TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 326, Number 2, August 1991

AN EQUIVARIANT TORUS THEOREM FOR INVOLUTIONS

W. H. HOLZMANN

ABSTRACT. A complete classification is given for equivariant surgery on incompressible tori with respect to involutions with possible 1- or 2-dimensional fixed sets.

1. INTRODUCTION

In [8] an equivariant torus theorem was proved for involutions which have at most isolated fixed points. The main result of this paper is an equivariant torus theorem (Theorem 4.5) for involutions with possible 1- or 2-dimensional fixed sets. If additional restrictions are imposed, various equivariant surgery theorems have been proved. If the manifold is closed, orientable, and irreducible with infinite first homology, a theorem was given in [12].

The proof of the equivariant torus theorem proceeds by isotoping a given incompressible torus to a new torus such that the new torus, its image under the involution, and the fixed point set of the involution are "almost pairwise transversal." The weaker "almost pairwise transversality" condition is used since, in general, pairwise transversality cannot be ensured (see Remark 3.3). The curves of intersection between the new torus and its image under the involution are of certain types. The curves are changed or removed by various surgeries, thus obtaining an equivariant torus or one of the exceptional cases as listed in the theorem.

An application of this theorem is the classification of involutions with 1- or 2-dimensional fixed sets on orientable torus bundles or unions of twisted I-bundles on Klein bottles. Surgery on an equivariant torus reduces these spaces to spaces on which the involutions are known. Details of this application are given in [4].

2. PRELIMINARIES

Throughout we use the piecewise linear category. A piecewise linear homeomorphism will be called an *isomorphism*.

Received by the editors January 20, 1988 and, in revised form, February 1, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 57A10, 57E25.
Key words and phrases. Involution, fixed sets, equivariant surgery.
The author wishes to thank the referee for his comments and suggestions.

0002-9947/91 \$1.00 + \$.25 per page

Let M be a connected compact 3-manifold. An *involution* $\imath$ is an isomorphism with $\imath \neq \mathrm{id}$ and $\imath^2 = \mathrm{id}$.

Let Fix denote the fixed set $\mathrm{Fix} = \mathrm{Fix}(\imath) = \{x : \imath(x) = x\}$. Let $\imath$ be an involution on a manifold M and $\imath'$ an involution on a manifold M'. $\imath$ and $\imath'$ are *conjugate* if there is an isomorphism $h: M \to M'$ with $\imath' = h \circ \imath \circ h^{-1}$. Call h a *conjugation* between $\imath$ and $\imath'$.

Lemma 2.1. *Given a simplicial subdivision K of M and an involution $\imath$ of M there is a subdivision L of K with $\imath$ simplicial with respect to L.*

Proposition 2.2. *Let $\imath$ be an involution on a manifold M. Let L be a subdivision of M with $\imath: L \to L$ simplicial, and let L' be the first barycentric subdivision of L. Then* $\mathrm{Fix} = \mathrm{Fix}(\imath)$ *is a subcomplex of L'.* Fix *is the union of disjoint 0-, 1- and 2-dimensional proper submanifolds. Write* Fix^0, Fix^1, *and* Fix^2 *respectively for the unions of the 0-, 1- and 2-dimensional components of* Fix.

If $v \in \mathrm{Fix}^0 \cup \mathrm{Fix}^2$ then $\imath$ is locally orientation reversing at v. If $v \in \mathrm{Fix}^1$ then $\imath$ is locally orientation preserving at v. In particular, if M is orientable then $\imath$ is orientation reversing if $\mathrm{Fix}^0 \cup \mathrm{Fix}^2 \neq \varnothing$ *and $\imath$ is orientation preserving if* $\mathrm{Fix}^1 \neq \varnothing$.

Proof. Use the following:

(1) Let Δ be a standard m-simplex (with standard subdivision) invariant under $\imath$. Then $\mathrm{Fix} \cap \Delta$ is a subcomplex of the first barycentric subdivision of Δ.

(2) If Fix contains a 3-simplex then $\imath = \mathrm{id}$.

If $v \in \mathrm{Fix}$ is a vertex of $\mathrm{int}(L)$ consider the link Lk of v.

(3) If $Lk \cap \mathrm{Fix}$ contains a 1-cell then $Lk \cap \mathrm{Fix}$ is one 1-sphere. So $v \in \mathrm{Fix}^2$.

(4) If $Lk \cap \mathrm{Fix}$ consists of $m \geq 0$ vertices then

$$\chi(Lk / \imath) - m = \tfrac{1}{2}(\chi(Lk) - m).$$

Since $Lk/\imath$ is a surface and Lk is a 2-sphere, it follows that $m = 2$ and hence $v \in \mathrm{Fix}^1$, or $m = 0$ and hence $v \in \mathrm{Fix}^0$. □

3. Equivariant transversality

In order to be able to perform surgeries on a surface F_0 in a 3-manifold M we would like to perform an ambient isotopy on F_0 such that the isotopic surface F has the property that F, $\imath F$, and Fix are pairwise transversal. This can be done if the manifold is orientable. If $\mathrm{Fix}^2 \neq \varnothing$ and M is nonorientable, however, pairwise transversality is not possible in general. This necessitates using a somewhat weaker form of transversality.

Lemma 3.1. *Let F be a proper surface in a 3-manifold with F, $\imath F$, and* Fix *pairwise transversal. Then the components of $F \cap \imath F$ are 1-spheres and proper 1-cells. If C is a component of $F \cap \imath F$ with $C \cap \mathrm{Fix}^2 \neq \varnothing$ then $C \subseteq \mathrm{Fix}^2$.*

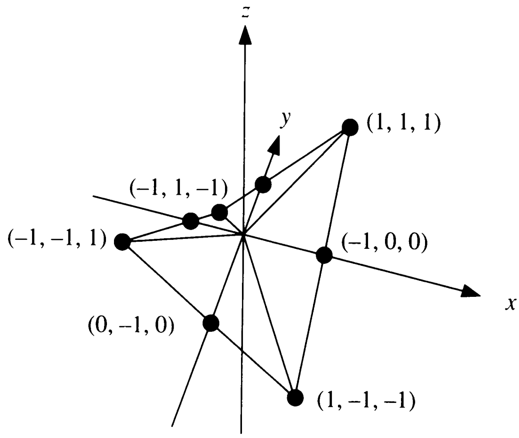

FIGURE 1

Proof. The first statement follows by transversality of F and ιF. The second statement follows by considering the star of a point in $C \cap \mathrm{Fix}^2$. $\square$

For the 3-cell $B^3 = \{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$ in $\mathbf{R}^3$ let $i: B^3 \to B^3$ be the map $i(x, y, z) = (-x, y, z)$. Then $\mathrm{Fix}(i)$ is the intersection of B^3 with the yz plane. Let S be the 1-sphere obtained as the join of $\{(1, 1, 1), (-1, -1, 1)\}$ with $\{(-1, 1, -1), (1, -1, -1)\}$ and let D be the cone from $(0, 0, 0)$ on S. D is a saddle shaped region (see Figure 1). (We could alternately take D defined by $\{z = xy/\sqrt{x^2+y^2}\} \cup \{(0, 0, 0)\}$.) Notice that $D \cap iD$ is the part of the x and y axis in B^3 while $D \cap \mathrm{Fix}(i)$ is part of the y axis. D and $\mathrm{Fix}(i)$ are transversal and iD and $\mathrm{Fix}(i)$ are transversal, but D and iD are not transversal at $(0, 0, 0)$. There is a subdivision making these spaces simplicial with all the vertices on $\partial B^3 \cup (0, 0, 0)$.

Definition 3.2. Let F be a proper surface in a 3-manifold M and ι an involution on M with fixed set Fix. Call a point v *a saddle point* if $v \in F \cap \mathrm{Fix}^2$ and if $(F, \iota F, \mathrm{Fix}) \cap \mathrm{star}(v)$ is isomorphic to $(D, iD, \mathrm{Fix}(i))$.

Remark 3.3. Saddle points exist since i is an involution with fixed set $\mathrm{Fix}(i)$. Although ∂D, ∂iD, $\partial \mathrm{Fix}(i)$ are pairwise transversal there is no 2-cell E with $\partial E = \partial D$ and E, iE, $\mathrm{Fix}(i)$ pairwise transversal. Otherwise, since $\partial E \cap \partial iE - \mathrm{Fix}(i) = (\pm 1, 0, 0)$ there is a 1-cell I of $E \cap iE$ with $(1, 0, 0) \in \partial I$ and this 1-cell must meet Fix, contradicting the previous lemma.

Let d denote the identification $(x, 1, z) \sim (x, -1, -z)$ for all x and z. Then D/d is an annulus in a solid Klein bottle B^3/d and no isotopy of D/d moves it to an annulus with F, iF, and $\mathrm{Fix}(i)/d$ pairwise transversal.

Definition 3.4. Let F be a proper surface in a 3-manifold and ι an involution on M with fixed set Fix. Then F, ιF, and Fix are *almost pairwise transversal* if:

(1) F, ιF, and Fix are pairwise transversal except at a finite number of saddle points, and

(2) the only components of $F \cap \mathrm{Fix}$ containing saddle points are 1-spheres and each such 1-sphere contains at most one saddle point.

