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RECURSIVE FUNCTIONALS AND QUANTIFIERS OF FINITE TYPES REVISITED, V

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ABSTRACT. This is the last in a sequence of papers that redoes the theory of recursion in finite types. A key feature of the theory is that a computation can succeed (or finish) even if some of its subcomputations do not, if these turn out to be irrelevant to the total computation. I give a detailed description of computations involving oracles for type 3 functionals. The computation may be viewed formally as a transfinite sequence of symbolic expressions, but I also describe a semantics in which each expression is given a concrete realization.

This is the fifth of the RFQFTR series of papers. The first four are cited in the references by their dates, and comprise the respective sections: 1978, §§1–3; 1980, §§4–8; 1982, §§9–11; 1985, §§12–13.[1]

The reader of this paper is not expected to have read all of the preceding four papers. But a substantial amount of material from them will have to be presupposed.

In 1985 the subject is approached from a different direction than in 1978, 1980, and 1982; and a fairly good overview of the project can be obtained by reading simply 1985.

To begin: §4 gives a general introduction, and §9 carries the development somewhat further and will be needed for what follows (§12 repeats and expands §4).

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[1]Also, in 1980: p. 4, line 2, for "1978" read "1959".

In 1982: p. 27, line 6 from below, the first " τ " should be " $\bar{\tau}$ "; and on p. 37 "Kierstead 1982" should be "Kierstead 1983" (also on pp. 2, 36) and "forthcoming" should be "276, 67–105."

In 1985: p. 129, line 5, after "until" insert "perhaps"; p. 130 in (1), " Ω_μ "; should be " Ω_{u} "; p. 132 in (2c), " $\underline{m}_{\surd}$ " should be " $\underline{m}\surd$ "; p. 133 near the middle, " $\overline{\overline{\Omega}}_\mu$ " should be " $\overline{\overline{\Omega}}_{\mathrm{u}}$ ", and near the bottom there should be a "1" under the slanted line bearing " $\underline{\overline{n}}_2$ "; and p. 135, line 4 from below, before "branches" insert "indexed".

Some clarifications and corrections to 1978 and 1980 are listed in footnote 2 of 1982.

14. Computation admitting type-$\dot{3}$ objects and assumed functions with type-$\dot{2}$ arguments

14.1. The computation rules of §2, as amended in §7, are presupposed here.

As viewed proceeding from left to right (end 2.4, 7.2), each computation (not necessarily completed) is of a 0-expression $\overline{E}$ at its leftmost vertex, under an assignment $\overline{\Omega}$ of worths to the free variables of $\overline{E}$ at least (cf. 8.1, paragraph 2) and to the symbols $\dot{\theta}_{\underline{t}}$ of the presupposed list Θ, with a choice of oracles for the functions in $\overline{\Omega}$ (cf. 7.5). Each vertex after the first is likewise embellished with an E (possibly, when E is a numeral, flagged), Ω, and a choice of oracles.[2]

In E4.2 Subcase $\overline{3 \cdot 3}^2$ (7.3, p. 22), after operating for a while in the first subcomputation to $\dot{\alpha}^2(B)$ of $B(\underline{c})$ under the assignment Ω_{u} (u assigned to $\underline{c}$), we continued (in a certain contingency) under the assignment $\Omega_{\underline{r}_0}$. In such a case, we can alternatively take the interpretation that $\underline{c}$ was assigned $\underline{r}_0$ all along, but that first we operated with no information about the assignment to $\underline{c}$ and later with the total information that its assignment is $\underline{r}_0$. (Similarly with E7 in 7.4.)

The like will happen with the interpretation of γ in certain subcomputations to $\dot{\alpha}^3(B)$ of $B(\gamma)$ in Subcase $\overline{3 \cdot 3}^3$ of the new rule E4.3 of 14.3 (and in the similar case of E7 as extended in 14.4). Thus, we may first assign to γ the worth u^1 with an oracle under Case $\overline{1}^1$ in 6.2, then u^1 with an oracle under Case $\overline{3}^1$,[3] then the worth $\{\langle \underline{r}_0, \underline{n}_0 \rangle\}$, then $\{\langle \underline{r}_0, \underline{n}_0 \rangle, \langle \underline{r}_1, \underline{n}_1 \rangle\}$, etc. It is essential to our purpose there that alternatively we can consider that γ is interpreted by a function $\dot{\alpha}^1$, with an oracle, about which we have increasing amounts of information: at first we know nothing about $\dot{\alpha}^1$ and her oracle, later only that the $\dot{\alpha}^1$-oracle opens envelopes, then further that $\dot{\alpha}^1(\underline{r}_0) = \underline{n}_0$, then also that $\dot{\alpha}^1(\underline{r}_1) = \underline{n}_1$, etc.[4]

It behooves us now to make it explicit that, in using E4.2 Subcase $\overline{3 \cdot 3}^2$ (7.3, p. 22), " $\cdots \dot{\alpha}^2(B)$ " followed by a well-ordered family of subcomputation. " $B(\underline{c}) \cdots \Omega_{\underline{r}_\eta} \cdots N_\eta \dagger$ " ($\eta < \zeta$) with no highest (ζ a limit ordinal) is a computation. (Cf. 13.3e, p. 131.) We can regard this as the union of the family of the computations

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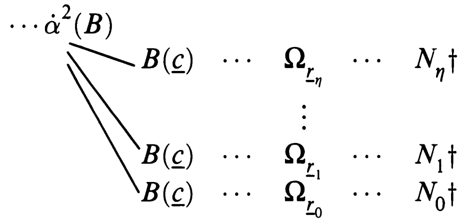
[2] The notation in §§2, 7 is confused between using " E " for the initial 0-expression (here I write that as " $\overline{E}$ ") and for a 0-expression already reached to which we are to make an application of one of our rules E1–E7. Under the definition of completed computations proceeding from right to left (2.4 middle p. 203, 7.1), the E of a rule application was at the same time the leftmost 0-expression, so there was no reason for using different notations.

[3] Note that the statement " $\dot{\alpha}^1 = \mathrm{u}^1$ " (in 6.2, p. 13, line 25 and 7.1, p. 20, lines 3–4) is not equivalent to the Case $\overline{1}^1$ hypothesis, since we can have $\dot{\alpha}^1 = \mathrm{u}^1$ also under Case $\overline{3}^1$ (6.2, next to last paragraph).

[4] This alternative would be impossible if γ were first assigned u^1 under Case $\overline{1}^1$, then u^1 under Case $\overline{3}^1$, then $\{\langle \underline{r}_0, \underline{n} \rangle\}$, then $\{\langle \underline{r}_0, \underline{n} \rangle, \langle \underline{r}_1, \underline{n} \rangle\}$, etc., and finally $\lambda \dot{\alpha}^0 \underline{n}$ (which extends all the preceding worths). For $\lambda \dot{\alpha}^0 \underline{n}$ has an oracle who answers " $\underline{n}$ " without opening the envelope, while the oracles for u^1 under Case $\overline{3}^1$, $\{\langle \underline{r}_0, \underline{n} \rangle\}$, $\{\langle \underline{r}_0, \underline{n} \rangle, \langle \underline{r}_1, \underline{n} \rangle\}$, etc., open envelopes.



one for each $\eta < \zeta$. After accepting the union as a computation, we may be able to take a step, either starting a still higher subcomputation of $B(\underline{c})$ under $\Omega_{\underline{r}_\zeta}$ or putting "$-M\dagger$" at the top (so ζ is the ξ of 7.3, p. 22). Likewise in the similar case of E7.

Here I am assuming that the steps in building the family of computations just described are not interspersed with other steps under Subcase $\overline{3 \cdot 3}^3$ of the new rule E4.3 of 14.3 or similar cases (old or new) of E7. I shall discuss that situation in 14.5.

We do not always take the union of a well-ordered family of computations, each properly extending those which precede, with no last, to be a computation. In computing the function $\phi(\mathfrak{a})$ of (XIV) in 3.1, p. 214, we have such a family whose union is a tree with an infinite branch. All our computations (now and later) will be *well-founded* trees, i.e., ones with only finite branches.[5]

As a computation is constructed, putting vertices in place in a well-ordered order, we may *index* the vertices by ordinals (constituting a segment of Cantor's first and second number classes). Thus, the first vertex (bearing $\overline{E}$ under $\overline{\Omega}$) is indexed by 0, the next one we add by 1,.... If, after vertices indexed by the natural numbers have been put in place, the union of the computations is accepted as a computation, and another vertex is then added, that one is indexed by ω; and so on. The computations thus constructed on the way to and including finally a given computation are *indexed* by the least ordinals greater than the ordinals indexing their vertices. Thus 0 indexes the empty computation (with no vertices), 1 the computation with one vertex (bearing $\overline{E}$ under $\overline{\Omega}$), ω the union of the computations indexed by the natural numbers, $\omega + 1$ the computation obtained from this union by adding one further vertex, and so on. The indices of these computations can also be called the *moments* in the construction of the final computation under consideration. I avoid saying the "stages", as that term will be employed in formulating rule E4.3 in 14.3 for certain nonconsecutive moments in the present sense (adapting to computations under E4.3 its use in 11.2 in articulating the behavior of $\dot{\alpha}^3$-oracles).

14.2. Let E_D be a 0-expression in which one or more free occurrences of a $\underline{j}$-expression D have been specified; call them a *tribe* of D's. E_D can be simply D.

[5]In 13.3e, for simplicity I said that the union of a well-ordered family of computations, each properly extending those which precede, with no last, shall be a computation iff it is well-founded. In the present paper, I instead provide for the union of such a family to be a computation in certain specific cases, and thence prove (for (XXV)(i) and (iii) in 15.2) that the union will be a computation iff it is well-founded.

Suppose we have an assignment Ω to, or are given information about the worths of, at least the variables occurring free in E_D and the $\dot{\theta}_{\underline{t}}$'s, under which computation starting or continuing rightward from E_D is carried out. (Here we are not interested in the case E_D is a flagged numeral, from which no computation can start or continue.)

We shall identify certain free occurrences of D in that computation as *descendants* of the specified occurrences in E_D. The tribe in E_D are the "0th generation" or "improper" descendants. For color, say the specified occurrences of D in E_D are printed in red, the other symbols in black.

As we carry out steps by E1–E7 in 2.4, 7.1–7.4, and 14.3, 14.4, the red may be propagated horizontally-rightward or downward-rightward along the branches of the tree being constructed. It should suffice to give a variety of illustrations.

If we have reached a vertex bearing $\phi_{\underline{i}}(A, B)$ (not itself a red occurrence of D) with some red parts in A and/or B, where $\phi_{\underline{i}}(\underline{a}, \underline{b}) = \underline{a}$ by S3, then a step by E2 takes us from $\phi_{\underline{i}}(A, B)$ to A. Any red in the shown A of $\phi_{\underline{i}}(A, B)$ carries into red in the resulting A. If instead $\phi_{\underline{i}}(\underline{a}, \underline{b}) = \phi_{\underline{j}}(\phi_{\underline{k}}(\underline{a}, \underline{b}), \underline{a}, \underline{b})$ by S4.0, then in the result of the E2 step there will be twice as many red parts D as in $\phi_{\underline{i}}(A, B)$.

A step by E4.λ from $\{\lambda\alpha^{\underline{j}} A\}(B)$ to $\underline{S}_B^{\alpha^{\underline{j}}} A|$ would not be performed without first changing bound variables inside A if necessary to make B free at the positions of all the free $\alpha^{\underline{j}}$'s. So red parts in B will give rise to red parts in each B entering $\underline{S}_B^{\alpha^{\underline{j}}} A|$ by the substitution for $\alpha^{\underline{j}}$ in A (none if there are no free $\alpha^{\underline{j}}$'s in A). Suppose $\lambda\alpha^{\underline{j}} A$ is not itself a red occurrence of D. If there are red parts in A, they cannot contain $\alpha^{\underline{j}}$ free (since the red occurrences of D in E_D, and all their descendants down to any generation to which descendants survive along any branch, are free parts of the whole 0-expressions at their vertices). If we have first to change bound variables in A to make the substitution of B free, none of these changes will be in red parts or cost the freedom of red parts. For, any $\lambda\beta^{\underline{k}}$ inside a red D would not need to be changed, because no free $\alpha^{\underline{j}}$ of A is in its scope; and if a $\lambda\beta^{\underline{k}}$ needing to be changed had a red D in its scope, $\beta^{\underline{k}}$ would not occur free in that red D and could only be changed to a $\overline{\beta}^{\underline{k}}$ not occurring free in D. So red D's in A will also give rise to red D's in $\underline{S}_B^{\alpha^{\underline{j}}} A|$.

A step by E4.$\underline{j}$ ($\underline{j} = 1, 2$, or 3) from $\dot{\alpha}^{\underline{j}}(B)$ in 7.1–7.3 and 14.3 for which there are lower next vertices (starting subcomputations) carries red forward from the B of $\dot{\alpha}^{\underline{j}}(B)$ into the B of each lower next vertex. Similarly with E7 in 7.4 and 14.4.

Red applies only to whole free occurrences of D. If D is dissected or altered (processed) by a step, the remains of D at next vertices are black.

Along each branch, red occurs consecutively from E_D at the vertex from which we began or continued the computation up to some last vertex having red, not bearing a flagged numeral unless D is simply 0 (nor always then). A flagged numeral can result only from either E5 or a step that does not carry red forward to it. (E6.$'$ and E6.$-$, even when A is a numeral, carry red forward only from in the A of A' or of $A-1$ to the lower next vertex.)

In an obvious way, each descendant in any formula of the tree can be identified as a descendant of a particular one of the red D's in E_D or of the red D's in any intervening formula along the path from E_D to the 0-expression in hand. That one is its *ancestor* in E_D or in the intervening formula.

If D is a red 0-expression and is the whole expression at its vertex, I say that D has *surfaced* there. If D is a $\underline{j}$-expression with $\underline{j} = 1, 2$, or 3, I say that D has *surfaced* at a vertex if the whole expression there is $D(B)$ (for some $(\underline{j}-1)$-expression B) with the initial D red. The occurrence of a 0-expression at a vertex where D surfaces I call a *D-face.*

14.3. I am now ready to formulate rule E4.3 for continuing computation rightward from a 0-expression E of the form $\dot{\alpha}^3(B)$, under an assignment Ω, with specific oracles for the functions in it, including one, the "$\dot{\alpha}^3$-oracle," of the sort described in 11.2 for the type-$\dot{3}$ function $\dot{\alpha}^3$ interpreting the variable $\dot{\alpha}^3$.

Persons who have read §13 will be able to recognize that what happens, in particular the final result $\underline{s}$ (if reached) and the series of stage charts in the principal case, is the same as in questioning the $\dot{\alpha}^3$-oracle under 11.2 with the $\dot{\alpha}^2$-oracle who is abstracted in 13.3b from the analysis in 13.3a of the computation of E_{γ^1} under $\Omega_{\dot{\alpha}^1}$ ($\dot{\alpha}^1$ interpreting γ^1 with a given $\dot{\alpha}^1$-oracle) when E_{γ^1}, γ^1 are the present $B(\gamma)$, γ.[6] This $\dot{\alpha}^2$-oracle is an oracle for the type-$\dot{2}$ interpretation of $\lambda\gamma B(\gamma)$ under #A in 8.1.

Case $\overline{1}^3$. The $\dot{\alpha}^3$-oracle, presented with any envelope, stands mute ($\dot{\alpha}^3 = \lambda\dot{\alpha}^2\mathrm{u}$). No further step can be taken after reaching $\dot{\alpha}^3(B)$.

Case $\overline{2}^3$. The $\dot{\alpha}^3$-oracle, without opening the envelope, declares that $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$ ($\dot{\alpha}^3 = \lambda\dot{\alpha}^2\underline{s}$). Our computation can be extended from $\dot{\alpha}^3(B)$ thus:

$$\cdots \dot{\alpha}^3(B) \text{---} S\,\dagger\,.$$

Case $\overline{3}^3$. The $\dot{\alpha}^3$-oracle opens envelopes. A subcomputation of $B(\gamma)$ (γ a variable not occurring free in B and not interpreted in Ω),

$$\cdots \quad \dot{\alpha}^3(B)$$
$$\searrow B(\gamma),$$

[6] This observation disposes of an objection that might otherwise trouble us in the treatment below. We assume that our $\dot{\alpha}^3$-oracle has been trained by Apollo in how to respond to the situations that could arise in her being questioned with one or another $\dot{\alpha}^2$-oracle, as described in 11.2. Confronted by a chart (or, in a limiting situation, by a union of charts) which could not so arise, she would be in a quandary ("Apollo didn't tell me what to do now!"). But all the charts she will face under the principal case below are charts that arise in her being questioned with the aforesaid $\dot{\alpha}^2$-oracle.

is started with no information provided about the interpretation $\dot\alpha^1$ of γ (equivalently, interpreting γ by $\dot\alpha^1 = \mathrm{u}^1$ with an oracle under Case $\overline{1}^1$ of 6.2, who does nothing.)[3]

Subcase $\overline{3\cdot 1}^3$. Neither of the following (Subcase $\overline{3\cdot 2}^3$ or $\overline{3\cdot 3}^3$) happens. Then no step can be taken horizontally-rightward from $\dot\alpha^3(B)$.

Subcase $\overline{3\cdot 2}^3$. Without γ surfacing, the subcomputation can be completed to give $\underline{m}$ as the value of $B(\gamma)$. According as $\dot\alpha^3(\lambda\dot\alpha^1\underline{m}) = \mathrm{u}$ or $\dot\alpha^3(\lambda\dot\alpha^1\underline{m}) = \underline{s}$ ($\underline{s}$ a natural number), no step can be taken horizontally-rightward from $\dot\alpha^3(B)$ or the step "— $S\dagger$" is taken thus:

$$\cdots\dot\alpha^3(B) \longrightarrow S\dagger, \quad \searrow B(\gamma)\cdots M\dagger$$

Subcase $\overline{3\cdot 3}^3$ (principal case). In the subcomputation of $B(\gamma)$, we come to a γ-face $\gamma(C)$. Now if our $\dot\alpha^3$-oracle clams up on finding that an $\dot\alpha^2$-oracle with whom she is questioned opens envelopes, no further step can be taken (indeed, by E4.1 Case $\overline{1}^1$ in 7.1). But if in that circumstance she embarks on further systematic questioning of the $\dot\alpha^2$-oracle, she now unrolls a blank chart and does one of two things (fixed for her in this subcase) aiming at Stage 0.

Option 1. The $\dot\alpha^3$-oracle supplies the new information that $\dot\alpha^1 = \lambda\dot\alpha^0\underline{n}$ (or replaces $\dot\alpha^1 = \mathrm{u}^1$ under Case $\overline{1}^1$ by $\dot\alpha^1 = \lambda\dot\alpha^0\underline{n}$) for some natural number $\underline{n}$. So by E4.1 Case $\overline{2}^1$ the subcomputation of $B(\gamma)$ can thereupon be continued from the γ-face $\gamma(C)$ by "— $N\dagger$", and likewise (with the same $\underline{n}$) from each subsequent γ-face. Eventually the subcomputation of $B(\gamma)$ thus pursued may be completed with the value $\underline{m}$. Iff so, Stage 0 is reached under Option 1 with the branch

$$\mathrm{u}?\overline{\underline{\dot\alpha^1 \text{ not opening says } \underline{n}}}\,\underline{m}\surd$$

entered on the chart and indexed by 0.

