in [M, §3, p. 299] concerning theories with selectors [M-V].)

Theorem. Given a *k-dimensional interpretation with parameters* $\alpha \mapsto \alpha^{(*)}$ of a first-order language L in L_{OF} and a closed formula φ of L such that $RCF \vdash \varphi^{(*)}$, there exists a *k-dimensional interpretation without parameters* $\alpha \mapsto \alpha^*$ of L in L_{OF} such that $RCF \vdash \varphi^*$.

In view of this, by an interpretation we shall mean henceforth an interpretation *without parameters*.

4. STRUCTURES AND DEFINABILITY

Given an L -structure $\mathcal{A}$, we shall denote by $L_{|\mathcal{A}|}$ the diagram language of L , i.e., the language obtained by adding to L the elements of the universe $|\mathcal{A}|$ (or rather, their names) as constants. If α is a formula in L with no free variables other than $x_1, \dots, x_n$ and $a_1, \dots, a_n \in |\mathcal{A}|$, then the formula $\alpha[a_1 \cdots a_n]$ is closed in $L_{|\mathcal{A}|}$, and thus it has a logical value in $\mathcal{A}$. We shall indicate the value *true* by prefixing the formula with $\mathcal{A} \models$.

Definition 4.1. Given an L -structure $\mathcal{A}$, a set S will be called *$\mathcal{A}$ -definable* if $S \subset |\mathcal{A}|^n$ for some $n \geq 1$ and there are $a_0, \dots, a_m \in |\mathcal{A}|$ and a formula α in L with no free variables other than $x_0, \dots, x_m, x_{m+1}, \dots, x_{m+n}$ such that

$$(s_1, \dots, s_n) \in S \Leftrightarrow \mathcal{A} \models \alpha[a_0, \dots, a_m, s_1, \dots, s_n]$$

for all $s_1, \dots, s_n \in |\mathcal{A}|$. We shall call $x_0, \dots, x_m$ the *parameter variables*, and we shall denote S by $\{\alpha[a_0, \dots, a_m, \cdot]\}$. An r -place relation on an $\mathcal{A}$ -definable subset of $|\mathcal{A}|^n$ will be called *$\mathcal{A}$ -definable* if its graph is an $\mathcal{A}$ -definable subset of $|\mathcal{A}|^{nr}$.

Now let L be a relational language such that $x_0, x_1, \dots$ are all the variables of L . We shall need the following theorem.

Theorem 4.1. *Given a k -dimensional interpretation $\alpha \mapsto \alpha^*$ of L in L'' , there corresponds to every L'' -structure $\mathcal{A}$ an L -structure $\mathcal{A}^*$ with $|\mathcal{A}^*| = |\mathcal{A}|^k$ such that*

- (i) $\mathcal{A}^* \models \alpha \Leftrightarrow \mathcal{A} \models \alpha^*$ for every closed α in L ,
- (ii) every $\mathcal{A}^*$ -definable set (relation) is $\mathcal{A}$ -definable.

We note that it makes sense to talk of $\mathcal{A}$ -definability of $\mathcal{A}^*$ -definable sets because $|\mathcal{A}^*| = |\mathcal{A}|^k$. We deal first with the case $k = 1$:

Lemma 4.1. *Given a one-dimensional interpretation $\alpha \mapsto \alpha^0$ of L in L' , there corresponds to every L' -structure $\mathcal{A}$ an L -structure $\mathcal{A}^0$ such that $|\mathcal{A}^0| = |\mathcal{A}|$ and*

- (i) $\mathcal{A}^0 \models \alpha \Leftrightarrow \mathcal{A} \models \alpha^0$ for every closed α in L ,
- (ii) every $\mathcal{A}^0$ -definable set (relation) is $\mathcal{A}$ -definable.