Let $\mathscr{E}$ be the closure of $(F \cap \iota F) - \mathrm{Fix}^2$. $\mathscr{E}$ consists of disjoint 1-spheres and proper 1-cells: in a neighborhood of a saddle point, $F \cap \iota F - \mathrm{Fix}^2$ corresponds to $[-1, 0) \times 0 \times 0 \cup (0, 1] \times 0 \times 0$ in the B^3 model for saddle points.

Let E be a component of $\mathscr{E}$ that contains a saddle point v. Then E has a fixed point and is invariant under ι. Therefore, either E is a 1-cell with no fixed points other than v or E is a 1-sphere with exactly two fixed points v and w. By transversality w is in Fix^1 or Fix^2. In the latter case w is a saddle point. Together with Lemma 3.1 this shows:

Proposition 3.5. *Let F, ιF, and* Fix *be almost pairwise transversal. Then the components of $F \cap \iota F$ are of one of the following forms*:

(1) *Components with no saddle points* (*standard components*):

(a) *proper* 1-*cell I with $I \cap \mathrm{Fix} = \varnothing$ or $I \subseteq \mathrm{Fix}^2$*,

(b) *proper* 1-*cell I with $I \cap \mathrm{Fix} = I \cap \mathrm{Fix}^1 = v$, v a point,*

(c) 1-*sphere S with $S \cap \mathrm{Fix} = \varnothing$ or $S \subseteq \mathrm{Fix}^2$*,

(d) 1-*sphere S with $S \cap \mathrm{Fix} = S \cap \mathrm{Fix}^1 = v_1 \cup v_2$, where v_1 and v_2 are points.*

(2) *Components with saddle points*:

Type I *component*: *$S_1 \cup I$ with $S_1 \cap I = \mathrm{Fix} \cap I = w$, $S_1 \subseteq \mathrm{Fix}^2$, and w is the only saddle point on $S_1 \cup I$.*

Type II *component*: *$S_1 \cup S$ with $S_1 \cap S = w$, $S_1 \subseteq \mathrm{Fix}^2$, $S \cap \mathrm{Fix} = v \cup w$, $v \in \mathrm{Fix}$, and w is the only saddle point on $S_1 \cup S$.*

Type III *component*: *$S_1 \cup S_2 \cup S$ with $S_1 \cap S_2 = \varnothing$, $S_i \cap S = w_i$, $S_i \subseteq \mathrm{Fix}^2$, $S \cap \mathrm{Fix} = w_1 \cup w_2$, and w_1 and w_2 are the only saddle points on $S_1 \cup S_2 \cup S$.*

Here S, S_1, and S_2 are 1-*spheres, the I are* 1-*cells and w_i are points, where $i = 1, 2$.*

Note that case (2) does not arise if the manifold is orientable because a regular neighborhood N of S_1 is always a solid Klein bottle. For if N is instead a solid torus instead, then $\mathrm{Fix} \cap N$, $F \cap N$, and $\iota F \cap N$ are all annuli or all Möbius bands. Consider annulus A, a component of $\partial N - \mathrm{Fix}$. $A \cap F$ and $A \cap \iota F$ are two 1-spheres intersecting transversally in A. One sees these 1-spheres have nonzero intersection number, a contradiction.

Corollary 3.6. *If F, ιF, and* Fix *are almost pairwise transversal, then they are pairwise transversal if one of the following holds*:

(a) *M is orientable.*

(b) *F is a 2-cell.*

(c) *F is an annulus with $\partial F \cap \iota \partial F = \varnothing$.*

Proof. In case (a) regular neighborhoods of 1-spheres are solid tori.

In cases (b) and (c) Type II or III components are excluded since the 1-sphere S is nonseparating. In case (c) Type I components are excluded a priori, while in case (b) the 1-sphere S_1 separates so a proper 1-cell C cannot intersect S_1 transversally at one point. □

In a surface F a proper 1-cell I *bounds* a disc D if $I = \overline{\partial D - \partial F}$.

Corollary 3.7. *Let* F, ιF, *and* Fix *be almost pairwise transversal and* C *a proper* 1*-cell or* 1*-sphere component of* $F \cap \iota F$, *that is, let* C *be a standard component. Then any disc in* F *or* ιF *bounded by* C *contains only standard components.*

Proof. The proof is similar to the proof of case (b) in the previous corollary. □

Equivariant Transversality Theorem 3.8. *Let* ι *be an involution on a* 3*-manifold* M *with* Fix = Fix(ι) *and let* F_0 *be a proper surface in* M. *Then there is an ambient* ε*-isotopy on* M *taking* F_0 *to a proper surface* F *such that* F, ιF, *and* Fix *are almost pairwise transversal. In* ∂M, *if* ∂F, $\iota\partial F$, *and* Fix *are pairwise transversal then the isotopy may be taken to be the identity on* $\partial M - N$, *where* N *is a given neighborhood of* $\partial \mathrm{Fix}^2 \cap \partial F$.

Proof. Let $F = F_0$ be a proper surface. By Proposition 2.2 and Lemma 2.1 subdivide M so that ι is simplicial with respect to the subdivision, Fix is a subcomplex of the subdivision, and Fix is a disjoint union of 0-, 1- and 2-dimensional components Fix^0, Fix^1, and Fix^2. All isotopies performed in the construction will be done in the star neighborhoods of certain simplexes. By taking a sufficiently fine subdivision ε-isotopies are obtained.

Step 1. *Adjust* F *near* Fix^2. By isotopies similar to those in the third step below we can assume F and Fix are transversal, the isotopy not moving ∂F unless ∂F and ∂Fix are nontransversal. In particular, $F \cap \mathrm{Fix}^0 = \varnothing$. Then $F \cap \mathrm{Fix}^2$ consists of disjoint 1-spheres and 1-cell components proper in M.

Let S be a 1-sphere component of $F \cap \mathrm{Fix}^2$. Let N' be a regular neighborhood of S with $N' \cap F$ and $N' \cap \mathrm{Fix}$ transversal and each an annulus or Möbius band. S has a regular neighborhood N contained in $\mathrm{int}(N')$ invariant under ι with no vertices on $\mathrm{int}(N) - S$ such that $N \cap \mathrm{Fix}$ is a regular neighborhood of S and $\mathrm{Fix} \cap \partial N$ has a regular neighborhood Q in ∂N which is invariant under ι and has no vertices except on $\mathrm{Fix} \cup \partial Q$.

Case 1. $F \cap N$ *and* $\mathrm{Fix} \cap N$ *are annuli.* Then N is a solid torus, ∂Q has four components, and $N - \mathrm{Fix}$ consists of two components N_1 and N_2 which are interchanged by ι. Let J_1 and J_2 be components of ∂Q with $J_i \subseteq N_i$ and $\iota J_1 \neq J_2$. Let A_i be the annulus with $\partial A_i = J_i \cup S$ having no vertices except on ∂A_i. F is isotopic to a surface F' by an ambient isotopy which is the identity on $M - N'$ and such that $F' \cap N' \cap \mathrm{Fix} \subseteq N$ and $F' \cap N = A_1 \cup A_2$. Since $\iota J_1 \neq J_2$ it follows $F' \cap N \cap \iota(F' \cap N) = S$ and $F' \cap N$, $\iota(F' \cap N)$, $\mathrm{Fix} \cap N$ are pairwise transversal.

Case 2. *$F \cap N$ and* $\mathrm{Fix} \cap N$ *are Möbius bands.* Then N is a solid torus, and ∂Q has two components that are interchanged by $\imath$. If J is one of these, then J and S determine a Möbius band A with $\partial A = J$. Proceed as in Case 1.

If M is orientable then Cases (3) and (4) do not arise. Only in these cases do saddle points arise.

Case 3. *$F \cap N$ is an annulus and* $\mathrm{Fix} \cap N$ *is a Möbius band.* Then N is a solid Klein bottle. Let A be one of the two (open) Möbius band components of $\partial N - \mathrm{Fix}$. There are two 1-spheres J_1 and J_2 which represent generators of $H_1(A) = \mathbf{Z}$ with J_1 and J_2 intersecting transversally and at only one point x. J_i and S bound an annulus A_i with $A_1 \cap A_2 = S \cup I$, where I is a 1-cell with $\partial I = x \cup y$, $y \in S$. Proceed as in Case 1 using $F' \cap N = A_1 \cup \imath A_2$. Then y is a saddle point and $F' \cap N$, $\imath(F' \cap N)$, and $\mathrm{Fix} \cap N$ intersect pairwise transversally elsewhere in N.

Case 4. *$F \cap N$ is a Möbius band and* $\mathrm{Fix} \cap N$ *is an annulus.* This case is similar to Case 3. Here $A = \partial N - Q$ is an invariant annulus under $\imath$. Find a curve J that bounds a Möbius band B with $B \cap \imath B = S$ by lifting (from annulus $A/\imath$) a curve J' which represents twice a generator and which is embedded in $A/\imath$ except for one transversal self-intersection. Now proceed as before.