Option 2. The $\dot\alpha^3$-oracle supplies the new information that the oracle for the interpretation $\dot\alpha^1$ of γ opens envelopes (or changes $\dot\alpha^1 = \mathrm{u}^1$ under Case $\overline{1}^1$ to $\dot\alpha^1 = \mathrm{u}^1$ under Case $\overline{3}^1$). So by E4.1 Case $\overline{3}^1$ a subcomputation of C to $\gamma(C)$ is started,

$$\gamma(C) \searrow C,$$

and similarly upon reaching further γ-faces. Iff it eventually happens that the C_0 downward-rightward from a γ-face $\gamma(C_0)$ is computed with the result $\underline{r}_0$, Stage 0 is reached under Option 2 with the chart branch

$$\mathrm{u}?\underline{\;\dot\alpha^1 \text{ opens}\;}\underline{r}_0?.$$

Continuation of the subcomputation of $B(\gamma)$ is blocked for the present (under E4.1 Case $\overline{3}^1$) for want of knowledge of a value of $\dot\alpha^1(\underline{r}_0)$.

Suppose a stage σ (≥ 0) has been reached in the computation and the chart. To each branch of the chart (an initial subtree of an $\dot\alpha^2$-oracle tree, after 10.1

and 11.2, p. 24) which is indexed (say by κ), there is a corresponding subcomputation of $B(\gamma)$. For a branch of the first kind (cf. 11.2),

$$\mathrm{u}?\frac{\dot{\alpha}^1 \text{ not opening says } \underline{n}_\kappa}{}\underline{m}_\kappa\surd,$$

the corresponding subcomputation was carried out (under Option 1) using $\lambda\dot{\alpha}^1\underline{n}_\kappa$ (for some $\underline{n}_\kappa \in \underline{N}^0$) as the interpretation of γ and ended with a flagged numberal $M\dagger_\kappa$ in the principal branch from $B(\gamma)$. For a branch of the second kind with more than one segment (with just one segment, the story is as I just told it for Option 2 toward Stage 0),

$$\mathrm{u}?\frac{\dot{\alpha}^1 \text{ opens}}{}\underline{r}_{\kappa 0}?\frac{\underline{n}_{\kappa 0}}{}\cdots\underline{r}_{\kappa\eta}?\frac{\underline{n}_{\kappa\eta}}{}\cdots(\eta < \zeta_\kappa)\underline{m}_\kappa\surd \text{ or } \underline{r}_{\kappa\zeta_\kappa}?$$

where $\underline{r}_{\kappa 0} = \underline{r}_0$ and $\zeta_\kappa > 0$, what happened in the corresponding subcomputation of $B(\gamma)$ is this. There were γ-faces $\gamma(C_{\kappa\eta})$, to each of which the subcomputation of $C_{\kappa\eta}$ led to a different $\underline{r}_{\kappa\eta}$ (besides maybe some γ-faces leading to $\underline{r}$'s which had already occurred), and for each the $\dot{\alpha}^3$-oracle eventually supplied the information that $\dot{\alpha}^1(\underline{r}_{\kappa\eta}) = \underline{n}_{\kappa\eta}$, so that the computation could be continued under E4.1 Case $\overline{3}^1$. After all of these events, either the subcomputation was completed with the value $\underline{m}_\kappa$ for $B(\gamma)$ (with " $\underline{m}_\kappa\surd$ " ending the chart branch) or we came to a new γ-face $\gamma(C_{\kappa\zeta_\kappa})$ for which $C_{\kappa\zeta_\kappa}$ received a new value $\underline{r}_{\kappa\zeta_\kappa}$ at which the computation was blocked (and the chart branch was then ended with " $\underline{r}_{\kappa\zeta_\kappa}$? "). If ζ_κ is infinite, there will be preceding each limit ordinal $\geq \zeta_\kappa$ a unionization of segment computations as will be explained below under Option 5 and for σ a limit ordinal.

Contemplating the Stage-σ chart, which alone determines her next action,[7] the $\dot{\alpha}^3$-oracle may shut the computation down (uncompleted); or she may declare that it justifies pronouncing (in what we call the *final* stage $\sigma+1$; cf. 11.2, p. 29) that $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$, whereupon the E4.3 application is completed with "— $S\dagger$ " added rightward from $\dot{\alpha}^3(B)$ (this only if all the indexed chart brachnes at Stage σ end with an " $\underline{m}_\kappa\surd$ "); or she may choose to aim for a Stage $\sigma + 1$ under one of Options 1–5.

Under *Option* 1, she starts a new subcomputation of $B(\gamma)$ just above those present at Stage σ, to correspond to a chart branch indexed by the next unused ordinal $\overline{\kappa}$, with γ interpreted by $\lambda\dot{\alpha}^0\underline{n}_{\overline{\kappa}}$ (with $\underline{n}_{\overline{\kappa}}$ different than in any Stage-σ branch of the first kind). Stage $\sigma+1$ is reached iff this leads to a value $\underline{m}_{\overline{\kappa}}$ for $B(\gamma)$.

Option 2 she may use only if it was not previously used; and the story then is as I told it above for the approach to Stage 0 (γ being interpreted by u^1 with

[7] An illustration of a Stage-6 chart is shown in 1985 13.4a, p. 133, Figure 1. Each indexed branch of chart shows its index on its last (or all sufficiently far out) line segments, or, if it just resulted from Option 5 below (ζ_κ infinite), on its newly added vertex. (In Figure 1, "1" should appear also under the segment from " $\underline{r}_2$?" to " $\overline{r}_3$?".)

an envelope-opening oracle), but for a next higher subcomputation of $B(\gamma)$ (corresponding to a chart branch indexed by $\overline{\kappa}$).

Under *Option* 3, she picks a Stage-σ chart branch of the second kind ending with "$\underline{r}_{\kappa\zeta_\kappa}$?" and unblocks the corresponding subcomputation of $B(\gamma)$ by supplying the new information that $\dot{\alpha}^1(\underline{r}_{\kappa\zeta_\kappa}) = \underline{n}_{\kappa\zeta_\kappa}$ ($\dot{\alpha}^1$ interpreting γ). Continuing that subcomputation, we may or may not reach a value (with "$\underline{m}_\kappa\surd$" going on the chart) or a new blockage ("$\underline{r}_{\kappa,\zeta_\kappa+1}$").

Under *Option* 4, she starts a next higher subcomputation of $B(\gamma)$ under $\dot{\alpha}^1 = \{\langle\underline{r}_\eta, \underline{n}_\eta\rangle \mid \eta < \theta\} \cup \{\langle\underline{r}_\theta, \overline{\underline{n}}_\theta\rangle\}$ as the interpretation of γ, where $\{\langle\underline{r}_\eta, \underline{n}_\eta\rangle \mid \eta < \theta\}$ is represented by proper initial segments of one or more Stage-σ chart branches (to which correspond identical proper initial segments of the Stage-σ computations corresponding to the branches), and $\overline{\underline{n}}_\theta$ is different from the $\underline{n}_\theta$ of the next argument-value pair $\langle\underline{r}_\theta, \underline{n}_\theta\rangle$ in each of them.[8] This new subcomputation will agree with the segments of the preexisting subcomputations in question up to the stage for each at which a value was needed for $\dot{\alpha}^1(\underline{r}_\theta)$, and thereafter will differ from them, leading or not leading to Stage $\sigma+1$ in either of two ways with a new chart branch indexed by $\overline{\kappa}$.

Under *Option* 5, she picks an (unindexed) branch with no last vertex in the Stage-σ chart, representing $\{\langle\underline{r}_\eta, \underline{n}_\eta\rangle \mid \eta < \zeta\}$ (ζ a limit ordinal). Every segment of it will belong to one or more indexed branches, each of which eventually diverges from it. (How such a branch could have arisen when σ is infinite is illustrated in 11.2, p. 28, Figure 4 or 13.4a, p. 135, Figure 2). So there must be identifiable (as initial segments of various Stage-σ subcomputations of $B(\gamma)$) subcomputations corresponding to each initial segment of $\{\langle\underline{r}_\eta, \underline{n}_\eta\rangle \mid \eta < \zeta\}$. The $\dot{\alpha}^3$-oracle now authorizes a next higher subcomputation of $B(\gamma)$ under $\dot{\alpha}^1 = \{\langle\underline{r}_\eta, \underline{n}_\eta\rangle \mid \eta < \zeta\}$ as the interpretation of γ.[8] This runs through the aforesaid segments, each extending its predecessors, without stopping after any one of them. The union of all these segment computations may or may not be a well-founded tree. Iff it is, the union of the whole computations obtained by pursuing the new subcomputation through all these segments is accepted as a computation. Then we may or may not be able to continue the new subcomputation so as to reach Stage $\sigma+1$ in either of two ways.

Suppose the computation has been carried through a succession of stages $\tau <$ a limit ordinal σ.

Consider the union, for all the stages $\tau < \sigma$, of the computations and the charts.

[8]Here, and similarly under Option 5, the information available about the interpretation of the variables of $B(\gamma)$ other than γ at the outset of the new subcomputation shall be what was used up through Stage σ of our E4.3 application. If, e.g., our $\dot{\alpha}^3(B)$ stands within a subcomputation of $\overline{B}(\overline{\gamma})$ to the $\overline{\alpha}^3(\overline{B})$ for an application of the principal case (Subcase $\overline{3\cdot 3^3}$) of E4.3 (or the like with E7) further to the left, information about the interpretation of $\overline{\gamma}$ may have progressively increased in the course of pursuing our application of E4.3 to $\dot{\alpha}^3(B)$. At Stage σ it will include the information about $\overline{\gamma}$ that was used in the aforesaid segments of Stage-σ subcomputations of $B(\gamma)$. The interpretations of the Θ are fixed from the outset.

Consider a given *aspiring* branch (if any) of the unionized chart, i.e., a union of branches established at various stages $\tau < \sigma$ with $\lim \tau = \sigma$ all having the same index κ; cf. 11.2, p. 27 (on line 6 from below, the first “ τ ” should be “ $\overline{\tau}$ ”). The union of the subcomputations of $B(\gamma)$ (each properly or improperly extending the preceding ones) which correspond to the stages τ of that chart branch may or may not be a well-founded tree. Iff it is for each aspiring branch (in particular, if there are no aspiring branches), then the union of the whole computations (for all $\tau < \sigma$) is accepted as a computation. Suppose this is the case.

Now the $\dot{\alpha}^3$-oracle, contemplating the unionized chart,[6] may stand mute. Or she may opt to seek a Stage σ. Suppose the latter.

If there are no aspiring branches in the unionized chart, the union of the Stage-τ computations is already the Stage-σ computation, and the just-unionized chart is the Stage-σ chart. If there are aspiring branches, she authorizes computation to be pursued in the respective subcomputations corresponding to the aspiring branches (just as in the one subcomputation after the unionization under Option 5), in the order of their indices κ from the lowest upward, hoping to *finish* each, i.e., to bring it either to completion (with an “ $\underline{m}\surd$ ” being added at the end of the respective aspiring chart branch) or to a blockage (with an “ $\underline{r}_\zeta$?” added). If for some aspiring branch one of these outcomes cannot be obtained, the computation for our E4.3 step ends with the failed attempt for that subcomputation. If we succeed in finishing all of the subcomputations corresponding to a family of aspiring branches with no highest, the union of the resulting whole computations beginning with “ $\cdots \dot{\alpha}^3(B)$ ” is accepted as a computation, either being the limit stage σ (if there are no higher aspiring branches) or setting us up to continue the subcomputation corresponding to the next higher aspiring branch. If eventually each aspiring branch of the unionized chart acquires either an “ $\underline{m}\surd$ ” or an “ $\underline{r}_\zeta$?” at its end, we have arrived at the limit stage σ of the chart and the computation.

As already stated, at some stage σ the $\dot{\alpha}^3$-oracle may have a Stage-σ chart meeting her demands for passing to her final stage as Stage $\sigma + 1$. The completion of the E4.3 step will thus give

$$\begin{array}{llll} \cdots \quad \dot{\alpha}^3(B) & \text{———} & S\dagger & \\ & \searrow & \vdots & \\ & & B(\gamma) \cdots \Omega_{\dot{\rho}_\kappa} \cdots M_\kappa \dagger & \quad (\kappa < \mu) \\ & & \vdots & \end{array}$$

where the functions $\dot{\rho}_\kappa$ are those finally represented by the chart branches (11.2 just before (1) on p. 29).

14.4. Now take E7 allowing for type-$\dot{2}$ arguments. For illustration:

E7: E is $\dot{\theta}_{\underline{t}}(A, B_1, B_2, C)$ where A, B_1, B_2, C are 0-, 1-, 1-, 2-expressions, respectively.

Case $\overline{1}^{(0:0,1,1,2)}$ and $\overline{2}^{(0:0,1,1,2)}$ (cf. 6.4, 7.4, 11.3) are obvious. In Case $\overline{3}^{(:0,1,1,2)}$ we have subcomputations started, and extended, corresponding to the steps our $\dot{\theta}_{\underline{t}}$-oracle takes when questioned with given $\underline{\dot{a}}$, $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\alpha}^2$ with particular oracles for $\dot{\alpha}_1$, $\dot{\alpha}_2$, $\dot{\alpha}^2$. The $\dot{\theta}_{\underline{t}}$-oracle keeps a chart if she has started a subcomputation of $C(\gamma)$ and γ surfaces in that, and she chooses not to give up then. We operate as under E4.1, E4.2, and E4.3, interpreting the computational behavior of A, B_1, B_2, C as giving information about $\underline{\dot{a}}$, $\dot{\alpha}_1$-, $\dot{\alpha}_2$-, and $\dot{\alpha}^2$-oracles for the use of our $\dot{\theta}$-oracle. The treatment here is related to that in 11.3 as 14.3 is to 11.2.

14.5. I return now to the discussion, begun in 14.1, paragraphs 5 and 6, of the approach to a limit ordinal ζ in using E4.2 Subcase $\overline{3\cdot 3^2}$. Suppose an application of the principal case (Subcase $\overline{3\cdot 3^3}$) of E4.3 is under way on, say, $\dot{\alpha}^3(\overline{B})$ at a vertex to the left of the $\dot{\alpha}^2(B)$ for our application of E4.2 Subcase $\overline{3\cdot 3^2}$. It could then happen that what is represented by "$\cdots\dot{\alpha}^2(B)$" is not fixed as η increases toward ζ owing to the computation under the E4.3 application being switched from pursuing the subcomputation to $\bar{\dot{\alpha}}^3(\overline{B})$ of $\overline{B}(\bar{\gamma})$ containing our $\dot{\alpha}^2(B)$ to some other subcomputation of $\overline{B}(\bar{\gamma})$, and later back again, maybe several times.

The like could happen with E7 instead of E4.3, not only as extended in 14.4 but also as in 7.4; cf. p. 23, lines 4–6 after the first display.

The like may obtain for the approaches of the stages τ toward a limit ordinal σ in an application of the principal case for an $\dot{\alpha}^3(B)$ under E4.3 (or the like under E7) with respect to an application of the principal case on an $\bar{\dot{\alpha}}^3(\overline{B})$ (or the like for E7) at a vertex to the left of that.[9]

Now the moments of completion of the subcomputations in our application of E4.2 to $\dot{\alpha}^2(B)$, or of the stages in our application of E4.3 to $\dot{\alpha}^3(B)$ (or the like with E7), may be an increasing sequence of ordinals that is cofinal with the moments in the approach to the limit ordinal for the stages in the application of E4.3 to $\bar{\dot{\alpha}}^3(\overline{B})$ further to the left (or the like with E7). In that case our application is *subordinate* to the one further to the left. We say it is *autonymous* if it is subordinate to none further to the left.

In the case of an autonymous application of E4.2 or E4.3 or E7, what the "$\cdots\dot{\alpha}^2(B)$" or "$\cdots\dot{\alpha}^3(B)$" or, e.g., "$\cdots\dot{\theta}_{\underline{t}}(A, B_1, B_2, C)$" represents is fixed after some moment, and we apply our rule for passing to the union in this more general case, as well as in the case considered for E4.2 in 14.1 (or the like considered implicitly for E4.3 in 14.3, or for E7 in 14.4).

If applications of E4.2, E4.3, or E7 are subordinate to applications of E4.3 or E7 to the left, they will all be subordinate to a leftmost such application, which

[9]After passing to the union when it is allowed, the subcomputations corresponding to the aspiring branches, and likewise later under Option 5 for an unaspiring branch, are not interrupted by appeals to oracles associated with E4.3 or E7 applications further to the left, since all the information called for was made available earlier in the segment computations preceding the unionization. [8]

is autonymous. The unionization, if allowed for that, will automatically effect the unionization of all the applications subordinate to it.

15. Some properties of computations

15.1.

(XXIV) *For the case the* B *of* E4.3 *in* 14.3 *is simply a type-* $\dot{2}$ *variable* $\dot{\alpha}^2$, $\dot{\alpha}^3(\dot{\alpha}^2)$ *computed under an assignment* Ω *with a choice of oracles for the functions interpreting* $\dot{\alpha}^3$ *and* $\dot{\alpha}^2$ *gives* "— S†" (S *the numeral for* S) *in the principal branch from* $\dot{\alpha}^3(\dot{\alpha}^2)$ *if and only if the* $\dot{\alpha}^3$*-oracle questioned with the* $\dot{\alpha}^2$*-oracle answers* $\underline{s}$ *under* 11.2.

Similarly for E4.1, E4.2, *and* E7.

The proof is straightforward.

Remark. (XXIV) holds whether the $\dot{\alpha}^3(\dot{\alpha}^2)$ begins the computation or occurs in a larger computation. The circumstances that in a larger computation might place restraints on pursuing the subcomputation of a 0-expression D surfacing in it do not apply if D is of the form $\dot{\alpha}^3(\dot{\alpha}^2)$. See the remark following the proof of (XXV)(iv) in 15.2.

15.2. Some of the properties of computations collected in (XXV) we have already been taking more or less for granted.

The statements of the rules E1–E7 (as given originally in 2.4 and subsequently modified and elaborated in 7.1–7.4, 14.1, 14.3–14.5), taken together with the understanding that only these rules allow acts in constructing computations, constitute an inductive definition of what are computations (each of a 0-expression $\overline{E}$ under an assignment $\overline{\Omega}$ with a choice of oracles, 14.1).

These acts, occurring in a well-ordered order, consist in supplying the initial vertex (bearing $\overline{E}$ under $\overline{\Omega}$ with a choice of oracles), in adding a line segment and vertex (which we call a *step*), and in taking the union of a family of computations, each properly extending the preceding ones, with no last. (If a vertex bearing 0 is supplied, the flagging of the 0 by E5 is considered part of the same act.)

Until we shall have established (v) of (XXV), we will not know that, for a given $\overline{E}$, $\overline{\Omega}$, and choice of oracles, the sequence of the acts in constructing a computation is uniquely determined. But, anyway, we know that, if we have a computation before us, it was constructed by a certain sequence of acts, each legitimated by our rules E1–E7. We can use the well-ordered series of the moments (end 14.1) in this sequence of acts in proving by induction properties of computations before we have established (v).

When we talk about a well-ordered family of computations each properly extending the preceding ones with no last (as in 14.1 where we discuss E4.2 Subcase $\overline{3 \cdot 3}^2$ and more generally in (XXV)(iii) below), we shall understand that the members of the family are reached through increasing moments (not necessarily consecutive) in this series; and of course the union of the family is the same as the union of the computations as they stood at the consecutive moments μ (end 14.1), μ ranging higher and higher as we go from one member of the family to another extending it.

