SEPARATING POINTS FROM CLOSED CONVEX SETS OVER ORDERED FIELDS AND A METRIC FOR $\tilde{R}^n$

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ABSTRACT. Let R be an arbitrary ordered field, let $\bar{R}$ be a real closure, and let $\tilde{R}$ and $\tilde{R}^n$ denote the real spectra of $\bar{R}[X]$ and $\bar{R}[X_1, \dots, X_n]$. We prove that a closed convex subset in R^n may be separated from a point not in it via a continuous "linear" functional taking values in $\tilde{R}$ and that there is a $\tilde{R}$ -valued metric on $\tilde{R}^n$. The methods rely on the ultrafilter interpretation of points in $\tilde{R}^n$ and on the existence of suprema and infima of sets in $\tilde{R}$.

INTRODUCTION

A basic result which uses the completeness and order relation on $\mathbb{R}$ is

Theorem I (for $\mathbb{R}$). *Let $C \subset \mathbb{R}^n$ be a closed convex set and let $\mathbf{p} \in \mathbb{R}^n \setminus C$. Then there exists a linear function*

$$L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{b})$$

such that $L(\mathbf{p}) < 0$ and $L(\mathbf{q}) > 0$ for all $\mathbf{q} \in C$. $\square$

Theorem I remains true if $\mathbb{R}^n$ is replaced by a locally convex real vector space and $\mathbf{p}$ is replaced by an arbitrary compact set [5, 3.3, Satz 4] but we are more interested in the remark that the *statement* of Theorem I makes sense if $\mathbb{R}$ is replaced by any ordered field R . It is false as stated for *every* $R \neq \mathbb{R}$, but if we replace L by an $\tilde{R}$ -valued linear function where $\tilde{R} := \text{Sper } R[X]$ is the *real spectrum* of $R[X]$, we obtain

Theorem I. *Let R be any ordered field, let $C \subset R^n$ be a closed convex set, and let $\mathbf{p} \in R^n \setminus C$. Then there exists a continuous $\tilde{R}$ -valued linear function $L : R^n \rightarrow \tilde{R}$ such that $L(\mathbf{p}) < 0$ and $L(\mathbf{q}) > 0$ for all $\mathbf{q} \in C$.*

In order to prove Theorem I for general R , we let $\bar{R}$ be a real closure of R and extend the Euclidean norm to the space

$$\tilde{R}^n := \text{Sper } \bar{R}[X_1, \dots, X_n],$$

which may be thought of as consisting of those points in $\text{Sper } R[X_1, \dots, X_n]$ which induce the given order on R . This extended norm does not give us

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a metric on $\tilde{R}^n$ because $\tilde{R}^n$ is not equipped with an addition, and we cannot define an inner product on $\tilde{R}^n$ as is discussed after Definition 2 in §3. However, with virtually no fudging of the definitions, we are able to prove

Theorem II. *Let R be any ordered field. Then there is a metric*

$$\mu : \tilde{R}^n \times \tilde{R}^n \rightarrow \tilde{R}$$

which extends the Euclidean distance function.

The point of this paper, then, is that the real spectrum provides the correct tool for both compactifying and completing general ordered fields with a view towards preserving at least some of the structure familiar from linear analysis over $\mathbb{R}$. We assume a certain degree of familiarity with the spaces $\tilde{R}$ and $\tilde{R}^n$. In particular we assume that the reader is familiar with

- (1) the various types of points in $\tilde{R}$ for general real closed R ,
- (2) the description given in [7 §2]
- (3) the “ultrafilter theorem” [3] and ultrafilter arguments, and
- (4) semialgebraic maps $f : R^n \rightarrow R$ and their extensions $\tilde{f} : \tilde{R}^n \rightarrow \tilde{R}$.

All of the necessary background is available in [2, Chapter 7], [1], and [7, §2].