When S is a 1-cell component of $F \cap \mathrm{Fix}^2$, use an isotopy similar to the one of Case 1 above. This isotopy may change ∂F in $N \cap \partial M$.

Step 2. *Adjust F near* Fix^1. By Step 1, $F \cap \mathrm{Fix}^1$ consists of a number of vertices in $\mathrm{int}(M)$. If $v \in F \cap \mathrm{Fix}^1$, let N' be a regular neighborhood of v and let N be the star neighborhood of v. Take the subdivision so that N is in the interior of N', $F \cap N$ is a proper 2-cell in N, and $\mathrm{Fix} \cap N$ is a proper 1-cell. Since F is transversal, $F \cap \partial N$ is a generator of $H_1(N - \mathrm{Fix})$. Let J' be a curve in the annulus $(\partial N - \mathrm{Fix})/\imath$ representing twice a generator of this annulus. Take J' embedded except for one transversal self-intersection. J' lifts to two 1-spheres J and $\imath J$, which on coning to v give 2-cells D and $\imath D$. D, $\imath D$, and Fix are pairwise transversal in int N. Proceed as in Case 1 of Step 1.

We obtain a surface F and a neighborhood N of Fix such that F has the required transversality properties in N. The following construction adjusts F only on star neighborhoods of simplexes of $\overline{F - N}$ where F and $\imath F$ are not already pairwise transversal. By subdividing sufficiently we may assume without loss that $\mathrm{Fix} = \varnothing$. For convenience assume also $\partial F = \varnothing$.

Let K be a subdivision of M with $\imath$ simplicial and F a subcomplex of K. Let Δ be an m-simplex of F in K with $m = 0$, 1, or 2. Define $\mathrm{St}(\Delta)$, the reduced star of Δ in K, to be all 3-simplexes σ of K with $\Delta \subseteq \sigma$ together with their faces. Let $\mathrm{St}_F(\Delta)$, the reduced star of Δ in F, be all 2-simplexes σ of K with $\Delta \subseteq \sigma \subseteq F$ together with their faces. Let $p: M \to M/\imath$ be the projection.

Step 3. There is a subdivision of M and a proper surface F' ε-isotopic to F such that for every simplex Δ of F' either $p^{-1}p(\Delta) \cap F' = \Delta$ or Δ is a 0- or 1-simplex with $\operatorname{int}(\mathrm{St}_{F'}(\Delta))$ and $\operatorname{int}(\mathrm{St}_{\iota F'}(\Delta))$ transversal.

Call a simplex exceptional if it fails to satisfy these conditions and is of the highest possible dimension $m = 0, 1$, or 2. Induct on the number of such simplexes. If there are no exceptional simplexes the theorem is established.

Suppose there is an exceptional simplex Δ. Adjoin all the vertices (and their translates under ι) of form $\frac{m+2}{m+3}b + \frac{1}{m+3}v$, where b is the barycenter of Δ and v is a vertex of $\mathrm{St}(\Delta) - \Delta$. This determines a refinement K' of K with the same number of exceptional simplexes; no m-simplexes are subdivided for $m = 1, 2$, while for $m = 0$ transversality already holds away from vertices of K. Consider the reduced stars in K'. $\partial\,\mathrm{St}'_F(\Delta)$ is a 1-sphere that decomposes $\partial\,\mathrm{St}'(\Delta)$ into two components D_+ and D_-. There is an ambient isotopy taking F to $F_1 = (F - \mathrm{St}'_F(\Delta)) \cup D_+$ which is the identity except on $\mathrm{St}'_F(\Delta)$. F_1 has fewer exceptional simplexes. When $m \neq 2$ this follows since $D_+ \cup D_-$ intersects the interior of any 2-simplex of $\mathrm{St}(\Delta)$ transversally. $\square$

Regular neighborhoods of the standard components of $F \cap \iota F$ can be taken in a special form.

Proposition 3.9. *Let F, ιF, and* Fix *be almost pairwise transversal and S be a 1-sphere component of $F \cap \iota F$. Suppose, in addition, that the regular neighborhood of S in F and ιF is an annulus. Then there exists a regular neighborhood $V \subseteq \operatorname{int}(M)$ of S, called a standard neighborhood of S, with the following properties*:

(1) *$V \cap F$ and $V \cap \iota F$ are annuli. Since these intersect transversally, V is a solid torus.*

(2) Fix *and ∂V intersect transversally,* $\mathrm{Fix} \cap F \cap V \subseteq S$, *and the closure of each component of* $(\mathrm{Fix} \cap V) - S$ *meets S and ∂V. In particular,* $\mathrm{Fix}^0 \cap V = \varnothing$.

(3) $\mathrm{Fix} \cap V$ *is an annulus, two proper 1-cells, or empty.*

(4) *If $\iota S = S$ then $\iota V = V$.*

(5) *If $\iota S \neq S$ then $\iota V \cap V = \varnothing$ and the above properties hold simultaneously for ιV.*

Property (3) can be arranged since if $\mathrm{Fix} \cap S \neq \varnothing$ then $\iota S = S$. ι is an involution on a 1-sphere so either $\iota = \mathrm{id}$ or ι has exactly two fixed points.

The four 1-spheres $(F \cup \iota F) \cap \partial V$ decompose ∂V into four (closed) annuli α_1, α_2, β_1, and β_2 with $\alpha_1 \cap \alpha_2 = \varnothing$ and $\beta_1 \cap \beta_2 = \varnothing$. Call these annuli the *standard annuli* corresponding to the standard neighborhood of V. Suppose $\iota S = S$. Relabeling, if necessary, we may assume $\iota(\alpha_1 \cap \beta_1) = (\alpha_1 \cap \beta_2)$. It follows that $\iota\alpha_1 = \alpha_1$. Then $\iota\beta_1 = \beta_2$ and $\iota\alpha_2 = \alpha_2$. When $\mathrm{Fix} \cap V \neq \varnothing$ we obtain $\mathrm{Fix} \cap \alpha_1 \neq \varnothing$, $\mathrm{Fix} \cap \alpha_2 \neq \varnothing$, $\mathrm{Fix} \cap \beta_1 = \varnothing = \mathrm{Fix} \cap \beta_2$, and each component of $\mathrm{Fix} \cap V$ meets both α_1 and α_2.

Proposition 3.10. *Let S be a 1-cell component of $F \cap \iota F$, where F, ιF,* Fix *are pairwise transversal (near S). Then there exists a regular neighborhood V of*

S with $V \cap \partial M$ a regular neighborhood of ∂S, called a standard neighborhood of S, with the following properties:

(1) *$V \cap F$ and $V \cap \iota F$ are 2-cells with $\partial M \cap V \cap F$ and $\partial M \cap V \cap \iota F$ each two 1-cells. Necessarily V is a 3-cell.*

(2), (4) *and* (5) *as for 1-sphere standard neighborhoods.*

(3) $\mathrm{Fix} \cap V$ *is a disc, one proper 1-cell, or empty.*

The four 1-cells $(F \cup \iota F) \cap \overline{\partial V - \partial M}$ subdivide $\overline{\partial V - \partial M}$ into four discs α_1, α_2, β_1, and β_2 with $\alpha_1 \cap \alpha_2 = \varnothing$, $\beta_1 \cap \beta_2 = \varnothing$, and the properties as in the previous situation. Call these discs the *standard discs* corresponding to V.

Remark 3.11. In the following theorem certain 1-sphere components S of $F \cap \iota F$ have standard neighborhoods because S bounds discs in F and ιF. In the disc theorem F is orientable so again there are standard neighborhoods. In the torus theorem the construction will be made so as to keep S in this form always.

Theorem 3.12. *Let M be an irreducible 3-manifold with involution ι and F_0 be an incompressible proper surface. Then there is an ambient isotopy of M which is an ε-isotopy on ∂M taking F_0 to a proper surface F such that F, ιF, and* Fix *are almost pairwise transversal and no 1-spheres in $F \cap \iota F$ bound 2-cells in F. On ∂M, if ∂F, $\iota\partial F$, and $\partial\mathrm{Fix}$ are pairwise transversal then the isotopy may be taken to be the identity on $\partial M - N$, where N is a given neighborhood of $\partial \mathrm{Fix}^2 \cap \partial F$.*

Proof. By the preceding transversality theorem there is an F with all the above properties except possibly 1-spheres in $F \cap \iota F$ bound 2-cells in F. By Proposition 3.5 those 2-cells contain no saddle components. Let S be a 1-sphere of $F \cap \iota F$ bounding an innermost 2-cell in ιF, that is, there is a 2-cell $D \subseteq \iota F$ with $D \cap F = \partial D = S$. Since F is compressible, S bounds a 2-cell B in F. If $\iota S = S$ then we may assume $\iota B = D$.