A step which for the first time from a given vertex adds a (line segment and) vertex downward-rightward from it I say *suspends* the given vertex, which remains *suspended* until it becomes unsuspended by a step horizontally-rightward from it.

In a given computation or "computation tree" (not necessarily completed) with $\overline{E}$ at the leftmost vertex (call it the *parent* tree), consider another vertex, bearing say D, determining *the subtree issuing from it*, consisting of all the vertices (together with their embellishments) at or to the right of it.[10] Let us take a copy of this subtree standing by itself. I shall say that in the copy we have *totaled* the assignments (*with respect to* the parent tree) if we have extended them (if necessary) from what they were in the subtree as contained in the parent tree so that the following is the case: For each $B(\gamma)$ beginning a subcomputation in the parent tree to an $\dot{\alpha}^3(B)$ for E4.3 Subcase $\overline{3 \cdot 3}^3$ to the left of D whose γ has descendants in the subtree (and thus in D), all the information about the interpretation of those descendants which in the parent tree was made available anywhere in the subtree is made available from the outset in the copy of the subtree considered by itself (for γ as a free variable of D). This results in an assignment to the γ because, as E4.3 is formulated, each time we change the assignment the change is compatible with the interpretation that simply new information is being supplied about the interpretation $\dot{\alpha}^1$ of γ and its oracle (cf. 14.1, paragraph 4).[4] Similarly with the $B(\underline{c})$ beginning the first subcomputation to an $\dot{\alpha}^2(B)$ for an application of E4.2 Subcase $\overline{3 \cdot 3}^2$ in the parent tree to the left of D, if in the parent tree a descendant of its $\underline{c}$ receives a value $\underline{r}$ somewhere in the subtree (so $\underline{c}$ occurs free in D), in the copy of the subtree it receives it from the outset. Likewise with applications of E7.

It may be observed that in formulating Option 4 under E4.3 Subcase $\overline{3 \cdot 3}^3$ in 14.3, we could have begun the new subcomputation by taking any one of certain identical preexisting subcomputations with the assignments totaled in the copy with respect to the whole computation as it stood at the stage when that segment subcomputation had just been finished. Similarly (but with a succession of increasing segments) with Option 5.

(XXV)(i) *Each computation is a well-founded tree.*

(ii) *If in a computation, the (part of) a branch running horizontally-rightward from a vertex bearing say F ends in a flagged numeral $E\dagger$, then, in the subtree issuing from our vertex bearing F, each branch ends in a flagged numeral and there are no suspended vertices.*

[10] Two places where we consider subtrees of a given tree which subtrees do not comprise all the vertices in the given tree at or to the right of a certain vertex in the given tree are in 11.2, where we considered "initial subtrees" of an $\dot{\alpha}^2$-oracle tree (10.1), and in the proof of (XXV)(iii) below, where we shall consider the "blued subtree" of a given computation tree.

(iii) *Whenever the union of a well-ordered family of computations, each properly extending the preceding ones, with no last, is well-founded, it is a computation.*

(iv) *In a given computation tree for* $\overline{E}$, *each proper subtree issuing from a vertex, except a single vertex bearing a flagged numeral other than* 0, *when taken by itself with the assignments totaled in it with respect to the whole tree, constitutes a computation.*

(v) *After each moment* > 0 *in constructing a computation, at most one next step is possible. Hence, given* $\overline{E}$ *and* $\overline{\Omega}$ *and a choice of oracles, the computation of* $\overline{E}$ *under* $\overline{\Omega}$ *with that choice of oracles is uniquely determined up through each moment which can be reached.*

(i) The proof will be by induction on a construction of the computation (on the moments in its construction if we please).

The induction step is trivial except for the cases involving the passage to the union of a family of computations, each extending the preceding ones, with no last.

When the passage to the union is within the principal case of an application of E4.2 or E4.3 or the like with E7, one such application further to the left will be autonymous.

If the autonymous application is of E4.2, we first apply the hypothesis of the induction (hyp. ind.) to the computation as it stands for any $\overline{\eta} < \zeta$ such that "$\cdots\dot{\alpha}^2(B)$" remains fixed for all η with $\overline{\eta} \geq \eta < \zeta$ ($\overline{\eta} = 0$ in 14.1). Such an $\overline{\eta}$ must exist, being the maximum of ones obtained with respect to each of the finitely many applications of the principal case of E4.3 or E7 applications further to the left. So all branches diverging before reaching $\dot{\alpha}^2(B)$ are finite, as well as all those in the subcomputations "$B(\underline{c})]\cdots\Omega_{\underline{r}_\eta}\cdots N_\eta\dagger$" for $\eta \geq \overline{\eta}$. And as η increases above $\overline{\eta}$ toward ζ, we only add more subcomputations, each completed in turn and well-founded by the hyp. ind. applied to the computation as it stands for the respective η.

Now consider an autonymous approach to the union of Stages $\tau <$ a limit stage σ in using E4.3 Subcase $\overline{3 \cdot 3}^3$. Because it is autonymous, the "$\cdots\dot{\alpha}^3(B)$" will remain fixed after some stage $\overline{\tau} < \sigma$, and the application of the hyp. ind. to the moment Stage $\overline{\tau}$ was reached gives the finiteness of all branches diverging from "$\cdots\dot{\alpha}^3(B)$" before reaching $\dot{\alpha}^3(B)$. For each branch in the unionized chart which, with its corresponding subcomputation to $\dot{\alpha}^3(B)$, was established at some stage $\tau < \sigma$, the subcomputation is well-founded by the hyp. ind. applied to that stage τ. For each of the other branches in the unionized chart (if any), i.e., the aspiring branches, it was a condition for accepting the union of the whole Stage-τ computaions as a computation that the union of the stages in the respective subcomputation be well-founded.

Next consider an approach to a union through finishing the subcomputations of $B(\gamma)$ corresponding to each of a family of aspiring branches with no highest.

Each of the branches of the whole tree diverging before the vertex bearing our $\dot{\alpha}^3(B)$, as well as the subcomputations of $B(\gamma)$ to our $\dot{\alpha}^3(B)$, except those of our family, is well-founded by the hyp. ind. applied at the moment just before we began the finishing of the subcomputations of $B(\gamma)$ of our family, and each of one of our family by the hyp. ind. applied at the moment when it was finished.

For accepting as a computation the union of the segment subcomputations in approaching a stage $\sigma + 1$ under Option 5, we required the union to be well-founded.

(ii) Consider what rule application can account for a line segment diverging downward-rightward from our vertex bearing F or from a vertex to the right of it along the branch running horizontally-rightward from it to the vertex bearing $E\dagger$. (Only under E6.cs can such a line segment be drawn downward-rightward earlier than just before the vertex bearing $E\dagger$.) Under each of the applicable rules, the branch running horizontally-rightward from the vertex, or each vertex, reached by such a segment also ends in a flagged numeral, and so that vertex is not suspended. Repeating the argument for each of the lower branches thus found running horizontally-rightward to a flagged numeral, etc., we see that the only possibility for finding a branch in the subtree issuing from the vertex bearing F which does not eventually run horizontally-rightward to a flagged numeral is for it to have no last vertex from which it goes downward-rightward. The existence of such a branch would contradict (i).

(iii) Suppose we have a well-ordered family of computations, each properly extending the preceding ones, with no last, whose union is well-founded. Say our family made consecutive consists of computations indexed by the ordinals $\mu <$ a limit ordinal ν (end 14.1 and 15.2, paragraph 5).

To picture the situation vividly, suppose the vertices and 0-expressions in the union are printed in black. For each $\mu < \nu$, we think of the part of this tree existing at the moment μ as in pink, printed over the black. Let us put a blue flag on each vertex in the black tree, beginning with the initial one bearing $\overline{E}$, to the right of which as μ increases toward ν no last status of the pinking is reached.

Each vertex of the black tree, occupied say by E, was introduced by the step which took us to some moment μ and thereafter remained in place. The form of E fixes the rule from among E1–E7 which governs any additions at later moments of vertices next rightward from the vertex bearing E, which once introduced remain in place thereafter. So from the applicable one of the rules E1–E7, we can picture what kind of an array of next vertices (if any) can issue from any vertex in the black tree.

Starting from the initial (blued) vertex bearing $\overline{E}$, let us see what happens as we endeavor to proceed rightward in the black tree on blued vertices. Suppose we have proceeded thus as far as a vertex bearing E.

If there are only finitely many next rightward vertices, at least one of them must be blued.

We can dispose rapidly of all of the applicable rules, except E4.2 Subcase $\overline{3 \cdot 3}^2$, E4.3 Subcase $\overline{3 \cdot 3}^3$, and similar cases of E7, by observing that exactly one of the possible one or two next rightward vertices is blued. Some situations can be ruled out as inapplicable. Thus, we cannot be sitting on a blued vertex coming under E4.1 with two next vertices (7.1, p. 20), since the subcomputation of B would have been completed before the upper next vertex bearing $N\dagger$ went into place (thus completing that branch). E6.cs (2.4, p. 202) could apply with just one next vertex (downward-rightward) and that blued, or with two next vertices and only the upper one blued.

Now consider in detail E4.2 Subcase $\overline{3 \cdot 3}^2$ (7.3, p. 22), our E being $\dot{\alpha}^2(B)$.

If there is a highest next vertex, it must bear the $B(\underline{c})$ for an uncompleted subcomputation, and this next vertex and no other is blued, since that $B(\underline{c})$ was put in place only after all the subcomputations below it were completed. It could not bear $M\dagger$ (by reasoning as for E4.1 above).

There remains the case of E4.2 Subcase $\overline{3 \cdot 3}^2$ with no highest next vertex $(*)$. So there are infinitely many next vertices, none of them blued (since each subcomputation was completed before those above it were started).

Now consider the case of E4.3 Subcase $\overline{3 \cdot 3}^3$, our E being $\dot{\alpha}^3(B)$. Let us analyze what is going on with respect to the stages as they were counted in the formulation of E4.3 in 14.3 (paralleling 11.2).

It cannot be that all the pinking to the right of our $\dot{\alpha}^3(B)$ constitutes what was done exactly up through reaching Stage 0 or a successor stage $\sigma + 1$ (final or nonfinal). For, the final vertex for the last stage reached would have gone into place by a certain μ (and all the others earlier).

Perhaps Stage 0 has not yet been reached, or the computation has run through some Stage σ and is continuing toward Stage $\sigma + 1$ (nonfinal) not yet reached. Then only that next vertex is blued which bears the $B(\gamma)$ beginning the subcomputation in which Stage 0 or Stage $\sigma + 1$, respectively, is being approached. In particular, if Stage $\sigma + 1$ is being approached under Option 5, the part of the black union to the right of this $B(\gamma)$ may be exactly the union of the family of segment computations described for Option 5.

Perhaps the computation (for $\mu < \nu$) ran exactly through all the stages $\tau <$ a limit stage σ. Then there are two subcases.

First, there may be no aspiring branches in the union of the Stage-τ charts $(*)$. Thus each subcomputation of $B(\gamma)$ was finished (i.e., completed or brought to a blockage) by a respective stage $\tau < \sigma$, and thus its $B(\gamma)$ is not blued, and so there must be infinitely many of them.

Alternatively, besides there being zero or more (maybe infinitely many) subcomputations which were finished at stages $\tau < \sigma$, with their respective $B(\gamma)$'s not blued, there may also be one or more aspiring branches in the unionized chart, whose corresponding $B(\gamma)$'s are blued in consequence of the definition of "aspiring" $(\circ)$.

Finally, suppose that the computation, as μ increases toward ν, has run past the unionization of the stages $\tau <$ a limit stage σ for the E4.3 application into subcomputations corresponding to aspiring branches in the chart resulting from that unionization.

If the subcomputations have been finished for certain aspiring branches and the subcomputation is under way but not finished for the next higher one, then the $B(\gamma)$ beginning the subcomputation for the latter is the only blued vertex next rightward from the one bearing $\dot{\alpha}^3(B)$.

There remains the case that the subcomputations have been finished exactly for each of a family of aspiring branches with no highest $(*)$. (If this is all the aspiring branches, unionization would put us at the limit stage σ.) Then no vertex next right of the one bearing $\dot{\alpha}^3(B)$ is blued, since the subcomputations corresponding to the aspiring branches of our family were each finished in turn at moments $\mu < \nu$, while all the other subcomputations were either finished earlier or (for remaining aspiring branches) have not been pursued further since the unionization of the stages $\tau < \sigma$.

Similar analysis into cases applies to E7.

Now we have an enumeration of cases in which, as we move rightward on blued vertices, there are exactly one, or in the case labeled with $(\circ)$ perhaps more than one, next blued vertex, plus just three cases labeled with $(*)$ in which there is no next blued vertex. Similar cases can arise under E7. Any blued path must end, since for (iii) we have assumed that the entire black tree is well-founded. Thus the blued tree has branchings at most at places labeled with $(\circ)$ and all its branches end at places labeled with $(*)$.

Now we quickly see that the black tree (i.e., the union of the pink trees for all moments $\mu < \nu$) is a computation under our rules. For, let us proceed rightward on the blued subtree of it, starting with the initial (blued vertex), until we first arrive at a place labeled either with $(\circ)$ or (without there having been any branching) with $(*)$. At the place thus reached, we have an autonymous approach to the union. If it is marked with $(\circ)$, then, because the whole black tree is well-founded, the unions for each of the families of subcomputations corresponding to the aspiring branches of the chart (if any) are well-founded; and thus E4.3 Subcase $\overline{3\cdot 3}^3$ for the approach to a limit stage, or the like with E7, authorizes taking the whole union, i.e., the black tree, as a computation. If it is marked with $(*)$, then again the relevant rule, namely E4.2 Subcase $\overline{3\cdot 3}^2$, or E4.3 Subcase $\overline{3\cdot 3}^3$ for the approach to a limit stage without aspiring branches or after finishing the subcomputations corresponding to family of aspiring branches with no highest, or the like with E7, says that the union is a computation.

(iv) I use induction over the construction of the parent computation to confirm that each step, and each passage to a union of an infinite family of computations, each extending the preceding ones, with no last, which does anything to the subtree issuing from the vertex bearing D, is valid also for constructing it as a computation begun with D with the assignments totaled.

For steps, nothing needs to be proved if the vertex from which the step is taken is outside the subtree. If it is inside, the only way the validation of the step could depend on anything to the left of the vertex bearing D is if it uses information about the worth of a variable $\underline{c}$ or the oracle for the worth of a variable γ supplied from an application of E4.2 Subcase $\overline{3 \cdot 3^2}$ or E4.3 Subcase $\overline{3 \cdot 3^3}$, or the like with E7, to the left. But, with the assignments totaled in the subtree when it is duplicated outside the parent tree, this information is made available in the subtree standing by itself.

For a unionization, applying (i) to the parent tree as it stood just after the unionization, we can apply the analysis used in proving (iii) to the family of trees which was unionized. Following the blued subtree until it runs into the subtree issuing from D (otherwise the unionization does nothing to the subtree), we pick the first place within the subtree labeled with $(\circ)$, or, if we come to none such, the one place there labeled with $(*)$, and reason as we did in proving (iii).

Remark. In (iv), we have that the subtree issuing from a vertex bearing D in a given computation is good as a computation taken by itself with the assignments totaled with respect to the parent tree. But if we take a D in a parent tree and construct a computation of it under the assignments totaled with respect to the parent tree, that computation might exceed the subtree issuing from it in the parent tree and indeed might not be capable of being used in extending the parent tree. For example, this could happen if the D there contains free a descendant of the γ of a subcomputation of $B(\gamma)$ to $\dot{\alpha}^3(B)$ corresponding (after unionization) to an aspiring branch of the chart in the approach to a limit stage for E4.3 Subcase $\overline{3 \cdot 3^3}$ (or the like with E7), when the subcomputation corresponding to some lower aspiring branch cannot be finished. (If the D doesn't contain such a γ free, or the like with E7, once D surfaced within the approach to a stage τ, its computation would not be prolonged beyond Stage τ.) Also it could happen if the D (containing such a γ free) is in the Stage-τ computation for some $\tau <$ a limit ordinal σ, in a subcomputation corresponding to an aspiring branch in the union of the Stage-τ charts, and the union of the subcomputations for that aspiring branch is well-founded but for some other aspiring branch is not.

(v) In proving the first sentence of (v), I shall be carrying out the induction step to a successor moment $\mu+1$ for proving the second sentence by induction on the moment. The basis is obvious, and the induction step to a limit moment μ simply uses the hyp. ind., by which the computations at all the moments $\nu < \mu$ (which are unionized to reach moment μ) are determined.

As a computation of $\overline{E}$ under $\overline{\Omega}$ is constructed, at each moment I classify the vertices then in place as follows: A vertex is *flagged* if it bears a flagged numeral. Otherwise, it is *open* if there is no vertex rightward from it, *suspended* (as I have already said) if there are vertices downward-rightward from it but none horizontally-rightward, and *locked* if there is a vertex horizontally-rightward from it. I assume that when a vertex is put in place bearing 0, the 0 is flagged at once (by E5). At each moment, each vertex is of one of these four sorts.

From the form of the rules E1–E7 governing steps, *if at a given moment a step can be taken from a given vertex, that step is uniquely determined.*

No step can be taken from a flagged or locked vertex.

I next established: *At any moment, there is at most one open vertex. If such exists, the next step, if a next step is possible, is from it. Otherwise, a next step is possible from at most one suspended vertex.*

To prove this by induction, I first remark that it is obvious for moment 1 (only one vertex in place).

Suppose our proposition is true at a moment μ and a step to moment $\mu+1$ is possible. Only the new vertex can be open at moment $\mu+1$, since at moment μ there was at most one open vertex, which (if it existed) became suspended or locked by the step. And, as we shall see, a step from a suspended vertex at moment $\mu+1$ is possible only if the step to moment $\mu+1$ introduced a flagged vertex. Thus it is again true at moment $\mu+1$ that, if there is an open vertex, the next step (if any) must be from it.

A step from a suspended vertex is possible only under a special circumstance in the case of each of our rules. This circumstance can be brought into existence at moment $\mu+1$ only if the step to moment $\mu+1$ introduces a flagged numeral and only for one suspended vertex. For, by the hyp. ind., at moment μ the circumstance existed for at most one suspended vertex (then, only if there was no open vertex at moment μ), in which case the circumstance no longer exists with respect to it at moment $\mu+1$. For, either the vertex has become locked, or it is still suspended but with a new subcomputation of $B(\underline{c})$ or $B(\gamma)$ or the like with E7, which does not set the stage immediately for a further step from the same vertex. So, if it exists, the suspended vertex newly satisfying the circumstance at moment $\mu+1$ as a result of the introduction of a new flagged numeral is the only one from which a step is possible at moment $\mu+1$.

By referring to the various rules, we can see how this works. Suppose then that the step to moment $\mu+1$ puts a flagged numeral $\underline{W}\dagger$ in place. Following back horizontally leftward as far as possible, if we reach the leftmost vertex of the tree, we know the computation is completed (with no more steps possible, since by (ii) every vertex is locked or flagged). Otherwise, going once leftward-upward we find a vertex bearing say E to which we have just finished a subcomputation. Depending on the circumstances, we may next be able to take a step from E, either horizontally-rightward (locking it) or downward-rightward to start another subcomputation to E, or (under E4.3 Subcase $\overline{3\cdot 3^3}$ or a like case of E7) perhaps a step elsewhere.