Proof. We define the L -structure $\mathcal{A}^0$ by requiring that for any n -place predicate symbol p of L , the corresponding n -place relation on $\mathcal{A}^0$, also denoted by p , satisfies

$$(4.1) \quad \mathcal{A}^0 \models p(a_1, \dots, a_n) \Leftrightarrow \mathcal{A} \models (p(x_1, \dots, x_n))^0[a_1, \dots, a_n]$$

for every $a_1, \dots, a_n \in |\mathcal{A}^0| = |\mathcal{A}|$. It is easy to see that (i) and (ii) are consequences of the following fact: for every formula α in L that has no free variables other than $x_{i_1}, \dots, x_{i_n}$,

$$(4.2) \quad \mathcal{A}^0 \models \alpha \left[\begin{smallmatrix} x_{i_1} & \cdots & x_{i_n} \\ a_1 & \cdots & a_n \end{smallmatrix} \right] \Leftrightarrow \mathcal{A} \models \alpha^0 \left[\begin{smallmatrix} x_{i_1} & \cdots & x_{i_n} \\ a_1 & \cdots & a_n \end{smallmatrix} \right]$$

for arbitrary $a_1, \dots, a_n \in |\mathcal{A}|$. To show (4.2), note that when α is $p(x_1, \dots, x_n)$, (4.2) coincides with (4.1). For other atomic formulas (i.e., α of the form $p(x_{i_1}, \dots, x_{i_n})$), (4.2) is a consequence of (4.1) and (ii) in Definition 2.1. Finally, to extend (4.2) to nonatomic formulas, we use (iii), Definition 2.1.

Lemma 4.2. *If $k \geq 1$ and $\alpha \mapsto \alpha^{(k)}$ is the k -spread of L in $L^{(k)}$, then to every $L^{(k)}$ -structure $\mathcal{A}$ there corresponds an L -structure $\mathcal{A}^{(k)}$ such that $|\mathcal{A}^{(k)}| = |\mathcal{A}|^k$ and*

- (i) $\mathcal{A}^{(k)} \models \alpha \Leftrightarrow \mathcal{A} \models \alpha^{(k)}$ for every closed α in L ,
- (ii) every $\mathcal{A}^{(k)}$ -definable set (relation) is $\mathcal{A}$ -definable.

The proof is analogous to the proof of Lemma 4.1. For example, the L -structure $\mathcal{A}^{(k)}$ is defined by the requirement that for any n -place predicate symbol p of L , the corresponding n -place relation on $\mathcal{A}^{(k)}$, also denoted by p , satisfies:

$$\mathcal{A}^{(k)} \models p(a_1, \dots, a_n) \Leftrightarrow \mathcal{A} \models p^{(k)}(a_{11}, \dots, a_{1k}, \dots, a_{n1}, \dots, a_{nk})$$

for any $a_i = (a_{i1}, \dots, a_{ik}) \in |\mathcal{A}^{(k)}| = |\mathcal{A}|^k$, $i = 1, \dots, n$.

Combining Lemmas 4.1 and 4.2, we obtain Theorem 4.1 with $\mathcal{A}^* = (\mathcal{A}^0)^{(k)}$. For a first application, let us denote by $\mathcal{R}$ the standard model of RCF, i.e., the field of real numbers viewed as an L_{OF} -structure with universe $|\mathcal{R}| = \mathbf{R}$. Let $\alpha \mapsto \alpha^*$ be the k -dimensional Cartesian interpretation of $L_{\beta\delta}$ in L_{OF} (see end of §2). Then Theorem 4.1 taken with $\mathcal{A} = \mathcal{R}$, yields at place of $\mathcal{A}^*$ the usual model $\mathcal{E}_k$ of E_k .

Definition 4.2. A structure $\mathcal{A}$ will be called *semialgebraic* if

- (s_1) $|\mathcal{A}| = \mathbf{R}^k$ for some $k \geq 1$,
- (s_2) every $\mathcal{A}$ -definable set (relation) is $\mathcal{R}$ -definable.

Thus, by the preceding construction from Theorem 4.1, $\mathcal{E}_k$ is semialgebraic. Now consider an *arbitrary* k -dimensional interpretation $\alpha \mapsto \alpha^*$ of $L_{\beta\delta}$ in L_{OF} , and let φ be a sentence in $L_{\beta\delta}$ such that $RCF \vdash \varphi^*$. Then $\mathcal{R} \models \varphi^*$ and Theorem 4.1 gives us a semialgebraic $L_{\beta\delta}$ -structure $\mathcal{R}^*$ such that $\mathcal{R}^* \models \varphi$. It follows from this that to establish the main result of this paper, i.e., Theorem 1.1, we only need to prove the following:

Theorem 4.2. *For every $n \geq 2$, there exists a theorem φ_n of E_n such that the sentence φ_n is false in every semialgebraic $L_{\beta\delta}$ -structure $\mathcal{A}$ with $|\mathcal{A}| = \mathbf{R}^k$ and $1 \leq k < n$.*