Let V be a standard neighborhood of S. Such a neighborhood exists since S bounds a disc in F and ιF. Let α be the standard annulus in ∂V meeting D but not B. Then $\iota\alpha \cap \alpha = \varnothing$. There is a bicollar $D \times [-1, 1]$ of $D = D \times 0$ with

$$\partial D \times [-1, 1] = D \times [-1, 1] \cap F = S \times [-1, 1]$$

and with $D \times 1 \cap \alpha \neq \varnothing$. Since D is innermost it follows that for a sufficiently thin collar $(D \times 1) \cap \iota(D \times 1) = \varnothing$ and $F \cap \iota(D \times 1) = \varnothing$. Consider $F' = (F - (B \cup S \times [-1, 1])) \cup D \times 1$. Then $F' \cap \iota F' \subseteq (F \cap \iota F) - S$ and F', $\iota F'$, and Fix are almost pairwise transversal. Since M is irreducible and $D \cup B$ is a 2-sphere, F' and F are ambient isotopic by an isotopy being the identity on ∂M. By induction, all 1-spheres bounding 2-cells can be removed. □

A 2-cell B in a 3-manifold is *essential* if it is proper and ∂B does not bound a 2-cell in ∂M. In an irreducible 3-manifold a nonseparating proper 2-cell is essential. The following theorem is well known (see [1, 2, 6, or 10] for instance).

Disc Theorem 3.13. *Let M be an irreducible 3-manifold with involution ι. Suppose M has an essential 2-cell B_0. Then there is an essential 2-cell $B \subseteq M$ such that B and* Fix *are transversal and either $B \cap \iota B = \varnothing$ or $\iota B = B$. In the former case $B \cap \mathrm{Fix} = \varnothing$ and in the latter case $B \cap \mathrm{Fix}$ is a proper 1-cell of B or one point in the interior of B. If $\partial B_0 \cap \iota\partial B_0 = \varnothing$ then one can take $\partial B = \partial B_0$ and B and B_0 are ambient isotopic by an isotopy that is the identity on ∂M.*

Proof. By Theorem 3.12 and Corollary 3.6 there is an essential 2-cell B with B, ιB, and Fix pairwise transversal, B and B_0 ambient isotopic, and $B \cap \iota B$ is either empty or consists of proper 1-cells only. Assume $B \cap \iota B \neq \varnothing$ (in particular, then $\partial B_0 \cap \iota\partial B_0 \neq \varnothing$). By induction it suffices to show how to obtain a new 2-cell B_i with fewer 1-cells in $B_i \cap \iota B_i$.

Let D be an outermost disc of B: $D \subseteq B$ with $D \cap \iota B = \partial D \cap \iota B = I$ a proper 1-cell of B and $\partial D - I \subseteq \partial B$. If $\iota I = I$ define $D' = \overline{\iota B - \iota D}$. If $\iota I \neq I$ define D' to be the closure of the component of $\iota B - I$ that does not contain ιI (see Figure 2). Let V be a standard neighborhood of I and let α_1, α_2, and β be standard discs of V with $\alpha_1 \cap \alpha_2 = \varnothing$, $\alpha_1 \cap \beta \cap D \neq \varnothing$, and $\beta \cap D' \neq \varnothing$. Consider

$$B_1 = (D \cup \beta \cup D') - \mathrm{int}(V) \quad \text{and} \quad B_2 = D \cup (\iota B - D').$$

Then $B_1 \cap \iota B_1 \subseteq (B \cap \iota B) - I$. If B_1 is essential we are done by induction or we arrive at the case $B_1 \cap \iota B_1 = \varnothing$. If B_1 is not essential then ∂B_1 bounds a 2-cell E of ∂M. Since M is irreducible, the 2-sphere $B_1 \cup E$ bounds a 3-cell. This 3-cell does not meet I, otherwise ιB would not be essential. Using the 3-cell construct an ambient isotopy taking B_2 to ιB.

So we may assume B_2 is essential. If $I = \iota I$ we have $\iota B_2 = B_2$ and note that $\mathrm{Fix} \cap B_2 \subseteq \mathrm{Fix} \cap I$, which is necessarily a point of I or all of I. If $I \cap \iota I = \varnothing$ consider a sufficiently thin bicollar $D \times [-1, 1]$ of $D = D \times 0$ such that $D \times [-1, 1] \cap I$ is a bicollar of I in ιB and $D \times 1$ meets α_1. Then $B_2' = (D \times 1 \cup \iota B) - (I \times [-1, 1) \cup D')$ is essential since it is isotopic to B_2 and $B_2' \cap \iota B_2' \subseteq B \cap \iota B - I$. $\square$

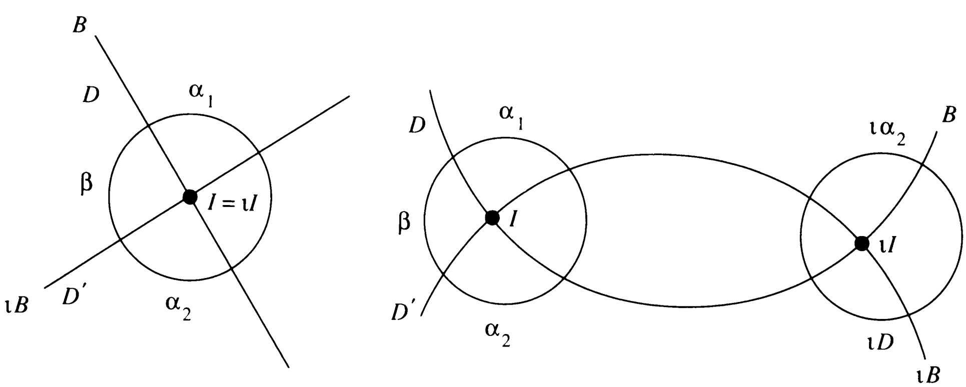

FIGURE 2

4. Equivariant torus theorem

A proper annulus A in a 3-manifold M is *trivial* if A decomposes M into a solid torus $V = D^2 \times S^1$ and a submanifold M_0 such that:

$$M = M_0 \cup V, \qquad M_0 \cap V = \partial M_0 \cap \partial V = A,$$

and there exists a nonseparating proper 2-cell $B \subseteq V$ with $B \cap A = \partial B \cap A$ a nonseparating 1-cell in A.

Otherwise call A *nontrivial.* Call V a solid torus that trivializes A.

Note that if A does not separate M or if ∂A is in different boundary components of M then A is nontrivial.

Call a nontrivial incompressible proper annulus an *essential annulus.* Call an incompressible proper Möbius band an essential Möbius band.

Let F be a surface and S a component of $F \cap \iota F$. In some surgeries performed later we will wish to replace F by $F' = F \times 1$, where $F \times [-1, 1]$ is a bicollar of F. To insure that F', $\iota F'$, and Fix are transversal at least in standard neighborhoods (see Propositions 3.9 and 3.10), the following lemma is used.

Lemma 4.1. *Let V be a solid torus and $\iota: V \to V$ an involution. Let A_0 and A_1 be annuli in ∂V with $\partial A_0 = S_0 \cup S_1$, $\partial A_1 = S_1 \cup S_2$, $A_0 \cap A_1 = S_1$, $\iota S_0 = S_0$, $(A_0 \cup A_1) \cap \iota(A_0 \cup A_1) = S_0$, and $\mathrm{Fix} \cap (A_0 \cup A_1) \subseteq S_0$. Then there is a proper annulus $A \subseteq V$ such that A, ιA, and Fix intersect transversally with $A \cap \iota A = S$ a 1-sphere and ∂A having one component in $\mathrm{int}(\iota A_0)$ and the other component is S_2.*

A similar statement holds if V is a 3-cell, A_0, A_1 are 2-cells, and the S_i are 1-cells (see Figure 3).

Proof. By transversality of Fix, by taking a sufficiently small regular neighborhood N of $A_0 \cup A_1 \cup \iota A_0 \cup \iota A_1$ we may assume one of the following holds:

(1) $\mathrm{Fix} \cap N = \varnothing$,

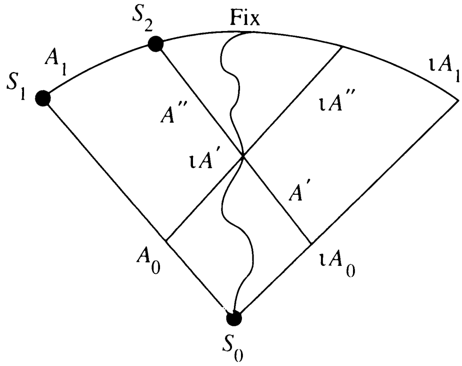

FIGURE 3

(2) $\operatorname{Fix} \cap N$ consists of two disjoint 1-cells I_i with exactly one point of ∂I_i in S_0 and the other in $\operatorname{int}(V) \cap \partial N$, or (3) $\operatorname{Fix} \cap N$ is an annulus such that one boundary component is S_0 and the other is in $\operatorname{int}(V) \cap \partial N$.

Further, there is a regular neighborhood N' of $S_0 \subseteq N$ such that $\iota N' = N'$, $A_0' = N' \cap A_0$ is an annulus, $A_0' \cup \iota A_0' = N' \cap \partial V$, and properties (1)–(3) hold with respect to N'.

