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ALTERNATING SEQUENCES AND INDUCED OPERATORS

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ABSTRACT. We show that when a positive L_p contraction is equipped with a norming function having full support, then it is related in a natural way to an operator on any other L_p space, $1 < p < \infty$. This construction is used to generalize a theorem of Rota concerning the convergence of alternating sequences.

1. INTRODUCTION

Let L_p be the usual Banach space of complex-valued functions. Denote by L_p^+ the class of L_p functions taking nonnegative values. An L_p operator T is positive if $TL_p^+ \subseteq L_p^+$. It is a contraction if $\|Tf\|_p \le \|f\|_p$ for every $f \in L_p$. We say u is *semi-invariant* for a positive L_p contraction T if both u and Tu have full support and $\|Tu\|_p = \|u\|_p$.

(1.1) **Theorem.** *Suppose $1 < p < \infty$ and $1 < r < \infty$. If T is a positive L_p contraction with a semi-invariant function u, then the formula*

$$T_r f = (Tu)^{p/r-1} T(u^{1-p/r} f),$$

where $f \in L_r$, defines a positive L_r contraction. This operator is independent of the choice of semi-invariant function. We call T_r the L_r operator ***induced*** *by T.*

We apply this notion of induced operators to the question of convergence of alternating sequences. For simplicity of notation, the following theorem is stated for L_p^+ only. The analogous result is proved for all of L_p. T^* denotes the adjoint of T; it is an operator on L_q where $q = p(p-1)^{-1}$. Whenever u is semi-invariant for an L_p operator T, then $(Tu)^{p-1}$ is semi-invariant for T^*.

(1.2) **Theorem.** *Suppose $1 < p < \infty$ and $1 < r < \infty$. Let $\langle T_n \rangle_{n=1}^{\infty}$ be a sequence of positive L_p contradictions with semi-invariant functions defined over a σ-finite*

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Lebesgue space. Then

$$(T_1^*)_r \cdots (T_n^*)_r (T_n \cdots T_1 f)^{p/r}$$

converges a.e. for every $f \in L_p^+$.

This theorem generalizes Rota's theorem of the alternating procedure [Rt]. We say an operator is bistochastic if $T\mathbf{1} = T^*\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the function taking the value 1 everywhere.

(1.3) **Theorem** (Rota). *If* $\langle T_n \rangle_{n=1}^{\infty}$ *is a sequence of positive bistochastic operators over a probability space, then*

$$T_1^* \cdots T_n^* T_n \cdots T_1 f \tag{1.4}$$

converges a.e. for every $f \in L_p$, *where* $1 < p < \infty$.

A positive bistochastic operator is a contraction of every L_p, where $1 \leq p \leq \infty$; thus the expression (1.4) is well defined for every p. A positive L_p contraction with a semi-invariant function does not necessarily have this property, but we may use the operator induced by T^* to define a "pseudo-adjoint" of T which operates on L_p.

In the finite measure case, $\mathbf{1}$ is semi-invariant for any bistochastic operator and for its adjoint. Furthermore, $T_r^* = T^*$ for any r, $1 < r < \infty$. Thus, Rota's theorem is a consequence of (1.1) with $r = p$.

2. Preliminaries

(2.1) **Definitions.** For any σ-finite measure space $(X, \mathscr{F}, \mu)$, let $\mathscr{M}(d\mu)$ be the vector space of $\mathscr{F}$-measurable complex-valued functions defined on X. Let $\mathscr{M}^+(d\mu)$ be the class of functions in $\mathscr{M}(d\mu)$ whose ranges are subsets of $\mathbb{R}^+ = [0, \infty)$. Let $\overline{\mathscr{M}}^+(d\mu)$ be the set of $\mathscr{F}$-measurable functions on X with values in the extended nonnegative reals, $[0, \infty]$.

The usual Banach space of functions in $\mathscr{M}(d\mu)$ for which $\int_X |f|^p \, d\mu < \infty$ is denoted by $L_p(d\mu)$, where $1 \leq p < \infty$, while $L_\infty(d\mu)$ denotes the space of essentially bounded functions $\mathscr{M}(d\mu)$. We also use $L_p^+(d\mu) = L_p(d\mu) \cap \mathscr{M}^+(d\mu)$. All of the relations between the functions in these classes are in the μ-a.e. sense, even when this is not made explicit. With the convention $0 \cdot \infty = 0$, functions in $\overline{\mathscr{M}}^+(d\mu)$ may be multiplied pointwise.

Let $(Y, \mathscr{G}, \nu)$ be another σ-finite measure space. Consider the class of all mappings

$$T: \overline{\mathscr{M}}^+(d\mu) \to \overline{\mathscr{M}}^+(d\nu)$$

which satisfy the following two conditions:

(2.2) T is "positive-linear"; that is, if $\alpha, \beta \in \mathbb{R}^+$ and $f, g \in \overline{\mathscr{M}}^+(d\mu)$, then

$$T(\alpha f + \beta g) = \alpha T f + \beta T g.$$

(2.3) T is "order-continuous" in the sense that $Tf_n \uparrow Tf$ ν-a.e. whenever $f_n \uparrow f$ μ-a.e. (the arrows indicate monotone nondecreasing pointwise convergence in $\mathbb{R}^+$).

If T is such a mapping, then its restriction to $\mathscr{M}^+(d\mu)$ need not be extendable linearly to $\mathscr{M}(d\mu)$. Thus, these mappings should not necessarily be associated with the usual class of linear operators. Nonetheless, it is convenient to make the following definition.

(2.4) Definition A mapping satisfying (2.2) and (2.3) will be called a *positive operator* on $\overline{\mathscr{M}}^+(d\mu)$ (or from $\overline{\mathscr{M}}^+(d\mu)$ to $\overline{\mathscr{M}}^+(d\nu)$).

(2.5) **Lemma.** *Given a positive operator* $T: \overline{\mathscr{M}}^+(d\mu) \to \overline{\mathscr{M}}^+(d\nu)$ *there exists a unique positive operator* $T^*: \overline{\mathscr{M}}^+(d\nu) \to \overline{\mathscr{M}}^+(d\mu)$ *such that*

$$\int_X fT^*g\,d\mu = \int_Y Tf\cdot g\,d\nu$$

for every $f \in \overline{\mathscr{M}}^+(d\mu)$ *and* $g \in \overline{\mathscr{M}}^+(d\nu)$.

Proof. Given $g \in \overline{\mathscr{M}}^+(d\nu)$, the mapping

$$f \in \overline{\mathscr{M}}^+(d\mu) \mapsto \int_Y Tf\cdot g\,d\nu \in \mathbb{R}^+$$

is integration with respect to some measure on $(X, \mathscr{F})$ which is absolutely continuous with respect to μ. This measure may be represented as $\rho\,d\mu$ for some $\rho \in \overline{\mathscr{M}}^+(d\mu)$. Define T^* by $T^*g = \rho$. □

(2.6) **Definition.** The operator T^* defined above is called the *adjoint* of T.

If $T: L_p(d\mu) \to L_p(d\nu)$ is a positive operator in the usual sense, then its restriction to $L_p^+(d\mu)$ can be extended to a positive operator on $\overline{\mathscr{M}}^+(d\mu)$, which will also be called T. It is unique by the requirement that it satisfy (2.3). If a positive operator on $\overline{\mathscr{M}}^+(d\mu)$ in the sense of (2.4) can be obtained in this way, then we will call it a positive L_p operator on $\overline{\mathscr{M}}^+(d\mu)$. The following definition states this in a different way.

(2.7) **Definition.** A positive operator T on $\overline{\mathscr{M}}^+(d\mu)$ is said to be a positive L_p operator if

$$\|T\|_p^p = \sup\left\{\int (Tf)^p\,d\nu \,\middle|\, f \in \overline{\mathscr{M}}^+(d\mu) \text{ and } \int f^p\,d\mu \le 1\right\}$$

is finite. If, furthermore, $\|T\|_p \le 1$, then T is called a positive L_p contraction.

Throughout this paper, whenever a number p with $1 < p < \infty$ is understood, then q denotes the adjoint index; that is, the number $p(p-1)^{-1}$. Note that T is a positive L_p operator if and only if T^* is a positive L_q operator. In this case, the definition of the adjoint operator agrees with the usual definition in the Banach space sense.

The following theorem is a standard result. Under the hypothesis one easily shows that the operator is a contraction of both L_1 and L_∞. The conclusion then follows by the Riesz convexity theorem.

(2.8) **Theorem.** *Let T be a positive operator such that $T\mathbf{1} \leq \mathbf{1}$ and $T^*\mathbf{1} \leq \mathbf{1}$. Then T is a positive L_p contraction for all p, $1 \leq p \leq \infty$.*

(2.9) **Definition.** If T is a positive L_p operator and $u \in L_p$ is a function satisfying $\|Tu\|_p = \|T\|_p \|u\|_p$, we say that u is a *norming function* for T. We say that u is *semi-invariant* for T if $\|Tu\|_p = \|u\|_p$ and both u and Tu are strictly positive a.e. A semi-invariant function for a contraction is clearly a norming function.

(2.10) **Lemma.** *If u is a norming function for a positive L_p operator T, then*

$$T^*(Tu)^{p-1} = \|T\|_p^p u^{p-1}.$$

Consequently, if u is semi-invariant for a positive contraction T, then $(Tu)^{p-1}$ is semi-invariant for T^*.

Proof.

$$\begin{aligned} \|Tu\|_p^p &= \int (Tu)(Tu)^{p-1}\, d\nu = \int uT^*(Tu)^{p-1}\, d\mu \\ &\leq \|u\|_p \|T^*(Tu)^{p-1}\|_q \leq \|u\|_p \|T^*\|_q \|(Tu)^{p-1}\|_q \\ &= \|u\|_p \|T\|_p \|Tu\|_p^{p-1} = \|Tu\|_p^p, \end{aligned}$$

where the first inequality follows from Hölder's inequality. Thus, we have equality in Hölder's inequality, and so $T^*(Tu)^{p-1}$ is a constant multiple of u^{p-1}. □

(2.11) **Definition.** Suppose T is a positive operator on $\overline{\mathscr{M}}^+(d\mu)$. A set $E \in \mathscr{F}$ is called a *reducing set* for T if $T(\chi_E) \cdot T(\mathbf{1} - \chi_E) = 0$, where χ_E is the characteristic function of the set E.

(2.12) **Lemma.** *The support of a norming function is a reducing set.*

Proof. Let u be a norming function for T, and E be the support of u. Then

$$\begin{aligned} \int (Tu)^{p-1} T(\mathbf{1} - \chi_E)\, d\nu &= \int T^*(Tu)^{p-1}(\mathbf{1} - \chi_E)\, d\mu \\ &= \|T\|_p^p \int u^{p-1}(\mathbf{1} - \chi_E)\, d\mu = 0. \end{aligned}$$

Hence $(Tu)^{p-1}T(\mathbf{1} - \chi_E) = 0$, and so $(Tu)T(\mathbf{1} - \chi_E) = 0$. Now approximate $\frac{1}{u}\chi_E$ from below by simple functions. Conclude by (2.3) and positivity that $T(\chi_E)T(\mathbf{1} - \chi_E) = 0$. □

The following lemma concerning functions of a real variable is needed. Observe that the conclusion of the lemma remains valid if we replace t^r in the hypothesis by any differentiable function which is strictly monotone almost everywhere.