The rest of this paper is devoted to establishing this result. We proceed as follows. In §5 we collect various known facts about $\mathcal{R}$ -definable (also called *semialgebraic*) sets and relations, and we introduce the concept of an *n-tower* in $\mathbf{R}^k$ (a system of subsets of $\mathbf{R}^k$, Definition 5.1). We also prove that if $1 \leq k < n$ then *there does not exist an n-tower in $\mathbf{R}^k$* (Theorem 5.1). In §6, we define the concept of an *n-grid* in a structure (Definition 6.1) and, if the structure $\mathcal{A}$ is semialgebraic, we *associate with every n-grid in $\mathcal{A}$ an n-tower in $|\mathcal{A}|$* (Theorem 6.1). Finally, in §7 we construct a theorem φ_n of E_n such that for every semialgebraic $L_{\beta\delta}$ -structure $\mathcal{A}$, $\mathcal{A} \models \varphi_n$ *implies the existence of an n-grid in $\mathcal{A}$* (Theorem 7.1). Obviously Theorems 5.1, 6.1, and 7.1 show that $\mathcal{A} \models \varphi_n$ cannot hold in a semialgebraic $L_{\beta\delta}$ -structure $\mathcal{A}$ with $|\mathcal{A}| = \mathbf{R}^k$ and $1 \leq k < n$.

5. SEMIALGEBRAIC SETS AND *n*-TOWERS

For every $n \geq 1$, the class of *semialgebraic* subsets of $\mathbf{R}^n$ is the smallest class of sets closed under forming complements, finite unions, and finite intersections, and containing all sets of the form $\{s \in \mathbf{R}^n : f(s) \geq 0\}$, where $s = (s_1, \dots, s_n)$ and f is a polynomial in n variables with real coefficients. The well-known Seidenberg-Tarski theorem states that *a set is semialgebraic if it is $\mathcal{R}$ -definable*. To prove this, one views $\mathcal{R}$ as a model of the theory RCF and applies Tarski's theorem on the elimination of quantifiers from formulas of RCF. See [B, K-K, Se, T1], and a recent discussion in [VdD3]. Thus, without

further reference to this result, we shall accept from now on $\mathcal{R}$ -definable as equivalent to *semialgebraic*.

A continuous map between semialgebraic subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$ is called *semialgebraic* if its graph is a semialgebraic subset of $\mathbf{R}^m \times \mathbf{R}^n$. According to Definition 4.1, an r -place relation on a semialgebraic subset of $\mathbf{R}^n$ is *semialgebraic* if its graph is a semialgebraic subset of $\mathbf{R}^{nr}$. The following properties of semialgebraic sets, maps, and relations will be needed later:

- I. The intersection of two semialgebraic sets is semialgebraic.
- II. The composite of semialgebraic maps (wherever defined) is semialgebraic.
- III. The direct and inverse image of a semialgebraic set under a semialgebraic map is semialgebraic.
- IV. Each semialgebraic equivalence relation on a semialgebraic set has a semialgebraic set of representatives.
- V. Each semialgebraic set has a finite semialgebraic triangulation.

Properties I, II, and III follow without difficulty from Definition 4.1 of $\mathcal{R}$ -definability. Property IV is shown in [VdD1, Proposition 4.3]. To state property V precisely, let us use Δ as a generic name for an open simplex in $\mathbf{R}^m$:

$$\Delta = \left\{ \sum_{i=0}^d t_i v_i : \sum_{i=0}^d t_i = 1; \text{ all } t_i > 0 \right\},$$

where the vertices $v_0, \dots, v_d \in \mathbf{R}^m$ are not all contained in some hyperplane of dimension $d - 1$. We call d the *dimension* of Δ . Then a *finite simplicial complex* in $\mathbf{R}^m$ is a set $K = \{\Delta_j : j \in J\}$, where J is finite and Δ_j are disjoint open simplexes in $\mathbf{R}^m$, such that all faces of each Δ_j also belong to K . In 1964, Łojasiewicz [L] proved the following:

Triangulation theorem. *Given a semialgebraic set S (in some $\mathbf{R}^n$), there exists a finite simplicial complex $K = \{\Delta_j : j \in J\}$ (in some $\mathbf{R}^m$) and, for some subset $J' \subset J$, a semialgebraic map*

$$h: S \rightarrow \bigcup \{\Delta_j : j \in J'\},$$

which is also a topological homeomorphism.

