ON CERTAIN PARTIAL DIFFERENTIAL OPERATORS OF FINITE ODD TYPE

A. ALEXANDROU HIMONAS

ABSTRACT. Let P be a linear partial differential operator of order $m \geq 1$ with real-analytic coefficients defined in Ω , an open set of $\mathbb{R}^n$, and let γ be in the cotangent space of Ω minus the zero section. If P is of odd finite type k and if the Hörmander numbers are $1 = k_1 < k_2$, k_2 odd, then P is analytic hypoelliptic at γ . These operators are not semirigid.

INTRODUCTION

In this paper we present a microlocal analytic hypoellipticity result for a class of linear partial differential operators of finite type with real-analytic coefficients. The assumption is that the type of the operator is an odd positive integer k and that there are two Hörmander numbers k_1, k_2 with $1 = k_1 < k_2$ and k_2 is odd. The main difference between this result and previous results obtained by the author is that the operators here may not be semirigid. Therefore we view it as a partial result of the more general problem of giving necessary and sufficient conditions for microlocal analytic hypoellipticity of operators of finite type. For CR structures a version of this problem has been recently solved by Tumanov [10] (sufficient conditions) and Baouendi-Rothschild [1] (necessary conditions). In the analytic case the necessity was proved in [2].

The condition used in [10] is a condition at a point in the base space called the minimality condition. An analytic CR structure is minimal at a point if it is of finite type at this point. For C^∞ CR structures finite type implies minimality but not vice versa. The finite type condition used in this paper is microlocal.

The new difficulty that arises in the proof of this result is that the phase function may contain terms of degree less than k which is the degree of the terms that result from the type of the operator. To make these lower degree terms negligible, we choose a special good contour and we use a more general sufficient condition than the one used in [6].

Received by the editors December 18, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 35H05; Secondary 58G15.

The author was partially supported by NSF.

The author would like to thank J. E. Fornaess for suggesting the proof of Lemma 2.3.

1. DEFINITIONS AND STATEMENT OF RESULT

Let $P = P(y, D)$ be a linear partial differential operator (p.d.o.) of order $m \geq 1$ with complex-valued real-analytic coefficients defined in an open neighborhood Ω of $y_0 \in \mathbb{R}^n$. We denote by $P_m(y, \eta)$ its principal symbol with $p_1(y, \eta) = \Re P_m(y, \eta)$ the real part and $p_2(y, \eta) = \Im P_m(y, \eta)$ the imaginary part. Also, let $\gamma = (y_0, \eta_0) \in T^*\Omega - 0$ be a characteristic point for P and assume that P is of *principal type* at γ , i.e., $P_m(\gamma) = 0$ and $\text{grad}_\eta P_m(\gamma) \neq 0$.

If $I = (i_1, \dots, i_j)$ is a sequence of $|I| = j$ elements which are either 1 or 2 then we denote by

$$p_I = \{p_{i_j}, \{p_{i_{j-1}}, \dots, \{p_{i_2}, p_{i_1}\} \dots\}\}$$

the repeated Poisson bracket of p_1 and p_2 associated to I .

The p.d.o. is said to be of *finite type* $k = k(\gamma)$ at γ (Egorov [4], Hörmander [7], Kohn [8]) if $p_I(\gamma) = 0$ for all I with $|I| < k$, and if $p_I(\gamma) \neq 0$ for all I with $|I| = k$.

Now we shall recall the definition of Hörmander numbers (see also [6]) for a p.d.o. P of finite type k , $k \geq 2$. Let $\lambda = \eta_1 \frac{\partial}{\partial \eta_1} + \dots + \eta_n \frac{\partial}{\partial \eta_n}$ and $V_0 = \text{span}\{\lambda(\gamma)\}$ in $T_\gamma(T^*\mathbb{R}^n)$. We define

$$k_1 = k_1(\gamma) = \min\{|I| : |I| \leq k/2 \text{ and } H_{p_I}(\gamma) \notin V_0\}.$$