As a simple example, if E is $\dot{\alpha}^1(B)$, the step to moment $\mu+1$ has just completed the subcomputation of B with the value $\underline{w}$. Then, if for the interpretation α^1 of the variable $\dot{\alpha}^1$ the quantity $\alpha^1(\underline{w})$ is defined, $=\underline{n}$, we take the step "— $N\dagger$" horizontally-rightward from $\dot{\alpha}^1(B)$. If it is undefined, in general the computation is frozen but there is the following possibility of a step. Let us change the notation, writing "$\gamma(C)$", "$\underline{r}$" instead of "$\dot{\alpha}^1(B)$",

"$\underline{w}$". Trace the ancestors (14.2) of the initial γ in the $\gamma(C)$ back as far as possible. If the earliest (leftmost) ancestor is in the $B(\gamma)$ beginning a subcomputation to an $\dot{\alpha}^3(B)$ under E4.3 Subcase $\overline{3 \cdot 3}^3$, the $\dot{\alpha}^3$-oracle, contemplating her chart, which just at moment $\mu + 1$ acquired "$\underline{r}$?" at the end of the branch corresponding to that subcomputation, will decide what (if anything) shall be done. If it is a stage-σ chart (all branches being finished, each with an "$\underline{m}\surd$" or an "$\underline{r}$?"), she will decide from the chart whether to approach stage $\sigma + 1$ (possibly final) and, if so, authorize the appropriate next step. Otherwise, it is a chart with some unfinished aspiring branches which arose in the unionization of the Stage-τ charts for $\tau <$ a limit ordinal σ, and she will authorize the next step to be made, if possible, in the subcomputation corresponding to the lowest still-unfinished aspiring branch. We shall say more on this possibility at the end of this proof of (v). The story is similar if instead of $\dot{\alpha}^3(B)$ we have a $\dot{\theta}_{\underline{t}}(\cdots B \cdots)$ for an application of a like case of E7.

We should observe that in the case the E is the $\dot{\alpha}^3(B)$ for an application E4.3 Subcase $\overline{3 \cdot 3}^3$ (let us write "E", "$\underline{w}$" as "$\dot{\alpha}^3(B)$", "$\underline{m}$"), the subcomputation then makes a certain branch of the chart end with "$\underline{m}\surd$", either a new branch or an extension of a previously existing branch, and the $\dot{\alpha}^3$-oracle determines on a step. If (at moment $\mu+1$) it is not a stage-σ chart but one in the approach to a limit stage σ with some aspiring branches still unfinished, the story is as just above.

Let us back up a bit to discuss the step, of one is possible, from an open vertex bearing a type-0 variable $\underline{c}$. If the assignment does not include a value for $\underline{c}$, we still have the possibility of getting a value by going back to the earliest ancestor of the $\underline{c}$. If it is in the $B(\underline{c})$ of a first subcomputation to $\dot{\alpha}^2(B)$ under E4.2 Subcase $\overline{3 \cdot 3}^2$, the $\dot{\alpha}^2$-oracle may supply a value $\underline{r}_0$. Similarly with $\dot{\theta}_{\underline{t}}(\cdots B \cdots)$ under a like case of E7.

Now consider moment ν for ν a limit, the union of the computations as they existed at the moments $\mu < \nu$ having just been accepted as a computation. In this union there is no open vertex, since each vertex open at a moment μ became suspended or locked at the moment $\mu+1$. Moreover, each of the special circumstances, produced by a flagged numeral being introduced in the step from a moment μ to the moment $\mu + 1$, for making a step from a suspended vertex expired with that step.

However, the act of unionization may have set up a special circumstance for a step to moment $\nu + 1$ from a vertex suspended at moment ν. To see how, consider the blued vertices which we explored in proving (iii). If there is a next step after taking the union, we will be led to it from the lowest end of a branch in the blued subtree, which will of course be labeled with $(*)$. This is the only vertex labeled with $(*)$, unless there are vertices labeled with $(\circ)$ at which there are two or more aspiring branches in the chart. But each time we come to one marked with $(\circ)$, by our statement of E4.3 Subcase $\overline{3 \cdot 3}^3$ the next computation step (if any) after the unionization must be in the lowest subcomputation corresponding to an aspiring branch, so we follow the lowest blued path if there is a branching at a (∘) (at the first (∘) we come to and at each subsequent one). Consider the cases that can apply at the lowest vertex marked with (∗) (which we thus reach), of which three were formulated explicitly in the proof of (iii) (besides which there are similar cases associated with E7). (a) If this (∗) goes with an application of E4.2 Subcase $\overline{3\cdot 3^2}$ with no highest next vertex, the $\dot{\alpha}^2$-oracle may authorize a new subcomputation of $B(\underline{c})$ under $\Omega_{\underline{r}_\zeta}$ or (with $\zeta = \xi$) the addition of "— $M\dagger$" rightward from $\dot{\alpha}^2(B)$ (cf. 14.1, paragraph 5). (b) If the (∗) goes with an application of E4.3 Subcase $\overline{3\cdot 3^3}$ in the approach to a limit stage σ when there are no aspiring branches in the unionized chart, the next step (if any) is that authorized by the $\dot{\alpha}^3$-oracle from the resulting stage-σ chart. If the (∗) goes with the finishing of each of a family of subcomputations corresponding to aspiring branches with no highest, and (c) there is no still higher aspiring branch, we have arrived at the stage-σ chart, from which the next step (if any) is determined by the $\dot{\alpha}^3$-oracle. (d) If there is still higher aspiring branch, we go to the lowest end of a blued branch in the subcomputation corresponding to the next higher aspiring branch as the blueing stood for the approach through the stages $\tau < \sigma$ in the approach to the limit stage σ of the E4.3 application to $\dot{\alpha}^3(B)$ for which we were engaged in doing the mandated computing for the family of aspiring branches. (By the hypothesis of our induction for (v), the entire history of the computation up to the moment ν is determined and thus in particular what the blueing was in this past approach to a limit.) There we have the same four possibilities (a)–(d) (or the like with E7). If it is (d), the E4.3 application has to be to an $\overline{\alpha}^3(\overline{B})$ within the respective subcomputation (corresponding to the next higher aspiring branch) of $B(\gamma)$ to the $\dot{\alpha}^3(B)$ at the previous vertex labeled (∗). So it cannot happen that we encounter (d) infinitely many successive times, as then there would be an infinite branch in the computation at our limit moment ν (contradicting (i)). So we may encounter (d) at most finitely many times before reaching one of (a)–(c), from which without further regress it will be determined by our $\dot{\alpha}^2$- or $\dot{\alpha}^3$-oracle (or the like with E7) what (if any) is the next step.

It is likewise from the lowest end of a blued branch (in the appropriate subcomputation) for the approach to the limit stage (as under (d)) that we seek to determine the next step (if any) at a successor moment $\mu+1$ which coincides with the finishing of a subcomputation corresponding to an aspiring branch not the highest. There one of (a)–(d) applies, and if it is (d) we will have to repeat, but this will not happen infinitely often.

16. Monotonicity

16.1. In the next theorem, I shall use the *normalized tree ordinal* (n.t.o.) of a completed computation, defined as follows by recursion over the computation as a well-founded tree. A vertex at the end of a branch (bearing a flagged numeral) has the n.t.o. 1. A (completed) computation of the form E—$F\ldots$, where

$F \ldots$ has the n.t.o. ϕ, has the n.t.o. $\phi + 1$. A computation of the form

$$\begin{array}{l} E \text{———} G \ldots\,, \\ \quad\searrow F \ldots \end{array}$$

where $F \ldots$ and $G \ldots$ have the n.t.o.'s ϕ and ψ, has the n.t.o. $\phi + \psi + 1$. A computation coming under E4.3 Subcase $\overline{3 \cdot 3}^3$ (cf. end 14.3) has the n.t.o. $(\sum'_{\kappa<\mu} \psi_\kappa) + 2$, where the ψ_κ's are the n.t.o.'s of the subcomputations of $B(\gamma)$ under $\Omega_{\dot{\rho}_\kappa}$ and $\sum'_{\kappa<\mu} \psi_\kappa$ is obtained by summing the ψ_κ's, with the omission of those for which $\dot{\rho}_\kappa$ is of the form $\lambda \dot{\alpha}^0 \underline{n}$ where there is a $\dot{\rho}_\lambda$ of the form $\{\langle \underline{r}_{\lambda\eta}, \underline{n}\rangle \mid \eta < \zeta_\lambda\}$ with the same $\underline{n}$, in an order which minimizes the sum. E4.2 Subcase $\overline{3 \cdot 3}^2$ (cf. 7.3, p. 22) is handled similarly, but without omitting the n.t.o.'s ψ_ζ of any of the subcomputations of $B(\underline{c})$ under $\Omega_{\underline{r}_\zeta}$ $(\zeta < \xi)$. Similar cases under E7 are handled similarly.

(XXVI) *Let E be a* 0*-expression and suppose that, under a given assignment Ω to its free variables* (*at least*) *and the $\dot{\theta}_{\underline{\iota}}$'s and a given choice of oracles to embody the functions in Ω, there is a completed computation of E with the result $\underline{w}$. Choose a free variable $\gamma^{\underline{j}}$ of E $(j = 0, 1, 2$ or $3)$, let $\dot{\delta}^{\underline{j}}$ be its worth in the assignment Ω, let $\bar{\delta}^{\underline{j}}$ be a type-$\underline{j}$ object with $\bar{\delta}^{\underline{j}} \supset \dot{\delta}^{\underline{j}}$, let $\overline{\Omega}$ be the assignment coming from Ω by substituting $\bar{\delta}^{\underline{j}}$ for $\dot{\delta}^{\underline{j}}$ as the worth of $\gamma^{\underline{j}}$, and* (*for $\underline{j} > 0$*) *choose an oracle to embody $\bar{\delta}^{\underline{j}}$. Then* (*with all the chosen oracles*) *there is a computation of E under $\overline{\Omega}$ with the same result $\underline{w}$, and there is a normalized tree ordinal smaller than or equal to that of the given computation of E under Ω.*

Similarly with a $\dot{\theta}_{\underline{\iota}}$ instead of a $\gamma^{\underline{j}}$.

Proof. I prove this proposition, with the choices of $\gamma^{\underline{j}}$ and $\bar{\delta}^{\underline{j}}$, and if $\underline{j} > 0$ of the oracle for $\bar{\delta}^{\underline{j}}$, variable, by induction on the normalized tree ordinal of the given computation of E under Ω with the chosen oracles.

I write out only the most challenging case, namely that the principal case under E4.3 applies to the given computation, which thus has the form

$$\begin{array}{lll} \dot{\alpha}^3(B) \text{———} S\dagger & & \\ \qquad\quad \vdots & & \\ \quad\searrow B(\gamma) \ldots \Omega_{\dot{\rho}_\kappa} \ldots M_\kappa{}^\dagger & & (\kappa < \mu) \\ \qquad\quad \vdots & & \end{array}$$

shown at the end of 14.3 (with 11.2).

Part of our discussion will be under two cases: (A) when $\gamma^{\underline{j}}$ is not $\dot{\alpha}^3$ and (B) when it is.

Is $\dot{\beta}^2 = \bigcup_{\kappa<\mu} \underline{S}_{\dot{\rho}_\kappa}$ for $\underline{S}_{\dot{\rho}_\kappa} = \{\langle \dot{\alpha}^1, \underline{m}_\kappa\rangle \mid \dot{\alpha}^1 \supset \dot{\rho}_\kappa\}$ a type-$\dot{2}$ object? By the analysis at the end of 10.1, the only way the chart branches represented by the $\dot{\rho}_\kappa$'s (that is, the branches of the final chart under our application of E4.3 Subcase $\overline{3 \cdot 3}^3$) could fail to be part of the chart for an $\dot{\alpha}^2$-oracle is if, for some branch representing $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}\rangle \mid \eta < \zeta_\kappa\}$ with the final entry "$\underline{m}_\kappa \surd$", there is a branch representing $\dot{\rho}_\lambda = \lambda\dot{\alpha}^0\underline{n}$ with a different final entry "$\underline{m}_\lambda \surd$" (which would frustrate our satisfying the monotonicity requirement, as analyzed on p. 14 at the end of 10.1). There being a branch representing $\dot{\rho}_\kappa$ would mean that there is a subcomputation "$B(\gamma) \ldots \Omega_{\dot{\rho}_\kappa} \ldots M_\kappa{}^\dagger$", which of course has an n.t.o. < the n.t.o. of our given computation of $\dot{\alpha}^3(B)$ under Ω. So using $\dot{\rho}_\lambda \supset \dot{\rho}_\kappa$, the hyp. ind. (applied to γ rather than to $\gamma^{\underline{j}}$) would say that there is a computation "$B(\gamma) \ldots \Omega_{\dot{\rho}_\lambda} \ldots M_\kappa{}^\dagger$", which rules out there being a branch representing $\dot{\rho}_\lambda$ with $\underline{m}_\lambda \neq \underline{m}_\kappa$.[11] Thus our chart is compatible with being a selection of branches from the oracle tree for a type-$\dot{2}$ object $\dot{\alpha}^2$. Let us put "MUM" at the end of each segment which diverges (before its end) from all the branches on our chart, except that, if any branch through the top second vertex representing a constant $\underline{n}$ on a subset of the natural numbers has "$\underline{m}\surd$" at its end, so must the branch representing $\lambda\dot{\alpha}^0\underline{n}$, if it is not in the chart (if it is, it already has "$\underline{m}\surd$"). This gives us an oracle tree exactly for $\dot{\beta}^2$. Our question is answered "yes", and $\dot{\alpha}^3(\dot{\beta}^2) = \underline{s}$.

By the hyp. ind., for each $\kappa < \mu$ for which the respective subcomputation is not omitted in calculating the n.t.o. of our given computation, there is a computation of $B(\gamma)$ under $\overline{\Omega}_{\dot{\rho}_\kappa}$ with the same result $\underline{m}_\kappa$ and an n.t.o. $\leq$ the one under $\Omega_{\dot{\rho}_\kappa}$.

Now it may happen that in one, and hence in all, of these there is no γ-face, and thus the interpretation $\dot{\rho}_\kappa$ of γ plays no role, so the result is a constant $\underline{m}$. So indeed $\dot{\beta}^2$ is this constant on its domain. Then $\lambda\dot{\alpha}^1\underline{m} \supset \dot{\beta}^2$, so by the monotonicity of $\dot{\alpha}^3$ (cf. (4) at the end of 11.2), $\dot{\alpha}^3(\lambda\dot{\alpha}^1\underline{m}) = \underline{s}$. (A) This gives us an E4.3 application under $\overline{\Omega}$ by Subcase $\overline{3 \cdot 2}^3$, using just one of the identical subcomputations of $B(\gamma)$ under $\overline{\Omega}_{\dot{\rho}_\kappa}$, indeed under $\overline{\Omega}_{u^1}$ with a Case $\overline{1}^1$ oracle for u^1, with the result $\underline{m}$, and an n.t.o. $\leq$ the given one under Ω. (B) By $\bar{\dot{\delta}}^3 \supset \dot{\delta}^3 = \dot{\alpha}^3$ with $\dot{\alpha}^3(\lambda\dot{\alpha}^1\underline{m}) = \underline{s}$, $\bar{\dot{\delta}}^3(\lambda\dot{\alpha}^1\underline{m}) = \underline{s}$. So if Case $\overline{3}^3$ applies to $\bar{\dot{\delta}}^3$ the computation described under (A) serves our purpose, whereas if Case $\overline{2}^3$ applies we have the computation "$\dot{\alpha}^3(B)$—$S\dagger$" under $\overline{\Omega}$, whose n.t.o. 2 is < the n.t.o. (at least 4) of our given computation under Ω.

Suppose instead that γ surfaces in one, and hence in all, of our computations of $B(\gamma)$ under $\overline{\Omega}_{\dot{\rho}_\kappa}$. While in computing $B(\gamma)$ under $\Omega_{\dot{\rho}_\kappa}$ in the case of a

[11] We can also see this by observing that if a computation of $B(\gamma)$ with a chart branch of the second kind for $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}\rangle\} \mid \eta < \zeta_\kappa\}$ gave the value $\underline{m}_\kappa$, the computation of $B(\gamma)$ with a chart branch of the first kind for $\dot{\rho}_\lambda = \lambda\dot{\alpha}^0\underline{n}$ would also give the value $\underline{m}_\kappa$ (and have a less than or equal n.t.o.), being in fact just the result of omitting the subcomputations of the $C_{\kappa\eta}$'s for the γ-faces $\gamma(C_{\kappa\eta})$ in the computation of $B(\gamma)$ for $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}\rangle \mid \eta < \zeta_\kappa\}$.

branch of the second kind (with $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}_{\kappa\eta}\rangle \mid \eta < \zeta_\kappa\}$, every argument-value pair $\langle \underline{r}_{\kappa\eta}, \underline{n}_{\kappa\eta}\rangle$ in $\dot{\rho}_\kappa$ is used, under $\overline{\Omega}_{\dot{\rho}_\kappa}$ maybe less than all (but anyway some) of them are used. Taking just the pairs which are used, we get a $\overline{\dot{\rho}}_\kappa \subset \dot{\rho}_\kappa$ with $B(\gamma) \simeq_{\overline{\Omega}_{\overline{\dot{\rho}}_\kappa}} \underline{m}_\kappa$. Of two $\overline{\dot{\rho}}_\kappa$'s, one $\overline{\dot{\rho}}_{\kappa_2}$ cannot properly extend the other $\overline{\dot{\rho}}_{\kappa_1}$, since if it did, then in computing $B(\gamma)$ under $\overline{\Omega}_{\overline{\dot{\rho}}_{\kappa_1}}$ we would get an answer good under $\overline{\Omega}_{\overline{\dot{\rho}}_{\kappa_2}}$ without using the additional pairs in $\overline{\dot{\rho}}_{\kappa_2}$. .The distinct such $\overline{\dot{\rho}}_\kappa$'s can be represented by chart branches formed using the pairs $\langle \underline{r}, \underline{n}\rangle$ in them in the order called for in the computation of $B(\gamma)$ under $\overline{\Omega}_{\overline{\dot{\rho}}_\kappa}$. For a branch of the first kind under Ω for which the respective subcomputation is not omitted in figuring the n.t.o. of the computation under $\overline{\Omega}$, let $\overline{\dot{\rho}}_\kappa = \dot{\rho}_\kappa$. We can index the distinct $\overline{\dot{\rho}}_\kappa$'s as $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$'s using the ordinals $\overline{\kappa} <$ some $\overline{\mu}$ $(B(\gamma) \simeq_{\overline{\Omega}_{\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}}} \overline{m}_{\overline{\kappa}})$.