Other proofs can be found in [C and H]. (The usually made assumption that S is bounded is not relevant for the previous version; see [H, Remark 1.10].) In a more general setting of 0-minimal Tarski systems, this result is stated in [VdD2] and proved in [VdD1]. As a corollary we obtain

Lemma 5.1. *An infinite semialgebraic set contains an arc (homeomorphic image of a nonzero interval in $\mathbf{R}$).*

We define now by recursion on n the n -towers under semialgebraic sets.

Definition 5.1. (1) The 1-towers under S are precisely the uncountable semialgebraic subsets of S .

- (2) If $n \geq 2$, then an n -tower $\mathbf{T}$ under S is a system $(\Sigma, \Pi, \mathbf{T}')$, where
- (a) Σ is any uncountable set,
 - (b) Π is a mapping that assigns to every $b \in \Sigma$ a semialgebraic set $\Pi(b) \subset S$ so that $\Pi(b) \cap \Pi(b') = \emptyset$ for every two distinct b, b' in Σ ,
 - (c) $\mathbf{T}'$ is a mapping that assigns to every $b \in \Sigma$ an $(n-1)$ -tower $\mathbf{T}'(b)$ under $\Pi(b)$.

An n -tower under a subset S of $\mathbf{R}^k$ will also be called simply an n -tower in $\mathbf{R}^k$. It is easily seen that a set S under which there exists an n -tower is uncountable. Thus, for $n \geq 2$, each $\Pi(b)$ is uncountable and we can regard $\Pi(b)$ as a 1-tower under itself. We conclude that if there is an n -tower in $\mathbf{R}^k$ with $n \geq 2$, then there is also a 2-tower in $\mathbf{R}^k$.

Lemma 5.2. *If $n \geq 2$, then there does not exist an n -tower in $\mathbf{R}$.*

Proof. By the preceding remarks, it is enough to show that there does not exist a 2-tower in $\mathbf{R}$. Now, if there were a 2-tower $(\Sigma, \Pi, \mathbf{T}')$ in $\mathbf{R}$, we would have an uncountable family $\{\Pi(b) : b \in \Sigma\}$ of uncountable, mutually disjoint, semialgebraic subsets of $\mathbf{R}$. By Lemma 5.1, each of these sets would have to contain an arc, thus also a rational number. However, we do not have uncountably many rational numbers.

Lemma 5.3. *If S, S_1, S_2 are semialgebraic such that $S \subset S_1 \cup S_2$ and there exists an n -tower under S , then there exists an n -tower under at least one of the intersections $S \cap S_1, S \cap S_2$.*

Proof. We use induction on n . If $n = 1$, we are given a 1-tower under S , i.e., an uncountable semialgebraic subset S' of S . Then at least one of the sets $S' \cap S_1, S' \cap S_2$ is uncountable, and is thus a 1-tower under itself.

We assume now that $n > 1$ and that the assertion holds for all $(n-1)$ -towers. Let $\mathbf{T} = (\Sigma, \Pi, \mathbf{T}')$ be an n -tower under $S \subset S_1 \cup S_2$. Then for each $b \in \Sigma$, $\mathbf{T}'(b)$ is an $(n-1)$ -tower under $\Pi(b) \subset S_1 \cup S_2$, so by the inductive assumption, there is an $(n-1)$ -tower under $\Pi(b) \cap S_1$ or under $\Pi(b) \cap S_2$. Renaming the S_i , if need be, we may assume that for each $b \in \Sigma'$, where Σ' is some uncountable subset of Σ , there is an $(n-1)$ -tower $\mathbf{T}''(b)$ under $\Pi(b) \cap S_1$. Denoting $\Pi(b) \cap S_1$ by $\Pi'(b)$, we obtain an n -tower $(\Sigma', \Pi', \mathbf{T}'')$ under $S \cap S_1$. $\square$

Given a semialgebraic injective map $h: S \rightarrow S'$, one can associate in an obvious fashion with every n -tower $\mathbf{T}$ under S an n -tower $h(\mathbf{T})$ under S' : the image of $\mathbf{T}$ by h . We shall use this observation several times in the subsequent proof.