The paper is organized as follows: In §1 we present a simple counterexample showing why Theorem I is false if we only allow R -valued linear functions, and in §2 we present an elementary proof of Theorem I for $\mathbb{R}$. In §3 we explain what we mean by a $\tilde{R}$ -valued linear function and prove a continuity result for these functions. In §4 we isolate two results needed to modify the proof given in §2, and in §5 we prove Theorem I for general R . In §6 we generalize the notion of “slices” from [7] to one called “rips”. Rips allow us to add points in $\tilde{R}^n$ and to compare the resulting sums so that we may interpret the triangle inequality. In §7 we define μ and prove Theorem II.

We point out right away that, for the purposes of our proofs, we will assume that R is real closed. This is no loss of generality since if $\bar{R}$ is a real closure of the ordered field R and $n \geq 1$, then $\tilde{R}^n = \widehat{\bar{R}}^n$ as we have defined it.

The motivation for this paper came from Bruce Reznick who conjectured that Theorem I (as stated for $\mathbb{R}$) failed for nonarchimedean R during a talk on blenders in May, 1989. A few weeks later Tom Craven pointed out the counterexample in §1 for the case $R = \mathbb{Q}$. The author wishes to thank both Reznick and Craven for their inspiration and an anonymous referee for suggesting some clarifications and simplifications which have been incorporated.

1. A COUNTEREXAMPLE

Suppose $R \neq \mathbb{R}$ and that α is a (finite) Dedekind cut of R which is not represented by an element of R . In keeping with notation from [7, §2], we will represent α by its *left set* α_l and its *right set* α_r . Thus α_l and α_r are

We use the word “interval” to include singletons, half-infinite, half-open, closed, infinite, and half-infinite intervals—in short, any semialgebraically connected subset of R .

two nonempty open half-infinite intervals¹ with the properties that $\alpha_l \leq \alpha_r$, meaning that $x \in \alpha_l, y \in \alpha_r \Rightarrow x \leq y$, and that $\alpha_l \cup \alpha_r = R$. In the case at hand, $\alpha_l < \alpha_r$.

Now consider the upper half-plane $H \subset R^2$ whose boundary is the non-existent line $x_2 = \alpha x_1$. Thus H consists of all points $(0, x_2)$ with $x_2 \geq 0$ plus all $(x_1, tx_1) \in R^2$ such that

$$(1) \quad x_1 \geq 0 \text{ and } t \in \alpha_r \quad \text{or} \quad x_1 \leq 0 \text{ and } t \in \alpha_l.$$

H is closed and convex, but there is no way to separate H from any $\mathbf{p} \notin H$ using a line. To see this, suppose $t \in \alpha_r$ and consider the line $x_2 = tx_1 + b$. Using the fact that α_r has no left endpoint, we may choose x_1 positive and large enough so that $tx_1 + b = t'x_1$ with t' still in α_r . Thus $(x_1, tx_1 + b) = (x_1, t'x_1) \in H$. The same idea works for $t \in \alpha_l$ and the lines $x_1 = c$. In other words, every line intersects H . This shows that Theorem I as stated for $\mathbb{R}$ needs some modification for other R .

2. A PROOF OF THEOREM I FOR $\mathbb{R}$

This section contains a very simple proof of Theorem I for $\mathbb{R}$ based upon the following lemma:

Lemma 1. *Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ and suppose that $\mathbf{p} \cdot \mathbf{q} < \|\mathbf{p}\|^2$. Then for all sufficiently small $t > 0$ we have*

$$(2) \quad \|t\mathbf{q} + (1-t)\mathbf{p}\|^2 < \|\mathbf{p}\|^2.$$

Proof. Use the fact that

$$\|t\mathbf{q} + (1-t)\mathbf{p}\|^2 - \|\mathbf{p}\|^2 = t\{t(\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 - 2\mathbf{q} \cdot \mathbf{p}) + 2(\mathbf{q} \cdot \mathbf{p} - \|\mathbf{p}\|^2)\}. \quad \square$$

Proof of Theorem I over $\mathbb{R}$. We may assume that $\mathbf{p} = \mathbf{0}$. Let $\mathbf{a} \in C$ be a point with minimal distance to $\mathbf{0}$, i.e., with minimal Euclidean norm. The existence of $\mathbf{a}$ is implied by the completeness of $\mathbb{R}^n$. Lemma 1 and convexity imply that

$$(3) \quad \mathbf{a} \cdot \mathbf{x} \geq \|\mathbf{a}\|^2$$

for all $\mathbf{x} \in C$. Thus $L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{a}/2)$ is the sought after function. $\square$

In §5 we carry out this proof for general R , but first we need to compactify the domain R^n , complete the range R , and define new linear functions. This is done via the real spectrum, starting in the next section.