(2.13) **Lemma.** *Let $\phi, \theta: \mathbb{R}^+ \to \mathbb{R}^+$ be measurable functions satisfying*

$$\begin{aligned} &\int_0^\infty \phi(t)\, dt = \int_0^\infty \theta(t)\, dt < \infty, \\ (2.14) \qquad &\int_0^\alpha \phi(t)\, dt \leq \int_0^\alpha \theta(t)\, dt, \end{aligned}$$

and

$$\int_0^\infty t^r\phi(t)\,dt = \int_0^\infty t^r\theta(t)\,dt \tag{2.15}$$

for every $\alpha \geq 0$ *and some* $r > 0$. *Then* $\phi = \theta$ *a.e.*

Proof.

$$\begin{aligned}\int_0^\infty t^r\phi(t)\,dt &= \int_0^\infty rs^{r-1}\left(\int_s^\infty \phi(t)\,dt\right)\,ds \\ &\geq \int_0^\infty rs^{r-1}\left(\int_s^\infty \theta(t)\,dt\right)\,ds = \int_0^\infty t^r\theta(t)\,dt.\end{aligned}$$

By (2.15), we have equality. Thus, the set of points at which inequality (2.14) is strict has measure zero. Since

$$\int_0^\alpha \phi(t)\,dt = \int_0^\alpha \theta(t)\,dt$$

for a.a. α, and ϕ and θ are positive functions, it follows that $\phi = \theta$ a.e., as desired. □

(2.16) **Definition.** A point transformation $\tau\colon X \to X$ is called an *automorphism* if it is invertible and both τ and τ^{-1} are measurable and nonsingular. An automorphism induces two measures, $\mu\circ\tau^{-1}$ and $\mu\circ\tau$, both absolutely continuous with respect to μ. Let ρ denote the Radon-Nikodým derivative of $\mu\circ\tau^{-1}$ with respect to μ. If $1 \leq p < \infty$, then define $Q\colon L_p \to L_p$ by

$$Qf = \rho^{1/p}(f\circ\tau^{-1})$$

for $f \in L_p$. We call Q the L_p *isometry induced by* τ.

(2.17) **Lemma.** *If* Q *is the* L_p *isometry induced by an automorphism* τ, *then* Q^{-1} *is the* L_p *isometry induced by* τ^{-1} *and* Q^* *is the* L_q*-isometry induced by* τ^{-1}.

Proof. This follows immediately from the definitions if one observes that when ρ is the Radon-Nikodým derivatives of $\mu\circ\tau^{-1}$ with respect to μ, then the Radon-Nikodým derivatives of $\mu\circ\tau$ with respect to μ is $1/(\rho\circ\tau)$. □

(2.18) **Definition.** Suppose $1 \leq p < \infty$ and $1 \leq r < \infty$. Define $\psi_{p,r}\colon L_p \to L_r$ by means of the equation

$$[\psi_{p,r}(f)](x) = \operatorname{sign}(f(x))|f(x)|^{p/r},$$

where $\operatorname{sign}(z)$ is the complex number of unit modulus having the same argument as z. When p and r are understood, we refer to this embedding simply as ψ. Usually f^* is used to represent $\psi_{p,q}f$. Perhaps the most important property of $\psi_{p,r}$ is that when $f \in L_p$, then $\|\psi_{p,r}f\|_r = \|f\|_p^{p/r}$.

(2.19) **Lemma.** *Let* $1 \leq p < \infty$ *and* $1 \leq r < \infty$. *Suppose* Q_p *and* Q_r *are, respectively, the* L_p *and* L_r *isometries induced by an automorphism* τ. *If* $\psi = \psi_{p,r}$ *and* $f \in L_p$, *then*

$$Q_r \psi f = \psi Q_p f .$$

Proof.

$$\begin{aligned} Q_r \psi f &= \rho^{1/r}[\operatorname{sign}(f)|f|^{p/r}] \circ \tau^{-1} \\ &= \operatorname{sign}(f \circ \tau^{-1}) \rho^{1/r} |f \circ \tau^{-1}|^{p/r} \\ &= \operatorname{sign}[\rho^{1/p}(f \circ \tau^{-1})] |\rho^{1/p}(f \circ \tau^{-1})|^{p/r} \\ &= \psi Q_p f . \quad \square \end{aligned}$$

(2.20) **Definition.** When $(X, \mathscr{F}, \mu)$ is a measure space and $\mathscr{F}'$ is a sub-σ-algebra of $\mathscr{F}$, then $E(\cdot|\mathscr{F}')$ denotes the conditional expectation operator with respect to $\mathscr{F}'$. We adopt the convention that $E(f|\mathscr{F}')$ is 0 on any atom of $\mathscr{F}'$ of infinite measure.

(2.21) **Theorem** (Martingale convergence theorem for finite σ-algebras). *Let* $(X, \mathscr{F}, \mu)$ *be a* σ*-finite measure space. For each* $k \geq 1$, *suppose* $\mathscr{G}_k$ *is a finite sub-*σ*-algebra of* $\mathscr{F}$ *and* $\mathscr{G}_k \subseteq \mathscr{G}_{k+1}$. *Let* $\mathscr{G}_\infty = \sigma(\bigcup_{k=1}^\infty \mathscr{G}_k)$, *the smallest* σ*-algebra containing the algebra* $\bigcup_{k=1}^\infty \mathscr{G}_k$. *Suppose* $1 \leq p < \infty$ *and* $f \in L_p(d\mu)$. *Let* $f_k = E(f|\mathscr{G}_k)$ *for* $1 \leq k \leq \infty$. *Then* $f_k \to k$ *a.e. and in* L_p *norm.*

If $p > 1$, *then the* f_k*'s have a maximal function; more precisely, there is a function* $g \in L_p^+$ *with* $|f_k| \leq g$ *for every* $k \geq 1$, *and* $\|g\|_p \leq q\|f\|_p$.

Proof. See any reference on martingales, e.g. [S, pp. 89-94].

(2.22) **Lemma.** *Let* $\langle \mathscr{G}_k \rangle_{k=1}^\infty$ *be as in the previous theorem and suppose* $\langle \mathscr{H}_k \rangle_{k=1}^\infty$ *is another monotone sequence of finite sub-*σ*-algebras of* $\mathscr{F}$. *Let*

$$\mathscr{H}_\infty = \sigma\left(\bigcup_{k=1}^\infty \mathscr{H}_k\right).$$

Let $f \in L_p^+(d\mu)$, *where* $1 < p < \infty$, *and* $f_k = E(f|\mathscr{G}_k)$. *Then*

$$E(f_k^p|\mathscr{H}_k) \to E(f_\infty^p|\mathscr{H}_\infty)$$

a.e. and in L_1 *norm.*

Proof. Let $\phi_k = f_k^p$ for each $k \geq 1$. Then $g = \sup f_k \in L_p$ by the martingale convergence theorem. Thus $0 \leq \phi_k \leq \theta = g^p \in L_1$, and $\phi_k \to \phi_\infty$ a.e. The proof is then completed by the following more general lemma.

(2.23) **Lemma.** *Let* $0 \leq \phi_k \leq \theta \in L_1$ *for* $k \geq 1$, *and let* $\phi_k \to \phi_\infty$ *a.e. Then* $E(\phi_k|\mathscr{G}_k) \to E(\phi_\infty|\mathscr{G}_\infty)$ *a.e. and in* L_1 *norm.*

Proof. Let

$$\xi_k = \inf_{n \geq k} \phi_n \quad \text{and} \quad \eta_k = \sup_{n \geq k} \phi_n .$$

Then $(\eta_k - \xi_k) \downarrow 0$ a.e. and in L_1 norm, by the dominated convergence theorem. We have, for any $n \geq k$,

$$E(\xi_k|\mathscr{G}_n) \leq E(\xi_n|\mathscr{G}_n) \leq E(\phi_n|\mathscr{G}_n) \\ \leq E(\eta_n|\mathscr{G}_n) \leq E(\eta_k|\mathscr{G}_n).$$

If $n \to \infty$ with k fixed, then

$$E(\xi_k|\mathscr{G}_\infty) \leq \underline{\lim}\, E(\phi_n|\mathscr{G}_n) \leq \overline{\lim}\, E(\phi_n|\mathscr{G}_n) \leq E(\eta_k|\mathscr{G}_\infty).$$

Thus

$$\|\overline{\lim}\, E(\phi_n|\mathscr{G}_n) - \underline{\lim}\, E(\phi_n|\mathscr{G}_n)\|_1 \\ \leq \|E(\eta_k|\mathscr{G}_\infty) - E(\xi_k|\mathscr{G}_\infty)\|_1 \leq \|\eta_k - \xi_k\|_1$$

which can be made arbitrarily small. This completes the proof. □

We will need the following four lemmas from [AS2], where they are numbered (2.2), (2.3), (2.5), and (2.8) respectively. L_p always refers to the case $1 < p < \infty$ over a σ-finite measure space.

(2.24) **Lemma.** *Let $f_k \in L_p$ for every k, $1 \leq k \leq n$. If $V: L_p \to L_p$ is a positive bounded linear operator, then*

$$\max_{1\leq k\leq n} |Vf_k| \leq V\left(\max_{1\leq k\leq n} |f_k|\right)$$

and, consequently,

$$\left\|\max_{1\leq k\leq n} |Vf_k|\right\|_p \leq \|V\| \cdot \left\|\max_{1\leq k\leq n} |f_k|\right\|_p.$$

(2.25) **Lemma.** *For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $E: L_p \to L_p$ is a conditional expectation operator, $f \in L_p$, and $\|f\|_p - \|Ef\|_p < \delta\|f\|_p$, then $\|f - Ef\|_p < \varepsilon\|f\|_p$.*

(2.26) **Lemma.** *Let $f_{km} \in L_p$ for every $m \geq 0$ and every k, $1 \leq k \leq n$. If $\lim_{m\geq 0} \|f_{km} - f_m\|_p = 0$ for each k, then*

$$\lim_{m\geq 0} \left\|\max_{1\leq k\leq n} |f_{km}| - \max_{1\leq k\leq n} |f_k|\right\|_p = 0.$$

(2.27) **Lemma.** *Let $\langle f_n\rangle_{n=0}^\infty$ be a sequence of functions in L_p such that $(\sup_{n\geq 0} |f_n|) \in L_p$. Then $\langle f_n\rangle_{n=0}^\infty$ converges a.e. if and only if*

$$\lim_{n\geq 0} \left\|\sup_{k\geq n} |f_k - f_n|\right\|_p = 0.$$

The following are analogous to Lemmas (2.6) and (2.7) in [AS2]. The first one follows from a result of Mazur [M], since the mapping $\psi_{p,r}$ may be regarded as a composition of his map F from L_1 to L_r and his map G from L_p to L_1, both uniformly continuous on the unit ball.