Let B be the annulus which is the closure of $\partial N' - A_0 \cup \iota A_0$. Then there is a 1-sphere S in $\operatorname{int}(B)$ with $\iota S = S$ and $\operatorname{Fix} \cap B \subseteq S$. There is an annulus A'' in $\overline{V - N'}$ such that $\partial A'' = A'' \cap \overline{V - N'} = S_2 \cup S$ and $A'' \cap \iota A'' = S$. Let A' be the component of $B - S$ that meets ιA_0. Then $A = A' \cup A''$ is the desired annulus. □

A solid Klein bottle is a twisted I-bundle over an annulus. The annulus is essential but it does not separate the boundary.

Lemma 4.2. *Let U be a solid torus or a solid Klein bottle. If U is a solid torus then U has no essential annuli. If U is a solid Klein bottle then U has no essential annuli that separate ∂U.*

Moreover, suppose A' is an annulus contained in ∂U such that a nonseparating proper disc D of U intersects A' in exactly one nonseparating 1-cell of A'. If A is an incompressible proper annulus disjoint from A' then the solid torus which trivializes A may be taken to be disjoint from A'.

Proof. Suppose A is an essential annulus. Then let D be any proper nonseparating 2-cell of U. (When A' is given, take D as in the statement.) Make A and D transversal. Since A is incompressible, adjust D so that $A \cap D$ consists of 1-cells only.

If $A \cap D = \varnothing$ then A is contained in a 3-cell obtained by removing a sufficiently small regular neighborhood of D from U. This contradicts incompressibility.

If $A \cap D \neq \varnothing$ let B be an outermost 2-cell of D (and disjoint from A' if A' is given): so $B \cap A = \partial B \cap A = I$ is a 1-cell and $B \cap \partial U = \overline{\partial B - I}$. If I bounds a 2-cell in A, then by an isotopy moving B, obtain a disc D' with fewer 1-cells in $A \cap D'$. Assume now that I does not bound a 2-cell in A. Then I does not separate A. Let V be the closure of the component of $U - A$ that meets $\operatorname{int}(B)$. ∂A decomposes ∂U into two annuli or possibly, in the case where U is a solid Klein bottle, into an annulus and two Möbius bands. However, in the latter case $\partial B \cap \partial U$ must meet the annulus. It follows that $\partial V \cap \partial U$ is an annulus and V is a solid torus with the properties making A trivial. □

In order to prove the torus theorem we will need to know the involutions up to conjugacy on a solid torus.

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Define maps on S^1 by $\kappa(z) = \overline{z}$ and $\alpha(z) = -z$. Let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$. Define maps on D^2 by $\hat{\kappa}(z) = \overline{z}$ and

$\hat{\alpha}(z) = -z$. Also define $\hat{\rho}$ on $D^2 \times S^1$ by $\hat{\rho}(z, w) = (zw, w)$. For later use, define τ on $I = [-1, 1]$ by $\tau(t) = -t$.

Define the following involutions on the solid torus $V = D^2 \times S^1 = \{(z, w) : |z| \leq 1, |w| = 1, z, w \in \mathbb{C}\}$:

$j_A = \hat{\kappa} \times \mathrm{id}$ having as fixed set the annulus $\mathrm{Re} \times S^1$.

$j_M = \hat{\rho} \circ (\hat{\kappa} \times \mathrm{id})$ having as fixed set the Möbius band $\{(s \cdot e^{\pi i t}, e^{2\pi i t}) : 0 \leq s \leq 1, -1 \leq t \leq 1\}$.

$j_{2D} = \mathrm{id} \times \kappa$ having as fixed set two 2-cells $D^2 \times \pm 1$.

$j_{DP} = \hat{\rho} \circ (\mathrm{id} \times \kappa)$ having as fixed set a 2-cell and a point $D^2 \times 1 \cup 0 \times -1$.

$j_S = \hat{\alpha} \times \mathrm{id}$ having as fixed set one 1-sphere $0 \times S^1$.

$j_{2C} = \hat{\kappa} \times \kappa$ having as fixed set two 1-cells $Re \times \pm 1$.

$j_{2P} = \hat{\alpha} \times \kappa$ having as fixed set two points $0 \times \pm 1$.

$j_N = \hat{\kappa} \times \alpha$, fixed point free and orientation reversing.

$j_O = \mathrm{id} \times \alpha$, fixed point free and orientation preserving.

The next theorem follows easily by applying the Disc Theorem 3.13 and using the fact (see [13]) that there are only three involutions up to conjugacy on a 3-cell (for details see [4]).

Theorem 4.3. *If ι and ι' are involutions on $V = D^2 \times S^1$ with nonempty isomorphic fixed point sets or if ι and ι' are fixed point free and of the same orientation type, then ι and ι' are conjugate. An involution on V is conjugate to one of the nine involutions listed above.*

Lemma 4.4. *Let M be an irreducible, P^2-irreducible 3-manifold containing an incompressible torus. Let F be a 1-sided Klein bottle in the interior of M and W a regular neighborhood of F in M with ∂W a torus. Then ∂W is an incompressible torus.*

Proof. Otherwise $M = W \cup U$, where $M - W = U$, $\partial U = U \cap (\overline{U - W})$, and U is a solid torus. Necessarily W is an orientable twisted I-bundle over T and M is orientable. The inclusion of U in M determines an index two subgroup of $\pi_1(M)$. Consider $p: \widetilde{M} \to M$, the 2-sheeted covering corresponding to that subgroup. Then $p^{-1}(W) = T \times [-1, 1]$, where T is a torus with $p(T \times 0) = F$, and $p^{-1}(U) = V_1 \cup V_2$, where V_1 and V_2 are two disjoint solid tori. $\widetilde{M}$ is a lens space. But M and hence $\widetilde{M}$ contains a 2-sided incompressible torus. □

Equivariant Torus Theorem 4.5. *Let M be an irreducible, P^2-irreducible 3-manifold with involution ι. Suppose M contains an incompressible torus. Then one of the following holds:*

(I) *There is a 2-sided incompressible torus or Klein bottle T in* $\mathrm{int}(M)$ *transversal to* Fix *with $T \cap \iota T = \emptyset$ or $\iota T = T$.*

(II) *$M = V_{-1} \cup V_1 \cup U_{-1} \cup U_1$, where V_i and U_i are solid tori and $\iota V_i = V_i$ and $\iota U_{-1} = U_1$. There are annuli A_i, $i = \pm 1$, with*

$$A_1 \cap A_{-1} = A_i \cap \iota A_i = \partial A_i = \partial \iota A_i = V_1 \cap V_{-1} = U_1 \cap U_{-1}$$

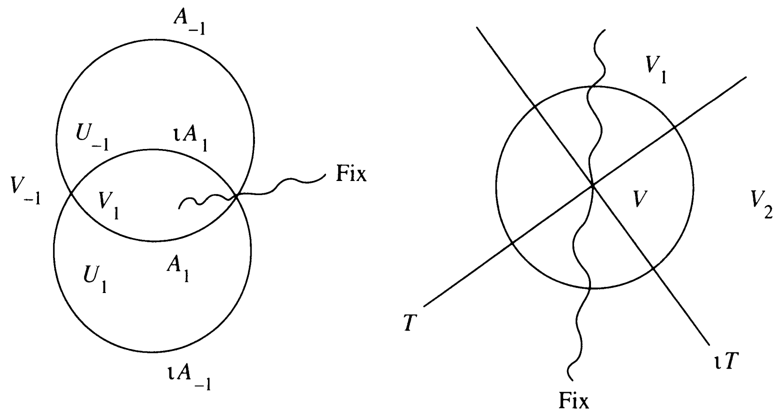

FIGURE 4

and $V_i \cap U_i = A_i$, $V_i \cap U_{-i} = \iota A_i$, $\partial V_i = A_i \cup \iota A_i$, $\partial U_i = A_i \cup \iota A_{-i}$, *(see Figure* 4). $A_1 \cup A_{-1}$ *is a* 2*-sided incompressible torus transversal to* Fix. $\iota | V_i$ *is orientation preserving.*

(III) $M = V_1 \cup V_2 \cup V$, *where* V_1, V_2, *and* V *are solid tori each invariant under* ι *such that* ι *is orientation preserving when restricted to any of* V_1, V_2, *and* V. *There is a* 1*-sided Klein bottle* T *with* $T \cap \iota T = S \subseteq \operatorname{int}(V)$ *a generator of* $\pi_1(V)$. $V_1 \cap V_2 = (T \cap \iota T) - \operatorname{int}(V)$ *are two annuli.* T, ιT, *and* Fix *are pairwise transversal and* $\text{Fix} \cap \partial V_2 = \varnothing$ *and* $\text{Fix} \cap S \neq \varnothing$. V *is a standard neighborhood of* S, *(see Figure* 4).

(IV) $M = W \cup V$, *where* W *is a twisted I-bundle over a torus* $T \subseteq W$ *and* V *is a solid torus with* $\partial W = \partial V = W \cap V$ *and* $\iota W = W$, $\iota T = T$, *and* $\iota V = V$. Fix *is transversal to* ∂W *and* T *except for a possible* 1*-sphere component* S *of* Fix^1 *contained in* T.

Proof. Let T_0 be an incompressible torus in $\operatorname{int}(M)$. By Theorem 3.12 assume T_0, ιT_0, and Fix are almost pairwise transversal and that no 1-spheres in $T_0 \cap \iota T_0$ bound 2-cells in T_0.