3. $\tilde{R}$ -VALUED LINEAR FUNCTIONS

The semialgebraic subsets of R^n form a boolean algebra and $\tilde{R}^n$ may be defined as the set of ultrafilters of this algebra. (Remember, we are assuming that R is real closed.) We will use this definition exclusively and denote points of $\tilde{R}^n$ by lower case Greek letters. The topology on $\tilde{R}^n$ is that generated by

the sets

$$\tilde{U} := \{ \alpha \in \tilde{R}^n \mid U \in \alpha \}$$

for open semialgebraic sets $U \subset R^n$. $\tilde{R}^n$ is quasicompact but not Hausdorff. If β is in the closure of α we say that α *specializes* to β and write $\alpha \rightarrow \beta$. The concepts of closed and bounded for points are defined by

Definition 1. A point $\alpha \in \tilde{R}^n$ is *closed* if it has no proper specializations and *bounded* if it contains a bounded set.

If $A \subseteq R^n$ and $B \subseteq R$ are semialgebraic sets, and f is a map with semialgebraic graph, then $f(A)$ and $f^{-1}(B)$ are also semialgebraic [2]. It is easy to verify that

$$(4) \quad \{ f(A) \mid A \in \alpha \}$$

generates an ultrafilter if $\alpha \in \tilde{R}^n$ is an ultrafilter. Thus f induces a function $\tilde{f} : \tilde{R}^n \rightarrow \tilde{R}$ where $\tilde{f}(\alpha)$ is defined to be the ultrafilter in $\tilde{R}$ generated by (4). If f is also continuous, then $\tilde{f}$ is continuous as a function from $\tilde{R}^n$ to $\tilde{R}$. Details appear in [1], [2, 7.2.8], and [2, 7.3].

As a particular example of this, fix $\mathbf{x} \in R^n$ and consider the dual linear function

$$(5) \quad L_{\mathbf{x}}(\mathbf{y}) := \mathbf{x} \cdot \mathbf{y}.$$

Then $L_{\mathbf{x}}$ is a continuous semialgebraic function, so we may extend it to a function

$$(6) \quad \tilde{L}_{\mathbf{x}} : \tilde{R}^n \rightarrow \tilde{R}$$

whose value at the ultrafilter α is computed by considering each semialgebraic subset $A \in \alpha$ and dotting each point in A with $\mathbf{x}$. This produces a semialgebraic subset of R , and the set of these sets forms an ultrafilter $\tilde{L}_{\mathbf{x}}(\alpha)$ in $\tilde{R}$.

This process may be dualized to produce a function

$$(7) \quad L_{\alpha} : R^n \rightarrow \tilde{R}, \quad \mathbf{x} \mapsto L_{\mathbf{x}}(\alpha).$$

Definition 2. An $\tilde{R}$ -valued linear function is a map $L : R^n \rightarrow \tilde{R}$ of the form $L(\mathbf{x}) = L_{\alpha}(\mathbf{x} - \mathbf{b})$ with L_{α} as in (7) and $\mathbf{b} \in R^n$.

We remark that extending L_{α} to all of $\tilde{R}^n$, i.e., defining an inner product on $\tilde{R}^n$, is too much to ask for. Indeed, for $n = 1$ this would define a multiplication on $\tilde{R}$, and this is hopeless if R is nonarchimedean. To see why, let η be the ultrafilter of semialgebraic sets which span the gap between the positive infinitesimals and the positive noninfinitesimal elements (with respect to $\mathbb{Q}$), and let ζ be the ultrafilter spanning the gap between the positive finite and the positive infinite elements. The product of any set from η with any set from ζ contains all positive, finite, noninfinitesimal elements. Thus “ $\eta \cdot \zeta$ ” is contained in an infinite number of ultrafilters.