I define $\overline{\dot{\beta}}^2 = \bigcup_{\overline{\kappa}<\overline{\mu}} \underline{S}_{\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}}$ where $\underline{S}_{\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}} = \{\langle \dot{\alpha}^1, \overline{\underline{m}}_{\overline{\kappa}}\rangle \mid \dot{\alpha}^1 \supset \overline{\overline{\dot{\rho}}}_{\overline{\kappa}}\}$, and I would like to be able to say that $\overline{\dot{\beta}}^2$ is a type-$\dot{2}$ object. But indeed the same reasoning can be applied as to $\dot{\beta}^2$, using now the hyp. ind. applied to a computation "$B(\gamma)\ldots\overline{\Omega}\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}\ldots\overline{M}_{\overline{\kappa}}{}^{\dagger}$" for $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}} = \{\langle \overline{\overline{\underline{r}}}_{\overline{\kappa}\eta}, \underline{n}\rangle \mid \eta < \overline{\zeta}_{\overline{\kappa}}\}$, whose n.t.o. is $\leq$ that of "$B(\gamma)\ldots\Omega_{\dot{\rho}_\kappa}\ldots M_\kappa{}^{\dagger}$" for any of the $\dot{\rho}_\kappa$'s (for branches of the second kind) from which we got $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$. So $\overline{\dot{\beta}}^2$ is of type $\dot{2}$. And clearly $\overline{\dot{\beta}}^2 \supset \dot{\beta}^2$.

Now by $\dot{\alpha}^3(\dot{\beta}^2) = \underline{s}$ with the monotonicity of $\dot{\alpha}^3$, $\dot{\alpha}^3(\overline{\dot{\beta}}^2) = \underline{s}$.

Not merely do we have $\overline{\dot{\beta}}^2$ as a type-$\dot{2}$ object. We have an oracle for it, with branches in its tree representing the functions $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$ plus maybe a few branches representing functions $\lambda\dot{\alpha}^0\underline{n}$ (each $\supset$ some $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$) added for monotonicity. In the case of each such added branch, by use of the hyp. ind. with respect to γ (as implicitly above for $\overline{\overline{\dot{\rho}}}_{\overline{\lambda}}$) there is a computation "$B(\gamma)\ldots\Omega_{\lambda\dot{\alpha}^0\underline{n}}\ldots\overline{M}_{\overline{\kappa}}{}^{\dagger}$" with an n.t.o. $\leq$ that of the computation corresponding to the branch representing that $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$. A branch of the second kind, representing $\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}$, has its argument-value pairs $\langle \overline{\overline{\underline{r}}}_{\overline{\kappa}\eta}, \overline{\overline{\underline{n}}}_{\overline{\kappa}\eta}\rangle$ $(\eta < \overline{\zeta}_{\overline{\kappa}})$ in the order in which they are used in computing $B(\gamma)$ under $\overline{\Omega}_{\overline{\overline{\dot{\rho}}}_{\overline{\kappa}}}$.

(A) Now consider in parallel the $\dot{\alpha}^3$-oracle's progress in answering the question "$\dot{\alpha}^3(\overline{\dot{\beta}}^2)$?" asked with the $\overline{\dot{\beta}}^2$-oracle just now described (we know she does answer "$\underline{s}$") and in masterminding the computation of $\dot{\alpha}^3(B)$ under $\overline{\Omega}$. Subcase $\overline{3\cdot 3}^3$ applies to both, since the $\dot{\alpha}^3$-oracle is under Case $\overline{3}^3$ and since respectively $\overline{\dot{\beta}}^2(\mathrm{u}^1)$ is undefined and γ surfaces in the computation of $B(\gamma)$ begun under $\overline{\Omega}_{\mathrm{u}^1}$.

Now indeed, because each branch in the $\overline{\dot{\beta}}^2$-oracle tree representing say $\overline{\dot{\rho}}$ is matched to a computation of $B(\gamma)$ under $\overline{\Omega}_{\overline{\dot{\rho}}}$ as described, the two processes will correspond exactly with the same charts at every stage and the same final result "$\underline{s}$". All the $\dot{\alpha}^3$-oracle has under her control is which chart branches from among those in the $\overline{\dot\beta}^2$-oracle tree she chooses to follow and in what order and with what pattern of skipping around from working on one to working on another. The questions "$\underline{r}$?" and answers "$\underline{m}\surd$" which the $\overline{\dot\beta}^2$-oracle, or the computation process, turns up at each stage for a given $\overline{\dot\rho}$ are determined by $\overline{\Omega}$ and B, as represented in the $\overline{\dot\beta}^2$-oracle tree. And because $\dot\alpha^3(\overline{\dot\beta}^2)$ is defined, we know the $\dot\alpha^3$-oracle will not venture outside of the $\overline{\dot\rho}$'s represented by chart branches of the $\overline{\dot\beta}^2$-oracle tree (as that would make $\dot\alpha^3(\overline{\dot\beta}^2)$ undefined). So, if we look only at what is shown in the charts from stage to stage, the two progressions will be indistinguishable.

(B) Using $\overline{\dot\delta}^3 \supset \dot\delta^3 = \dot\alpha^3$ with $\dot\alpha^3(\overline{\dot\beta}^2) = \underline{s}$, $\overline{\dot\delta}^3(\overline{\dot\beta}^2) = \underline{s}$, and the same reasoning applies. The $\overline{\dot\delta}^3$-oracle may explore a different selection of chart branches, in a different order, but confining herself to the $\overline{\dot\beta}^2$-oracle tree. So she will be doing the same things at each stage, whether she is answering "$\overline{\dot\delta}^3(\overline{\dot\beta}^2)$?" or computing $\dot\alpha^3(B)$ under $\overline{\Omega}$.

It remains to be shown (under each of (A) and (B)) that the n.t.o. of the computation of $\dot\alpha^3(B)$ under Ω is $\leq$ that of the given computation of $\dot\alpha^3(B)$ under Ω. To do this, I shall pair each of the subcomputations of $B(\gamma)$ under $\overline{\Omega}_{\overline{\dot\rho}}$, which is actually used in the computation of $\dot\alpha^3(B)$ under $\overline{\Omega}$ and whose n.t.o. is not omitted in calculating the n.t.o. of that computation, with one or several subcomputations under $\Omega_{\dot\rho}$ whose n.t.o.'s are not omitted in calculating the n.t.o. under Ω. This pairing will be chosen so that the first member of each pair has an equal or smaller n.t.o. than the second member. No $\dot\rho$ will thus "pay for" more than one $\overline{\dot\rho}$.

Remember: we started with the collection of the $\dot\rho_\kappa$ $(\kappa < \mu)$ representing the chart branches in the computation of $\dot\alpha^3(B)$ under Ω; second, we possibly threw out some of them of the form $\lambda\dot\alpha^0\underline{n}$ (for calculating the n.t.o.); third, we went to the resulting collection of the $\overline{\overline{\dot\rho}}_{\overline\kappa}$ $(\overline\kappa < \overline\mu)$; fourth, maybe we added some of the form $\lambda\dot\alpha^0\underline{n}$ to get the oracle tree for $\overline{\overline{\dot\beta}}^2$; fifth, we let the $\dot\alpha^3$-oracle or the $\overline{\dot\delta}^3$-oracle do her thing when questioned with the $\overline{\dot\beta}^2$-oracle, which in general entails a selection from the fourth collection; and sixth, maybe we omitted some of form $\lambda\dot\alpha^0\underline{n}$ (for calculating the n.t.o.).

Each subcomputation of $B(\gamma)$ under $\overline{\Omega}_{\overline{\dot\rho}}$ for which $\overline{\dot\rho} = \overline{\overline{\dot\rho}}_{\overline\kappa}$ corresponds to a chart branch of the second kind is paired one-many with the subcomputations under $\Omega_{\dot\rho_\kappa}$ for the $\dot\rho_\kappa \supset \overline{\overline{\dot\rho}}_{\overline\kappa}$, each of which has an equal or greater n.t.o. What about a subcomputation corresponding to a chart branch of the first kind (for which $\overline{\dot\rho} = \lambda\dot\alpha^0\underline{n}$) which is not omitted in figuring the n.t.o. under $\overline{\Omega}$? There are three cases. It may be that $\overline{\dot\rho} = \overline{\overline{\dot\rho}}_{\overline\kappa} = \overline{\dot\rho}_\kappa = \dot\rho_\kappa$ for some κ with the subcomputation under $\Omega_{\dot\rho_\kappa}$ not omitted in figuring the n.t.o. under Ω. Then our subcomputation under $\overline{\Omega}_{\overline{\dot\rho}}$ is simply paired with the subcomputation under $\Omega_{\dot\rho_\kappa}$ (with an equal or greater n.t.o.). Or it may be the subcomputation under $\Omega_{\dot\rho}$ for $\dot\rho = \lambda\dot\alpha^0\underline{n}$ is omitted because there is also a subcomputation under $\Omega_{\dot\rho_\kappa}$

with $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}\rangle \mid \eta < \zeta_\kappa\}$, but the $\dot{\alpha}^3$-oracle or the $\dot{\overline{\delta}}{}^3$-oracle did not select the chart branch for $\overline{\rho}_\kappa$ under $\overline{\Omega}_{\overline{\rho}_\kappa}$ corresponding to the one for $\dot{\rho}_\kappa$ under $\Omega_{\dot{\rho}_\kappa}$. Then our subcomputation under $\overline{\Omega}_{\overline{\rho}}$ can be paired with the subcomputation under $\Omega_{\dot{\rho}_\kappa}$ (which we are free to use because it is then not needed to be paired with a subcomputation under $\overline{\Omega}_{\overline{\rho}_\kappa}$), since the subcomputation under $\overline{\Omega}_{\overline{\rho}_\kappa}$ for the spurned $\overline{\rho}_\kappa$-chart branch has an n.t.o. $\leq$ that under $\Omega_{\dot{\rho}_\kappa}$, and by footnote 11 above our subcomputation under $\overline{\Omega}_{\overline{\rho}}$ has an n.t.o. $\leq$ that of the one under $\overline{\Omega}_{\overline{\rho}_\kappa}$. Third, it may be that there is no subcomputation under $\Omega_{\dot{\rho}_\kappa}$ for $\dot{\rho}_\kappa = \lambda\dot{\alpha}^0\underline{n}$ or for $\dot{\rho}_\kappa = \{\langle \underline{r}_{\kappa\eta}, \underline{n}\rangle \mid \eta < \zeta_\kappa\}$ but that for one or more $\dot{\rho}_\kappa$'s for chart branches of the second kind not of the latter form, $\overline{\rho}_\kappa$ (the same for all) is of that form but the $\dot{\alpha}^3$- or the $\dot{\overline{\delta}}{}^3$-oracle did not select it. This is handled like the second case (our subcomputation under $\overline{\Omega}_{\overline{\rho}}$ being paired with the ones under those $\Omega_{\dot{\rho}_\kappa}$'s).

16.2. We can see now that the charts which will arise in the application of E4.3 are ones that arise "naturally", i.e., in the questioning of the $\dot{\alpha}^3$-oracle by an $\dot{\alpha}^2$-oracle under 11.2.[6] We simply apply the reasoning we used in paragraph 4 of the proof of (XXVI), where monotonicity (given there by our hyp. ind. applied to γ, but now available by (XXVI) as proved) was used to show that the final chart (but it applies equally to any Stage-σ chart and to the union of Stage-τ charts for $\tau <$ a limit ordinal σ) is one that is compatible with arising in the questioning of our $\dot{\alpha}^3$-oracle by an $\dot{\alpha}^2$-oracle.

16.3. Now I extend the result of 7.5 to the presence of the new rules. For we can conclude that the result of a completed computation is independent of the choice of the oracles for the functions in Ω simply by applying (XXVI) successively to each function in Ω with $\dot{\overline{\delta}}{}^{\underline{j}} = \dot{\delta}^{\underline{j}}$, only the oracles for the worth of $\gamma^{\underline{j}}$ differing (two different oracles for the same type-$\underline{j}$ object), and similarly for $\dot{\theta}_{\underline{t}}$ with $\dot{\overline{\theta}}_{\underline{t}} = \dot{\theta}_{\underline{t}}$.[12]

17. Interpretation of j-expressions by type-$\underline{j}$ objects

17.1. I now state #A of 8.1, but extended to include $\underline{j} = 3$ and type-$\dot{2}$ arguments for a $\dot{\theta}_{\underline{t}}$.

(XXVII) *To each* 0*-expression* E *and each assignment* Ω *to its free variables and the* $\dot{\theta}_{\underline{t}}$*'s, there is a unique type-*$\dot{0}$ *object* $\dot{\alpha}^0$ *such that*

$$E \simeq_\Omega \dot{\alpha}^0.$$

[12]This justifies the use of the notation "$E \simeq_\Omega \underline{w}$" without specifications of oracles for the functions in Ω (cf. footnote 4 in 2.4, p. 203, and 8.1, p. 25).

For $j = 1, 2,$ *or* 3, *to each* 0-*expression* $E_{\underline{\gamma}^{j-1}}$ *and each assignment* Ω *to its free variables and the* $\dot{\theta}_{\underline{t}}$ *'s except to the type-* $(j-1)$ *variable* $\gamma^{\underline{j}-1}$, *there is a unique type-* $\underline{j}^{\dot{}}$ *object* $\dot{\alpha}^{j}$ *such that, for each* $\dot{\alpha}^{j-1}$ *of type* $(\underline{j}-1)^{\dot{}}$,

$$E_{\gamma^{\underline{j}-1}} \simeq_{\Omega_{\dot{\alpha}^{j-1}}} \dot{\alpha}^{\underline{j}}(\dot{\alpha}^{\underline{j}-1})$$

where $\Omega_{\dot{\alpha}^{\underline{j}-1}}$ *is the assignment coming from* Ω *by assigning* $\dot{\alpha}^{\underline{j}-1}$ *to* $\gamma^{\underline{j}-1}$. *Applying this to* $\{\lambda\gamma^{\underline{j}-1}E\}(\gamma^{\underline{j}-1})$ *as the* $E_{\gamma^{\underline{j}-1}}$, *each type-* $\underline{j}$ λ*-functor* $\lambda\gamma^{\underline{j}-1}E$ *is interpretable, under any assignment* Ω *to its free variables and the* $\dot{\theta}_{\underline{t}}$ *'s, as a type-* $\underline{j}^{\dot{}}$ *object* $\dot{\alpha}^{\underline{j}}$.

For instance, to each 0-*expression* $E_{\underline{c}, \gamma_1, \gamma_2, \gamma^2}$ *and each assignment* Ω *to its free variables and the* $\dot{\theta}_{\underline{t}}$ *'s except to the variables* $\underline{c}, \gamma_1, \gamma_2, \gamma^2$, *there is a unique type-* $(\dot{0}: \dot{0},\ \dot{1},\ \dot{1},\ \dot{2})$ *object* $\dot{\theta}$ *such that, for each* $\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2$ *of types* $\dot{0},\ \dot{1},\ \dot{1},\ \dot{2}$,

$$E_{\underline{c}, \gamma_1, \gamma_2, \gamma^2} =_{\Omega_{\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2}} \dot{\theta}(\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$$

where $\Omega_{\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2}$ *is the assignment coming from* Ω *by assigning* $\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2$ *to* $\underline{c}, \gamma_1, \gamma_2, \gamma^2$.

The first statement (for $\underline{j} = 0$) is all but immediate.

The second statement for $\underline{j} = 1$ is established by the way 13.2a leads to 13.2b; for $\underline{j} = 2$, by 13.3a to 13.3b; for $\underline{j} = 3$, by 13.4a to 13.4b.

The third statement is established similarly.

17.2. For $\underline{j} = 3$ (as indeed we stated in 14.3, paragraph 2): An $\dot{\alpha}^3$-oracle under 11.2, questioned with the $\dot{\alpha}^2$-oracle given by the proof of (XXVII) for $\underline{j} = 2$ (i.e., by 13.3b) with a given Ω when E_{γ^1}, γ^1 are the $B(\gamma)$, γ for an application of E4.3, declares that $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$ if and only if $\dot{\alpha}^3(B)$ computed by E4.3 under Ω gives the value $\underline{s}$, and the series of charts is the same in the principal case.

Similarly for $\underline{j} = 2$ and $\underline{j} = 1$.

17.3. In (XXVII) for $\underline{j} = 3$ with a given Ω, if E_{γ^2} is simply $\gamma^3(\gamma^2)$, the $\dot{\alpha}^3$ which it gives is just the function assigned to γ^3 in Ω, indeed by (XXIV) in 15.1. Similarly for $\underline{j} = 2$, $\underline{j} = 1$, and $\underline{j} = 0$.

18. Replaceability of $\underline{j}$-expressions by equivalent $\underline{j}$-expressions

18.1. We recall that, for Ω an assignment to at least the free variables of two 0-expressions E and F and to the $\dot{\theta}_{\underline{t}}$'s, "$E \simeq_\Omega F$" means that E and F have the same worth under Ω, i.e., that under Ω E and F have the same value $\underline{w}$ (a natural number) ($E \simeq_\Omega \underline{w}$ and $F \simeq_\Omega \underline{w}$) or both are undefined.[12]

Now I extend the notation to $\underline{j}$-expressions for $\underline{j} = 1, 2$, or 3. Indeed, if we write $(E)_\Omega$ for the worth of E under Ω (in the sense of (XXVII) if E is not just a type-$\underline{j}$ variable), then for $\underline{j} = 0, 1, 2$, or 3, $E \simeq_\Omega F$ shall mean $(E)_\Omega \simeq (F)_\Omega$ where $\simeq$ is equality in the domain $\underline{j}^{\bullet}$. Furthermore, I use $E \supset_\Omega F$ for $\underline{j} = 0, 1, 2, 3$ to mean that $(E)_\Omega \supset (F)_\Omega$ with $\supset$ as defined in 5.2, bottom of p. 7. Thus for $\underline{j} = 0$,

$$E \supset_\Omega F \equiv (\underline{w})[F \simeq_\Omega \underline{w} \rightarrow E \simeq_\Omega \underline{w}]$$

and $E \simeq_\Omega F \equiv (E \supset_\Omega F)$ and $(F \supset_\Omega E)$. For $\underline{j} = 1, 2$, or 3,

$$\begin{aligned} E \supset_\Omega F &\equiv (\dot{\alpha}^{\underline{j}-1})[E(\gamma^{\underline{j}-1}) \supset_{\Omega_{\dot{\alpha}^{\underline{j}-1}}} F(\gamma^{\underline{j}-1})] \\ &\equiv (\dot{\alpha}^{\underline{j}-1})(\underline{w})[F(\gamma^{\underline{j}-1}) \simeq_{\Omega_{\dot{\alpha}^{\underline{j}-1}}} \underline{w} \rightarrow E(\gamma^{\underline{j}-1}) \simeq_{\Omega_{\dot{\alpha}^{\underline{j}-1}}} \underline{w}] \end{aligned}$$

where $\gamma^{\underline{j}-1}$ is a variable not assigned a value in Ω and $\Omega_{\dot{\alpha}^{\underline{j}-1}}$ comes from Ω by assigning $\dot{\alpha}^{j-1}$ to $\gamma^{\underline{j}-1}$, and $E \simeq_\Omega F \equiv (E \supset_\Omega F)$ and $(F \supset_\Omega E) \equiv (\dot{\alpha}^{\underline{j}-1})[E(\gamma^{\underline{j}-1}) \simeq_{\Omega_{\dot{\alpha}^{\underline{j}-1}}} F(\gamma^{\underline{j}-1})]$.

18.2.