(2.28) **Lemma** (Uniform continuity of $\psi_{p,r}$). *Let $1 \le p < \infty$ and $1 \le r < \infty$. Given $\varepsilon > 0$ and $M > 0$, there is a $\delta > 0$ depending only on ε, M, p, and r such that $\|\psi f - \psi g\|_r < \varepsilon$ whenever $\|f\|_p \le M$, $\|g\|_p \le M$, and $\|f - g\|_p < \delta$.*

(2.29) **Lemma.** *Given $\varepsilon > 0$ and $M > 0$ there is a $\delta > 0$ depending only on ε, M, p, and r such that if $\langle f_k\rangle_{k=0}^{\infty}$ is a sequence in L_p with $\|\sup_{k\ge 0}|f_k|\|_p \le M$ and $\|\sup_{k\ge 0}|f_k - f_0|\|_p < \delta$ then*

$$\left\|\sup_{k\ge 0}|\psi f_k - \psi f_0|\right\|_r < \varepsilon.$$

Proof. Let δ be as given in the uniform continuity of ψ corresponding to $\varepsilon/2$, M, p, and r. Let $n \ge 1$ be given. Fix a partition $\{A_1, \dots, A_n\}$ of X such that

$$\max_{0\le k\le n}|\psi f_k - \psi f_0| = \sum_{m=1}^{n}|\psi f_m - \psi f_0|\chi_{A_m}.$$

Let $f = \sum_{m=1}^{n} f_m \chi_{A_m}$, so that

$$\max_{0\le k\le n}|\psi f_k - \psi f_0| = |\psi f - \psi f_0|.$$

We have $\|f\|_p \le M$, $\|f_0\|_p \le M$, and $\|f - f_0\| \le \|\sup_{k\ge 0}|f_k - f_0|\|_p$. Therefore, if this last norm is less than δ, the uniform continuity of ψ implies that $\|\psi f - \psi f_0\|_r < \varepsilon/2$. This completes the proof. □

We also need the following, which is an immediate consequence of

$$\|T_n f_n - Tf\|_p \le \|T_n\| \cdot \|f_n - f\|_p + \|T_n f - Tf\|_p.$$

(2.30) **Lemma.** *Suppose $\langle T_n\rangle_{n=1}^{\infty}$ and T are L_p contractions and*

$$\lim_{n\ge 1}\|T_n f - Tf\|_p = 0$$

whenever $f \in L_p$. If $f_n \to f$ in L_p norm, then

$$\lim_{n\ge 1}\|T_n f_n - Tf\|_p = 0.$$

3. Induced operators

In this section, we will be interested primarily in positive L_p operators with strictly positive norming functions. We begin, however, with two more general lemmas.

(3.1) **Lemma.** *Let T be a positive operator on $\overline{\mathscr{M}}^{+}(d\mu)$. Suppose $u \in \mathscr{M}^{+}(d\mu)$ is strictly positive. If there is a $\lambda \in \mathbb{R}^{+}$ such that*

$$T^*(Tu)^{p-1} \le \lambda^p u^{p-1}, \tag{3.2}$$

then T is a positive L_p operator with $\|T\|_p \le \lambda$.

(3.3) *Remarks.* In the Borel case, this follows from a result in [AS1] concerning dilations. The general case was considered in [K1]. We have included the following short proof to make this paper more self-contained.

Proof. If $\lambda = 0$, it is easy to see that $T = 0$, since $\int (Tu)^{p-1}(Tf)\,d\mu = 0$ for every $f \in \overline{\mathscr{M}}^+(d\mu)$.

Suppose $\lambda > 0$ and let $v = Tu$. Because of (3.2), the σ-finiteness of μ and the fact that u is finite a.e., one argues that v is finite a.e. (The proof is essentially contained in [AS1, p. 391].)

Let $d\mu' = u^p\,d\mu$ and $d\nu' = (v/\lambda)^p\,d\nu$. Define an operator $R\colon \overline{\mathscr{M}}^+(d\mu') \to \overline{\mathscr{M}}^+(d\nu')$ by $Rf = \chi_G \frac{1}{v} T(uf)$ for $f \in \overline{\mathscr{M}}^+(d\mu')$, where G is the support of v. This is clearly a positive operator in the sense of (2.4). A routine computation shows that the adjoint, $R^*\colon \overline{\mathscr{M}}^+(d\nu') \to \overline{\mathscr{M}}^+(d\mu')$, is given by

$$R^* g = \frac{1}{\lambda^p u^{p-1}} T^*(v^{p-1} g)$$

for $g \in \overline{\mathscr{M}}^+(d\nu')$. Thus $R\mathbf{1} \le \mathbf{1}$ and $R^*\mathbf{1} \le \mathbf{1}$, so by Theorem (2.8), R is an L_p contraction. This means that if $f \in \overline{\mathscr{M}}^+(d\mu')$, then

$$\int (Rf)^p\,d\nu' \le \int f^p\,d\mu'.$$

If $f \in \overline{\mathscr{M}}^+(d\mu)$, then $f = u\tilde{f}$ for some $\tilde{f} \in \overline{\mathscr{M}}^+(d\mu')$. Hence

$$\begin{aligned}\int (Tf)^p\,d\nu &= \int [T(u\tilde{f})]^p\,d\nu = \lambda^p \int (R\tilde{f})^p\,d\nu' \\ &\le \lambda^p \int \tilde{f}^p\,d\mu' = \lambda^p \int f^p\,d\mu.\end{aligned}$$

This shows that T is an L_p operator with $\|T\|_p \le \lambda$. □

(3.4) **Lemma.** *Let T be a positive operator on $\overline{\mathscr{M}}^+(d\mu)$. Suppose $u \in \mathscr{M}^+(d\mu)$ is strictly positive, and that there is a $\lambda \in \mathbb{R}^+$ such that*

$$T^*(Tu)^{p-1} \le \lambda^p u^{p-1}.$$

Let $v = Tu$ and let G be the support of v. Let r be any exponent, $1 < r < \infty$. Then

$$Sf = \chi_G \left(\frac{v}{\lambda}\right)^{p/r-1} T(u^{1-p/r} f),$$

for $f \in \overline{\mathscr{M}}^+(d\mu)$, defines a positive L_r operator $S\colon \overline{\mathscr{M}}^+(d\mu) \to \overline{\mathscr{M}}^+(d\nu)$ with $\|S\|_r \le \lambda$.

Proof. $S^*\colon \overline{\mathscr{M}}^+(d\nu) \to \overline{\mathscr{M}}^+(d\mu)$ is easily calculated; one sees that for $g \in \overline{\mathscr{M}}^+(d\nu)$,

$$S^* g = (\lambda u)^{1-p/r} T^*(v^{p/r-1} \chi_G g).$$

Let $\tilde{u} = u^{p/r}$. Then $\tilde{u}$ is strictly positive a.e., and $S^*(Su)^{r-1} \le \lambda^r \tilde{u}^{r-1}$. Thus, Lemma (3.1) completes the proof. □

(3.5) **Lemma.** *Suppose u_1 and u_2 are strictly positive norming functions for a positive L_p operator T on $\overline{\mathscr{M}}^+(d\mu)$. For any $\alpha \in \mathbb{R}^+$, the set*

$$E_\alpha = \left\{ x \in X \,\middle|\, \frac{u_2(x)}{u_1(x)} > \alpha \right\}$$

is a reducing set for T.

Proof. As in the proof of Lemma (3.1), let $d\mu' = u_1^p\, d\mu$ and $d\nu' = (v_1/\lambda)^p\, d\nu$, where $v_1 = Tu_1$ and $\lambda = \|T\|_p$. Observe that even if v_1 is not strictly positive a.e., its support is equal to the support of $v_2 = Tu_2$ a.e. Without loss of generality then, we may replace the set Y with this common support. Define $R: \overline{\mathscr{M}}^+(d\mu') \to \overline{\mathscr{M}}^+(d\nu')$ for $f \in \overline{\mathscr{M}}^+(d\mu')$ by $Rf = T(u_1 f)/v_1$.

$R\mathbf{1} = R^*\mathbf{1} = \mathbf{1}$, so R is an L_p contraction. $\mathbf{1}$ is a norming function for R; we now show that $u = u_2/u_1$ is another. One may verify that $R^*(Ru)^{p-1} = u^{p-1}$, from which $\|Ru\|_p = \|u\|_p$ easily follows. Let $v = Ru$.

Let $\alpha \geq 0$ be arbitrary. Let $u_\alpha = u \wedge \alpha$, the function u truncated at the value α. Observe that E_α is the support of $u - u_\alpha$. Also note that $Ru_\alpha \leq v_\alpha = v \wedge \alpha$, hence

$$\int u_\alpha\, d\mu' = \int Ru_\alpha\, d\nu' \leq \int v_\alpha\, d\nu'. \tag{3.6}$$

Let $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be the distribution of u; that is, $\phi(t) = \mu'\{x: u(x) \geq t\}$. Let θ be the distribution of v, similarly defined with respect to ν'. Inequality (3.6) has the equivalent form

$$\int_0^\alpha \phi(t)\, dt \leq \int_0^\alpha \theta(t)\, dt. \tag{3.7}$$

Since $\|u\|_p = \|v\|_p$, we have

$$\int_0^\infty t^{p-1}\phi(t)\, dt = \int_0^\infty t^{p-1}\theta(t)\, dt. \tag{3.8}$$

Finally, $u \in L_1(d\mu')$, since $p > 1$ and μ' is a finite measure. Since $\|u\|_1 = \|v\|_1$, we have

$$\int_0^\infty \phi(t)\, dt = \int_0^\infty \theta(t)\, dt < \infty. \tag{3.9}$$

Conditions (3.7)–(3.9) allow us to invoke Lemma (2.13) and conclude that $\phi = \theta$ a.e. in Lebesgue measure. Since

$$\|u - u_\alpha\|_p^p = p \int_\alpha^\infty t^{p-1}\phi(t)\, dt,$$

we have

$$\|u - u_\alpha\|_p = \|v - v_\alpha\|_p \leq \|R(u - u_\alpha)\|_p,$$

where the inequality follows because $Ru_\alpha \leq v_\alpha$. As R is a contraction, we conclude that the norms are in fact equal. Thus, $u - u_\alpha$ is a norming function. By Lemma (2.12), then, its support is a reducing set for R. It easily follows that E_α also reduces T. □

(3.10) *Remarks.* One may replace the "less than" in the definition of E_α by any other inequality, simply by considering complements or reversing the roles of u_1 and u_2. The complement of a reducing set is a reducing set; it is also easy to show that the intersection of reducing sets is a reducing set. In fact, the class of reducing sets of a bounded L_p operator is a sub-σ-algebra of the underlying measure space. This is shown in [K2], which also includes a different proof of the above lemma.