Lemma 5.4. *If $k, n \geq 2$ and there exists an n -tower in $\mathbf{R}^k$, then there exists an $(n-1)$ -tower in $\mathbf{R}^{k-1}$.*

Proof. Let $\mathbf{T} = (\Sigma, \Pi, \mathbf{T}')$ be an n -tower in $\mathbf{R}^k$, where $k, n \geq 2$. As we observed earlier, each $\Pi(b)$ is uncountable. Thus, we have an uncountable

family $\{\Pi(b): b \in \Sigma\}$ of nonempty, mutually disjoint, semialgebraic subsets of $\mathbf{R}^k$. Obviously at least one of them *has an empty interior* (contains no open subset). Let this be $\Pi(b)$. By the triangulation theorem, we have a semialgebraic map and topological homeomorphism

$$h: \Pi(b) \rightarrow \bigcup \{\Delta_j: j \in J'\},$$

where Δ_j are open simplexes in some $\mathbf{R}^m$. Consider the $(n-1)$ -tower $\mathbf{T}'(b)$ under $\Pi(b)$. Since $\Pi(b) = \bigcup \{h^{-1}(\Delta_j): j \in J'\}$ and Δ_j and hence $h^{-1}(\Delta_j)$ are semialgebraic sets, we infer from Lemma 5.3 that there exists an $(n-1)$ -tower under one of the sets $h^{-1}(\Delta_j)$. Applying h , which is injective, we get an $(n-1)$ -tower under Δ_j . The restriction of h^{-1} to Δ_j is a topological homeomorphism onto $h^{-1}(\Delta_j) \subset \Pi(b) \subset \mathbf{R}^k$, and since $\Pi(b)$ does not contain an open subset of $\mathbf{R}^k$, the dimension of Δ_j is less than k , by the “invariance of domain theorem.” See [E-S, p. 303]. Let $d = \dim \Delta_j \leq k-1$. We have $\Delta_j \subset \mathbf{R}^m$, so there is a linear map $g: \mathbf{R}^m \rightarrow \mathbf{R}^d$ such that the restriction $g|_{\Delta_j}$ is injective. Clearly g is semialgebraic and maps the just-obtained $(n-1)$ -tower under Δ_j onto an $(n-1)$ -tower under $g(\Delta_j) \subset \mathbf{R}^d$. Taking the image of this tower under the linear embedding $\mathbf{R}^d \rightarrow \mathbf{R}^{k-1}$, we obtain an $(n-1)$ -tower in $\mathbf{R}^{k-1}$.

Theorem 5.1. *If $n > k$, then there does not exist an n -tower in $\mathbf{R}^k$.*

Proof. We use induction on k . The case $k = 1$ is Lemma 5.2. If $k \geq 2$, and there exists an n -tower in $\mathbf{R}^k$ for some $n > k$, then by Lemma 5.4, there also exists an $(n-1)$ -tower in $\mathbf{R}^{k-1}$. This provides the inductive step.

6. n -GRIDS IN STRUCTURES

If we choose a rectangular coordinate system in $\mathcal{E}_n$, then the set of hyperplanes perpendicular to the coordinate axes is an example of an n -grid in $\mathcal{E}_n$ (see the example in this section). In general, we have:

Definition 6.1. Let $\mathcal{A}$ be a structure for a first-order language. A system $(S, \Sigma_1, \dots, \Sigma_n, P_1, \dots, P_n)$, where $n \geq 2$, will be called an n -grid under S in $\mathcal{A}$ if

- (1) S is an $\mathcal{A}$ -definable subset of $|\mathcal{A}|$;
- (2) $\Sigma_1, \dots, \Sigma_n$ are any uncountable sets;
- (3) each P_i is a map from Σ_i into the set of $\mathcal{A}$ -definable subsets of S , such that $P_i(b) \cap P_i(b') = \emptyset$ for every two distinct $b, b' \in \Sigma_i$ and every $i = 1, \dots, n$;
- (4) for arbitrary $b_i \in \Sigma_i$ ($i = 1, \dots, n$),

$$P_1(b_1) \cap P_2(b_2) \cap \dots \cap P_n(b_n) \neq \emptyset.$$

Let us observe that in every n -grid all the sets $P_i(b)$ are *uncountable*. Indeed, for any $j \neq i$, $P_i(b)$ contains all the disjoint sets $P_i(b) \cap P_j(a)$, where $a \in \Sigma_j$.