(XXVIII) (Lemma for (XXIX)). *Let E be a 0-expression and D a $\underline{j}$-expression ($\underline{j} = 0, 1, 2$, or 3). Let Ω be an assignment to at least the free variables of E and of D, to the variable $\gamma^{\underline{j}}$ not occurring free in E or in D, and to the $\dot{\theta}_{\underline{t}}$'s. Suppose that $D \simeq_\Omega \gamma^{\underline{j}}$. Pick any way of construing E as containing a tribe of free occurrences of D; call E then E_D. Suppose that, replacing each of the occurrences of the tribe by $\gamma^{\underline{j}}$ to get $E_{\gamma^{\underline{j}}}$, the resulting occurrences of $\gamma^{\underline{j}}$ are free.*

(a) *If $E_D \simeq_\Omega w$, then $E_{\gamma^{\underline{j}}} \simeq_\Omega \underline{w}$.*
(b) *If $E_{\gamma^{\underline{j}}} \simeq_\Omega \underline{w}$, then $E_D \simeq_\Omega \underline{w}$.*

Proof. I prove the theorem successively for $\underline{j} = 0, 1, 2$, and 3, using for $\underline{j} > 0$ the theorem as already proved for $\underline{j} - 1$. I write out some of the details for $\underline{j} = 3$.

(a) According to the hypothesis that $E_D \simeq_\Omega \underline{w}$, there is a completed computation tree for E_D under Ω with the result $\underline{w}$. I use induction over this computation tree, with the choice of the tribe of D's variable. I want to show that there is a computation tree for $E_{\gamma^{\underline{j}}}$ under Ω with the same result $\underline{w}$.

E4.λ: E_D is $\{\lambda\beta^{\underline{k}}A\}(B)$. *Case* 1: the initial $\lambda\beta^{\underline{k}}A$ is not one of the tribe of D's in E_D. I write A and B as A_D and B_D, since each may contain members of the tribe. Now there is a completed computation tree for $\underline{S}^{\beta^{\underline{k}}}_{B_{\gamma^3}} A_{\gamma^3}|$ under Ω with the result $\underline{w}$, by the hyp. ind. and E_{γ^3} gives $\underline{S}^{\beta^{\underline{k}}}_{B_{\gamma^3}} A_{\gamma^3}|$ by E4.λ. So $E_{\gamma^3} \simeq_\Omega \underline{w}$. *Case* 2: $\lambda\beta^{\underline{k}}A$ is one of the tribe, so $\underline{k} = 2$. I write B as B_D, since it may contain members of the tribe. As in Case 1 (indeed immediately if B_D contains no member of the tribe), there is a computation of $\underline{S}^{\beta^2}_{B_{\gamma^3}}A|$ under Ω with the result $\underline{w}$. Using E4.λ, followed by this computation, (1) $\{\lambda\beta^2 A\}(B_{\gamma^3}) \simeq_\Omega \underline{w}$. Now (2) $B_{\gamma^3} \simeq_{\Omega_{\dot\alpha^2}} \gamma^2$ where $\dot\alpha^2$, assigned to γ^2, is the worth of B_{γ^3} under Ω as given by (XXVII). (In detail: by (XXVII), $(\dot\alpha^1)[B_{\gamma^3}(\gamma^1) \simeq_{\Omega_{\dot\alpha^1}} \dot\alpha^2(\dot\alpha^1)]$ and by 17.3 $(\dot\alpha^1)[\gamma^2(\gamma^1) \simeq_{\Omega_{\dot\alpha^2,\dot\alpha^1}} \dot\alpha^2(\dot\alpha^1)]$, so $(\dot\alpha^1)[B_{\gamma^3}(\gamma^1) \simeq_{\Omega_{\dot\alpha^2,\dot\alpha^1}} \gamma^2(\gamma^1)]$.) By the hypothesis that $D \simeq_\Omega \gamma^3$, (3) $D(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \gamma^3(\gamma^2)$. (In detail: from $D \simeq_\Omega \gamma^3$, $(\dot\alpha^2)[D(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \gamma^3(\gamma^2)]$.) From (1) and (2) by (a) for $\underline{j} = 2$ (remembering that D is $\lambda\beta^2 A$), $D(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \underline{w}$. Thence with (3), $\gamma^3(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \underline{w}$. Thence with (2) and (b) for $\underline{j} = 2$,

$$\gamma^3(B_{\gamma^3}) \simeq_\Omega \underline{w}.$$

E4.3: E_D is $\dot\alpha^3(B_D)$. Suppose Subcase $\overline{3\cdot 3}^3$ applies to the given computation by E4.3. (Case $\overline{2}^3$ and Subcase $\overline{3\cdot 2}^3$ are simpler.) *Case* 1: the initial $\dot\alpha^3$ is not one of the tribe of D's. The completed computation has the form shown at the end of 14.3, with subcomputations of $B_D(\gamma)$ under $\Omega_{\dot\rho_\kappa}$ with results $\underline{m}_\kappa$ $(\kappa < \mu)$. The fact that the application of E4.3 gave $\underline{w}$ (written "$\underline{s}$" in 14.3) means that the $\dot\alpha^3$-oracle, questioned with an envelope-opening oracle for $\dot\alpha^2$ with $\dot\alpha^2 \supset \dot\beta^2 = \{\langle \dot\rho_\kappa, \underline{m}_\kappa\rangle \mid \kappa < \mu\}$, gives $\underline{w}$ (cf. 14.3 and 11.2). By the monotonicity of $\dot\alpha^3$ (as imposed on her oracle at the end of 11.2, in particular in the form (3) there), she must also give the same result $\underline{w}$ when questioned with an envelope-(not-opening) $\dot\alpha^2$-oracle with $\dot\alpha^2 \supset \dot\beta^2$. So she gives $\underline{w}$ when questioned with any $\dot\alpha^2$-oracle with $\dot\alpha^2 \supset \dot\beta^2$. By 17.2, the result of computing $\dot\alpha^3(B_{\gamma^3})$ under Ω by E4.3 is the same as what the $\dot\alpha^3$-oracle gives when questioned with the $\dot\alpha^2$-oracle for B_{γ^3} described in the proof of (XXVII) for $\underline{j} = 2$ with $B_{\gamma^3}(\gamma)$, γ as the E_{γ^1}, γ^1. And this oracle gives for each of the assignments $\dot\rho_\kappa$ to γ the value $\underline{m}_\kappa$ by the definition of $\dot\alpha^2$ in (XXVII), since $B_{\gamma^3}(\gamma) \simeq_{\Omega_{\dot\rho_\kappa}} \underline{m}_\kappa$ (by the hyp. ind.). So $\dot\alpha^2 \supset \dot\beta^2$; so $\dot\alpha^3(\dot\alpha^2) \simeq_\Omega \underline{w}$, as remarked above. So $\dot\alpha^3(B_{\gamma^3})$ gets this value by E4.3 under Ω; i.e., $\dot\alpha^3(B_{\gamma^3}) \simeq_\Omega \underline{w}$, which in this case is $E_{\gamma^3} \simeq_\Omega \underline{w}$. *Case* 2: the initial $\dot\alpha^3$ is one of the tribe of D's. Applying Case 1 to just the members of the tribe in B (i.e., to that subtribe), $\dot\alpha^3(B_{\gamma^3}) \simeq_\Omega \underline{w}$. Now $B_{\gamma^3} \simeq_{\Omega_{\dot\alpha^2}} \gamma^2$ where $\dot\alpha^2$, assigned to γ^2, is the worth of B_{γ^3} under Ω as given by (XXVII) (cf. (2) under E4.λ Case 2). So by (a) for $\underline{j} = 2$, $\dot\alpha^3(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \underline{w}$. By the hypothesis that $D \simeq_\Omega \gamma^3$ with our case hypothesis, $\dot\alpha^3 \simeq_\Omega \gamma^3$; so $\dot\alpha^3(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \gamma^3(\gamma^2)$ (cf. (3) under E4.λ Case 2). So $\gamma^3(\gamma^2) \simeq_{\Omega_{\dot\alpha^2}} \underline{w}$. By (b) for $\underline{j} = 2$, $\gamma^3(B_{\gamma^3}) \simeq_\Omega \underline{w}$; i.e., $E_{\gamma^3} \simeq_\Omega \underline{w}$.

(b) By hypothesis, $E_{\gamma^3} \simeq_\Omega \underline{w}$. I use induction over a given computation tree for E_{γ^3} under Ω.

E4.λ: E_{γ^3} is $\{\lambda\beta^{\underline{k}}A_{\gamma^3}\}(B_{\gamma^3})$. By hyp. ind., there is a completed computation tree for $\underline{S}^{\beta^{\underline{k}}}_{B_D}A_D|$ with the result $\underline{w}$, and E_D, i.e., $\{\lambda\beta^{\underline{k}}A_D\}(B_D)$, gives the latter by E4.λ.

E4.3: E_{γ^3} is $\dot{\alpha}^3(B_{\gamma^3})$. We treat Subcase $\overline{3\cdot 3}^3$. *Case* 1: the initial $\dot{\alpha}^3$ is not one of the γ^3's replacing D's of the tribe. In the completed computation by E4.3, we have subcomputations of $B_{\gamma^3}(\gamma)$ under $\Omega_{\dot{\rho}_\kappa}$ with results $\underline{m}_\kappa$. By the hyp. ind., there are computations of $B_D(\gamma)$ under $\Omega_{\dot{\rho}_\kappa}$ with the same results $\underline{m}_\kappa$. So in view of the application of E4.3 to $\dot{\alpha}^3(B_{\gamma^3})$ and the monotonicity of $\dot{\alpha}^3$, the $\dot{\alpha}^3$-oracle, questioned with any $\dot{\alpha}^2 \supset \dot{\beta}^2 = \{\langle\dot{\rho}_\kappa, \underline{m}_\kappa\rangle \mid \kappa < \mu\}$, gives the value $\underline{w}$. By 17.2, $\dot{\alpha}^3(B_D)$ is computed under E4.3 with the same result as $\dot{\alpha}^3$ gives in responding to the $\dot{\alpha}^2$-oracle which we obtain by applying (XXVII) for $\underline{j} = 2$ to $B_D(\gamma)$ as the E_γ. And that $\dot{\alpha}^2$-oracle gives for each $\dot{\rho}_\kappa$ the value $\underline{m}_\kappa$, since $(\dot{\alpha}^1)[B_D(\gamma) \simeq_{\Omega_{\dot{\alpha}^1}} \dot{\alpha}^2(\dot{\alpha}^1)]$. Thus $\dot{\alpha}^2 \supset \dot{\beta}^2$, so $\dot{\alpha}^3(\dot{\alpha}^2) \simeq_\Omega \underline{w}$. So $\dot{\alpha}^3(B_D)$ gets the value $\underline{w}$; i.e., $E_D \simeq_\Omega \underline{w}$. *Case* 2: the initial $\dot{\alpha}^3$ replaces a D of the tribe. I first apply Case 1, using as the tribe for it only the γ^3's occurring in B_{γ^3} as replacements for D's in B_D, to conclude that the $\dot{\alpha}^3$-oracle, questioned with any $\dot{\alpha}^2 \supset \dot{\beta}^2$, gives the value $\underline{w}$. And the $\dot{\alpha}^2$-oracle we obtain by applying (XXVII) for $\underline{j} = 2$ to $B_D(\gamma)$ as the E_γ satisfies $\dot{\alpha}^2 \supset \dot{\beta}^2$; so using 17.3, (1) $\dot{\gamma}^3(\gamma^2) \simeq_{\Omega_{\dot{\alpha}^2}} \underline{w}$ with $\dot{\alpha}^2$ interpreting γ^2. And (2) $B_D \simeq_{\Omega_{\dot{\alpha}^2}} \gamma^2$ (like (2) for (a) E4.λ Case 2). By the hypothesis that $D \simeq_\Omega \gamma^3$, (3) $D(\gamma^2) \simeq_{\Omega_{\dot{\alpha}^2}} \dot{\gamma}^3(\gamma^2)$ (like (3) for E4.λ Case 2). Now by (3) and (1), $D(\gamma^2) \simeq_{\Omega_{\dot{\alpha}^2}} \underline{w}$. Thence by (2) with (b) for $\underline{j} = 2$, $D(B_D) \simeq_\Omega \underline{w}$; i.e., $E_D \simeq_\Omega \underline{w}$.

18.3.

(XXIX) *Let E_D be a* 0*-expression containing a tribe of specified free occurrences of a $\underline{j}$-expression D $(j = 0, 1, 2,$ or $3)$, and let $\overline{D}$ be a $\underline{j}$-expression such that replacing in E_D each of the occurrences of the tribe by $\overline{D}$ to obtain $E_{\overline{D}}$, the resulting occurrences of $\overline{D}$ in $E_{\overline{D}}$ are free. If, under an assignment Ω to at least the free variables of D, $\overline{D}$, and E_D and to the $\dot{\theta}_{\underline{t}}$'s, $\overline{D} \supset_\Omega D$, then $E_{\overline{D}} \supset_\Omega E_D$.*

Proof follows the corollaries.

Corollary 1. *Likewise with $\simeq$ instead of $\supset$.*

This corollary is the #B of 8.1, but extended to include $\underline{j} = 3$. It follows from the theorem, applied as stated and also interchanging $\overline{D}$ with D.

Corollary 2. *If instead E_D is a k-expression $(k = 1, 2,$ or $3)$, then: if $\overline{D} \supset_\Omega D$, then $E_{\overline{D}} \supset_\Omega E_D$. If $\overline{D} \simeq_\Omega D$, then $E_{\overline{D}} \simeq_\Omega E_D$.*

Proof. For each type-$(k-1)^\cdot$ object $\dot{\alpha}^{\underline{k}-1}$ as interpretation of $\gamma^{\underline{k}-1}$, apply the theorem, or its Corollary 1, to $E_D(\gamma^{\underline{k}-1})$, $\Omega_{\dot{\alpha}^{\underline{k}-1}}$ as its E_D, Ω.

Corollary 3. *For* E_D *a given* 0*-expression, if for each type-* $(\underline{k}-1)^{\cdot}$ *object* $\dot{\alpha}^{\underline{k}-1}$ *as the interpretation of a free variable* $\gamma^{\underline{k}-1}$ *of* E_D *and/or* $E_{\overline{D}}$ $(\underline{k}>0)$, $\overline{D} \supset_{\Omega_{\dot{\alpha}^{\underline{k}-1}}} D$, *then* $\lambda\gamma^{\underline{k}-1}E_{\overline{D}} \supset_\Omega \lambda\gamma^{\underline{k}-1}E_D$. *Similarly with* $\simeq$ *instead of* $\supset$.

Proof. $\{\lambda\gamma^{\underline{k}-1}E_D\}(\gamma^{\underline{k}-1})$ reduces by E4.λ to E_D (and similarly with $\overline{D}$), to which the theorem (or its Corollary 1) applies with $\Omega_{\dot{\alpha}^{\underline{k}-1}}$ as its Ω.

Proof of (XXIX). Assume the tribe of D's in E_D is nonempty, as otherwise the conclusion is trivial. Pick a variable $\gamma^{\underline{j}}$ not occurring free in E_D or $E_{\overline{D}}$ and such that, replacing the D's of the tribe of $\gamma^{\underline{j}}$, the resulting occurrences of $\gamma^{\underline{j}}$ are free. By (XXVII), there are type-$\underline{j}$ objects $\dot{\alpha}^{\underline{j}}$ and $\bar{\dot{\alpha}}^{\underline{j}}$ such that $D \simeq_\Omega \dot{\alpha}^{\underline{j}}$ and $\overline{D} \simeq_\Omega \bar{\dot{\alpha}}^{\underline{j}}$. To prove that $E_{\overline{D}} \supset_\Omega E_D$, suppose that $E_D \simeq_\Omega \underline{w}$. By (XXVIII) (a), $E_{\gamma^{\underline{j}}} \simeq_{\Omega_{\dot{\alpha}^{\underline{j}}}} \underline{w}$. Using the hypothesis that $\overline{D} \supset_\Omega D$, $\bar{\dot{\alpha}}^{\underline{j}} \supset \dot{\alpha}^{\underline{j}}$. So from $E_{\gamma^{\underline{j}}} \simeq_{\Omega_{\dot{\alpha}^{\underline{j}}}} \underline{w}$ by monotonicity (XXVI), $E_{\gamma^{\underline{j}}} \simeq_{\Omega_{\bar{\dot{\alpha}}^{\underline{j}}}} \underline{w}$. Finally, by (XXVIII) (b), $E_{\overline{D}} \simeq_\Omega \underline{w}$. Thus $E_{\overline{D}} \supset_\Omega E_D$.

19. Representation of types 0, 1, 2, 3 within types $\dot{0}$, $\dot{1}$, $\dot{2}$, $\dot{3}$. Semantics

9.1. We shall now see that our type-0, -1, -2, -3 objects are included, or at least naturally represented, among the type-$\dot{0}$, -$\dot{1}$, -$\dot{2}$, -$\dot{3}$ objects, respectively, obtaining #C in 8.1 extended to include $\underline{j}=3$.[13]

The type-1 objects α^1 are the type-$\dot{1}$ objects which are defined exactly on $\underline{N}^0$ = type 0; and each $\alpha^0 \in \underline{N}^0$ is its own basis with respect to any α^1 as a type-$\dot{1}$ object (by Case 3^1 in 5.3, reading "*all* $\underline{r}$'s" for "*some* $\underline{r}$'s").

The type-2 objects α^2 are by definition the total one-place functions from type-1 into $\underline{N}^0$. As such, they are partial one-place functions from type-$\dot{1}$ into $\underline{N}^0$. But these partial functions are not monotone. For, if α^1 is the constant $\underline{n}$ on $\underline{N}^0$, $\lambda\underline{\dot{a}n} \supset \alpha^1$ with $\alpha^2(\lambda\underline{\dot{a}n})$ undefined. So they are not of type $\dot{2}$. However, the following holds for the specific correlation $\alpha^2 \to \dot{\alpha}^2$ given in its proof.

(XXX.2) (type 2 represented within type $\dot{2}$). *To the type-*2 *objects* α^2 *we correlate* 1–1 *type-*$\dot{2}$ *objects* $\dot{\alpha}^2$ *so that, for each* α^2, α^1 (*of types* 2, 1), $\dot{\alpha}^2(\alpha^1)=\alpha^2(\alpha^1)$ *and* α^1 *is its own basis with respect to* $\dot{\alpha}^2$.

Proof. Let $\dot{\alpha}^2(\dot{\alpha}^1)$ be defined, and $=\alpha^2(\alpha^1)$, exactly when $\dot{\alpha}^1$ is an extension of α^1 for some $\alpha^1 \in$ type 1. What this does is simply make $\dot{\alpha}^2(\lambda\underline{\dot{a}n}) = \alpha^2(\lambda\underline{\dot{a}n} \restriction \underline{N}^0) = \alpha^2(\lambda\underline{an})$ for each $\underline{n} \in \underline{N}^0$ (besides making $\dot{\alpha}^2(\alpha^1)=\alpha^2(\alpha^1)$ for each $\alpha^1 \in$ type 1, with $\dot{\alpha}^2$ otherwise undefined). Not only each α^1, but also each $\lambda\underline{\dot{a}n}$, is its own basis with respect to $\dot{\alpha}^2$ (cf. 5.4 Case 3^2). To establish the intrinsicality of the determination of the bases, observe that in 6.3 our $\dot{\alpha}^2$-oracle

[13] The last part of #C is extended to allow type-$\dot{2}$ arguments and moreover is now stated (as indeed was needed for #D) allowing $\dot{\theta}$ to be a nontotal partial function.

can be programmed to operate under Case $\overline{3}^2$, standing mute except in Subcase $\overline{3\cdot 2}^2$, when she declares that $\dot{\alpha}^2(\dot{\alpha}^1) = \underline{m}$ where $\underline{m} = \alpha^2(\dot{\alpha}^1 \restriction \underline{N}^0)$, and in Subcase $\overline{3\cdot 3}^2$, when she picks as $\underline{r}_0, \underline{r}_1, \ldots, \underline{r}_\zeta, \ldots$ all the natural numbers $0, 1, \ldots, \zeta, \ldots$ $(\zeta < \omega)$ in their natural order, declaring that $\dot{\alpha}^2(\dot{\alpha}^1) = \underline{m}$ exactly if $\dot{\alpha}^1$ is defined for all of them, indeed is an α^1, with $\alpha^2(\alpha^1) = \underline{m}$.