Also, the function $L_\eta(x)$ jumps from η to ζ as the argument $x \in R$ crosses the gap ζ . L_η is therefore *not a slice* [7] and certainly cannot be extended to a continuous function $\tilde{R}^n \rightarrow \tilde{R}$. In general, L_α need not even be continuous as function from $R^n \rightarrow \tilde{R}$ as can be seen by taking $n = 1$ and $\alpha = +\infty$. But we do have

Proposition 1. *If α is closed and bounded, then the function L in Definition 2 is continuous. For any α we have the following “sublinearity” property: If $s_1 \leq L(\mathbf{x}) \leq s_2$ and $t_1 \leq L(\mathbf{y}) \leq t_2$ for $s_1, s_2, t_1, t_2 \in R$, then $s_1 + s_2 \leq L(\mathbf{x} + \mathbf{y}) \leq t_1 + t_2$.*

Proof. We may assume that $L = L_\alpha$. A subbasic open set in $\tilde{R}$ consists of the set $\tilde{I}$ of ultrafilters containing an open interval $I = (a, b) \subset R$. We have

$$(8) \quad L_\alpha(\mathbf{x}) \in \tilde{I} \text{ if and only if } \mathbf{x} \cdot A \subseteq I \text{ for some } A \in \alpha.$$

Now, if $L_\alpha(\mathbf{x}) \in \tilde{I}$, there must be some closed bounded $B \in \alpha$ such that $\mathbf{x} \cdot B$ is a finite union of (necessarily closed and bounded) subintervals of (a, b) . For otherwise every closed bounded set $B \in \alpha$ would contain points in the semialgebraic set

$$K := \{\mathbf{y} \mid \mathbf{x} \cdot \mathbf{y} \leq a \text{ or } \mathbf{x} \cdot \mathbf{y} \geq b\}.$$

Since every closed semialgebraic set in α contains a closed bounded semialgebraic set in α , the set of $B \cap K$ with $B \in \alpha$ and B closed would then have the finite intersection property, from which the existence of a specialization β of α with $L_\beta(\mathbf{x}) \notin (a, b)$ would follow. Since $L_\beta(\mathbf{x}) \neq L_\alpha(\mathbf{x})$, we see $\beta \neq \alpha$, contradicting the assumption that α is closed.

We have seen that there is a closed bounded $B \in \alpha$ such that $\mathbf{x} \cdot B \subseteq [c, d]$ with

$$(9) \quad a < c < d < b.$$

Using (9) and a bound on the norm of points in B it is straightforward to find an $\varepsilon > 0$ so that $\mathbf{z} \cdot B \subseteq I$ whenever $\|\mathbf{x} - \mathbf{z}\| < \varepsilon$. This implies $L_\alpha(\mathbf{z}) \in \tilde{I}$ whenever $\|\mathbf{x} - \mathbf{z}\| < \varepsilon$ and establishes continuity.

For the last statement, we note that there is a set $A \in \alpha$ such that for every $\mathbf{a} \in A$ we have $s_1 \leq \mathbf{a} \cdot \mathbf{x} \leq s_2$ and $t_1 \leq \mathbf{a} \cdot \mathbf{y} \leq t_2$ and hence $s_1 + s_2 \leq \mathbf{a} \cdot (\mathbf{x} + \mathbf{y}) \leq t_1 + t_2$. $\square$

Finally, we point out that if R is an arbitrary ordered field and $\alpha = \mathbf{x}$ is a rational point in $\overline{R}^n$ over a real closure of R , then L_α is just the restriction of the dot product with $\mathbf{x}$ to R^n .

4. SOME GROUNDWORK

The purpose of this section is to recall some key results on $\tilde{R}$ from [7] and to isolate a slightly technical but trivial lemma on abstract functions.