As a first step we handle the cases where saddle components arise. Only Type III and Type II components are possible. In both cases since S and S_1 intersect transversally at one point, there can be only one component in $T_0 \cap \iota T_0$.

Suppose $T_0 \cap \iota T_0$ is a Type III component $S \cup S_1 \cup S_2$. Then S_1 and S_2 bound an annulus A in T_0 since $S_1 \cap S_2 = \varnothing$ and both intersect S transversally once. Then $T = A \cup \iota A$ is a torus with $\iota T = T$ and T and Fix are transversal. T is 1-sided since a regular neighborhood of S_1 is a solid Klein bottle. Let N be a regular neighborhood of T invariant under N. If ∂N is incompressible then it is a 2-sided torus satisfying (I). If ∂N is compressible we arrive at (IV).

Suppose $T_0 \cap \iota T_0$ is a Type II component $S \cup S_1$. First we construct a torus T' isotopic to $T = T_0$ with $\iota T' = T'$. Let $N(S)$ and $N(S_1)$ be regular neighborhoods of S and S_1 respectively, both invariant under ι such that $N = N(S) \cup N(S_1)$ is a regular neighborhood of $S \cup S_1$ and such that $T \cap N(S)$

and $T \cap N(S_1)$ are annuli, $N(S_1) \cap \mathrm{Fix}$ is a Möbius band, $N(S) \cap \mathrm{Fix}^2$ is a proper 2-cell, and $N(S) \cap \mathrm{Fix}^1$ is a proper 1-cell. Both $N(S)$ and $N(S_1)$ are solid Klein bottles. By transversality there are two disjoint open 2-cell components K_1 and K_2 of $N(S) - (T \cup \imath T)$ that meet Fix^1 and there are two disjoint open 2-cell components L_1 and L_2 of $N(S_1) - (T \cup \imath T)$ that do not meet Fix^2. By considering the effect of $\imath$ near saddle points we see $A = (K_1 \cup K_2 \cup L_1 \cup L_2) \cap \partial N$ is an annulus with $\partial A = C \cup \imath C$, where $C = \partial N \cap T$. The closure of $A \cup (T - N) \cup \imath(T - N)$ is a 2-sphere which by the irreducibility of M bounds a 3-cell E. E cannot contain the proper punctured torus $T \cap N$ so $E \cap \mathrm{int}(N) = \varnothing$. Since Fix^1 is transversal to ∂E and $\imath \partial E = \partial E$ it follows $\imath E = E$. In particular, $\imath | E$ is conjugate to j_1, the standard involution of a 3-cell with fixed set one 1-cell. A is invariant and contains $\mathrm{Fix}^1 \cap \partial E$. Hence one shows there is a proper 2-cell D with ∂D a generator of $H_1(A)$ such that $\mathrm{Fix}^1 \cap E$ is a proper 1-cell of D and $\imath D = D$. Since $\imath \partial D = \partial D$, by taking N sufficiently small we can construct a proper punctured torus P in N with $\partial P = \partial D$ and $\imath P = P$ (namely isotope $T \cap N$). Consider the torus $T' = P \cup D$. Fix^2 intersects T' transversally at S_1. The component of Fix^1 meeting N is contained in T'. T' is 1-sided. Let W be a regular neighborhood of T' invariant under $\imath$. If ∂W is incompressible then it is a 2-sided torus satisfying (I). If ∂W is compressible we arrive at (IV).

We may now assume $T_0 \cap \imath T_0$ has no saddle components. T_0, $\imath T_0$, and Fix are pairwise transversal and $T_0 \cap \imath T_0$ consists of disjoint 1-spheres bounding annuli in T_0 and $\imath T_0$. We successively construct incompressible tori or Klein bottles T with fewer 1-spheres in $T \cap \imath T$, but always keep $T \cap \imath T$ consisting of 1-spheres bounding annuli in T and $\imath T$ and always retain the property that each component of $T \cap \imath T$ has a standard neighborhood, (see Propositions 3.9 and 3.10). It also follows then that any 1-sided Klein bottle arising from such a construction has a regular neighborhood W with ∂W a torus. So Lemma 4.4 is applicable.

Note: Suppose T satisfies all the conditions of (I) except that T is 1-sided instead of 2-sided. Let W be a regular neighborhood of T. We can take W so that ∂W and Fix are transversal and $\imath W = W$ or $W \cap \imath W = \varnothing$. ∂W is 2-sided. If ∂W is incompressible, ∂W satisfies (I). Assume now that ∂W is compressible. T cannot be a Klein bottle since the previous paragraph shows that ∂W is a torus and thus Lemma 4.4 would show ∂W to be incompressible. So T is a torus. Now $V = \overline{M - W}$ is a solid torus. If $\imath T = T$ we have (IV). If $\imath T \cap T = \varnothing$ then the solid torus V contains an embedded 1-sided torus $\imath T$, a contradiction.

We give constructions that reduce the number of components of $T \cap \imath T$, producing a new torus or Klein bottle. Repeat these constructions with the new torus or Klein bottle produced until eventually one satisfying condition (I), (II), (III), or (IV) is constructed. There are four cases depending on the number of 1-spheres of $T \cap \imath T$ and the compressibility of certain surfaces.

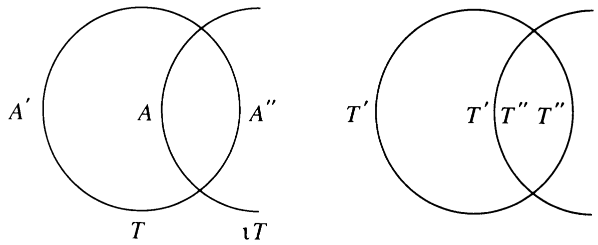

FIGURE 5

Assume $T \cap \imath T$ consists of at least two 1-spheres. Let $A \subsetneq \imath T$ be an innermost annulus: $A \cap T = \partial A$. ∂A decomposes T into two annuli A' and A'' with $T = A' \cup A''$ and $\partial A = \partial A' = \partial A'' = A' \cap A''$. $T' = A' \cup A$ and $T'' = A'' \cup A$ are tori or Klein bottles (see Figure 5).

Case (1). *T' is incompressible.*

Case (1.1): *$\imath\partial A = \partial A$ and $\imath A = A'$.* Then $\imath T' = T'$. One sees Fix is transversal to T' by considering the standard neighborhoods of ∂A. We arrive at (I) or (IV).

Case (1.2): *Either $\imath\partial A = \partial A$ and $\imath A = A''$ or $\imath\partial A \cap \partial A$ is a single* 1*-sphere S and $\imath A \subsetneq A''$.* In the latter case $\imath S = S$. Let V_1 and V_2 be distinct standard neighborhoods of ∂A and let γ_1 and γ_2 be the two distinct standard annuli that meet both A and A'. Let $T_1 = (A' \cup A \cup \gamma_1 \cup \gamma_2) - \operatorname{int}(V_1 \cup V_2)$. Then $T_1 \cap \imath T_1 \subsetneq (T \cap \imath T) - \partial A$ and $T_1 \cap \imath T_1 \neq T \cap \imath T$ because $(\gamma_1 \cup \gamma_2) \cap \imath(\gamma_1 \cup \gamma_2) = \varnothing$. T' and T_1 are ambient isotopic so T_1 is incompressible. $\mathrm{Fix} \cap (\gamma_1 \cup \gamma_2) = \varnothing$ and A is innermost so T_1, $\imath T_1$, and Fix are pairwise transversal. Proceed with T_1.

Case (1.3): *Either $\imath\partial A \cap \partial A$ is a single* 1*-sphere S and $\imath A \subsetneq A'$ or $\imath\partial A \cap \partial A = \varnothing$.* Let $\partial A = S \cup S'$. Let V be a standard neighborhood of S and let γ be the standard annulus that meets both A and A'. Let $A \times [0, \varepsilon]$ be a sufficiently thin collar of $A = A \times 0$ in M such that

$$S' \times [0, \varepsilon] \subsetneq A', \quad S \times [0, \varepsilon] \subsetneq T, \quad (A \cap \partial V) \times [0, \varepsilon] = (A \times [0, \varepsilon]) \cap \partial V.$$

The collar exists since V is a solid torus. In the first case, $\imath S = S$ and $\imath\gamma = \gamma$. By Lemma 4.1 if $(A \times \varepsilon) \cap \gamma \neq \varnothing$ we may assume $(A \times \varepsilon) \cap V$ and $\imath((A \times \varepsilon) \cap V)$ intersect transversally in a 1-sphere S_1 and that both are transversal to Fix. In all other cases set $S_1 = S$. Define

$$T_1 = (A \times \varepsilon) \cup \overline{A' - ((S' \cup S) \times [0, \varepsilon])} \cup ((S \times [0, \varepsilon]) - A').$$

Then $T_1 \cap \imath T_1 \subsetneq ((T \cap \imath T) - \partial A) \cup S_1$. T_1 is incompressible since it is ambient isotopic to T'. T_1, $\imath T_1$, and Fix are pairwise transversal. Proceed with T_1.