(3.11) **Theorem.** *Suppose T is a positive L_p operator on $\overline{\mathcal{M}^+}(d\mu)$, and u_1 and u_2 are strictly positive norming functions for T. Let $v_i = Tu_i$ for $i = 1, 2$ and let G be the support of the v_i's. Let $1 < r < \infty$, and define positive operators S_1 and S_2 on $\overline{\mathcal{M}}^+(d\mu)$ by*

$$S_i f = \chi_G \|T\|_p^{1-p/r} v_i^{p/r-1} T(u_i^{1-p/r} f)$$

for $f \in \overline{\mathcal{M}}^+(d\mu)$ and $i = 1, 2$. Then $S_1 f = S_2 f$ a.e. for every $f \in \overline{\mathcal{M}}^+(d\mu)$.

Proof. By (2.3), it suffices to consider $f \in \mathcal{M}^+(d\mu)$.

Let $s = p/r - 1$. If $s = 0$, there is nothing to prove. Otherwise, let $\varepsilon > 0$ be given, and choose a positive integer $N > 1/\varepsilon$.

For each $n \geq 1$, let

$$E_n = \left\{ x \in X \,\middle|\, \frac{N+n-1}{N} < \frac{u_2(x)}{u_1(x)} \leq \frac{N+n}{N} \right\}$$

and

$$E_{-n} = \left\{ x \in X \,\middle|\, \frac{N+n-1}{N} < \frac{u_1(x)}{u_2(x)} \leq \frac{N+n}{N} \right\}.$$

Also, let E_0 be the set of points in A where $u_1(x) = u_2(x)$. Then $\{E_n | n \in \mathbb{Z}\}$ is a partition of X into reducing sets.

Let $f \in \overline{\mathcal{M}}^+(d\mu)$ be given and let $f_n = f\chi_{E_n}$ for every $n \in \mathbb{Z}$. The f_n's have disjoint support, as do the functions $T(u_1^{-s} f_n)$ and $T(u_2^{-s} f_n)$.

Now suppose $n \geq 1$ and $s > 0$. Since T is positive, we have

$$\left(\frac{N}{N+n}\right)^s T\left(\frac{f_n}{u_1^s}\right) \leq T\left(\frac{f_n}{u_2^s}\right) \leq \left(\frac{N}{N+n-1}\right)^s T\left(\frac{f_n}{u_1^s}\right). \tag{3.12}$$

Let $u_{in} = u_i \chi_{E_n}$ and $v_{in} = T(u_{in})$ for every $n \in \mathbb{Z}$ and $i = 1, 2$. The functions $T(u_i^{-s} f_n)$ and v_{mi} will have disjoint supports unless $m = n$; thus $S_i f_n$ depends only on $T(u_i^{-s} f_n)$ and v_{ni}^s. We have

$$\left(\frac{N+n-1}{N}\right)^s v_{n1}^s \leq v_{n2}^s \leq \left(\frac{N+n}{N}\right)^s v_{n1}^s. \tag{3.13}$$

Therefore,

$$\left(\frac{N+n-1}{N+n}\right)^s S_1 f_n \leq S_2 f_n \leq \left(\frac{N+n}{N+n-1}\right)^s S_1 F_N. \tag{3.14}$$

If $(S_1 f_n)(x) = 0$, then $(S_2 f_n)(x)$ must be zero as well. Otherwise,

$$\left| \left(\frac{(S_2 f_n)(x)}{(S_1 f_n)(x)}\right)^{1/s} - 1 \right| \leq \frac{1}{N+n-1} < \varepsilon. \tag{3.15}$$

If $s < 0$, then the order of the terms in (3.14) is reversed, but (3.15) remains valid.

If $n \leq -1$, the argument is symmetric, with the conclusion

$$\left| \left(\frac{(S_1 f_n)(x)}{(S_2 f_n)(x)} \right)^{1/s} - 1 \right| \leq \frac{1}{N+n-1} < \varepsilon .$$

It is clear that $S_1 f_0 = S_2 f_0$. Since $\varepsilon > 0$ is arbitrary, we conclude that $S_1 f_n = S_2 f_n$ a.e. for each $n \in \mathbb{Z}$. Thus $S_1 f = S_2 f$ a.e., as desired. □

(3.16) **Theorem.** *Suppose $1 < p < \infty$ and $1 < r < \infty$. Let T be a positive L_p operator with a strictly positive norming function u. Let $v = Tu$ and let G be the support of v. Then*

$$T_r f = \chi_g \|T\|^{1-p/r} v^{p/r-1} T(u^{1-p/r} f),$$

for $f \in \overline{\mathscr{M}}^+(d\mu)$, defines a positive L_r operator $T_r\colon \overline{\mathscr{M}}^+(d\mu) \to \overline{\mathscr{M}}^+(d\nu)$ such that $\|T_r\|_r = \|T\|_p$. This operator, called the L_r operator induced by T, is independent of the choice of u.

Proof. Whether T is given as an L_p operator in the Banach space sense or in the sense of Definition (2.4), it is clear that T_r is a positive operator in the sense of (2.4). Lemmas (2.10) and (3.4) combine to show that T_r is in fact an L_r operator with norm bounded by $\|T\|_p$. To see that this norm is actually achieved, let $f = u^{p/r}$. Theorem (3.11) demonstrates that T_r does not depend on the choice of norming function. □

(3.17) **Corollary.** *Suppose T is an L_p contraction with a semi-invariant function where $1 < p < \infty$. For every r, $1 < r < \infty$,*

$$T_r f = v^{p/r-1} T(u^{1-p/r} f)$$

defines a positive contraction of L_r.

(3.18) *Remarks.* If T is an L_p isometry induced by an automorphism τ (as in (2.16)), then T_r is simply the L_r isometry induced by τ. When the underlying space has finite measure, we may take $u = \mathbf{1}$ and $v = \rho^{1/p}$. The general σ-finite case is not much harder to check.

A larger and more important class of operators has the form EQE, where Q is an L_p isometry induced by an automorphism and E is a conditional expectation operator of finite rank. Such operators where crucial to the proof of the pointwise ergodic theorem for positive L_p contractions (see [A]). Thus, the following lemma is of some general interest as well as being necessary for §5 of this paper.

(3.19) **Lemma.** *Suppose $1 < p < \infty$, $1 < r < \infty$, and that Q_p and Q_r are, respectively, the L_p and L_r isometries induced by an automorphism τ over a measure space $(X, \mathscr{F}, \mu)$. Let $\widetilde{\mathscr{F}}$ be a sub-σ-algebra of $\mathscr{F}$ and let $\tilde{\mu}$ be the* *restriction of* μ *to* $\widetilde{\mathscr{F}}$. *Let* E *be conditional expectation with respect to* $\widetilde{\mathscr{F}}$ *and suppose*

$$T\colon L_p(X, \widetilde{\mathscr{F}}, \tilde{\mu}) \to L_p(X, \widetilde{\mathscr{F}}, \tilde{\mu})$$

is given by $T = EQ_pE$. *If* T *has a semi-invariant function* u, *then* $T_r = EQ_rE$.

Proof. Let $v = Tu$. For $f \in L_r(X, \widetilde{\mathscr{F}}, \tilde{\mu})$, we have

$$\begin{aligned} T_r f &= v^{p/r-1} T(u^{1-p/r} f) \\ &= v^{p/r-1} E(\rho^{1/p} (u \circ \tau^{-1})^{1-p/r} [(Ef) \circ \tau^{-1}]) \\ &= v^{p/r-1} E[\rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})], \end{aligned}$$

where the third line follows because f is already $\widetilde{\mathscr{F}}$-measurable. Because $\|v\|_p = \|u\|_p$, Q is an isometry and $p > 1$, we conclude that $Q_p u$ must already be $\widetilde{\mathscr{F}}$-measurable, lest some norm be lost in taking the conditional expectation. Thus $v = \rho^{1/p}(u \circ \tau^{-1})$ and

$$\begin{aligned} T_r f &= E[v^{p/r-1} \rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})] \\ &= E[(\rho^{1/p})^{p/r-1} (u \circ \tau^{-1})^{p/r-1} \rho^{1/p} (u \circ \tau^{-1})^{1-p/r} (f \circ \tau^{-1})] \\ &= E[\rho^{1/r} (f \circ \tau^{-1})] = EQ_r f = EQ_r Ef. \quad \square \end{aligned}$$

(3.20) **Lemma.** *Let* $1 < p < \infty$, $1 < r < \infty$, *and let* Q *be the* L_p *isometry induced by an automorphism* τ. *Let* $T = EQE$ *for some conditional expectation operator* E. *If* T *has a semi-invariant function and* $R = R_r$ *is the* L_r *isometry induced by* τ^{-1}, *then* $(T^*)_r = ERE$.

Proof. $(T^*)_r = (EQ^*E)_r = (ER_qE)_r = ERE$. We have used the self-adjointness of E and Lemmas (3.19) and (2.17) for the fact that Q^* is the L_r isometry induced by τ^{-1}. $\square$

4. Finite-dimensional approximation

In [AK], it was shown that all positive contractions over the unit interval are induced by a point mapping of some type, followed by a conditional expectation. For positive contractions with semi-invariant functions, the argument is easier and does not require the underlying space to be interval. However, we will want to extract a point mapping from a set mapping, so we will require our measure spaces to be Lebesgue spaces. That is, a measure space $(X, \mathscr{F}, \mu)$ where X is a complete metric space and $\mathscr{F}$ is the Borel σ-algebra. We allow the space to have σ-finite measure. Since a separable metric space is second countable, the σ-algebra of measurable sets in a Lebesgue space can always be generated by a countable algebra of sets.

The details of the construction give us a family of finite-dimensional operators $\langle T^n \rangle_{n=1}^{\infty}$ (these are ordinary superscripts, not powers), each with a semi-invariant function u_n, where $u_n \to u$ a.e. Furthermore, these operators have the property that $(T^n)_r f \to T_r f$ a.e. and in L_r norm for every $f \in L_r$. These finite-dimensional approximations to the induced operator provide the key to the proof of the Theorem (1.2).

(4.1) **Definitions.** Let $\mathbf{X} = (X, \mathscr{F}, \mu)$ be a σ-finite Lebesgue space and suppose $T: L_p(d\mu) \to L_p(d\mu)$ has a semi-invariant function u. Let $\mathbf{I} = (I, \mathscr{B}, m)$ be the usual Lebesgue space of the unit interval. Let $\mathbf{W} = (W, \mathscr{H}, \omega) = \mathbf{X} \times \mathbf{I}$.

Let $\mathscr{I} = \{F \times I | F \in \mathscr{F}\}$, the "vertical" sub-$\sigma$-algebra of $\mathscr{H}$, and let v be the $\mathscr{I}$-measurable function given by $v(x, y) = (Tu)(x)$ for every y in the unit interval.