In [7, §2] it is shown that the points $\alpha \in \tilde{R}$ for R real closed may be represented as pairs (α_l, α_r) of subsets of R , called *slices*, satisfying $\alpha_l \leq \alpha_r$,

and $\alpha_l \cup \alpha_r = R$. Specifically, for a point $x \in R$ we have

$$(10) \quad \begin{aligned} x \in \alpha_l & \text{ if and only if } (-\infty, x] \in \alpha, \\ x \in \alpha_r & \text{ if and only if } [x, +\infty) \in \alpha. \end{aligned}$$

The crucial result for us is

Proposition 2. *$\tilde{R}$ is totally ordered with*

$$(11) \quad \alpha \leq \beta \text{ if and only if } \alpha_l \subseteq \beta_l \text{ and } \beta_r \subseteq \alpha_r.$$

Every subset of $\tilde{R}$ has both a supremum and an infimum in $\tilde{R}$.

Proof. [7, §2]. $\square$

The technical result we need, which we state in more generality than necessary, is a souped-up version of the result that the extension of a continuous semialgebraic function to $\tilde{R}^n$ assumes a minimum on a closed subset: Let $K \subseteq \tilde{R}^n$ be closed and let $f: R^n \rightarrow R$ be a continuous semialgebraic function. Let

$$\lambda := \inf_{\alpha \in K} \tilde{f}(\alpha).$$

In addition, let $\{g_s\}_{s \in \mathcal{S}}$ be a family of continuous semialgebraic functions from R^n to R and let $\{\kappa_s\}_{s \in \mathcal{S}}$ be a family of closed points in $\tilde{R}$ such that for any finite subset $s_1, \dots, s_m \in \mathcal{S}$ and any $\varepsilon \in R$ with $\varepsilon \geq \lambda$ there are points $\alpha \in K$ such that simultaneously $\tilde{f}(\alpha) \leq \varepsilon$ and $\tilde{g}_{s_i}(\alpha) \geq \kappa_{s_i}$ for $i = 1, \dots, m$.

Lemma 2. *Under the hypotheses just given there is a closed point $\zeta \in K$ with $\tilde{f}(\zeta) = \lambda$ and $g_s(\zeta) \geq \kappa_s$ for all $s \in \mathcal{S}$.*

Proof. The sets

$$K_{\varepsilon, s_1, \dots, s_m} := \{ \alpha \in K \mid \tilde{f}(\alpha) \leq \varepsilon \text{ and } \tilde{g}_{s_i}(\alpha) \geq \kappa_{s_i} \text{ for } i = 1, \dots, m \}$$

are closed and have the finite intersection property. Therefore there is a closed point ζ in their intersection. $\square$

5. PROOF OF THEOREM I

Let $C \subset R^n$ be closed and convex and assume that $0 \notin C$ with the intention of proving Theorem I. Let

$$N^2(\mathbf{x}) = \|\mathbf{x}\|^2 = \sum_{i=1}^n x_i^2$$

be the square of the Euclidean norm, which is a continuous semialgebraic function. The following is a version of Lemma 1 from §2:

Lemma 3. *Let $\mathbf{p}$ and $\mathbf{q}_1, \dots, \mathbf{q}_m$ be finitely many points in C . Then there is a point $\mathbf{a} \in C$ such that $N(\mathbf{a}) \leq N(\mathbf{p})$ and such that for $i = 1, \dots, m$ we have*

$$(12) \quad \mathbf{q}_i \cdot \mathbf{a} \geq N^2(\mathbf{a}).$$

Proof. We might as well assume that C is the convex hull of the $\mathbf{q}_i$. This is a closed bounded semialgebraic set, so there is a point $\mathbf{a}$ in it of minimal norm [2, 2.5.8]. The calculation used to prove Lemma 1 is valid over any R , from which (12) follows. $\square$

The same argument works in the case of an arbitrary closed bounded semialgebraic set C and (using the fact that a semialgebraic function achieves a maximum on C as well) yields

Lemma 4. *If C is a closed bounded semialgebraic set in R^n which does not contain 0, then there is a vector $\mathbf{x} \in R^n$ such that $\mathbf{x} \cdot C$ is contained in a closed interval $[a, b]$ with $0 < a \leq b < \infty$. $\square$*