Here H_{p_I} is the Hamilton vector field of p_I . We let $V_1 = \text{span}\{H_{p_I}(\gamma) : |I| \leq k_1, \lambda(\gamma)\}$, and we define the multiplicity d_1 of k_1 by $d_1 = \dim V_1 - \dim V_0$. We proceed inductively. We assume that the numbers $k_1, \dots, k_\nu$, their multiplicities $d_1, \dots, d_\nu$, and the spaces $V_1, \dots, V_\nu$ have been defined and we define

$$k_{\nu+1} = \min\{|I| : |I| \leq k/2 \text{ and } H_{p_I}(\gamma) \notin V_\nu\},$$

$$V_{\nu+1} = \text{span}\{H_{p_I}(\gamma) : |I| \leq k_{\nu+1}, \lambda(\gamma)\} \quad \text{and} \quad d_{\nu+1} = \dim V_{\nu+1} - \dim V_\nu.$$

The Hörmander numbers $k_1, \dots, k_r$ are the numbers defined above, counted with their multiplicities. It is easy to see that the assumption that P is of principal type at γ implies that $k_1 = 1$.

Now we shall state our result.

Theorem 1.1. *Let P be a linear p.d.o. with real-analytic coefficients defined in Ω , an open set of $\mathbb{R}^n$, and $\gamma \in T^*\Omega - 0$. If P is of odd finite type k and if the Hörmander numbers are $1 = k_1 < k_2$, k_2 odd, then P is analytic hypoelliptic at γ .*

We recall that P is *analytic hypoelliptic* at γ iff u is analytic at γ whenever Pu is analytic at γ . A distribution u defined near $y_0 \in \mathbb{R}^n$ is said to be analytic at γ (see Sjöstrand [9]) iff

$$\left| \int_{\mathbb{R}^n} e^{-ix\eta - |\eta|(x-y)^2} \chi(x) u(x) dx \right| \leq ce^{-|\eta|/c}$$

for all y near y_0 and all η in a conic neighborhood of η_0 . Here χ is in $C_0^\infty(\mathbb{R}^n)$ and equal to 1 near y_0 .

The Hörmander numbers k_1 , k_2 , and k are invariant under a symplectic change of coordinates, since they are defined by the repeated Poisson brackets of p_1 and p_2 and since symplectic changes of coordinates preserve Poisson brackets. By straightforward computations one can show that these numbers are also invariant under a multiplication by an elliptic symbol.

If P is a vector field in $\mathbb{R}^3$ then a typical example that Theorem 1.1 applies is

$$P = \frac{\partial}{\partial y_1} + i \left[y_1^{k_2-1} \frac{\partial}{\partial y_2} + p_{k-1}(y_1, y_2) \frac{\partial}{\partial y_3} \right],$$

where k and k_2 are positive odd integer with $1 < k_2 < k/2$, and $p_{k-1} \neq 0$ is a homogeneous polynomial of weight $k-1$ for the weights (k_1, k_2) . Here $k_1 = 1$ and it corresponds to the variable y_1 , k_2 corresponds to the variable y_2 , and k corresponds to the variable y_3 . The point $\gamma = (y_0, \eta_0)$, where $y_0 = (0, 0, 0)$ and $\eta_0 = (0, 0, \pm 1)$. By Theorem 1.1 it follows that P is analytic hypoelliptic at γ . In fact, the vector field P above is analytic hypoelliptic at 0 since at the characteristic points of the form $\gamma = (0, \eta_0)$, $\eta_0 = (0, \eta_2, \eta_3)$ with $\eta_2 \neq 0$, P is analytic hypoelliptic by Theorem 1.4 in [6]. Note when $\eta_0 = (0, 0, \pm 1)$ then P is not semirigid at γ .