Suppose $\langle \mathscr{F}_n \rangle_{n=1}^{\infty}$ is an increasing sequence of finite sub-σ-algebras of $\mathscr{F}$ such that $\sigma(\bigcup_{n=1}^{\infty} \mathscr{F}_n) = \mathscr{F}$. That is, $\mathscr{F}$ is the smallest σ-algebra containing all the $\mathscr{F}_n$'s. Let $\mathscr{I}_n = \{F \times I | F \in \mathscr{F}_n\}$.

For each $n \geq 1$, fix an enumeration $\{F_{n,i}\}_{i=1}^{k_n}$ of the atoms of $\mathscr{F}_n$. Let $\gamma_{n,0} = 0$, and for each i, $1 \leq i \leq k_n$, let

$$\gamma_{n,i} = T\left(u \sum_{j=1}^{i} \chi_{F_{n,j}}\right),$$

and

$$H_{n,i} = \left\{(x, y) \in W \,\middle|\, \frac{\gamma_{n,i-1}(x)}{(Tu)(x)} < y \leq \frac{\gamma_{n,i}(x)}{(Tu)(x)}\right\}.$$

Let $\mathscr{H}_n$ be the finite sub-σ-algebra of $\mathscr{H}$ generated by the partition $\{H_{n,i}\}_{i=1}^{k_n}$ of W. Let Π_n be the set mapping from $\mathscr{F}_n$ to $\mathscr{H}_n$ determined by $\Pi_n F_{n,i} = H_{n,i}$ for each i, $1 \leq i \leq n$.

(4.2) **Lemma.** *There is a point mapping $\pi: W \to X$ such that $\pi^{-1} F_{n,i} = H_{n,i}$ for every $n \geq 1$ and every i, $1 \leq i \leq k_n$.*

Proof. The family of set mappings Π_n determines a unique set mapping of the algebra $\bigcup_{n=1}^{\infty} \mathscr{F}_n$, because of $\mathscr{F}_n$'s form a monotone sequence. This mapping preserves unions and complements, and it extends to a homomorphism of the measure algebras of $(X, \mathscr{F})$ and $(W, \mathscr{H})$. Since the sets underlying both spaces are complete metric spaces, there is a point mapping π defined from almost all of W onto almost all of X which induces the set mapping (see [Ry, p. 329]). Thus if $\Pi F = H$, then $\pi^{-1} F = H$. Since $\Pi F_{n,i} = \Pi_n F_{n,i}$, the desired result follows. □

(4.3) **Lemma.** *For every $F \in \mathscr{F}$, $\int_F u^p \, d\mu = \int_{\pi^{-1} F} v^p \, d\omega$.*

Proof. If $F = F_{n,i} \in \mathscr{F}$ for some $n \geq 1$ and some i, $1 \leq i \leq k_n$, then

$$\begin{aligned}\int_{\pi^{-1}F} v^p \, d\omega &= \int_X (Tu)^p \left[\frac{\gamma_{n,i} - \gamma_{n,i-1}}{Tu}\right] d\mu \\ &= \int_X (Tu)^{p-1} T(u\chi_F) \, d\mu \\ &= \int_F uT^*(Tu)^{p-1} \, d\mu = \int_F u^p \, d\mu .\end{aligned}$$

The lemma is true for a generating subalgebra of $\mathscr{F}$. The proof is easily completed. □

(4.4) **Lemma.** *Suppose ϕ is an $\mathscr{F}$-measurable function and θ is a $\mathscr{K}$-measurable function with $\phi > 0$ μ-a.e. and $\theta > 0$ ω-a.e. such that*

$$\int_F \phi \, d\mu = \int_{\pi^{-1}F} \theta \, d\omega$$

for every $F \in \mathscr{F}$. Then, if $1 \leq p < \infty$,

$$Sf = \left(\frac{\theta}{\phi \circ \pi}\right)^{1/p} (f \circ \pi),$$

for $f \in L_p(d\mu)$, defines an isometry $S\colon L_p(d\mu) \to L_p(d\omega)$.

Proof. First suppose $f = \phi^{1/p}\chi_F$ for some $F \in \mathscr{F}$. Then

$$\begin{aligned}\|Sf\|_p^p &= \int_Z \frac{\theta}{\phi \circ \pi} (\phi^{1/p}\chi_F)^p \circ \pi \, d\omega \\ &= \int_{\pi^{-1}F} \theta \, d\omega = \int_F \phi \, d\mu = \|f\|_p^p .\end{aligned}$$

In the general case, approximate $f\phi^{-1/p}$ by $\mathscr{F}$-simple functions. □

This isometry yields a result analogous to the theorem of Akcoglu and Koop [AK].

(4.5) **Theorem.** *Define $Q\colon L_p(d\mu) \to L_p(d\omega)$ by*

$$Q = \frac{v}{u \circ \pi}(f \circ \pi) \quad \textit{for } f \in L_p(d\mu).$$

If ω' is ω restricted to $\mathscr{I}$, and we identify $\mathbf{X}$ with $(W, \mathscr{I}, \omega')$, then $Tf = E(Qf|\mathscr{I})$ for every $F \in L_p(d\mu)$.

Proof. By the two previous lemmas, we see that Q is an isometry of the indicated spaces. Suppose $f = u\chi_F$ for some $F \in \mathscr{F}$. Then

$$\begin{aligned}[E(Qf|\mathscr{I})](x) &= \int_0^1 (Qf)(x, y) \, dy = \int_0^1 v(x, y)\chi_{\pi^{-1}F}(x, y) \, dy \\ &= (Tu)(x)\left(\frac{T(u\chi_F)(x)}{(Tu)(x)}\right) = (Tf)(x).\end{aligned}$$

For a general $\mathscr{F}$-measurable f, approximate fu^{-1} by $\mathscr{F}$-simple functions. □

(4.6) **Lemma.** *Suppose $\widetilde{\mathscr{F}}$ is a finite sub-σ-algebra of $\mathscr{F}$ and $\tilde{\mu}$ is the restriction of μ to $\widetilde{\mathscr{F}}$. If $\tilde{v} \in L_p(d\omega)$ satisfies $\tilde{v} > 0$ a.e., then there is a unique $\widetilde{\mathscr{F}}$-measurable function $\tilde{u}$ such that, for every $F \in \widetilde{\mathscr{F}}$,*

$$\int_F \tilde{u}^p \, d\mu = \int_{\pi^{-1}F} \tilde{v}^p \, d\omega .$$

Furthermore, the mapping

$$f \in L_p(X, \widetilde{\mathscr{F}}, \tilde{\mu}) \mapsto \frac{\tilde{v}}{\tilde{u}\circ\pi}(f\circ\pi) \in L_p(d\omega)$$

is an isometry which transforms $\tilde{u}$ to $\tilde{v}$.

Proof. Let $\{F_i\}_{i=1}^k$ be an enumeration of the atoms of $\widetilde{\mathscr{F}}$. Let $H_i = \pi^{-1}F_i$ for each i, $1 \le i \le k$. Then

$$\tilde{u} = \sum_{i=1}^k \left(\frac{1}{\mu F_i} \int_{H_i} \tilde{v}^p \, d\omega \right)^{1/p} \chi_{F_i} .$$

If $f = \sum_{i=1}^k c_i \chi_{F_i} \in L_p(X, \widetilde{\mathscr{F}}, \tilde{\mu})$, then

$$\left\| \frac{\tilde{v}}{\tilde{u}\circ\pi}(f\circ\pi) \right\|_p^p = \sum_{i=1}^k c_i^p \int_{\pi^{-1}F_i} \frac{\tilde{v}^p}{(\tilde{u}\circ\pi)^p} \, d\omega = \sum_{i=1}^k c_i^p \mu F_i = \|f\|_p^p ,$$

as desired. □

(4.7) **Theorem.** *For each $n \ge 1$, let $v_n = E(v|\mathscr{I}_n)$. Let u_n be the corresponding $\mathscr{F}_n$-measurable functions as given by Lemma* (4.6). *Then $u_n \to u$ μ-a.e.*

Proof. Let $u_n = \sum_{i=1}^{k_n} u_{n,i} \chi_{F_{n,i}}$. Then

$$\begin{aligned} u_{n,i}^p &= \frac{1}{\mu F_{n,i}} \int_{H_{n,i}} v_n^p \, d\omega \\ &= \left(\frac{1}{\mu F_{n,i}} \int_{F_{n,i}} u^p \, d\mu \right) \frac{(\omega H_{n,i})^{-1} \int_{H_{n,i}} v_n^p \, d\omega)}{(\omega H_{n,i})^{-1} \int_{H_{n,i}} v^p \, d\omega)} . \end{aligned}$$

Thus

$$u_n^p \circ \pi = \frac{E(v_n^p|\mathscr{H}_n)}{E(v^p|\mathscr{H}_n)} [E(u^p|\mathscr{F}_n)\circ\pi] .$$

By the martingale convergence theorem, with $p = 1$, we have $E(u^p|\mathscr{F}_n) \to u^p$ μ-a.e., and $E(v^p|\mathscr{H}_n) \to E(v^p|\mathscr{H})$ ω-a.e. By Lemma (2.22), we also have $E(v_n^p|\mathscr{H}_n) \to E(v^p|\mathscr{H})$ ω-a.e. Therefore

$$u_n^p \circ \pi \to E(u^p|\mathscr{F}_n)\circ\pi ,$$

and so $u_n^p \to u^p$ μ-a.e., by the martingale convergence theorem. This completes the proof. □

(4.8) **Definition.** For each $n \geq 1$, let u_n and v_n be as defined in the hypothesis of the previous lemma. Define

$$Q^n : L_p(X, \mathscr{F}_n, \mu_n) \to L_p(d\omega),$$

where μ_n is the restriction of μ to $\mathscr{F}_n$, by

$$Q^n f = \frac{v_n}{u_n \circ \pi}(f \circ \pi)$$

for $f \in L_p(X, \mathscr{F}_n, \mu_n)$.

By Lemma (4.6), this is an isometry. If ω_n is the restriction of ω to $\mathscr{I}_n$, and we make the obvious identification of $(W, \mathscr{I}_n, \omega_n)$ with $(X, \mathscr{F}_n, \mu_n)$, then define $T^n : L_p(d\mu) \to L_p(X, \mathscr{F}_n, \mu_n)$ by

$$T^n f = E(Q^n E(f|\mathscr{F}_n)|\mathscr{I}_n)$$

for $f \in L_p(\mathbf{X})$. Each T^n is a positive contraction, and it is easy to see that if $f \in L_p(d\mu)$, then $T^n f \to Tf$ μ-a.e.

Observe that u_n is a semi-invariant function for each T^n; the reason is that v_n is already $\mathscr{I}_n$-measurable. (In fact, it is easy to see that u_n is the only normalized semi-invariant function for T^n.) Thus, the induced operator $(T^n)_r$ is defined for any r, $1 < r < \infty$. For brevity, denote it R_n.