Definition 6.2. Given an n -grid as above and a fixed $a \in \Sigma_n$, we shall call $(P_n(a), \Sigma_1, \dots, \Sigma_{n-1}, P'_1, \dots, P'_{n-1})$, where $P'_i(b) = P_i(b) \cap P_n(a)$ for each $b \in \Sigma_i$; $i \leq n-1$, the *associated* $(n-1)$ -grid under $P_n(a)$.

Example. For $\mathcal{A} = \mathcal{E}_n$, let us define $\Sigma_1, \dots, \Sigma_n$ and $P_i(b)$ as follows: Σ_i is the i th coordinate axis

$$\Sigma_i = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_j = 0 \text{ for all } j \neq i\},$$

and for each $b \in \Sigma_i$, $P_i(b)$ is the $(n-1)$ -dimensional hyperplane passing through b and perpendicular to Σ_i :

$$P_i(b) = \{(x_1, \dots, x_n) \in \mathbf{R}^n : (0, \dots, 0, x_i, 0, \dots, 0) = b\}.$$

Putting $S = |\mathcal{E}_n| = \mathbf{R}^n$, we check easily that the conditions of Definition 6.1 are satisfied, except perhaps for $\mathcal{E}_n$ -definability; however, that will be obtained subsequently as a by-product of the proof of Theorem 7.1. Henceforth, we shall consider n -grids in semialgebraic $L_{\beta\delta}$ -structures (see Definition 4.2).

Theorem 6.1. *Given a semialgebraic $L_{\beta\delta}$ -structure $\mathcal{A}$, there is for each $n \geq 2$ a mapping that assigns to every n -grid in $\mathcal{A}$ an n -tower in $|\mathcal{A}|$, so that to a grid under S there corresponds a tower under S .*

Proof. We define the mappings by recursion on n . Let us consider an n -grid $(S, \Sigma_1, \dots, \Sigma_n, P_1, \dots, P_n)$ in $\mathcal{A}$, where $n \geq 2$.

We assign to this n -grid an n -tower $\mathbf{T} = (\Sigma, \Pi, \mathbf{T}')$ under S , where $\Sigma = \Sigma_n$, $\Pi(b) = P_n(b)$ for each $b \in \Sigma$, and $\mathbf{T}'$ is yet to be defined. Obviously Σ is uncountable and $\Pi(b)$ is semialgebraic and also uncountable, as we observed just after Definition 6.1. Moreover:

$$\Pi(b) \cap \Pi(b') = P_n(b) \cap P_n(b') = \emptyset \quad \text{for } b \neq b'.$$

Hence, if $n = 2$, we may take $\mathbf{T}'(b) = \Pi(b)$, which is a 1-tower under itself, and we are done. If $n > 2$ then, by recursive assumption, there is already assigned to every $(n-1)$ -grid under S' in $\mathcal{A}$ an $(n-1)$ -tower under S' . In particular, consider for each $b \in \Sigma_n = \Sigma$, the *associated* (see Definition 6.2) $(n-1)$ -grid under $P_n(b)$, i.e., under $\Pi(b)$. By assumption, to this $(n-1)$ -grid there is assigned an $(n-1)$ -tower under $\Pi(b)$. We denote this tower by $\mathbf{T}'(b)$, thus completing the description of the n -tower $\mathbf{T}$.

7. EXISTENCE OF n -GRIDS

To complete the proof of Theorem 4.2 (and hence of Theorem 1.1; see end of §4) we still need to show the following:

Theorem 7.1. *For each $n \geq 2$, there is a theorem φ_n of E_n such that if $\mathcal{A}$ is a semialgebraic $L_{\beta\delta}$ -structure and $\mathcal{A} \models \varphi_n$, then there exists an n -grid under $S = |\mathcal{A}|$ in $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a semialgebraic $L_{\beta\delta}$ -structure (fixed for the rest of this proof). Let us associate with any $a_0, a_1, a_2 \in |\mathcal{A}|$ certain $\mathcal{A}$ -definable sets, which we

shall call by the names “line” and “hyperplane,” since such names are normally used if a_0, a_1, a_2 are distinct and $\mathcal{A}$ is $\mathcal{E}_n$. For this, we need two formulas of $L_{\beta\delta}$ which we shall denote by λ and π .