Proof of Theorem I. Let K be the closure of C in $\tilde{R}^n$ and let

$$\lambda := \inf_{\mathbf{q} \in C} N(\mathbf{q}).$$

Since C is closed, we have $0^+ < \lambda$, so we may choose a $z \in R$ with $0 < z \leq \lambda$. We now consider the semialgebraic function N , the family $\{L_{\mathbf{q}}\}_{\mathbf{q} \in C}$ of semialgebraic functions, and the constant family $\kappa_{\mathbf{q}} = z^2$. If $\varepsilon \geq \lambda$, there is a point $\mathbf{p} \in C$ with $N(\mathbf{p}) \leq \varepsilon$. Lemma 3 says that this setup satisfies the hypotheses of Lemma 2, so we find a closed $\zeta \in K$ such that

$$(13) \quad \tilde{N}(\zeta) = \lambda \quad \text{and} \quad \tilde{L}_{\mathbf{q}}(\zeta) \geq z^2 \text{ for all } \mathbf{q} \in C.$$

Let $\tilde{L}_{\zeta}$ be defined as in (7). From (13) we see both that ζ is bounded and that $\tilde{L}_{\zeta}(\mathbf{q}) \geq z^2$ for $\mathbf{q} \in C$. By Proposition 1 the function

$$L(\mathbf{x}) = \tilde{L}_{\zeta}(\mathbf{x} - \mathbf{b})$$

fulfills the requirements of Theorem I for any $\mathbf{b}$ with $-z^2/2 < \tilde{L}_{\zeta}(-\mathbf{b}) < 0$.

To see that such a $\mathbf{b}$ exists, note that ζ contains a closed bounded B which does not contain 0. We now apply Lemma 4 to B and let $\mathbf{b} = -\frac{z^2}{2b}\mathbf{x}$ with $\mathbf{x}$ and b as in the lemma. $\square$

6. SLICES AND RIPS

In the description of $\tilde{R}$ from [7], the set α_l is called the *left set* of α and consists of all $x \in R$ with $x \leq \alpha$, while α_r is called the *right set* of α and consists of all x with $\alpha_r \leq x$. If we try to add two points α and β by adding their left and right sets, we see that

$$\alpha_l + \beta_l \leq \alpha_r + \beta_r$$

and also that

$$(14) \quad \begin{aligned} x \in \alpha_l + \beta_l \text{ and } y < x &\Rightarrow y \in \alpha_l + \beta_l, \\ x \in \alpha_r + \beta_r \text{ and } y > x &\Rightarrow y \in \alpha_r + \beta_r, \end{aligned}$$

but in general there is a gap between the set sums in (14). For example, the point 2 is in neither the sum of the left sets nor in the sum of the right sets of “ $1^- + 1^+$ ”. In the nonarchimedean case entire intervals may be left out.

For this reason we are led to consider more general pairs

$$(15) \quad \zeta = (\zeta_l, \zeta_r)$$

of left and right sets and define

Definition 3. A *rip* is a pair (15) satisfying

$$(17) \quad \begin{aligned} &\zeta_l \leq \zeta_r, \\ &x \in \zeta_l \text{ and } y < x \Rightarrow y \in \zeta_l, \text{ and} \\ &x \in \zeta_r \text{ and } y > x \Rightarrow y \in \zeta_r. \end{aligned}$$

Thus points in $\tilde{R}$ are special types of rips. If ζ and ξ are rips, we define

$$\zeta + \xi = (\zeta_l + \xi_l, \zeta_r + \xi_r),$$

which is again a rip. Next, we borrow from John Conway's philosophy (used to define the ordering on the surreal numbers in [JC, Chapter 1]) and define $\zeta \leq \xi$ *unless there is an obstruction to this inequality*. An obstruction is a point $x \in R$ with either $\xi < x \leq \zeta$ or $\xi \leq x < \zeta$. We interpret $\xi \leq x$ to mean that $x \in \xi_r$ and $\xi < x$ to mean that $x \in \xi_r \setminus \xi_l$. There is no such obstruction x if

$$\zeta_l \cap \xi_r \setminus \xi_l = \zeta_l \cap \xi_r \setminus \zeta_r = \emptyset,$$

which may be rephrased as

Definition 4. Let ζ and ξ be two rips. We define $\zeta \leq \xi$ if and only if and $(\zeta_l \cap \xi_r) \subseteq (\zeta_r \cap \xi_l)$.