2. PROOF OF THEOREM 1.1

We start by reducing the p.d.o. P microlocally near γ to a classical analytic pseudodifferential operator of order one, which we will denote again with P , with principal symbol in the form

$$p(y, \eta) = \eta_1 + ip_2(y, \eta'), \quad \eta = (\eta_1, \eta'),$$

where p_2 is a holomorphic function, positively homogeneous of degree one in η' and

$$(2.1) \quad \gamma = (y_0, \eta_0), \quad y_0 = (0, \dots, 0) \in \mathbb{R}^n, \quad \eta_0 = (0, \dots, 1) \in \mathbb{R}^n.$$

This reduction is standard for operators of principal type (see [5]) and at the level of principal symbols it is done by using the implicit function theorem and a multiplication by an elliptic symbol.

If, in addition, the Hörmander numbers of the operator P are as in the assumptions of Theorem 1.1 then by Proposition 2.2 in [6] we obtain the following normal form for the principal symbol of our p.d.o. P .

Lemma 2.1. *If P is as in Theorem 1.1 then its principal symbol P_m can be reduced, after a multiplication by an elliptic symbol and a symplectic change of coordinates, to*

$$(2.2) \quad p(y, \eta) = \eta_1 + i[y_1^{k_2-1} \eta_2 + p_{k-1}(y_1, y_2) \eta_n + R(y, \eta')],$$

near $\gamma = (y_0, \eta_0)$, $y_0 = 0 \in \mathbb{R}^n$ and $\eta_0 = (0, \dots, 1) \in \mathbb{R}^n$, where $p_{k-1} \neq 0$ is a real homogeneous polynomial of weight $k-1$ for the weights $K = (k_1, k_2, k/2, \dots, k/2, k)$ and the remainder $R = O_K(-\frac{1}{2})$.

We recall that a function $f(y, \eta)$ is homogeneous of weight w for the weights K if, for $t > 0$ and $(y, \eta) \in \mathbb{R}^{2n}$, it is

$$f(t^{k_1}y_1, t^{k_2}y_2, t^{k/2}y_3, \dots, t^k y_n; t^{-k_1}\eta_1, \dots, t^{-k}\eta_n) = t^w f(y, \eta).$$

If h is a holomorphic function defined near γ , then we write $h = O_K(w)$ if every term in the Taylor expansion of h is of weight greater than or equal to w for the weights K .

Next, for any $\delta \in (0, 1)$ we consider the following change of coordinates:

$$\begin{aligned} y_1 &= \delta \tilde{y}_1, \\ y_2 &= \delta^{k_2} \tilde{y}_2, \\ y_j &= \delta^{k/2} \tilde{y}_j, \quad 3 \leq j \leq n-1, \\ y_n &= \delta^k \tilde{y}_n. \end{aligned}$$

Then the symbol $p(y, \eta)$ in (2.2) is transformed to

$$p(\tilde{y}, \tilde{\eta}) = \frac{1}{\delta} [\tilde{\eta}_1 + i[\tilde{y}_1^{k_2-1} \tilde{\eta}_2 + p_{k-1}(\tilde{y}_1, \tilde{y}_2) \tilde{\eta}_n + \delta^{1/k} R_\delta(\tilde{y}, \tilde{\eta}')]],$$

where R_δ consists of all terms of R each multiplied by a positive power of δ . If we multiply p by δ , replace $\delta^{1/k}$ by a new δ , and drop the tildes then we obtain the following final normal form for the principal symbol of our operator P :

$$(2.3) \quad p(y, \eta) = \eta_1 + i[y_1^{k_2-1} \eta_2 + p_{k-1}(y_1, y_2) \eta_n + \delta R(y, \eta)].$$

Construction of phase function. Let L be the vector field with symbol

$$L(y, \eta) = \eta_1 + i[y_1^{k_2-1} \eta_2 + p_{k-1}(y_1, y_2) \eta_n].$$

The function

$$Z = y_2 + i \frac{1}{k_2} y_1^{k_2}$$

satisfies the equation $LZ = 0$. Also we can find real homogeneous polynomials $A(y_1, y_2)$ and $B(y_1, y_2)$ of weight k for the weights (k_1, k_2) such that $LW = 0$ with

$$W = y_n + B(y_1, y_2) + iA(y_1, y_2),$$

where $A \neq 0$ and A is of the form

$$(2.4) \quad A(y_1, y_2) = c_0 y_1^k + c_1 y_1^{k-k_2} y_2 + \dots + c_\lambda y_1^{k-\lambda k_2} y_2^\lambda, \quad k - \lambda k_2 > 0.$$