(XXX.3) (type 3 represented within type $\dot{3}$). *To the type-*3 *objects* α^3 *we correlate* 1-1 *type-*$\dot{3}$ *objects* $\dot{\alpha}^3$ *so that, for each* α^3, α^2 *(of types* 3, 2*),* $\dot{\alpha}^3(\dot{\alpha}^2) = \alpha^3(\alpha^2)$ *and* $\dot{\alpha}^2$ *is its own basis with respect to* $\dot{\alpha}^3$.

Proof. Let $\dot{\alpha}^3(\dot{\alpha}^2)$ be defined, and $= \alpha^3(\alpha^2)$, exactly when $\dot{\alpha}^2$ is an extension of $\dot{\alpha}^2$ for some $\alpha^2 \in$ type 2. For a given α^3, this does define $\dot{\alpha}^3$ as a partial one-place function from type $\dot{2}$ into $\underline{N}^0$, since any $\dot{\alpha}^2$ can be an extension of at most one $\dot{\alpha}^2$ (because all $\dot{\alpha}^2$'s have the same domain), and the $\dot{\alpha}^2$'s are correlated 1-1 to the α^2's. Since $\dot{\alpha}^2$ extends itself, $\dot{\alpha}^3(\dot{\alpha}^2) = \alpha^3(\alpha^2)$; thence, since the $\dot{\alpha}^2$'s are correlated 1-1 to the α^2's, our definition correlates the $\dot{\alpha}^3$'s 1-1 to the α^3's.

Clearly, $\dot{\alpha}^3$ is monotone (since an extension $\overline{\dot{\alpha}}^2$ of an $\dot{\alpha}^2$ extending an $\dot{\alpha}^2$ is an extension of the same $\dot{\alpha}^2$).

Toward establishing that there are unique intrinsically determined bases, let type-1 be well-ordered as $\alpha_0, \alpha_1, \ldots, \alpha_\zeta, \cdots$ $(\zeta < \xi)$. If $\dot{\alpha}^3(\dot{\alpha}^2)$ is defined, i.e., if $\dot{\alpha}^2 \supset \dot{\alpha}^2$ for a unique $\dot{\alpha}^2$, then, for each ζ, $\dot{\alpha}^2(\alpha_\zeta)$ is defined and $= \dot{\alpha}^2(\alpha_\zeta)$ (since $\dot{\alpha}^2(\alpha_\zeta)$ is defined). Conversely, if $\dot{\alpha}^2(\alpha_\zeta)$ is defined for each $\zeta < \xi$, then the collection of those values makes up an α^2 whose correlated $\dot{\alpha}^2$ is $\subset \dot{\alpha}^2$, so $\dot{\alpha}^3(\dot{\alpha}^2)$ is defined and its value $\alpha^3(\alpha^2)$ is determined by that collection.

So our $\dot{\alpha}^3$-oracle, to determine for any $\dot{\alpha}^2$ whether $\dot{\alpha}^3(\dot{\alpha}^2)$ is defined, and its value if so, needs to ellicit from the $\dot{\alpha}^2$-oracle exactly the values of $\dot{\alpha}^2$ for all α_ζ $(\zeta < \xi)$, standing mute if any one is not forthcoming, and otherwise declaring that $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$ for $\underline{s} = \alpha^3(\dot{\alpha}^2 \restriction \text{type } 1) = \alpha^3(\alpha^2)$.

In questioning the $\dot{\alpha}^2$-oracle in the manner of 11.2, Case $\overline{1}^3$ does not apply, since $\dot{\alpha}^3(\dot{\alpha}^2)$ is defined for the $\dot{\alpha}^2$'s extending $\dot{\alpha}^2$'s. Case $\overline{2}^3$ does not apply, since this would make $\dot{\alpha}^3(\dot{\alpha}^2)$ defined for the $\dot{\alpha}^2$'s not extending $\dot{\alpha}^2$'s.

In Case $\overline{3}^3$ Subcase $\overline{3\cdot 2}^3$ with $\dot{\alpha}^2 = \lambda\dot{\alpha}^1\underline{m}$, the $\dot{\alpha}^3$-oracle declares that $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$ where $\underline{s} = \alpha^3(\lambda\alpha^1\underline{m})$. For, $\{(\lambda\alpha^1\underline{m})^0\}(\dot{\alpha}^1)$ is defined exactly when $\dot{\alpha}^1$ is an extension of some α^1, and then its value is $\{\lambda\alpha^1\}(\alpha^1) = \underline{m}$. So $\lambda\dot{\alpha}^1\underline{m}$ extends $(\lambda\dot{\alpha}^1\underline{m})^0$, and hence $\dot{\alpha}^3(\lambda\dot{\alpha}^1, \underline{m}) = \alpha^3(\lambda\alpha^1\underline{m})$.

In Subcase $\overline{3\cdot 3}^3$, the $\dot{\alpha}^3$-oracle can pursue the values of $\dot{\alpha}^2$ on $\alpha_0, \alpha_1, \ldots, \alpha_\zeta, \ldots$ in turn thus. Toward Stage 0, she first questions the $\dot{\alpha}^2$-oracle under Option 2 with an envelope-opening u^1-oracle. If $\dot{\alpha}^2$ (under Case $\overline{3}^2$) extends an $\dot{\alpha}^2$, we know the $\dot{\alpha}^2$-oracle will ask "$\dot{\alpha}^1(\underline{r}_{00})$?". The $\dot{\alpha}^3$-oracle supplies the answer $\underline{n}_{00} = \alpha_0(\underline{r}_{00})$ $(\underline{r}_{00} = \underline{r}_0)$. Continuing if necessary, under Option 3 (and maybe Option 5), she persists in supplying $\underline{n}_{0\eta} = \alpha_0(\underline{r}_{0\eta})$, until the $\mathring{\alpha}^2$-oracle fails to respond (making $\mathring{\alpha}^3(\mathring{\alpha}^2)$ undefined) or answers with $\overline{\underline{m}}_0 = \mathring{\alpha}^2(\alpha_0) = \alpha^2(\alpha_0) = \dot{\alpha}^2(\overline{\rho}_0)$ for $\overline{\rho}_0 = \{\langle \underline{r}_{0\eta}, \underline{n}_{0\eta}\rangle \mid \eta < \theta\}$ ($\overline{\rho}_0$ the basis for α_0 with respect to $\mathring{\alpha}^2$). Then using Options 4 and 3 (and maybe 5), she gives values not from α_0 but from the first of $\alpha_1, \dots, \alpha_\zeta, \dots$ not $\supset \overline{\rho}_0$; and so on, until she has learned $\overline{\underline{m}}_\zeta = \mathring{\alpha}^2(\alpha_\zeta) = \alpha^2(\alpha_\zeta) = \dot{\alpha}^2(\overline{\rho}_\zeta)$ ($\overline{\rho}_\zeta$ the basis for α_ζ with respect to $\mathring{\alpha}^2$) for every $\zeta < \xi$, and thus has learned an $\dot{\alpha}^2$ with $\mathring{\alpha}^2 \supset \dot{\alpha}^2$, whereupon she gives $\underline{s} = \alpha^3(\alpha^2)$ as the value of $\mathring{\alpha}^3(\mathring{\alpha}^2)$. The distinct $\overline{\rho}_\zeta$'s in the order in which the $\mathring{\alpha}^3$-oracle explores them we can index as $\dot{\rho}_\kappa$'s for $\kappa < \mu$, to agree with the notation in 11.2.

It should be clear that this meets the program requirements of 11.2. And it gives monotonicity, since for $\mathring{\alpha}^2$ under Case $\overline{3}^2$ (Subcase $\overline{3 \cdot 3}^3$) $\mathring{\alpha}^3(\mathring{\alpha}^2) = \alpha^3(\mathring{\alpha}^2 \restriction \text{type } 1)$, while if $\mathring{\alpha}^2$ comes under Case $\overline{2}^2$ the same is true ($\mathring{\alpha}^2 \restriction$ type 1 in this case being the constant $\underline{m}$ on type 1).

In Subcase $\overline{3 \cdot 3}^3$ with a successful outcome ($\mathring{\alpha}^2 \supset \dot{\alpha}^2$), the basis for $\mathring{\alpha}^2$ with respect to $\mathring{\alpha}^3$ is of course $\dot{\beta}^2 = \bigcup_{\kappa<\mu} \underline{S}_{\dot{\rho}_\kappa} = \bigcup_{\zeta<\xi} \underline{S}_{\overline{\rho}_\zeta}$, while in Subcase $\overline{3 \cdot 2}^3$ it is $\dot{\alpha}^2 = \lambda\dot{\alpha}^1\underline{m}$. So indeed, as we stated in (XXX.3), the basis for $\mathring{\alpha}^2$ is $\mathring{\alpha}^2$ itself. Thus, in Subcase $\overline{3 \cdot 3}^3$, it has sufficed for her to learn every value of $\mathring{\alpha}^2$ (but less would not suffice). In Subcase $\overline{3 \cdot 2}^3$ with $\mathring{\alpha}^2 = \lambda\dot{\alpha}^1\underline{m}$, it is likewise the case that she needs to know only the value of $\mathring{\alpha}^3$ on $\dot{\alpha}^2 = (\lambda\alpha^1\underline{m})^0$ which $\lambda\dot{\alpha}^1\underline{m}$ extends (and again less would not suffice).

If we define $\mathring{\alpha}^1 = \alpha^1$ and $\mathring{\alpha}^0 = \alpha^0$ (or $\mathring{\underline{a}} = \underline{a}$), then, for $\underline{j} = 2$ and 1, (XXX.$\underline{j}$) and the definition of the object $\mathring{\alpha}^{\underline{j}}$ can read exactly like those for $\underline{j} = 3$.

(XXX.θ) (θ's represented by $\mathring{\theta}$'s). *For example*: *to the type-*$(0 : 0, 1, 1, 2)$ *functions* θ *we correlate* 1-1 *type-*$(\dot{0} : \dot{0}, \dot{1}, \dot{1}, \dot{2})$ *functions* $\mathring{\theta}$ *so that, for each* $\theta, \underline{a}, \alpha_1, \alpha_2, \alpha^2$ (*of types* $(0 : 0, 1, 1, 2), 0, 1, 1, 2$), $\mathring{\theta}(\mathring{\underline{a}}, \mathring{\alpha}_1, \mathring{\alpha}_2, \mathring{\alpha}^2) \simeq \theta(\underline{a}, \alpha_1, \alpha_2, \alpha^2)$ *and, if* $\theta(\mathring{\underline{a}}, \mathring{\alpha}_1, \mathring{\alpha}_2, \mathring{\alpha}^2)$ *is defined,* $(\mathring{\underline{a}}, \mathring{\alpha}_1, \mathring{\alpha}_2, \mathring{\alpha}^2)$ *is its own basis with respect to* $\mathring{\theta}$.

Proof. Let $\mathring{\theta}(\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$ be defined, and $= \theta(\underline{a}, \alpha_1, \alpha_2, \alpha^2)$, exactly when $\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2$ respectively extend $\mathring{\underline{a}}, \mathring{\alpha}_1, \mathring{\alpha}_2, \mathring{\alpha}^2$ for some $\underline{a}, \alpha_1, \alpha_2, \alpha^2$ of types $0, 1, 1, 2$ for which $\theta(\underline{a}, \alpha_1, \alpha_2, \alpha^2)$ is defined. For such $\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2$, $\dot{\alpha}^2$, we have $\dot{\underline{a}} = \mathring{\underline{a}} = \underline{a}$, $\mathring{\alpha}_1 = \alpha_1$, $\mathring{\alpha}_2 = \alpha_2$, and $\dot{\alpha}^2(\alpha_\zeta)$ is defined and $= \mathring{\alpha}^2(\alpha_\zeta) = \alpha^2(\alpha_\zeta)$ for all $\zeta < \xi$ (the α_ζ being a well-ordering of type 1). Thus $\mathring{\theta}(\dot{\underline{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$ is undefined, if $\dot{\underline{a}} = \mathrm{u}$, or not all of $\dot{\alpha}_1(\underline{r})$ for $\underline{r} = 0, 1, 2, \dots$ are defined, or not all of $\dot{\alpha}_2(\underline{r})$ for $\underline{r} = 0, 1, 2, \dots$ are defined, or not all of $\alpha^2(\alpha_\zeta)$ for $\zeta < \xi$ are defined. If the $\mathring{\theta}$-oracle, in seeking the value of

$\underline{\dot{a}}$, probing $\dot{\alpha}_1$ for its values for $\underline{r} = 0, 1, 2, \ldots$ if $\dot{\alpha}^1 \neq \lambda\underline{\dot{a}n}$, likewise $\dot{\alpha}_2$, and probing $\dot{\alpha}^2$ for its values on α_ζ for $\zeta < \xi$ if $\dot{\alpha}^2 \neq \lambda\dot{\alpha}^1\underline{m}$, is rebuffed, she correctly makes $\mathring{\theta}(\underline{\dot{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$ undefined; and, if she gets answers to all her probes (for a $\bar{\rho}_\zeta \subset \alpha_\zeta$ in the case of each α_ζ), she has all the information she needs to assemble $\underline{a}, \alpha_1, \alpha_2, \alpha^2$, and she can then answer that $\mathring{\theta}(\underline{\dot{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2) = \theta(\underline{a}, \alpha_1, \alpha_2, \alpha^2)$ if $\theta(a, \alpha_1, \alpha_2, \alpha^2)$ is defined. In each case with $\mathring{\theta}(\underline{\dot{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$ defined, she has learned everything about $\underline{\dot{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2$; so $(\underline{\dot{a}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}^2)$ is the basis.

19.2. We recall that under the computation rules of 2.4, Ω had to be an assignment of members of types 0, 1, 2, 3 to the free variables of E, as well as of partial functions of variables ranging on the types 0, 1, 2 to the $\theta_{\underline{t}}$'s (cf. 2.2 and paragraph 1 of footnote 2 to 1982). Now we get #D of 8.1 extended to include $\underline{j} = 3$.

(XXXI) (the theory of types 0, 1, 2, 3 within the theory of types $\dot{0}$, $\dot{1}$, $\dot{2}$, $\dot{3}$). *$E \simeq_\Omega \underline{w}$ under the computation rules of* 2.4 *if and only if $E \simeq_{\mathring{\Omega}} \underline{w}$ under the computation rules of* 2.4 *together with* 7.1, 7.3, 7.4, 14.3, *and* 14.4, *where $\mathring{\Omega}$ comes from Ω by replacing each $\alpha^{\underline{j}}$ by $\dot{\alpha}^{\underline{j}}$ and each θ by $\mathring{\theta}$.*

Proof. Each of the two implications "if" (←) and "only if" (→) is proved by induction over a computation tree of the one sort to show there is a tree of the other sort producing the same value of E and with exactly the same variables surfacing.

Only the cases for the five rules which were modified in 7.1, 7.2, 7.4, 14.3, and 14.4 compared to 2.4 are not automatic. The treatments of E1, E4.1, and E4.2 are similar (but easier than the following).

E4.3: E is $\gamma^3(B)$.[14]

(←) We start with a computation as in 14.3, under $\mathring{\Omega}$. Since $\dot{\alpha}^3(\mathrm{u}^2)$ is undefined, Case $\overline{3}^3$ of E4.3 applies. *Subcase* $\overline{3 \cdot 2}^3$. We have as subordinate computation "$B(\gamma) \ldots M\dagger$" under $\mathring{\Omega}_{\mathrm{u}^1}$ (without γ surfacing) and in the principal branch "$\gamma^3(B)$—$S\dagger$" with $\underline{s} = \dot{\alpha}^3(\lambda\dot{\alpha}^1\underline{m})$, $= \alpha^3(\lambda\alpha^1\underline{m})$ as we saw in paragraph 6 of the proof of (XXX.3). By the hyp. ind., there is a computation of $B(\gamma)$ under Ω_{u^1} with the same result $\underline{m}$ and with exactly the same variables, and thus not γ, surfacing. Using a well-ordered series of copies of this with the ordinal ξ, and with $\alpha_0, \ldots, \alpha_\zeta, \ldots$ $(\zeta < \xi)$ as the respective values of γ, we have just what we need to complete the computation of $\gamma^3(B)$ under Ω via E4.3 in 2.4 by "$\gamma^3(B)$—$S\dagger$" with $\underline{s} = \alpha^3(\lambda\alpha^1\underline{m})$. *Subcase* $\overline{3 \cdot 3}^3$. As we embodied $\dot{\alpha}^3$ in an oracle in proving (XXX.3), the $\dot{\rho}_0, \ldots, \dot{\rho}_\kappa, \cdots$ for the $\mathring{\Omega}_{\dot{\rho}_\kappa}$ of the subcomputations with results $\underline{m}_\kappa$ for $\kappa < \mu$ are a selection

[14] I prefer "$\gamma^3(B)$" here to the "$\alpha^3(B)$" of 2.4 and the "$\dot{\alpha}^3(B)$" of 14.3, as I wish to make explicit the alternative interpretations of one and the same variable γ^3 by α^3 of type 3 and $\dot{\alpha}^3$ of type $\dot{3}$. (In 13.4e I used "γ^3.")

to avoid repetitions of subfunctions of the functions $\alpha_0, \dots, \alpha_\zeta, \cdots$. In our well-ordering of type 1. Let us use these subcomputations without suppressing the repetitions, so that we have for each $\zeta < \xi$ a computation of $B(\gamma)$ under $\dot{\Omega}_{\bar{\rho}_\zeta}$ where $\bar{\rho}_\zeta \subset \alpha_\zeta = \dot{\alpha}_\zeta$, and hence under $\dot{\Omega}_{\dot{\alpha}_\zeta}$, with result $\overline{\underline{m}}_\zeta$. By the hyp. ind., there are computations of $B(\gamma)$ under Ω_{α_ζ} with the same respective results $\overline{\underline{m}}_\zeta$ and the same variables surfacing. In the given computation under $\dot{\Omega}$, we had in the principal branch "$\gamma^3(B)$—$S\dagger$", where $\dot{\alpha}^3(\dot{\alpha}^2) = \underline{s}$ for any $\dot{\alpha}^2 \supset \{\langle \dot{\rho}_\kappa, \underline{m}_\kappa \rangle \mid \kappa < \mu\} = \{\langle \bar{\rho}_\zeta, \overline{\underline{m}}_\zeta \rangle \mid \zeta < \xi\}$, in particular, for $\dot{\alpha}^2 = \bigcup_{\zeta<\xi} \underline{S}_\zeta$ where $\underline{S}_\zeta = \{\langle \dot{\alpha}^1, \overline{\underline{m}}_\zeta \rangle \mid \dot{\alpha}^1 \supset \bar{\rho}_\zeta\}$. But this $\dot{\alpha}^2 \supset \dot{\alpha}^2$ for $\alpha^2 = \{\langle \alpha_\zeta, \overline{\underline{m}}_\zeta \rangle \mid \zeta < \xi\}$ (using the values $\overline{\underline{m}}_\zeta$ just on the α_ζ's, rather than on all extensions of the $\bar{\rho}_\zeta$'s) so $\underline{s} = \dot{\alpha}^3(\dot{\alpha}^2) = \alpha^3(\alpha^2)$. Thus the assemblage of the computations of $B(\gamma)$ under Ω_{α_ζ} for $\zeta < \xi$ meets the requirements of E4.3 in 2.4 for giving "$\gamma^3(B)$—$S\dagger$" under Ω with the same $\underline{s}$ as under $\dot{\Omega}$.