Proposition 3. The " $\leq$ "-relation on rips extends the total ordering on $\tilde{R}$. Given two rips ζ and ξ , either $\zeta \leq \xi$, or $\xi \leq \zeta$ or both.

Proof. The first statement follows from the development of Definition 4 and can be checked by looking at cases. To prove the second statement, we need to exclude the possibility that there are $x, y \in R$ with

$$(18) \quad \begin{aligned} &x \in \zeta_l \cap \xi_r \text{ and } x \notin \zeta_r \cap \xi_l, \\ &y \in \zeta_r \cap \xi_l \text{ and } y \notin \zeta_l \cap \xi_r. \end{aligned}$$

But if (18) holds, then $x \in \zeta_l$ and $y \in \zeta_r$ imply that $x \leq y$, while $x \in \xi_r$ and $y \in \xi_l$ imply that $x \geq y$. Hence $x = y$, but now (18) is clearly contradictory. $\square$

7. THE METRIC μ

To define μ , consider $\alpha, \beta \in \tilde{R}^n$. If $A \in \alpha$ and $B \in \beta$, we define

$$(19) \quad d(A, B) := \inf_{\substack{\mathbf{a} \in A \\ \mathbf{b} \in B}} \|\mathbf{a} - \mathbf{b}\|.$$

Note that the infimum in (19) is to be taken not in R (where it need not exist) but in $\tilde{R}$. We then go on to define

$$(20) \quad \mu(\alpha, \beta) := \sup_{\substack{A \in \alpha \\ B \in \beta}} d(A, B)$$

where the supremum is taken in $\tilde{R}$ as well. Interpreting the values of μ as rips for the triangle inequality, we have:

Theorem II. *The function $\mu: \tilde{R}^n \times \tilde{R}^n \rightarrow \tilde{R}$ is a positive definite symmetric function satisfying the triangle inequality. Moreover, if $\mathbf{a}, \mathbf{b} \in R^n \subset \tilde{R}^n$, then $\mu(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$.*

Proof. Symmetry is obvious from the definition as is the fact that $\mu(\alpha, \beta) \geq 0$. For the rest of the proof we let $\alpha, \beta, \gamma \in \tilde{R}^n$.

Suppose $\mu(\alpha, \beta) = 0$. Since $\{x \in R \mid x > 0\}$ has the infimum 0^+ in $\tilde{R}$, every set $A \in \alpha$ must have zero distance to every set $B \in \beta$. In other words, $A \cap B \neq \emptyset$ for every $A \in \alpha$ and $B \in \beta$. Thus $\alpha \cup \beta$ is a filter, which implies $\alpha = \beta$ since both are ultrafilters. Thus μ is positive definite.

The triangle inequality states that

$$\mu(\alpha, \gamma) \leq \mu(\alpha, \beta) + \mu(\beta, \gamma).$$

To verify this we need to show that

$$(21) \quad (\mu(\alpha, \gamma)_l \cap (\mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r)) \subseteq (\mu(\alpha, \gamma)_r \cap (\mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l)).$$

So suppose there is a point z in the left-hand set given in (21). Since $z \in \mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r$, there are points $x, y \in R$ such that $x + y = z$ and such that given any $A \in \alpha$, $B \in \beta$, and $C \in \gamma$ there are points $\mathbf{a} \in A$, $\mathbf{b} \in B$ and $\mathbf{b}' \in B$, $\mathbf{c} \in C$ with

$$(22) \quad \|\mathbf{a} - \mathbf{b}\| \leq x \quad \text{and} \quad \|\mathbf{b}' - \mathbf{c}\| \leq y.$$

Fixing A , B , and C , consider the set

$$B' := \{\mathbf{b} \in B \mid d(A, \{\mathbf{b}\}) \leq x\}.$$

We must have $B' \in \beta$, for otherwise its complement would be in β and (22) could not hold for any $\mathbf{b} \in B'$. Now consider the set

$$(23) \quad B'' := \{\mathbf{b}' \in B' \mid d(\{\mathbf{b}'\}, C) \leq y\}.$$

Again, we must have $B'' \in \beta$. But this implies that $d(A, C) \leq z$ for any $A \in \alpha$ and $C \in \gamma$. Thus we conclude that $z \in \mu(\alpha, \gamma)_r$.