We let

$$(2.5) \quad \begin{aligned} \varphi_0(x, y) = & iC(x_1 - y_1)^2 + \frac{i}{2}(x_2 - Z)^2 \\ & + \frac{i}{2} \sum_{j=3}^{n-1} (x_j - y_j)^2 + \frac{i}{2}(x_n - W - ax_2^{k/k_2})^2, \end{aligned}$$

where C is a large constant to be chosen, $a = 0$ if k/k_2 is not an integer, and $a \in \mathbb{C}$ will be chosen later if k/k_2 is an integer.

The function φ_0 satisfies the equation

$$(2.6) \quad \frac{\partial \varphi_0}{\partial x_1} + L\varphi_0 = 0.$$

Let $(x_0, y_0) \in \mathbb{C}^{2n}$, $x_0 = y_0 - i\eta_0$, and $(y_0, \eta_0) = \gamma$ be as in (2.1). By the Cauchy-Kowalevski theorem, there is a holomorphic function $\varphi(x, y)$ defined near (x_0, y_0) and such that

$$(2.7) \quad \frac{\partial \varphi}{\partial x_1} = p \left(y_1, -\frac{\partial \varphi}{\partial y} \right) = -L\varphi + i\delta R \left(y_1, -\frac{\partial \varphi}{\partial y} \right),$$

$$\varphi(0, x', y) = \varphi_0(0, x', y),$$

where p is given by (2.3). By (2.5), (2.6), and (2.7) we have

$$(2.8) \quad \varphi(x, y) = \varphi_0(x, y) + O(\delta).$$

Let

$$\varphi_1(x, y) = -\Im \varphi(x, y), \quad y \in \mathbb{R}^n, \quad x \in \mathbb{C}^n.$$

Then by (2.5) and (2.8) we have

$$(2.9) \quad \begin{aligned} \varphi_1(x, y) = & -C(y_1 - \Re x_1)^2 + C[\Im x_1]^2 \\ & - \frac{1}{2}(\Re x_2 - y_2)^2 + \frac{1}{2} \left(\Im x_2 - \frac{1}{k_2} y_1^{k_2} \right)^2 \\ & - \frac{1}{2} \sum_{j=3}^{n-1} (\Re x_j - y_j)^2 + \frac{1}{2} \sum_{j=3}^{n-1} [\Im x_j]^2 \\ & - \frac{1}{2} (\Re x_n - y_n - B(y_1, y_2) - \Re[ax_2^{k/k_2}])^2 \\ & + \frac{1}{2} (\Im x_n - A(y_1, y_2) - \Im[ax_2^{k/k_2}])^2 + O(\delta). \end{aligned}$$

The critical points of $\varphi_1(x, y)$ as a function of $y \in \mathbb{R}^n$ are of the form

$$(2.10) \quad \begin{aligned} y_1 &= \Re x_1 + O\left(\frac{1}{C}\right), \\ y_2 &= \Re x_2 - \Im x_n \frac{\partial A}{\partial y_2}(\Re x_1, \Re x_2) + \text{higher order} + O(\delta) + O\left(\frac{1}{C}\right), \\ y_j &= \Re x_j + O(\delta), \quad 3 \leq j \leq n-1, \\ y_n &= \Re x_n + B(\Re x_1, \Re x_2) + \text{higher order} + O(\delta) + O\left(\frac{1}{C}\right). \end{aligned}$$