(4.9) **Theorem.** *$\|R_n f - T_r f\|_r \to 0$ as $n \to \infty$, for every $f \in L_r(d\mu)$.*

Proof. If f is $\mathscr{F}_n$-measurable, then

$$\begin{aligned} R_n f &= v_n^{p/r-1} T^n (v_n^{1-p/r} f) \\ &= v_n^{p/r-1} E\left[\frac{v_n}{u_n \circ \pi}(u_n^{1-p/r} \circ \pi)(f \circ \pi)|\mathscr{I}_n\right] \\ &= E\left[\left(\frac{v_n}{u_n \circ \pi}\right)^{p/r}(f \circ \pi)|\mathscr{I}_n\right], \end{aligned}$$

since v_n is $\mathscr{I}_n$-measurable. Whether or not f is $\mathscr{F}_n$-measurable, define

$$\phi_n = \left(\frac{v_n}{u_n \circ \pi}\right)^{p/r} [E(f|\mathscr{F}_n) \circ \pi].$$

Then $R_n f = E(\phi_n|\mathscr{I}_n)$ for any $f \in L_r(d\mu)$. Similarly, if $\phi = (v/u \circ \pi)^{p/r}(f \circ \pi)$, then $T_r f = E(\phi|\mathscr{I})$.

Clearly $\phi_n \to \phi$ a.e.; if we can show that $\|\phi_n\|_r \to \|\phi\|_r$, we may conclude that $\phi_n \to \phi$ in L_r norm (see [Ry, p. 118]):

$$\begin{aligned} \|\phi_n\|_r^r &= \|Q^n([E(f|\mathscr{F}_n)]^{r/p})\|_p^p \\ &= \|[E(f|\mathscr{F}_n)]^{r/p}\|_p^p = \|E(f|\mathscr{F}_n)\|_r^r \to \|f\|_r^r, \end{aligned}$$

as $E(f|\mathscr{F}_n)$ is an L_r martingale. The second line follows because Q^n is an isometry. Also $\|\phi\|_r = \|f\|_r$ by a similar calculation. This tells us that

$$\|\phi_n - \phi\|_r \to 0.$$

To conclude the proof, observe that

$$\|R_n f - T_r f\|_r \le \|E(\phi_n|\mathscr{I}_n) - E(\phi_n|\mathscr{I})\|_r + \|E(\phi_n|\mathscr{I}) - E(\phi|\mathscr{I})\|_r .$$

The first term tends to zero by the martingale convergence theorem and the second term is dominated by $\|\phi_n - \phi\|_r$. □

5. Convergence of the alternating sequence

This section is in many ways analogous to §§3 and 4 of [AS2], and so the reader will often be referred there for details. Where we follow [AS2] closely, every effort is made to keep the notation consistent.

(5.1) **Definitions.** Suppose $1 < p < \infty$, $1 < r < \infty$, and let $\psi = \psi_{p,r}$. Let $\langle T_n \rangle_{n=1}^{\infty}$ be a sequence of positive linear contractions with semi-invariant functions operating on the L_p space of a σ-infinite Lebesgue space. Call such a sequence of operators a *norming sequence*. Call a norming sequence *special* if all operators are finite dimensional.

Let V_0 and U_0 be the identities on L_p and L_r respectively, and make the following definitions for each $n \ge 1$:

$$V_n = T_n \cdots T_1 , \qquad U_n = (T_1^*)_r \cdots (T_n^*)_r .$$

For a given $f \in L_p$ and an $n \ge 0$, let $g_n = U_n \psi(V_n f)$. Observe that $g_0 = \psi f$ and that $\|g_0\|_r = \|f\|_p^{p/r}$.

We say that *Estimate* A holds for a norming sequence $\langle T_n \rangle_{n=1}^{\infty}$ if

$$\left\| \sup_{n \ge 0} |g_n| \right\|_r \le (q\|f\|_p)^{p/r} (= q^{p/r} \|g_0\|_r)$$

for every $f \in L_p$.

We say that *Estimate* B holds for a norming sequence $\langle T_n \rangle_{n=1}^{\infty}$ if for every $\varepsilon > 0$ there is a $\delta > 0$, depending only on ε, p, and r, such that

$$\left\| \sup_{n \ge 0} |g_n - g_0| \right\|_r < \varepsilon \|f\|_p^{p/r} \quad (= \varepsilon \|g_0\|_r)$$

whenever $f \in L_p$ is such that

$$\|f\|_p - \lim_{n \ge 0} \|V_n f\|_p < \delta \|f\|_p .$$

Given a norming sequence $\langle T_n \rangle_{n=1}^{\infty}$, a fixed $n \ge 1$, and a function $f \in L_p$, let $\tilde{f} = V_n f$. For every $k \ge 1$, let $\widetilde{T}_k = T_{n+k}$. Let $\widetilde{V}_0$ and $\widetilde{U}_0$ be the identities on L_p and L_r respectively. For each $k \ge 1$, let

$$\widetilde{V}_k = \widetilde{T}_k \cdots \widetilde{T}_1 , \qquad \widetilde{U}_k = (\widetilde{T}_1^*)_r \cdots (\widetilde{T}_k^*)_r ,$$

and for each $k \ge 0$, let $\tilde{g}_k = \widetilde{U}_k \psi(\widetilde{V}_k \tilde{f})$. Observe that $g_{n+k} = U_n \tilde{g}_k$ for every $k \ge 0$.

(5.2) **Theorem.** *Suppose Estimates* A *and* B *are satisfied for every norming sequence. Then given a norming sequence* $\langle T_n\rangle_{n=1}^{\infty}$ *and an* $f \in L_p$, $\langle g_n\rangle_{n=0}^{\infty}$ *converges a.e.*

Proof. Because of Estimate A, $(\sup_{n\geq 0}|g_n|) \in L_r$. Therefore, by Lemma (2.27), it suffices to show that

$$\lim_{n\geq 0}\left\|\sup_{k\geq n}|g_{n+k} - g_n|\right\|_r = 0.$$

Let $\beta = \lim_{n\geq 0}\|V_n f\|_p$ and distinguish two cases:

Case 1: $\beta = 0$. Given $\varepsilon > 0$, find $n_0 \geq 1$ such that

$$\|V_{n_0} f\|_p < \frac{1}{q}\left(\frac{\varepsilon}{2}\right)^{r/p}.$$

Fix $n \geq n_0$ and define $\tilde{f}$ and $\tilde{g}_k$ as above. Observe that $\|\tilde{f}\|_p \leq \|V_{n_0} f\|_p$. We have

$$\begin{aligned}\left\|\sup_{k\geq 0}|g_{n+k} - g_n|\right\|_r &= \left\|\sup_{k\geq 0}|U_n\tilde{g}_k - U_n\tilde{g}_0|\right\|_r \\ &\leq \left\|\sup_{k\geq 0}|\tilde{g}_k - \tilde{g}_0|\right\|_r \leq 2\left\|\sup_{k\geq 0}|\tilde{g}_k|\right\|_r \\ &\leq 2(q\|\tilde{f}\|_p)^{p/r} < \varepsilon,\end{aligned}$$

where the first inequality follows from Lemma (2.24) and the third follows from Estimate A for the sequence $\langle \tilde{T}_k\rangle_{k=1}^{\infty}$.

Case 2: $\beta > 0$. Given $\varepsilon > 0$, choose $\delta > 0$ as given by Estimate B, corresponding to $\varepsilon/\|f\|_p^{p/r}$. Choose $n_0 \geq 1$ such that $\|V_{n_0} f\|_p < (1+\delta)\beta$. Fix $n \geq n_0$ and define $\tilde{f}$ and $\tilde{g}_k$ as above. Observe that $\beta \leq \|\tilde{f}\|_p$, since the $\|V_n f\|_p$'s form a monotone sequence. We have

$$\|\tilde{f}\|_p - \lim_{k\geq 0}\|\tilde{V}_k\tilde{f}\|_p = \|V_n f\|_p - \beta < (1+\delta)\beta - \beta \leq \delta\|\tilde{f}\|_p.$$

Now apply Estimate B for $\langle \tilde{T}_k\rangle_{k=1}^{\infty}$; we conclude

$$\left\|\sup_{k\geq 0}|g_{n+k} - g_k|\right\|_r \leq \left\|\sup_{k\geq 0}|\tilde{g}_k - \tilde{g}_0|\right\|_r < \left(\frac{\varepsilon}{\|f\|_p^{p/r}}\right)\|\tilde{f}\|_p^{p/r} \leq \varepsilon,$$

where the first inequality follows as in Case 1. □

(5.3) **Lemma.** *If Estimate* A *holds for every special norming sequence, then it holds for every norming sequence.*

Proof. Suppose $\langle T_n\rangle_{n=1}^{\infty}$ is a uniform norming sequence for which Estimate A fails. Then there is a function $f \in L_p$ and an $n \geq 1$ such that

$$\left\|\max_{0\leq k\leq n}|g_k|\right\|_r > q^{p/r}\|g_0\|_r.$$

Suppose $\langle \mathscr{F}_m \rangle_{m=1}^{\infty}$ is a monotone sequence of finite sub-σ-algebras of $\mathscr{F}$ with $\mathscr{F} = \sigma(\bigcup_{m=1}^{\infty} \mathscr{F}_m)$, the smallest σ-algebra containing the algebra $\bigcup_{m=1}^{\infty} \mathscr{F}_m$. For each k and m, $1 \le k \le n$ and $m \ge 1$, let T_k^m be the finite-dimensional operator as defined in (4.8). Let $f^m = E(f|\mathscr{F}_m)$.

Let $m \ge 1$ be arbitrary. Let V_0^m and U_0^m be $E(\cdot|\mathscr{F}_m)$ operating on L_p and L_r respectively. For each k, $1 \le k \le n$, let

$$V_k^m = T_k^m \cdots T_1^m, \qquad U_k^m = ((T_1^*)^m)_r \cdots ((T_k^*)^m)_r.$$

For $f \in L_p$, $m \ge 1$, and each k, $0 \le k \le n$, let

$$g_{km} = U_k^m \psi(V_k^m f) = U_k^m \psi(V_k^m f^m).$$

By the martingale convergence theorem, $\lim_{m \ge 1} \|f - f^m\|_p = 0$. We will show that

$$\lim_{m \ge 1} \|g_{km} - g_k\|_r = 0 \tag{5.4}$$

as well. Therefore, by applying Lemma (2.26),

$$\lim_{m \ge 1} \left\| \max_{0 \le k \le n} |g_{km}| - \max_{0 \le k \le n} |g_k| \right\|_r = 0.$$

Thus, for a suitably large integer m_0,

$$\left\| \max_{0 \le k \le n} |g_{km_0}| \right\|_r > (q\|f^{m_0}\|_p)^{p/r},$$

since the same inequality holds for f and the g_k's. Because $\mathscr{F}_{m_0}$ is finite, the operators $\langle T_1^{m_0}, \dots, T_n^{m_0} \rangle$ are essentially finite dimensional. Therefore, they form the initial portion of a special norming sequence for which Estimate A fails, contradicting the hypothesis of the lemma.