By assumption, $z \in \mu(\alpha, \gamma)_l$. Thus there are sets $A \in \alpha$ and $C \in \gamma$ with $d(A, C) \geq z$. Fix these sets. Suppose there were an $E \in \beta$ with $d(A, \{\mathbf{e}\}) < x$ for all $\mathbf{e} \in E$. By intersecting E with B'' from (23) we would obtain $d(A, C) < z$. Thus the set

$$E := \{\mathbf{b} \in B' \mid d(A, \{\mathbf{b}\}) = x\}$$

is in β , from which we see that $x \in \mu(\alpha, \beta)_l$. Similarly, $y \in \mu(\beta, \gamma)_l$, and so $z = x + y \in \mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l$, establishing (21). $\square$

Some examples. In order to illustrate μ , we state (without proof) some values in a few simple cases. First, a table of values for pairs of points in $\tilde{R}$ with the first point specializing to 1 and the second to 2:

$$\begin{aligned} \mu(1^-, 2^-) &= 1^-, & \mu(1, 2^-) &= 1^-, & \mu(1^+, 2^-) &= 1^+, \\ \mu(1^-, 2) &= 1^+, & \mu(1, 2) &= 1, & \mu(1^+, 2) &= 1^-, \\ \mu(1^-, 2^+) &= 1^+, & \mu(1, 2^+) &= 1^+, & \mu(1^+, 2^+) &= 1^-. \end{aligned}$$

For a slightly more complicated case, consider the two-dimensional point $\alpha \in \tilde{\mathbb{R}}^2$ consisting of all semialgebraic subsets A containing a set of the form $\{(x, e^x) \mid 0 < x < \varepsilon\}$ for some $\varepsilon > 0$. Let β be the one-dimensional point consisting of all semialgebraic sets containing some piece of the algebraic half-branch $y = 0, x > 0$ at $(0, 0)$. Then $\mu(\alpha, \beta) = 1^+$.

In order to clarify the nature of μ , we point out

Proposition 4. Suppose $\alpha, \beta \in \tilde{R}^n$ and α is bounded. Then $\mu(\alpha, \beta) = 0^+$ if and only if α and β are distinct but have a common specialization.

Proof. Suppose $\mu(\alpha, \beta) = 0^+$. Let $A \in \alpha$ be bounded, let $B \in \beta$ be arbitrary, and let $\bar{A}$ and $\bar{B}$ be their closures in R^n . If we fix $\mathbf{x}$, then the function of $\mathbf{x}$ which measures the distance to $\bar{B}$ is a continuous semialgebraic function [2 2.5.8]. This takes a minimum on the closed bounded set $\bar{A}$. If $\mu(\alpha, \beta) = 0^+$, this minimum cannot be strictly positive, so $\bar{A} \cap \bar{B} \neq \emptyset$. We conclude that

$$(24) \quad \{\bar{A} \cap \bar{B} \mid A \in \alpha, B \in \beta\}$$

has the finite intersection property and is contained in at least one ultrafilter γ . Since any closed set in α is in γ , $\alpha \rightarrow \gamma$. Similarly, $\beta \rightarrow \gamma$.

If α and β have no common specialization, we can find disjoint closed sets $A \in \alpha$ and $B \in \beta$ [7], and since we may choose A to be bounded, it is immediate that $\mu(\alpha, \beta) \geq d(A, B) = x > 0$ for some $x \in R$, so $\mu(\alpha, \beta) > 0^+$. $\square$

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