Let $\Phi(x)$ be the critical values of $\varphi_1(x, y)$, $y \in \mathbb{R}^n$. By (2.9) and (2.10) for $\Im x_1 = 0$, $x_j = 0$, $j = 3, \dots, n-1$, and $x_n = -i$ we have

$$(2.11) \quad \begin{aligned} \Phi(\Re x_1, x_2, 0, \dots, -i) &= \frac{1}{2} \left(\Im x_2 - \frac{1}{k_2} [\Re x_1]^{k_2} \right)^2 \\ &+ \frac{1}{2} + A(\Re x_1, \Re x_2) + \Im[ax_2^{k/k_2}] \\ &+ \text{higher order terms} + O(\delta) + O\left(\frac{1}{C}\right). \end{aligned}$$

We let

$$(2.12) \quad \Re x_1 = (k_2 \Im x_2)^{1/k_2}.$$

By (2.11) and (2.12) we have

$$(2.13) \quad \begin{aligned} &\Phi([k_2 \Im x_2]^{1/k_2}, x_2, 0, \dots, -i) \\ &= \Phi(x_0) - Q(x_2, \bar{x}_2) + \text{higher order terms} + O(\delta) + O\left(\frac{1}{C}\right), \end{aligned}$$

where

$$Q(x_2, \bar{x}_2) = -A([k_2 \Im x_2]^{1/k_2}, \Re x_2) + \Im[ax_2^{k/k_2}].$$

Then by (2.4) we have

$$(2.14) \quad \begin{aligned} Q(x_2, \bar{x}_2) &= [\Im x_2]^{k/k_2 - \lambda} (d_0 [\Im x_2]^\lambda + d_1 [\Im x_2]^{\lambda-1} \Re x_2 + \dots + d_\lambda [\Re x_2]^\lambda) \\ &+ \Im[ax_2^{k/k_2}]. \end{aligned}$$

We have the following

Lemma 2.2. *Let $Q(x_2, \bar{x}_2)$ be as in (2.14). Then there exists an open sector S in $\mathbb{C}$ with*

$$\text{angle of } S \geq \frac{\pi}{k/k_2} \quad \text{and} \quad Q(x_2, \bar{x}_2) > 0 \text{ on } S$$

except possibly on a ray inside S , where $Q(x_2, \bar{x}_2) = 0$.

Proof. Assume first that k/k_2 is an integer. Then we choose $a \in \mathbb{C}$ such that the polynomial $Q(x_2, \bar{x}_2)$ does not have harmonic terms. Thus

$$Q(x_2, \bar{x}_2) = |x_2|^2 q(x_2, \bar{x}_2), \quad \deg q = k/k_2 - 2.$$

The polynomial $Q(x_2, \bar{x}_2)$ vanishes on at most $k/k_2 - 2$ lines passing through the origin. Therefore there are at most $2(k/k_2 - 2)$ open sectors on each of which Q has the same sign. Since k and k_2 are odd integers, there exists an open sector S such that

$$\text{angle of } S \geq \frac{2\pi}{2(k/k_2 - 2)} > \frac{\pi}{k/k_2} \quad \text{and} \quad Q > 0 \text{ in } S.$$

Next we assume that k/k_2 is not an integer. Then let $a = 0$ to get

$$Q(x_2, \bar{x}_2) = (\Im x_2)^{k/k_2 - \lambda} q_\lambda(x_2, \bar{x}_2),$$

where q_λ is a polynomial of degree λ . In this case we have that q_λ vanishes on at most λ lines passing through the origin. These lines form at most 2λ open sectors on each of which q_λ has the same sign. Taking into account the form of Q and the fact that k_2, k are odd numbers we conclude that there exists an open sector S , which might contain a ray on the line $\Im x_2 = 0$, such that angle of $S \geq 2\pi/2\lambda > \pi/(k/k_2)$ and $Q > 0$ on S except possibly on $\Im x_2 = 0$, where $Q = 0$. This completes the proof of Lemma 2.2.

The following lemma describes a condition on Q which, as we will show later, is a sufficient condition for the operator P to be analytic hypoelliptic at γ . It is an extension of Lemma III.1 in [3]. The proof given here was suggested by J. E. Fornæss.