To prove (5.4), we first prove

$$\lim_{m \ge 1} \|V_k^m f - V_k f\|_p = 0 \tag{5.5}$$

for every k, $0 \le k \le n$. When $k = 0$, this is simply the martingale convergence theorem. For the inductive step, observe that

$$\|V_{k+1}^m f - V_{k+1} f\|_p = \|T_{k+1}^m V_k^m f - T_{k+1} V_k f\|_p,$$

where $\lim_{m \ge 1} \|V_k^m f - V_k f\|_p = 0$ by the inductive hypothesis. We apply Theorem (4.9) with $r = p$ and Lemma (2.30) to conclude that

$$\lim_{m \ge 1} \|V_{k+1}^m f - V_{k+1} f\|_p = 0,$$

completing the induction.

Because of the uniform continuity of ψ,

$$\lim_{m \ge 1} \|\psi V_k^m f - \psi V_k f\|_p = 0$$

for each k, $0 \le k \le n$.

We now perform another induction similar to the proof of (5.5) to show that when $g \in L_r$,

$$\lim_{m\geq 1} \|U_k^m g - U_k g\|_r = 0,$$

for each k, $0 \leq k \leq n$. This completes the proof. □

(5.6) **Lemma.** *Suppose that for every $\xi > 0$, there is an $\eta > 0$ depending only on ξ, p, and r such that*

$$\left\| \max_{0\leq k\leq n} |g_k' - g_0'| \right\|_r < \xi \|f'\|_p^{p/r}$$

whenever $\langle T_n' \rangle_{n=1}^{\infty}$ is a special norming sequence, $n \geq 1$, and $f' \in L_p$ is such that $\|f'\|_p - \|V_n' f'\|_p < \eta \|f'\|_p$, where V_n' and g_n' are defined exactly as V_n and g_n in (5.1), *relative to $\langle T_n' \rangle_{n=1}^{\infty}$. Then Estimate* B *holds for every norming sequence.*

Proof. Let $\langle T_n \rangle_{n=1}^{\infty}$ be a norming sequence and suppose $\xi > 0$ is given. Choose $\eta > 0$ from the hypothesis of the lemma, corresponding to $\xi/2$. If Estimate B fails for $\langle T_n \rangle_{n=1}^{\infty}$, then there is a function $f \in L_p$ with $\|f\|_p - \|V_n f\|_p < \eta \|f\|_p$, but for which

$$\left\| \max_{0\leq k\leq n} |g_k - g_0| \right\|_r > \frac{\xi}{2} \|f\|_p^{p/r}.$$

As in the proof of the previous lemma, we approximate the operators T_k with the operators T_k^m from (4.8). Define g_{km} as before, for each $m \geq 1$ and each k, $0 \leq k \leq n$, and let $h_k = g_k - g_0$ and $h_{km} = g_{km} - g_{0m}$ for the same set of indices. Then

$$\|h_{km} - h_k\|_r \leq \|g_{km} - g_k\|_r + \|g_{0m} - g_0\|_r,$$

and we have seen that both of these terms tend to zero as m increases. Thus $\lim_{m\geq 1} \|h_{km} - h_k\|_r = 0$, and we may apply Lemma (2.26) to conclude

$$\lim_{m\geq 1} \left\| \max_{0\leq k\leq n} |g_{km} - g_{0m}| - \max_{0\leq k\leq n} |g_k - g_0| \right\|_r = 0.$$

At the same time, we have

$$\lim_{m\geq 1} \|f - f^m\|_p = 0 \quad \text{and} \quad \lim_{m\geq 1} \|V_n f - V_n^m f^m\|_p = 0.$$

Thus, we may choose an m_0 sufficiently large that we maintain the relations

$$\|f^{m_0}\|_p - \|V_n^{m_0}\|_p < \eta \|f^{m_0}\|_p$$

and

$$\left\| \max_{0\leq k\leq n} |g_{km_0} - g_{0m_0}| \right\|_r > \frac{\xi}{2} \|f^{m_0}\|_p^{p/r}.$$

As $\mathscr{F}_m$ is finite, $\langle T_1^{m_0}, \ldots, T_n^{m_0} \rangle$ form the initial portion of a special norming sequence for which the hypothesis of the lemma fails. □

We have reduced the proof of Theorem (1.2) to verifying that finitary versions of Estimates A and B hold for every special norming sequence. In order to show that this is true, we introduce a dilation of these operators similar to the one given in [A].

(5.7) **Definitions.** Let $(X, \mathcal{F}, \mu)$ be a measure space in which $\mathcal{F}$ is a finite set. Let $\{F_i\}_{i=1}^{d}$ be an enumeration of the atoms of $\mathcal{F}$ of positive measure. Let the indices i and j range through the integers $\{1, \dots, d\}$. If T is a positive operator with a semi-invariant function u, let $u = \sum_i \alpha_i \chi_{F_i}$ and $Tu = \sum_i \beta_i \chi_{F_i}$. We have $\alpha_i > 0$ and $\beta_i > 0$ for each i. Let $m_i = \mu(F_i)$ and let $a_{ij} = \omega[\pi^{-1}F_i \cap (F_j \times [0, 1])]$, with π and ω as given in §4. Observe that $\sum_i a_{ij} = m_j$ for each j, and that for each i,

$$\alpha_i^p m_i = \int_{F_i} u^p \, d\mu = \int_{\pi^{-1}F_i} v^p \, d\omega = \sum_j \beta_j^p a_{ij}.$$

Let

$$b_{ij} = \left(\frac{\beta_j}{\alpha_i}\right)^p \frac{a_{ij}}{m_i}.$$

It is easy to verify that $\sum_j b_{ij} = 1$. Observe also that $a_{ij} = 0$ if and only if $b_{ij} = 0$.

We are going to construct a set Z in the coordinate plane $\mathbb{R}^2$ and an isometry of its L_p space. The construction is virtually identical to the one given in [A] and used in [AS2], except that some of the subrectangles may have measure zero. However, because of the last observation, this will cause no problems.

Let $\langle I_i \rangle_{i=1}^{d}$ be disjoint intervals on the x-axis of the coordinate plane, each of length m_i. Let $\langle J_i \rangle_{i=1}^{d}$ be disjoint intervals on the y-axis, each of unit length. Let $P_i = I_i \times J_i$ and $Z = \bigcup_i P_i$. Let $\mathbb{Z} = (Z, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra of Z and λ is the restriction of Lebesgue measure on $\mathbb{R}^2$ to Z. let L_p denote $L_p(\mathbf{Z})$, and let $\mathcal{P}$ be the partition $\{P_i\}_{i=1}^{d}$ of Z. Let $E = E(\cdot|\mathcal{P})$ and let $l_p = EL_p$.

Define a further partitioning of Z as follows. Each I_j is partitioned into d subintervals $\langle I_{ij} \rangle_{i=1}^{d}$, each of length a_{ij}. Each J_i is partitioned into d subintervals $\langle J_{ij} \rangle_{j=1}^{d}$, each of length b_{ij}. Let $R_{ij} = I_i \times J_{ij}$, a horizontal strip of P_i, and $S_{ij} = I_{ij} \times J_j$, a vertical strip of P_j.

Define a point transformation $\tau \colon Z \to Z$ by mapping each R_{ij} of nonzero measure to the corresponding S_{ij}, in such a way that the Radon-Nikodým derivative for the mapping of these rectangles is constant. Thus, τ "squeezes" the width of R_{ij} from m_i to a_{ij} and "stretches" its height from b_{ij} to 1; this deformation determines the constant value of

$$\rho = \frac{d(\lambda \circ \tau^{-1})}{d\lambda}$$

on S_{ij}.

$\lambda(R_{ij}) = 0$ if and only if $\lambda(S_{ij}) = 0$, because of the corresponding property of a_{ij} and b_{ij}, and so τ is an automorphism of Z. An automorphism of Z determined in this manner by any pair of sequences of a_{ij}'s and b_{ij}'s satisfying $\sum_i a_{ij} = m_j$, $\sum_j b_{ij} = 1$, and $a_{ij} = 0$ if and only if $b_{ij} = 0$, is called an *admissible* automorphism. Each admissible automorphism induces an *admissible L_p isometry Q* in the usual manner by $Qf = \rho^{1/p}(f \circ \tau^{-1})$.

(5.8) **Theorem.** *The action of EQ on l_p is isomorphic to the action of the original operator T on $L_p(\mathbf{X})$.*

Proof. Let i range through $\{1, \dots, d\}$. Let Φ be given by

$$\sum_i c_i \chi_{P_i} \in l_p \mapsto \sum_i c_i \chi_{F_i} \in L_p(\mathbf{X}).$$

This is an isometric isomorphism since $\lambda(P_i) = \mu(F_i) = m_i$.

Let $\mathbf{W} = (W, \mathscr{K}, \omega)$, π, $\mathscr{I}$, and v be as given in Theorem (4.5). According to that theorem, if we define $R\colon L_p(d\mu) \to L_p(d\omega)$ by

$$Rg = \frac{v}{u \circ \pi}(g \circ \pi),$$

then $Tg = E(Rg|\mathscr{I})$ for every $g \in L_p(d\mu)$. Since $u = \sum_i \alpha_i \chi_{F_i}$ and $Tu = \sum_i \beta_i \chi_{F_i}$, we have $Rg = (\beta_j/\alpha_i)c_i$ on each $\pi^{-1}F_i \cap (F_j \times [0, 1]) \subseteq F_j \times I$.

When $f \in l_p$, then $Qf = \rho_{ij}^{1/p} c_i$ on each $S_{ij} \subseteq P_j$, where ρ_{ij} is the constant value of the Radon-Nikodým derivatives ρ on the rectangle S_{ij}. Observe that

$$\rho_{ij} = \frac{\lambda(R_{ij})}{\lambda(S_{ij})} = \frac{m_i b_{ij}}{a_{ij}} = \left(\frac{\beta_j}{\alpha_i}\right)^p .$$

We also have $\omega[\pi^{-1}F_i \cap (F_j \times [0, 1])] = \lambda(S_{ij}) = a_{ij}$. This means that Qf and Rg are simple functions taking the same range of values over sets of identical measure. Therefore, $Tg = E(Rg|\mathscr{I}) = \Phi(EQf)$ as desired. □

The proof of the convergence of the alternating sequence is now reduced to an examination of the actions of admissible isometries of $\mathbf{Z}$, intertwined with the conditional expectation operator with respect to $\mathscr{P}$.

(5.9) **Definitions.** Let G be a subset $\mathbb{R}^2$. A subset F of G is called a *vertical subset* of G if

$$F = (F' \times \mathbb{R}) \cap G$$

for some subset F' of the x-axis. Similarly, if

$$H = (\mathbb{R} \times H') \cap G$$

for some subset H' of the y-axis, then H is called a *horizontal subset* of G.

We say that a function f is *constant on vertical lines* if $f(x_1, y_1) = f(x_2, y_2)$ whenever $x_1 = x_2$. We say that f is *constant on horizontal lines* if $f(x_1, y_1) = f(x_2, y_2)$ whenever $y_1 = y_2$.