λ is $\beta(x_0, x_1, x_2) \vee \beta(x_1, x_2, x_0) \vee \beta(x_2, x_0, x_1)$ and, if $a_0 \neq a_1$, the set $\{\lambda[a_0, a_1, \cdot]\}$ (see Definition 4.1, with parameter variables x_0, x_1) gets the name “the line through a_0, a_1 .”

π is chosen so that for $a_0 \neq a_1$, $\{\pi[a_0, a_1, a_2, \cdot]\}$ should be called “the $(n-1)$ -dimensional hyperplane containing a_2 and perpendicular to the line $\{\lambda[a_0, a_1, \cdot]\}$ ”; explicitly, we may take for π the formula:

$$\begin{aligned} \exists x_4 \exists x_5 (\neg e(x_4, x_5) \& \lambda[x_0, x_1, x_4] \& \lambda[x_0, x_1, x_5] \\ \& \delta(x_2, x_4, x_2, x_5) \& \delta(x_3, x_4, x_3, x_5)) . \end{aligned}$$

Further, let $\bar{\beta}$ be the formula expressing “strict betweenness”:

$$\beta(x_1, x_2, x_3) \& \neg e(x_1, x_2) \& \neg e(x_2, x_3) ,$$

and let σ be the conjunction of these five formulas:

- (1) $\forall x_0 e(x_0, x_0)$,
- (2) $\forall x_0 \forall x_1 (e(x_0, x_1) \rightarrow e(x_1, x_0))$,
- (3) $\forall x_0 \forall x_1 \forall x_2 (e(x_0, x_1) \& e(x_1, x_2) \rightarrow e(x_0, x_2))$,
- (4) $\forall x_0 \forall x_1 \exists x_2 (\neg e(x_0, x_1) \rightarrow \bar{\beta}[x_0, x_2, x_1])$,
- (5) $\forall x_0 \forall x_1 \forall x_2 \forall x_3 (\bar{\beta}[x_0, x_2, x_1] \& \bar{\beta}[x_0, x_3, x_2] \rightarrow \bar{\beta}[x_0, x_3, x_1])$.

Lemma 7.1. *Suppose $\mathcal{A} \models \sigma$ and let a_0, a_1 be such that $\mathcal{A} \models \neg e(a_0, a_1)$. Then there exists an uncountable set $\Sigma \subset \{\lambda[a_0, a_1, \cdot]\}$ such that $\mathcal{A} \models \neg e(b, b')$ whenever $b, b' \in \Sigma$ and $b \neq b'$.*

Proof. It follows from $\mathcal{A} \models \sigma$ that e is an equivalence relation on $|\mathcal{A}|$. We show first that $\{\lambda[a_0, a_1, \cdot]\}$ contains infinitely many pairwise not e -equivalent elements. Let $a_0, a_1, a_2, \dots$ be obtained from (4), so that $\mathcal{A} \models \bar{\beta}[a_0, a_{i+1}, a_i]$ for $i = 1, 2, \dots$. By the transitivity (5), we conclude that $\mathcal{A} \models \bar{\beta}[a_0, a_{i+1}, a_j]$ for all $j = 1, 2, \dots, i$, which implies $\mathcal{A} \models \neg e(a_{i+1}, a_j)$, by the definition of $\bar{\beta}$. Also, putting $j = 1$, we get $\mathcal{A} \models \bar{\beta}[a_0, a_{i+1}, a_1]$ whence all a_i are on the required line.

Obviously e is an $\mathcal{A}$ -definable relation, so it is semialgebraic. Restricting e to $\{\lambda[a_0, a_1, \cdot]\}$ we get a semialgebraic equivalence relation on a semialgebraic set. Thus, property IV in §5 yields a semialgebraic set of representatives $\Sigma \subset \{\lambda[a_0, a_1, \cdot]\}$ of the e -equivalence classes. As we have already shown, each of the $a_0, a_1, \dots$ is in a different equivalence class. Hence, Σ is infinite. By Lemma 5.1, Σ is uncountable. $\square$

Evidently there exists a sentence τ_n in $L_{\beta\delta}$ such that the meaning of $\mathcal{A} \models \tau_n$ is as follows:

THERE EXIST $a_0, a_1, \dots, a_n$ such that:
 $\neg e(a_i, a_j)$ for all distinct $i, j = 0, \dots, n$
 AND for any $1 \leq i \leq n$ and every $b, b' \in \{\lambda[a_0, a_i, \cdot]\}$ such that
 $\neg e(b, b')$ one has $\{\pi[a_0, a_i, b, \cdot]\} \cap \{\pi[a_0, a_i, b', \cdot]\} = \emptyset$
 AND for every $b_1, \dots, b_n$, if $b_i \in \{\lambda[a_0, a_i, \cdot]\}$ for $1 \leq i \leq n$
 then $\{\pi[a_0, a_1, b_1, \cdot]\} \cap \dots \cap \{\pi[a_0, a_n, b_n, \cdot]\} \neq \emptyset$.