Lemma 2.3. *Let $Q(x_2, \bar{x}_2)$ be as in (2.14). Then there exist a domain $D \subset \mathbb{C}$, $0 \in D$, and a holomorphic function $f(x_2)$ on D and continuous on $\bar{D}$ with*

$$[Q(x_2, \bar{x}_2) + \Re f(x_2)]_{\partial D} > 0, \quad f(0) = 0.$$

Proof. Let $x_2 = z = x + iy$. If S is as in Lemma 2.2, then we can choose μ such that angle of $S > \pi/\mu > k/k_2$. Then after a rotation we can assume that

$$(2.15) \quad \begin{aligned} Q(z, \bar{z}) &> 0 \quad \text{if } |\arg z| \leq \frac{\pi}{2\mu}, \quad z \neq 0, \\ Q(z, \bar{z}) &\geq 0 \quad \text{if } \arg z = \phi \text{ for some } \phi, \quad |\phi| < \frac{\pi}{2\mu}. \end{aligned}$$

Let

$$G = \{z \in \mathbb{C} : |\arg z| \leq \pi/2\mu, |z| \leq 1\} \cup \{0\}.$$

For $\varepsilon > 0$ we let

$$G_\varepsilon = \{z - \varepsilon : z \in G\}.$$

If ε is small enough then by (2.15)

$$(2.16) \quad Q(z, \bar{z}) \geq 0 \quad \text{on } \{z \in G_\varepsilon : |z + \varepsilon| = 1\}.$$

To make things clear we draw a picture (Figure 1).

The region G is enclosed by the segments O_1A_1 , $O_1A'_1$ and the arc $A_1A'_1$. G_ε is the region enclosed by the segments O_2A_2 , $O_2A'_2$ and the arc $A_2A'_2$. The polynomial $Q(z, \bar{z})$ is nonnegative in the region G' enclosed by the segments O_1A_3 , $O_1A'_3$ and the arc $A_3A'_3$.

Let h be the harmonic extension of $-Q$ from the boundary of G_ε to its interior, i.e., h is the solution to the following Dirichlet problem:

$$\Delta h = 0, \quad h = -Q \text{ on } \partial G_\varepsilon.$$

We shall show that if ε is small enough then

$$(2.17) \quad h(0) < 0.$$

If (2.17) holds then we let $f = u + iv$, where $u = h - h(0)$ and v is the conjugate harmonic function of u with $v(0) = 0$. The function f satisfies

FIGURE 2

Next we map the region G_ε onto the half unit disc by the map $z \mapsto z + \varepsilon$ followed by the map $z \mapsto z^\mu$. This is shown in Figure 2.

The origin is mapped to the point ε^μ , $\mathbf{b}_1$ is mapped onto the segment NN' with $|N| = |N'| = c\varepsilon^\mu$, $\mathbf{b}_2$ is mapped onto the two segments NA and NA' , and $\mathbf{b}_3$ is mapped onto the half unit circle C' .

If we denote the transformed Q again by Q then by (2.18) we have

$$(2.21) \quad |Q(z, \bar{z})| \leq c\varepsilon^{k/k_2} \quad \text{on } NN'.$$

By (2.19) we have

$$(2.22) \quad |Q(z, \bar{z})| \leq c\varepsilon y^{(k/k_2-1)/\mu} \quad \text{and} \quad Q < 0 \quad \text{on } NA \text{ and } N'A'.$$

By (2.20) there exists an arc Γ' of the half circle C' such that

$$(2.23) \quad 0 < c \leq Q(z, \bar{z}) \quad \text{on } \Gamma' \quad \text{and} \quad Q \geq 0 \quad \text{on } C'.$$

We shall again denote by h the harmonic extension of $-Q$ from the boundary of the half unit disc to the half unit disc. To show (2.17) it suffices to show that

$$(2.24) \quad h(\varepsilon^\mu) < 0.$$

To prove (2.24) we map the half unit disc conformally to the unit disc and then we use the Poisson integral

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|\zeta - z|^2} u(\zeta) d\theta, \quad \zeta = e^{i\theta},$$

for the unit disc to estimate $h(\varepsilon^\mu)$.