By Case (1) we may now assume T' and T'' are compressible.

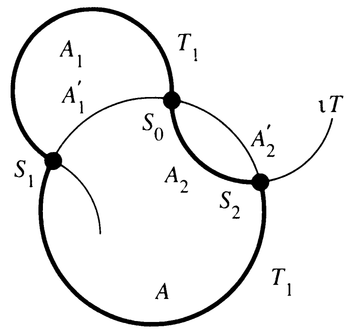

FIGURE 6

Case (2). *For every annulus $A \subseteq \iota T$ with $A \cap T = \partial A$, both corresponding surfaces T' and T'' are compressible and $T \cap \iota T$ contains more than two* 1-*spheres*. Then let A_1 and A_2 in ιT be annuli with $A_i \cap T = \partial A_i$ and with $\partial A_i = S_0 \cup S_i$, where S_0, S_1, and S_2 are 1-spheres with $S_1 \neq S_2$. Let A, A_1', and A_2' be the three annuli of T that these 1-spheres decompose T into: $\partial A = S_1 \cup S_2$ and $\partial A_i' = S_0 \cup S_i$ for $i = 1, 2$ (see Figure 6).

Define $T_1 = A \cup A_1 \cup A_2$. T_1 is incompressible. Otherwise T_1 bounds a solid torus or a Klein bottle U. Say $A_1' \subseteq U$. A_1' is trivial in U by Lemma 4.2. If $A_1' \cup A_1$ bounds the trivializing torus then the incompressible $T = A_1' \cup A_2' \cup A$ is ambient isotopic to $A_1 \cup A_2' \cup A$ which was compressible by hypothesis. If $A_1' \cup A_2 \cup A$ bounds the trivializing torus then, since A_1' and A_2 meet on S_0, A_2 must also be trivial in $A_1' \cup A_2 \cup A$. So T is ambient isotopic to $A_2' \cup A_2$ which was assumed compressible.

We have five cases:

Case (2.1): $\iota(S_1 \cup S_2) = S_1 \cup S_2$ *and* $\iota S_0 \subseteq A$. Then $\iota(A_1 \cup A_2) = A$.

Case (2.2): $\iota(S_1 \cup S_2) = S_1 \cup S_2$ *and* $\iota S_0 \subseteq A'$. Then $\iota(A_1 \cup A_2) = A_1' \cup A_2'$ and $\iota S_0 = S_0$.

Case (2.3): $\iota S_1 = S_1$ *and* $\iota S_0 = S_0$. Then $\iota A_1 = A_1'$ and $\iota S_2 \subseteq A_2'$.

Case (2.4): $\iota(S_1 \cup S_2) \cap (S_1 \cup S_2) = \varnothing$.

Case (2.5): $\iota(S_1 \cup S_2) \cap (S_1 \cup S_2)$ *is one* 1-*sphere*.

These cases cover all possibilities. In each case we find a T_i with fewer 1-spheres.

Case (2.5) follows from the other cases. After relabeling assume S_1 is the 1-sphere in the intersection. Then $\iota S_1 = S_1$. By Case (2.3) we assume $\iota S_0 \neq S_0$. Let A_3 be the innermost annulus adjacent to A_1: $A_3 \subseteq \iota T_1$ with $A_3 \cap T_1 = \partial A_3 = S_1 \cup S_3$, where $S_3 \neq S_0$. By Case (2.3) again we may assume $\iota S_3 \neq S_3$. By Cases (2.1) and (2.2) we may assume $\iota S_0 \neq S_3$. So we have $\iota(S_0 \cup S_3) \cap (S_0 \cup S_3) = \varnothing$ and Case (2.4) gives the reduction.

In Case (2.1) use $T_1 = \iota T_1$. Fix is transversal to T_1 since the standard annulus meeting A_1 and A is invariant.

In Case (2.2) and Case (2.3): For $i = 1, 2$ let V_i be the standard neighborhoods of S_i with γ_i the standard annuli that meet both A and A_i. In all cases $(\gamma_1 \cup \gamma_2) \cap \iota(\gamma_1 \cup \gamma_2) = \varnothing$. Define T_2 to be the incompressible surface ambient isotopic to T_1 given by

$$T_2 = T_1 \cup \gamma_1 \cup \gamma_2 - \operatorname{int}(V_1 \cup V_2).$$

Then T_2, ιT_2, and Fix are pairwise transversal and

$$T_2 \cap \iota T_2 \subseteq T_1 \cap \iota T_1 - (S_1 \cup S_2).$$

In Case (2.4): First assume $\iota S_0 \neq S_0$. By symmetry assume $\iota S_0 \neq S_1$. Let $(A_1 \cup A_2) \times [0, \varepsilon]$ be a sufficiently thin collar of $A_1 \cup A_2 = (A_1 \cup A_2) \times 0$ in M such that $(S_1 \cup S_2) \times [0, \varepsilon] \subseteq T$ and $S_0 \times [0, \varepsilon] \subseteq A'$. Define

$$T_2 = (A_1 \cup A_2) \times \varepsilon \cup \overline{A - ((S_1 \cup S_2) \times [0, \varepsilon])} \cup ((S_1 \cup S_2) \times [0, \varepsilon] - A).$$

Then $T_2 \cap \iota T_2 \subseteq \iota(T \cap \iota T) - S_0$. Also T_2 is ambient isotopic to incompressible T_1. T_2, ιT_2, and Fix are pairwise transversal.

If $\iota S_0 = S_0$, proceed as above but replace the condition $S_0 \times [0, \varepsilon] \subseteq A'$ by $S_1 \times [0, \varepsilon] \subseteq A$. Use Lemma 4.1 on a standard neighborhood of S_0 to adjust the collar so that $(A_1 \cup A_2) \times \varepsilon$ and $\iota(A_1 \cup A_2) \times \varepsilon$ intersect transversally in one 1-sphere S_3. Then $T_2 \cap \iota T_2 \subseteq (\iota(T \cap \iota T) - (S_0 \cup S_1)) \cup S_3$.

Case (3). *For each annulus $A \subseteq \iota T$ with $A \cap T = \partial A$, both corresponding surfaces T' and T'' are not incompressible and $T \cap \iota T$ is exactly two* 1*-spheres.* Set $\iota T = A_{-1} \cup A_1$ with $A_{-1} \cap A_1 = \partial A_1 = \partial A_{-1} = T \cap \iota T = S_1 \cup S_2$. Then $T = \iota A_{-1} \cup \iota A_1$. There are solid tori or Klein bottles U_i and V_i, for $i = \pm 1$, pairwise disjoint on their interiors with $\partial V_i = A_i \cup \iota A_i$ and $\partial U_i = A_i \cup \iota A_{-i}$. None of U_i or V_i are solid Klein bottles. Otherwise, if, say, V_1 is a solid Klein bottle, then since $S_1 \cup S_2$ decomposes ∂V_1 into two annuli it follows that S_1 bounds a disc in V_1. This contradicts the incompressibility of T. By considering the standard annuli of a standard neighborhood of S_1 we see $\iota V_i = V_i$ and $\iota U_i = U_{-i}$. That $A_1 \cup A_{-1}$ is a torus and not a Klein bottle follows from the fact that V_1 and U_{-1} are solid tori.

If both $\iota|V_1$ and $\iota|V_{-1}$ are orientation preserving then we arrive at (II). So assume that $\iota|V_1$, say, is orientation reversing. We will show $\iota|V_1$ is conjugate to j_M and arrive at (I). By Theorem 4.3, $\iota|V_1$ is conjugate to j_A, j_{2D}, j_N, j_M, or j_{DP}, the standard involutions on a solid torus. j_{2D} and j_{DP} are not possible since S_1 or S_2 would bound a disc contradicting the incompressibility of T.

If $\iota|V_1$ is conjugate to j_M then, say, $S_1 \subseteq \text{Fix}$ and $S_2 \cap \text{Fix} = \varnothing$. Then $\iota|V_{-1}$ has a 2-dimensional fixed set component that has only one boundary component. It follows that $\iota|V_{-1}$ is also conjugate to j_M. So Fix contains a Klein bottle K. There is a regular neighborhood W of K with $\iota\partial W = \partial W$ and $W \cap \text{Fix} = \varnothing$. Since V_i are solid tori and $K \cap V_i$ is a Möbius band, ∂W is a torus. By Lemma 4.4, ∂W is incompressible. We arrive at (I).

If $\iota|V_1$ is conjugate to j_A, then $[S_1]$ represents a generator of $H_1(V_1)$ and hence there is an ambient isotopy taking ιT to ∂U_{-1} (move A_1 to ιA_1). This contradicts that ιT is incompressible.