($\rightarrow$) In 2.4, we have subordinate computations "$B(\gamma) \dots \Omega_{\alpha_\zeta} \dots \overline{M}_\zeta \dagger$," and as the principal branch "$\gamma^3(B)$—$S\dagger$" where $\underline{s} = \alpha^3(\alpha^2)$ for $\alpha^2 = \{\langle \alpha_\zeta, \overline{\underline{m}}_\zeta \rangle \mid \zeta < \xi\}$. In 14.3, we must try under Case $\overline{3}^3$ (as $\dot{\alpha}^3(\mathrm{u}^2)$ is undefined). *Case* A: In one (and therefore all) of the subcomputations in 2.4, γ does not surface. Therefore $\overline{\underline{m}}_\zeta = \underline{m}$ say for all ζ, and $\alpha^2 = \lambda\alpha^1 m$. By the hyp. ind., we have for 14.3 respective computations giving $B(\gamma)$ the same value $\underline{m}$ under $\dot{\Omega}_{\dot{\alpha}_\zeta}$, with exactly the same variables, and thus not γ, surfacing. All are alike, except for the unused value $\dot{\alpha}_{\dot{\alpha}_\zeta}$ of γ carried in the $\dot{\Omega}_\zeta$. So using one copy of them (with γ uninterpreted) as the subordinate computation for Subcase $\overline{3 \cdot 2}^3$ in 14.3, we are called upon to complete the E4.3 step there by "$\gamma^3(B)$—$S\dagger$" for the value $\underline{s}$ (if it exists) of $\dot{\alpha}^3(\lambda\dot{\alpha}^1\underline{m})$. But $\lambda\dot{\alpha}^1\underline{m}$ extends $(\lambda\alpha^1\underline{m})^0$ (paragraph 6 of the proof of (XXX.3)); so $\dot{\alpha}^3(\lambda\dot{\alpha}^1\underline{m}) = \alpha^3(\lambda\alpha^1\underline{m}) = \underline{s}$, the same $\underline{s}$ as we started with in 2.4 in this case. *Case* B: γ does surface (in all). By the hyp. ind., there are computations of $B(\gamma)$ under $\dot{\Omega}_{\dot{\alpha}_\zeta}$ with results $\overline{\underline{m}}_\zeta$ for each $\zeta < \xi$. Actually, each is good under $\dot{\Omega}_{\bar{\rho}_\zeta}$ for some $\bar{\rho}_\zeta \subset \alpha_\zeta$ $(= \dot{\alpha}_\zeta)$ embodying the values of $\dot{\alpha}_\zeta$ actually used in it. We can assemble these without repetitions as under $\dot{\Omega}_{\dot{\rho}_\kappa}$ with results $\underline{m}_\kappa$ $(\kappa < \mu)$; and indeed, if we take the $\dot{\rho}_\kappa$'s in the order in which they first occur as $\bar{\rho}_\zeta$'s, we have just the family of subcomputations to $\gamma^3(B)$ under E4.3 which the $\dot{\alpha}^3$-oracle would mastermind as we programmed her in paragraph 7 of the proof of (XXX.3). For any $\dot{\alpha}^2 \supset \{\langle \dot{\rho}_\kappa, \underline{m}_\kappa \rangle \mid \kappa < \mu\} = \{\langle \bar{\rho}_\zeta, \overline{\underline{m}}_\zeta \rangle \mid \zeta < \xi\}$, $\dot{\alpha}^3(\dot{\alpha}^2) = \alpha^3(\alpha^2)$; so the computation under E4.3 in 14.3 is completed by "$\gamma^3(B)$—$S\dagger$" for the same $\underline{s}$ as under Ω.

19.3. Let us see how the present theory enables us to fill the lacunae in some of the proofs in §3 (1978, pp. 205 ff).

I begin with the proof of (VII). The intended meaning is that, under any assignment Ω (à la 2.2) to the variables $\mathfrak{a}$ and the $\theta_{\underline{\iota}}$'s, $\phi(\mathfrak{a}) = \psi(\mathscr{L}^*)$; i.e., if either side is defined, so is the other with the same value. If we can prove this under any $\mathring{\Omega}$, by (XXXI) it will follow for any Ω as desired in §3.

In Case 0 for (VII), the construction of $\chi(\mathfrak{a}) \simeq \underline{a}$ by (S3) with (VI) does give that, for any $\mathring{\Omega}$ (including a value $\underline{a}_0 = \mathring{\underline{a}}_0$ of $\underline{a}$), $\chi(\mathfrak{a}) \simeq_{\dot{\Omega}} \underline{a}$. So by (XXIX) Corollary 1 with $\underline{j} = 0$, $\rho(\chi(\mathfrak{a}), \mathfrak{a}) \simeq_{\dot{\Omega}} \rho(\underline{a}, \mathfrak{a})$.

In Case 1 for (VII), the construction of the χ assures that, for any $\mathring{\Omega}$ and any given worth $\dot{\underline{b}}_0$ of $\underline{b}$, $\{\lambda\underline{b}\chi(\underline{b}, \mathfrak{a})\}(\underline{b}) \simeq_{\dot{\Omega}_{\dot{b}_0}} \{\lambda\underline{b}\alpha(\underline{b})\}(\underline{b})$; so $\lambda\underline{b}\chi(\underline{b}, \mathfrak{a}) \simeq_{\dot{\Omega}} \lambda\underline{b}\alpha(\underline{b})$. Likewise, $\lambda\underline{b}\alpha(\underline{b}) \simeq_{\dot{\Omega}} \alpha$. So (XXIX) Corollary 1 with $\underline{j} = 1$ assures the same results in computing $\rho(B, \mathfrak{a})$ under $\mathring{\Omega}$, whichever of $\lambda\underline{b}\chi(\underline{b}, \mathfrak{a})$, $\lambda\underline{b}\alpha(b)$, and α is the B.

In this supplement to Case 1 for (VII), we have used any given worth $\dot{\underline{b}}_0$ of $\underline{b}$, even though in §3 we were entertaining only type-0, -1, -2, and -3 values of the variables. However, to apply the replacement theorem (XXIX) Corollary 1 for $\underline{j} > 0$, we need that $\overline{D}$ and D have the same worth in the sense of (XXVII) (cf. 18.1), where $\gamma^{\underline{j}-1}$ ($\underline{b}$ in the present application) has to be interpreted by every type-$(\underline{j} - 1)^{\bullet}$ object. This is in addition to our transposing Ω into $\mathring{\Omega}$, to apply the present theory, and then transposing back again. All new variables introduced in 2.4 by E4.2, E4.3, and E7 will there be interpreted by values from types 0, 1, and 2, so we can transpose from Ω^{μ} to $\mathring{\Omega}^{\mu}$ and back again, at any moment μ in the computation via 2.4.

Continuing to (VIII), (IX), and (X), the replacements can be vindicated similarly.

A general principle is involved. We want to replace expressions by others ostensibly "equivalent." The "equivalences" may, e.g., rest on the use of (IV) to rewrite functions with more variables, or (VI) which suffixes to a derivation applications of S6.$\underline{j}$ to change the order of the variables, or of S3 used (before using (VI) and S4.0) to write $\underline{a}$ as $\chi(\mathfrak{a})$ where $\underline{a}$ is one of the $\mathfrak{a}$. Expressions ostensibly "equivalent" are built up in complicated ways. But in our practice in §3, the recognition of each "equivalence" is "uniform" in the sense that the computation steps to reduce one expression to the other, or the alteration of computation steps to compute one instead of the other as in the case of (IV), do not depend on the values of the variables. Therefore, from our extended viewpoint, the "equivalences" hold also for $\mathfrak{u}$ as the worth of a type-0 variable and above type 0 for objects not of the form $\dot{\alpha}^{\underline{j}}$. So when we apply λ-prefixes, the requirements of (XXVII) to use (XXIX) are met, as, e.g., in the proof for (VIII) that $\lambda\underline{b}\chi_2(\underline{b}, \mathfrak{F}, \mathfrak{a})$ has the same worth as $\lambda\underline{b}\chi_2(\underline{b}, \mathfrak{a})$ with $\mathfrak{u}$ in the range of $\underline{b}$. Similarly, in the proof of (X) Case E2, $\lambda\underline{c}\chi(\underline{c}, \mathfrak{a})$ has the same worth as $\lambda\underline{c}B$, because $\chi(\underline{c}, \mathfrak{a})$ is "equivalent" to B by previous steps holding irrespective of any value of $\underline{c}$.

An example of a "nonuniform equivalence" is that of $\chi_1(\underline{b}, \underline{a}) \simeq 0$ (by S0) to $\chi_2(\underline{b}, \underline{a}) \simeq \underline{b} \dot{-} \underline{b}$ for $\dot{-}$ as in 5.1, paragraph 2; $\chi_2(\underline{b}, \underline{a})$ is only equivalent to 0 by computations which have a different form for each of 0, 1, 2, ... as the value of $\underline{b}$ (and in fact there is no such computation for $\mathfrak{u}$ as the worth of $\underline{b}$).

19.4. To give the supplementary argument for the proof of (V) (cf. paragraph 1 and 3 of footnote 2 of 1982), I prove by induction over completed computations of 0-expressions E defined relative to a derivation $\phi_1, \ldots, \phi_{\underline{p}}$ of ϕ $(= \phi_{\underline{p}})$ from Θ (cf. 2.2, 2.3) that, if E contains none of the symbols $\overline{\Theta}$ and is computed under 2.4 via $\phi_1, \ldots, \phi_{\underline{p}}$ from Θ as $\underline{w}$, then its counterpart $\overline{E}$ (obtained by replacing each $\phi_{\underline{i}}$ by $\overline{\phi}_{\overline{\underline{i}}}$ as explained in the proof of (V)) receives the same value $\underline{w}$ when computed under 2.4 via $\overline{\phi}_1, \ldots, \overline{\phi}_{\overline{\underline{p}}}$ from Σ. This, of course, is relative to a given choice of Θ, Σ, as stated in the proof of (V), paragraph 2. Applying this result to the case of computing simply $\phi(\mathfrak{a})$ for a given set of values of the variables $\mathfrak{a}$ gives us what we need. (The supplementary argument called for at the end of paragraph 3 enters in the course of doing this.)

As a completed computation is built up from right to left (under 2.4), the only case in which we have just put an expression E not containing any $\theta_{\underline{t}}$ at a vertex when not all of the vertices next rightward (horizontally or downward) also bear Θ-free expressions (so that the hyp. ind. does not apply to all of them) is when the step is by E2 with $\phi_{\underline{i}}$ introduced by S0, so that we have, e.g.,

$$\phi_{\underline{i}}(A, B_1, B_2, C, Z) \text{---} \theta_{\underline{t}}(A, B_1, B_2, C).$$

Next downward-rightward from $\theta_{\underline{t}}(A, B_1, B_2, C)$ under E7 are subcomputations of A, of $B(\underline{c}_1)$ under all natural numbers $\underline{b}_1$ as the values of $\underline{c}_1$, of $B(\underline{c}_2)$ likewise, and of $C(\gamma)$ under all type-1 functions as the values of γ. Since E, i.e., $\phi_{\underline{i}}(A, B_1, B_2, C, Z)$, is Θ-free, so are A, $B_1(\underline{c}_1)$, $B_2(\underline{c}_2)$, and $C(\gamma)$. So by the hyp. ind., $\overline{A}$, $\overline{B}_1(\underline{c}_1)$, $\overline{B}_2(\underline{c}_2)$, and $\overline{C}(\gamma)$ receive the same values via $\overline{\phi}_1, \ldots, \overline{\phi}_{\overline{\underline{p}}}$ from Σ in the respective subcomputations as A, $B_1(\underline{c}_1)$, $B_2(\underline{c}_2)$, and $C(\gamma)$ did via $\phi_1, \ldots, \phi_{\underline{p}}$ from Θ. Say that the assignment for $\overline{E}$ is Ω, and beginning the subcomputations the assignments are Ω^{A}, $\Omega^{\overline{B}_1}_{\underline{b}_1}$, $\Omega^{\overline{B}_2}_{\underline{b}_2}$, $\Omega^{\overline{C}}_{\beta}$ for various $\underline{b}_1$, $\underline{b}_2$, β as values of $\underline{c}_1$, $\underline{c}_2$, γ. By (XXXI), we get the same respective values by the rules of 7.1, 7.3, 7.4, 14.3, and 14.4 under each of $\mathring{\Omega}^{\overline{A}}$, $\mathring{\Omega}^{\overline{B}_1}_{\underline{b}_1}$, $\mathring{\Omega}^{\overline{B}_2}_{\underline{b}_2}$, $\mathring{\Omega}^{\overline{C}}_{\beta}$. Remember that, as remarked in the proof of (V), the original subcomputations for the E7 step via $\phi_1, \ldots, \phi_{\underline{p}}$ from Θ determine type-0, -1, -1, and -2 objects $\underline{a}$, α_1, α_2, α^2 such that $\theta_{\underline{t}}(\underline{a}, \alpha_1, \alpha_2, \alpha^2) = \underline{w}$ (written "$\underline{v}$" there); so by the theorem hypothesis, $\theta_{\underline{t}}(\varepsilon^0, \varepsilon_1, \varepsilon_2, \varepsilon^2)$ can be computed as $\underline{w}$

from Σ under the assignment $\underline{a}, \alpha_1, \alpha_2, \alpha^2$; so by (XXXI) $\theta_{\underline{t}}(\varepsilon^0, \varepsilon_1, \varepsilon_2, \varepsilon^2)$ can be computed as $\underline{w}$ from $\dot{\Sigma}$ under the assignment $\underline{a}, \alpha_1, \alpha_2, \dot{\alpha}^2$; and thus $\theta_{\underline{t}}(\varepsilon^0, \varepsilon_1, \varepsilon_2, \varepsilon^2, \zeta)$ as $\underline{w}$ under $\underline{a}, \alpha_1, \alpha_2, \dot{\alpha}^2, \mathrm{u}^1$.

Now consider what we may get by 7.1, 7.3, 7.4, 14.3, and 14.4 via $\overline{\phi}_1, \ldots, \overline{\phi}_{\underline{p}}$ from $\dot{\Sigma}$ for $\overline{B}_1(\underline{c}_1), \overline{B}_2(\underline{c}_2), \overline{C}(\gamma)$ for worths of the variables $\underline{c}_1, \underline{c}_2, \gamma$ not of the above forms $\underline{b}_1, \underline{b}_2, \beta$ (as, e.g., $\dot{\alpha}^2$ is defined only on extensions of β's, and is determined fully by the values of the β's). But throwing open the ranges of $\underline{c}_1, \underline{c}_2, \gamma$ to all of types $\dot{0}$, $\dot{0}$, $\dot{1}$ leads to $\overline{B}_1, \overline{B}_2, \overline{C}$ receiving worths (under the assignment $\dot{\Omega}$ reduced to them) for (XXVII) which are extensions of $\alpha_1, \alpha_2, \dot{\alpha}^2$. The worth of $\overline{Z}$ is in any case an extension of u^1. So, since $\theta_{\underline{t}}(\varepsilon^0, \varepsilon_1, \varepsilon_2, \varepsilon^2, \zeta)$, when renamed $\overline{\phi}_{\underline{\bar{t}}}(\varepsilon^0, \varepsilon_1, \varepsilon_2, \varepsilon^2, \zeta)$, receives the value $\underline{w}$ via $\overline{\phi}_1, \ldots, \overline{\phi}_{\underline{p}}$ from $\dot{\Sigma}$ under $\underline{a}, \alpha_1, \alpha_2, \dot{\alpha}^2, \mathrm{u}^1$, we can use five successive applications of (XXIX) for $\underline{j} = 0$ to replace $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon^2, \zeta$ by $\overline{A}, \overline{B}_1, \overline{B}_2, \overline{C}, \overline{Z}$ to obtain a computation of $\overline{\phi}_{\underline{\bar{t}}}(\overline{A}, \overline{B}_1, \overline{B}_2, \overline{C}, \overline{Z})$ under $\dot{\Omega}$ via $\overline{\phi}_1, \ldots, \overline{\phi}_{\underline{p}}$ from Σ with the same result w. By (XXXI), there is hence one under Ω via $\overline{\phi}_1, \ldots, \overline{\phi}_{\underline{p}}$ from Σ under the rules of 2.4, which is what remained to be shown for the induction step.

19.5. We have now reached a major objective. *Even if* we are only interested in evaluating, e.g., $\phi(\mathfrak{a}) \simeq \phi_{\underline{p}}(\mathfrak{a})$ for an assignment of values of the types 0, 1, 2, 3 of 1.1 to $\mathfrak{a}$ (and of functions of variables of the types 0, 1, 2 to the Θ), we may in pursuing computation encounter, e.g., $\phi_{\underline{i}}(A_1, \ldots, D_{\underline{n}_{\underline{i}_3}})$ with quite complicated $A_1, \ldots, D_{\underline{n}_{\underline{i}_3}}$. If we start with the free variables $\mathfrak{a}$ assigned values of types 0, 1, 2, 3, the free variables $\mathscr{L}$ in $A_1, \ldots, D_{\underline{n}_{\underline{i}_3}}$ will also be assigned values of the types 0, 1, 2, 3. But the expressions $A_1, \ldots, D_{\underline{n}_{\underline{i}_3}}$ may not all of them be interpretable using just those types.

However, if we operate in the world of types $\dot{0}$, $\dot{1}$, $\dot{2}$, $\dot{3}$ of §5 ff. and construe each $\alpha^{\underline{j}}$ of type $\underline{j}$ as represented in that world by $\dot{\alpha}^{\underline{j}}$ of 19.1 (and now if we wish we may start with some of $\mathfrak{a}$ interpreted by $\dot{\alpha}^{\underline{j}}$'s that are not $\dot{\alpha}^{\underline{j}}$'s, i.e., not representing type-$\underline{j}$ objects), then by (XXVII) $A_1, \ldots, D_{\underline{n}_{\underline{i}_3}}$ will have interpretations $\dot{\mathscr{L}}$. By (XXIX) Corollary 1, the worth of $\phi_{\underline{i}}(A_1, \ldots, D_{\underline{n}_{\underline{i}_3}})$ is the same as when we replace $A_1, \ldots, D_{\underline{n}_{\underline{i}_3}}$ by variables having those worths $\dot{\mathscr{L}}$. Thus $\phi_{\underline{i}}(A_1, \ldots, D_{\underline{n}_{\underline{i}_3}})$ expresses the value of $\phi_{\underline{i}}(\mathscr{L})$, for $\dot{\mathscr{L}}$ as the worths of $\mathscr{L}$, of a partial recursive function of those variables on the extended types.

With our extended types we are no longer *restricted* to thinking formally in computing, but we have semantical interpretations of all the expressions under which the schemata are true and the computation rules correct.

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