By (2.23) the contribution I_3 to $h(\varepsilon^\mu)$ due to the values of Q on the half circle C' is

$$(2.25) \quad I_3 = -c\varepsilon^\mu, \quad c > 0.$$

By (2.21) the contribution to $h(\varepsilon^\mu)$ due to the values of Q on the segment NN' is I_1 with

$$(2.26) \quad |I_1| \leq c\varepsilon^{k/k_2}.$$

Finally by (2.22) the contribution I_2 to $h(\varepsilon^\mu)$ due to the values of Q on the segments NA and NA' is

$$|I_2| \leq c \int_{\varepsilon^\mu}^1 \frac{\varepsilon^\mu}{y^2 + \varepsilon^{2\mu}} \varepsilon y^{(k/k_2-1)/\mu} dy \leq c\varepsilon^{1+\mu} \int_{\varepsilon^\mu}^1 y^{(k/k_2-1)/\mu-2} dy.$$

Since μ can be chosen such that $k/k_2 = \mu + \alpha$, with $0 < \alpha < 1$, the last integral is equal to

$$\int_{\varepsilon^\mu}^1 \frac{1}{y^{1+(1-\alpha)/\mu}} dy = -\frac{\mu}{1-\alpha} \left[1 - \frac{1}{\varepsilon^{1-\alpha}} \right].$$

Therefore,

$$(2.27) \quad |I_2| \leq c_1 \varepsilon^{1+\mu} + c_2 \varepsilon^{\mu+\alpha}, \quad 0 < \alpha < 1.$$

By (2.25), (2.26), and (2.27) we see that I_3 is the dominant term for $\varepsilon > 0$ and small enough. Therefore inequality (2.24) holds and this completes the proof of Lemma 2.3.

Lemma 2.3 provides a sufficient condition for the p.d.o. P to be analytic hypoelliptic at γ (see [3, 6]). In the following we give an outline of the proof of this fact.

Let $(x_0, y_0) \in \mathbb{C}^{2n}$ with

$$(2.28) \quad x_0 = y_0 - i\eta_0, \quad (y_0, \eta_0) = \gamma \text{ is given by (2.1).}$$

We can find an elliptic classical analytic symbol a near (x_0, y_0) (see [9]) such that the Fourier-Bros-Iagolnitzer transformation defined by

$$(Tu)(x, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(x, y)} a(x, y, \lambda) \chi(y) u(y) dy, \quad \lambda > 0,$$

with $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ near y_0 , and φ given by (2.7) reduces the operator P , with principal symbol given by (2.3), to complex D_{x_1} ; i.e., there exists an $\varepsilon_1 > 0$ such that

$$(2.29) \quad \left| \left(\frac{1}{\lambda} D_{x_1} T - TP \right) u(x, \lambda) \right| \lesssim e^{\lambda[\Phi(x) - \varepsilon_1]} \quad \text{near } x_0.$$

Now let u be a distribution defined near y_0 and such that Pu is analytic at γ . By (2.29) it follows that there exists $\varepsilon_2 > 0$ such that

$$(2.30) \quad |D_{x_1} Tu(x, \lambda)| \lesssim e^{\lambda[\Phi(x) - \varepsilon_2]} \quad \text{near } x_0.$$

To prove Theorem 1.1 it suffices to show that there exists $\varepsilon_3 > 0$ such that

$$(2.31) \quad |Tu(x, \lambda)| \lesssim e^{\lambda[\Phi(x) - \varepsilon_3]} \quad \text{near } x_0.$$

Relation (2.31) follows from (2.30) and Lemma 2.3. In fact, by (2.13) and Lemma (2.3) we have that there exists $\varepsilon_4 > 0$ such that

$$(2.32) \quad [\Phi([k_2 \Im x_2]^{1/k_2}, x_2, 0, \dots, -i) - \Re f(x_2)]_{\partial D} < \Phi(x_0) - \varepsilon_4,$$

for C large enough and δ small enough. Notice also that for (2.32) to hold we may need to replace D by a dilation of D .