Let φ_n be σ & τ_n . We have $\mathcal{E}_n \models \tau_n$ (taking a_0 above to be the origin of a rectangular coordinate system in $\mathcal{E}_n$ and a_i on the i th coordinate axis). Hence, $E_n \vdash \tau_n$, and since also $E_n \vdash \sigma$, we get $E_n \vdash \varphi_n$, as required.

Suppose $\mathcal{A} \models \varphi_n$, and let $a_0, a_1, \dots, a_n \in |\mathcal{A}|$ have the properties assured by $\mathcal{A} \models \tau_n$. For each $i = 1, \dots, n$ we denote by Σ_i an uncountable subset of $\{\lambda[a_0, a_i, \cdot]\}$ such that $\mathcal{A} \models \neg e(b, b')$ for every two distinct $b, b' \in \Sigma_i$ (see Lemma 7.1). Further, let $S = |\mathcal{A}|$ and let $P_i(b) = \{\pi[a_0, a_i, b, \cdot]\}$ for every $b \in \Sigma_i$ and $i = 1, \dots, n$. Then, by $\mathcal{A} \models \tau_n$, the system $(S, \Sigma_1, \dots, \Sigma_n, P_1, \dots, P_n)$ satisfies all the conditions of Definition 6.1. Thus we have obtained an n -grid in $\mathcal{A}$.

REFERENCES

- [B] M. Boffa, *Elimination des quantificateurs en algèbre*, Bull. Soc. Math. Belg. **32** (1980), 107–133.
- [C-K] C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.
- [C] M. Coste, *Ensembles semi-algébriques* (Proc. Conf. on Semi-algebraic Geometry and Quadratic Forms, Rennes, 1981), Lecture Notes in Math., vol. 959, Springer-Verlag, Berlin and New York, 1982.
- [VdD1] L. van den Dries, *Definable sets in 0-minimal structures*, Lectures at University of Konstanz, spring 1985.
- [VdD2] —, *A generalization of the Tarski-Seidenberg theorem, and some nondefinability results*, Bull. Amer. Math. Soc. **15** (1986), 189–193.
- [VdD3] —, *Alfred Tarski's elimination theory for real closed fields*, J. Symbolic Logic **53** (1988), 7–19.
- [E-S] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952.
- [K-K] G. Kreisel and J. L. Krivine, *Elements of mathematical logic*, North-Holland, Amsterdam, 1967.
- [H] H. Hironaka, *Triangulation of algebraic sets*, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R.I., 1975, pp. 165–185.
- [L] S. Lojasiewicz, *Triangulation of semialgebraic sets*, Ann. Sci. École Norm. Sup. (3) **18** (1964), 449–474.
- [M-V] R. Montague and R. L. Vaught, *A note on theories with selectors*, Fund. Math. **47** (1959), 243–247.
- [M] J. Mycielski, *A lattice of interpretability types of theories*, J. Symbolic Logic **42** (1977), 297–305.

- [M-P-S] J. Mycielski, P. Pudlák, and A. S. Stern, *A lattice of chapters of mathematics*, Mem. Amer. Math. Soc. (to appear).
- [Se] A. Seidenberg, *A new decision method for elementary algebra*, Ann. of Math. (2) **60** (1954), 365–374.
- [Sz] L. W. Szczerba, *Dimensions of interpretations (abstract)*, J. Symbolic Logic **49** (1984), 677.
- [T1] A. Tarski, *Decision method for elementary algebra and geometry*, 2nd ed., Berkeley, 1951.
- [T2] ———, *What is elementary geometry?*, Symposium on the Axiomatic Method with Special Reference to Geometry and Physics, North-Holland, Amsterdam, 1959, pp. 16–29.

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