The following is a summary of Lemmas (4.5) through (4.12) from [AS2].

(5.10) **Lemma.** *Let τ be an admissible automorphism, and let Q be the induced L_p isometry.*

(a) *Suppose $\mathscr{G}$ is a finite partition of Z in which each atom is a vertical subset of some P_i. Let f be an L_p function which is constant on vertical lines. Then*

$$QE(f|\mathscr{G}) = E(Qf|\mathscr{P} \vee \tau\mathscr{G}).$$

(a$'$) *Suppose $\mathscr{H}$ is a finite partition of Z in which each atom is a horizontal subset of some P_i. Let f be an L_p function which is constant on horizontal lines. Then*

$$Q^{-1}E(f|\mathscr{H}) = E(Q^{-1}f|\mathscr{P} \vee \tau\mathscr{H}).$$

(b) *If f_1 and f_2 are L_p functions that are constant on vertical lines and $Ef_1 = Ef_2$, then also $EQf_1 = EQf_2$.*

(b$'$) *If f_1 and f_2 are L_p functions that are constant on horizontal lines and $Ef_1 = Ef_2$, then also $EQ^{-1}f_1 = EQ^{-1}f_2$.*

(c) *If f is constant on vertical lines, then Qf is constant on vertical lines.*

(c$'$) *If f is constant on horizontal lines, then $Q^{-1}f$ is constant on horizontal lines.*

(5.11) **Definitions.** Let n be a fixed integer, $n \geq 1$, and let k range through $\{0, 1, \dots, n\}$. If $1 \leq k \leq n$, let τ_k be an admissible isometry of Z, let Q_k be the L_p isometry induced by τ_k, and let R_k be the L_r isometry induced by τ_k^{-1}. Let Q_0 and R_0 be the identities on L_p and L_r respectively. Let

$$\begin{aligned} T_k &= EQ_kE, & V_k &= T_k \cdots T_0, & W_k &= Q_k \cdots Q_0, \\ S_k &= ER_kE, & U_k &= S_0 \cdots S_k, & D_k &= R_0 \cdots R_k. \end{aligned}$$

Observe that $S_k = (T_k^*)_r$ by Lemma (3.20).

Let f be a fixed but arbitrary function in L_p. Let $g_k = U_k\psi(V_kf)$ and $\phi_k = W_k^{-1}EW_kEf$. Observe that $g_0 = \psi\phi_0 = \psi Ef$.

(5.12) **Lemma.** *For any $f \in L_p$, $V_kf = EW_kEf$.*

Proof. This is Lemma (4.14) of [AS2]. When $k = 0$ this is immediate from the definitions. The inductive step is given by Lemma (5.10)(b) and (c). □

(5.13) **Lemma.** *For any $g \in L_r$, $U_kg = ED_kEg$.*

Proof. We will show that

$$S_i \cdots S_jg = ER_i \cdots R_jEg \tag{5.14}$$

for every pair i, j with $0 \leq i \leq j \leq n$. This will prove the lemma, since the desired identity is (5.14) with $i = 0$ and $j = k$. The proof is by induction on $j - i$. When $i = j$, (5.14) is simply the definition of S_ig.

Now suppose (5.14) holds for some pair $i+1$, $j+1$ with $0 \leq i \leq j < n$. We have

$$ER_{i+1} \cdots R_{j+1} Eg = ES_{i+1} \cdots S_{j+1} g,$$

by the inductive hypothesis and the idempotence of E, the outermost operator in S_{i+1}. $R_{i+1} \cdots R_{j+1} Eg$ is constant on horizontal lines, by repeated application of Lemma (5.10)(c′). Thus, by Lemma (5.10)(b′), we have

$$ER_i R_{i+1} \cdots R_{j+1} Eg = ER_i S_{i+1} \cdots S_{j+1} = S_i \cdots S_{j+1} g.$$

This completes the induction. □

(5.15) **Lemma.** $g_k = E\psi(\phi_k)$.

Proof.

$$\begin{aligned} g_k &= U_k \psi(V_k f) = ED_k E\psi(EW_k Ef) \\ &= ED_k \psi(EW_k Ef) = E\psi[(R_0)^p \cdots (R_k)^p EW_k Ef]. \end{aligned}$$

The second line follows from the two previous lemmas. The third line follows because ψ maps $\mathscr{P}$-measurable functions to $\mathscr{P}$-measurable functions. For the fourth line, we use $(R_i)^p$ to denote the L_p isometry induced by τ_i^{-1}. Thus, this line follows by an application of Lemma (2.19). By Lemma (2.17), that isometry is Q_i^{-1}. Thus

$$g_k = E\psi(W_k^{-1} RW_k Ef) = E\psi(\phi_k),$$

as desired. □

(5.16) **Lemma.** *There exists a monotone sequence $\mathscr{G}_n \subseteq \mathscr{G}_{n-1} \subseteq \cdots \subseteq \mathscr{G}_0$ of finite σ-algebras such that*

$$\phi_k = W_n^{-1} E(W_n Ef|\mathscr{G}_k).$$

Proof. This is Lemma (4.16) of [AS2]. We may take $\mathscr{G}_n = \mathscr{P}$. Lemma (5.10)(a) provides the induction step needed to show that we may take

$$\mathscr{G}_{n-k} = \mathscr{P} \vee \tau_n \mathscr{P} \vee \cdots \vee \tau_n \cdots \tau_{n-k+1} \mathscr{P}$$

when $1 \leq k \leq n$. □

(5.17) **Definition.** Let $u_k = E(W_n Ef|\mathscr{G}_k)$, where the $\mathscr{G}_k$'s are as in the previous lemma. Observe that $\phi_k = W_n^{-1} u_k$.

(5.18) **Theorem.** *The sequence $\langle u_0, \ldots, u_n \rangle$ is an L_p martingale. Furthermore,*

$$\left\| \max_{0 \leq k \leq n} |u_k| \right\|_p \leq q \|u_0\|_p$$

and

$$\left\| \max_{0 \leq k \leq n} |u_k - u_n| \right\|_p \leq q \|u_0 - u_n\|_p.$$

Proof.

$$u_k = E(W_n Ef|\mathscr{G}_k) = E(E(W_n Ef|\mathscr{G}_0)|\mathscr{G}_k) = E(u_0|\mathscr{G}_k),$$

since $\mathscr{G}_k \subseteq \mathscr{G}_0$ for every k, $0 \le k \le n$. As well,

$$u_k - u_n = E(u_0|\mathscr{G}_k) - E(u_n|\mathscr{G}_k) = E(u_0 - u_n|\mathscr{G}_k).$$

In the first case, this follows from the above computation. In the second case, $u_n = E(u_n|\mathscr{G}_k)$ because u_n is already constant on the atoms of $\mathscr{G}_k$.

The lemma now follows by an application of the martingale convergence theorem for L_p. □

(5.19) **Theorem.** $\|\max_{0\le k\le n} |g_k|\|_r \le (q\|f\|_p)^{p/r}$.

Proof. Since $\phi_k = W_n^{-1} u_k$ and W_n^{-1} is a positive isometry, we have $|\phi_k| = W_n^{-1}|u_k|$ and $\max_{0\le k\le n} |\phi_k| = W_n^{-1}(\max_{0\le k\le n} |u_k|)$ and so

$$\left\|\max_{0\le k\le n} |\phi_k|\right\|_p = \left\|\max_{0\le k\le n} |u_k|\right\|_p \le q\|u_0\|_p \le q\|f\|_p. \tag{5.20}$$

The inequalities follow by an application of Theorem (5.18) and the fact that $\|u_0\|_p = \|Ef\|_p$.

Since $g_k = E\psi(\phi_k)$, we have

$$\max_{0\le k\le n} |g_k| \le E\left(\max_{0\le k\le n} |\psi(\phi_k)|\right) = E\psi\left(\max_{0\le k\le n} |\phi_k|\right),$$

where Lemma (2.24) was used for the inequality. Thus

$$\begin{aligned}\left\|\max_{0\le k\le n} |g_k|\right\|_r &\le \left\|\psi\left(\max_{0\le k\le n} |\phi_k|\right)\right\|_r \\ &= \left\|\max_{0\le k\le n} |\phi_k|\right\|_p^{p/r} \le (q\|f\|_p)^{p/r}. \quad \square\end{aligned}$$

(5.21) **Theorem.** *For any $\xi > 0$ there is an $\eta > 0$ depending only on ξ, p, and r such that*

$$\left\|\max_{0\le k\le n} |g_k - g_0|\right\|_r < \xi\|f\|_p^{p/r}$$

whenever $\|f\|_p - \|V_n f\|_p < \eta\|Ef\|_p$.

Proof. Since $u_n = E(u_0|\mathscr{G}_n)$, we may apply Lemma (2.25) to choose an $\eta > 0$, depending only on δ (which will be specified later) and p so that

$$\|u_0\|_p - \|u_n\|_p < \eta\|u_0\|_p$$

implies

$$\|u_0 - u_n\|_p < \frac{\delta}{2q}\|u_0\|_p.$$

We have already observed that $\|u_0\|_p = \|Ef\|_p$. As well, we note that $\|u_n\|_p = \|V_n f\|_p$. Thus, if $\|f\|_p - \|V_n f\|_p < \eta\|Ef\|_p$, we have

$$\begin{aligned}\left\|\max_{0\le k\le n} |u_k - u_0|\right\|_p &\le 2\left\|\max_{0\le k\le n} |u_k - u_n|\right\|_p \\ &\le 2q\|u_0 - u_n\|_p < \xi\|u_0\|_p,\end{aligned}$$

where the second inequality follows from Theorem (5.18).

As in the proof of the previous theorem, we deduce

$$\max_{0\leq k\leq n} |\phi_k - \phi_0| = W_n^{-1}\left(\max_{0\leq k\leq n} |u_k - u_0|\right),$$

and so $\|\max_{0\leq k\leq n} |\phi_k - \phi_0|\|_p \leq \delta\|Ef\|_p$.

Since the inequality $\|\max |\phi_k|\|_p \leq q\|Ef\|_p$ is simply a restatement of (5.20), we are in a position to apply Lemma (2.29). Choose δ from that lemma corresponding to ξ, q (which depends only on p), p and r, and conclude that

$$\left\|\max_{0\leq k\leq n} |\psi(\phi_k) - \psi(\phi_0)|\right\|_r < \xi\|Ef\|_p^{p/r}$$

whenever $\|f\|_p - \|V_n f\|_p < \eta\|Ef\|_p$.

Now apply Lemma (2.24):

$$\begin{aligned}\left\|\max_{0\leq k\leq n} |g_k - g_0|\right\|_r &\leq \left\|E\left(\max_{0\leq k\leq n} |\psi(\phi_k) - \psi(\phi_0)|\right)\right\|_r \\ &< \xi\|Ef\|_p^{p/r}.\end{aligned}$$

This completes the proof of this theorem, and hence of Theorem (1.2). □

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