By (2.32) we have

$$(2.33) \quad \Phi([k_2 \Im x_2]^{1/k_2}, x') - \Re f(x_2) < \Phi(x_0) - \varepsilon_4/2$$

if $x_2 \in \partial D$ and $|x'' - x_0''| < r$ for some $r > 0$, where $x'' = (x_3, \dots, x_n)$.

By (2.30) and the fact that $f(0) = 0$ we have

$$(2.34) \quad |D_{x_1}[e^{-\lambda f(x_2)} Tu(x, \lambda)]| \lesssim e^{\lambda[\Phi(x) - \varepsilon_2/2]} \quad \text{near } x_0.$$

By (2.33) we have

$$(2.35) \quad |e^{-\lambda f(x_2)} T([k_2 \Im x_2]^{1/k_2}, x', \lambda)| \lesssim e^{\lambda[\Phi(x_0) - \varepsilon_4/2]}$$

if $x_2 \in \partial D$ and $|x'' - x_0''| < r$.

By (2.34), (2.35), and an integration along x_1 we have that there exists ε_5 such that

$$(2.36) \quad |e^{-\lambda f(x_2)} Tu(0, x', \lambda)| \lesssim e^{\lambda[\Phi(x_0) - \varepsilon_5]},$$

where $x_2 \in \partial D$ and $|x'' - x_0''| < r_1$, for some $r_1 > 0$.

Now we apply the maximum principle in x_2 in D and we use (2.36) to obtain

$$(2.37) \quad |e^{-\lambda f(x_2)} Tu(0, x', \lambda)| \lesssim e^{\lambda[\Phi(x_0) - \varepsilon_5]}, \quad x_2 \in D, \quad |x'' - x_0''| < r_1.$$

By (2.34), (2.37), and an integration along x_1 we get

$$(2.38) \quad |e^{-\lambda f(x_2)} Tu(x, \lambda)| \lesssim e^{\lambda[\Phi(x_0) - \varepsilon_6]} \quad \text{near } x_0 \text{ for some } \varepsilon_6 > 0.$$

Since $f(0) = 0$, relation (2.38) implies that

$$(2.39) \quad |Tu(x, \lambda)| \lesssim e^{\lambda[\Phi(x_0) - \varepsilon_6/2]} \quad \text{near } x_0.$$

By (2.39) we obtain (2.31) which completes the proof of Theorem 1.1.

REFERENCES

1. M. S. Baouendi and L. P. Rothschild, *Cauchy-Riemann functions on manifolds of higher codimension in complex space*, preprint, 1989.
2. ———, *Normal forms for generic manifolds and holomorphic extension of CR functions*, J. Differential Geom. **25** (1987), 431–467.
3. M. S. Baouendi, L. P. Rothschild, and F. Trèves, *CR structures with group action and extendability of CR functions*, Invent. Math. **82** (1985), 359–467.
4. Y. V. Egorov, *Subelliptic pseudo-differential operators*, Dokl. Akad. Nauk SSSR **188** (1969), 20–22.
5. A. A. Himonas, *On analytic microlocal hypoellipticity of linear partial differential operators of principal type*, Comm. Partial Differential Equations **11** (1986), 1539–1574.
6. ———, *Semirigid partial differential operators and microlocal analytic hypoellipticity*, Duke Math. J. **59** (1989), 265–287.
7. L. Hörmander, *Subelliptic operators*, Ann. of Math. Studies, no. 91, Princeton Univ. Press, Princeton, N.J., 1979, pp. 127–208.
8. J. J. Kohn, *Boundary behavior of $\bar{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. **6** (1972), 523–542.
9. J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque 95, Soc. Math. France, 1982.
10. A. E. Tumanov, *Extending CR functions on manifolds of finite type to a wedge*, Mat. Sb. **136** (1988), 128–139. (Russian)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556