Finally suppose $\iota|V_1$ is conjugate to the involution $j_N = \hat{\kappa} \times \alpha$ on $D^2 \times S^1$. If $\iota S_1 = S_1$ then $S_1' = 1 \times S^1$ and $S_2' = -1 \times S^1$ determine annuli A' and $\iota A'$ of $\partial D^2 \times S^1$. It is possible to construct a conjugation $\partial V_1 \to \partial D^2 \times S^1$ taking A_1 to A'. This conjugation extends to a conjugation $V_1 \to D^2 \times S^1$. But $[S_1']$ is a generator of $H_1(D^2 \times S^1)$ and we get a contradiction as for the j_A case above. If $\iota S_1 = S_2$ then use $S_1' = \partial D^2 \times 1$ and $S_2' = \partial D^2 \times -1$ and proceed as above but this time obtaining a contradiction as for j_{2D} above.

Case (4). *$T \cap \iota T$ is a single 1-sphere S*. Then $\iota S = S$. Let V be a standard neighborhood of S and let α_1, α_2, β_1 and β_2 be the standard annuli with $\alpha_1 \cap \alpha_2 = \varnothing$, $\beta_1 \cap \beta_2 = \varnothing$, $\iota\alpha_1 = \alpha_1$, $\iota\alpha_2 = \alpha_2$, and $\iota\beta_1 = \beta_2$. Define

$$T_1 = (T \cup \iota T \cup \alpha_1 \cup \alpha_2) - \operatorname{int}(V)$$

and

$$T_2 = (T \cup \iota T \cup \beta_1 \cup \beta_2) - \operatorname{int}(V).$$

If T is 2-sided then T_1 is 2-sided. Also $\iota T_1 = T_1$. Since T is 2-sided it follows that a sufficiently thin collar $T \times [0, \varepsilon]$ of $T = T \times 0$ can intersect only one of $\operatorname{int}(\alpha_1)$ and $\operatorname{int}(\alpha_2)$. Hence T_1 cannot separate and therefore T_1 is incompressible. We arrive at (I).

From now on assume T is 1-sided. T_1 and T_2 are tori. This follows since V is a solid torus and either both of the annuli $T - \operatorname{int}(V)$ and $\iota T - \operatorname{int}(V)$ are "twisted" relative to V (if T is a Klein bottle) or neither is (if T is a torus).

If either of T_1 or T_2 is incompressible we arrive at (I). Assume then that T_1 and T_2 are compressible. Then T_i bounds a solid torus V_i. If $S \subseteq V_i$ then V_i contains a 1-sided torus or Klein bottle, a contradiction. So $M = V \cup V_1 \cup V_2$ with $\operatorname{int}(V)$, $\operatorname{int}(V_1)$, and $\operatorname{int}(V_2)$ pairwise disjoint.

By the choice of α_1 and α_2, ι interchanges the components of $\partial\alpha_i$. Therefore $\iota|\alpha_i$ is conjugate to one of $\mathrm{id} \times \tau$, $\kappa \times \tau$, or $\alpha \times \tau$, the standard involutions of $S^1 \times I$, where $I = [-1, 1]$ and $\tau(t) = -t$. Let S_i be a 1-sphere of α_i that is the image of $S^1 \times 0$ under some conjugation. Note that S_i does not bound a disc D in V_1, otherwise T would be compressible.

If there is an annulus $A \subseteq V_1$ with $\partial A = S_1 \cup S_2$ and $\iota A = A$ then we arrive at (IV) as follows. Torus V_1 is separated by A. Since ι interchanges the components of $\partial\alpha_1$, ι interchanges the components of $A - V_1$. A is trivial in V_1 so it follows that V_1 can be given a trivial I-bundle structure over A. There is an annulus $B \subseteq V$ with $\partial B = S_1 \cup S_2$ and $\iota B = B$. V is an I-bundle over B. Consider $T_3 = A \cup B$. It follows that $V \cup V_1$ is an I-bundle over T_3 with $\partial(V \cup V_1) = T_2$ a torus. Moreover T_3 does not separate T so T_3 is 1-sided. Since T_2 is compressible, T_3 is not a Klein bottle in view of Lemma 4.4. Thus T_3 is a torus and we arrive at (IV).

Assume now that such an annulus A does not exist. Since V is a standard neighborhood, $\mathrm{Fix} \cap \alpha_1 = \varnothing$ if and only if $\mathrm{Fix} \cap \alpha_2 = \varnothing$. Therefore $\iota|\alpha_1$ and $\iota|\alpha_2$ are conjugate.

Case (4.1): *$\iota|\alpha_1$ is conjugate to* $\mathrm{id}\times\tau$. Then $\iota|V_1$ has a 2-dimensional fixed set that meets ∂V_1 in two fixed 1-spheres. It follows that S_1 bounds a disc or $S_1 \cup S_2$ bounds an annulus A fixed by $\iota|V_1$. By the above comments, we arrive at (IV).

Case (4.2): *$\iota|\alpha_1$ is conjugate to $\alpha\times\tau$*. Then $\iota|V_1$ is orientation reversing. $\iota|\partial V_1$ is conjugate to $\alpha\times\kappa$ on $S^1\times S^1$ by a conjugation taking S_i to $S^1\times(-1)^i$. Then $\iota|V_1$ is conjugate to $\hat{\alpha}\times\kappa$ or $\alpha\times\hat{\kappa}$ by a conjugation extending the one given on the boundaries. In the first case S_1 bounds a disc and in the second case $S_1\cup S_2$ bounds an annulus with $\iota A = A$. Again by the above comments we arrive at (IV).

Case (4.3): *$\iota|\alpha_1$ is conjugate to $\kappa\times\tau$*. Then $\iota|V_1$ and $\iota|V$ are orientation preserving. Now $\iota|V_2$ is orientation reversing if and only if T is a torus. To see this let $S_1 = \alpha_1\cap\beta_1$ and without loss say $S_1\subseteq T$. Orient S_1. S_1 and ιS_1 bound two annuli A_1 and A_2 of ∂V_2 with $\iota A_1 = A_2$. Consider the ways of inducing an orientation on ιS_1. The orientation induced by A_1 and the orientation induced by α_1 are the same if and only if T is a torus. Since $\iota|\alpha_1$ is orientation reversing the orientation induced by α_1 and the orientation induced by ι are opposite. So ι and A_1 induce opposite orientations on ιS_1 if and only if T is a torus. Since $\iota A_1 = A_2$ the claim follows.

If T is a torus then $\iota|V_2$ is orientation reversing, so $\iota|\partial V_2$ is conjugate to the involution $\alpha\times\kappa$ on $S^1\times S^1$ by a conjugation taking S_1 to $\mathrm{i}\times S^1$. As in Case (4.2) we arrive at (I) or (IV).

If T is a Klein bottle then we arrive at (III). $\iota|\partial V_2$ is fixed point free so $\iota|V_2$ is conjugate to j_S or j_O while $\iota|V_1$ is conjugate to j_{2C}. □

Corollary 4.6. *Let M be an irreducible, P^2-irreducible 3-manifold with involution ι. Suppose M contains an incompressible torus. Suppose M is neither an orientable Seifert fiber space over the 2-sphere with four exceptional fibers nor a nonorientable Seifert fiber space over the projective plane with at most one exceptional fiber. Then there is a 2-sided incompressible torus or Klein bottle T in $\mathrm{int}(M)$ transversal to* Fix *with $T\cap\iota T=\varnothing$ or $\iota T = T$.*

Proof. Case (II) in the Equivariant Torus Theorem can only occur if M is an orientable Seifert fiber space over the 2-sphere with four exceptional fibers, while Cases (III) and (IV) can only occur if M is a nonorientable Seifert fiber space over the projective plane with at most one exceptional fiber. Since these possibilities have been excluded, only Case (I) of the Equivariant Torus Theorem remains. □

References

1. A. L. Edmonds, *A topological proof of the equivariant Dehn lemma*, Trans. Amer. Math. Soc. **297** (1986), 605–615.

2. C. McA. Gordon and R. A. Litherland, *Incompressible surfaces in branched coverings*, *The Smith Conjecture*, Pure and Appl. Math, vol. 112, Academic Press, 1984, pp. 139–152.

3. J. Hempel, 3-*manifolds*, Ann. of Math. Stud., no. 86, Princeton Univ. Press, Princeton, N.J., 1976.

4. W. H. Holzmann, *Involutions with* 1- *or* 2-*dimensional fixed point sets on orientable torus bundles over a* 1-*sphere and on unions of orientable twisted I-bundles over a Klein bottle*, Ph.D. thesis, University of British Columbia, Vancouver, B. C., 1984.

5. P. K. Kim and J. Tollefson, *PL involutions of fibered* 3-*manifolds*, Trans. Amer. Math. Soc. **232** (1977), 221–237.

6. ——, *Splitting the PL involutions of nonprime* 3-*manifolds*, Michigan Math. J. **27** (1980), 259–274.

7. K. Kwun and J. Tollefson, *PL involutions of* $S^1 \times S^1 \times S^1$, Trans. Amer. Math. Soc. **203** (1975), 97–106.

8. E. Luft, *Equivariant surgery on incompressible tori and Klein bottles in* 3-*manifolds with respect to involutions*, Math. Ann. **272** (1985), 519–544.

9. E. Luft and D. Sjerve, *Involutions with isolated fixed points on orientable flat* 3-*dimensional space forms*, Trans. Amer. Math. Soc. **285** (1984), 305–336.

10. W. H. Meeks and S. T. Yau, *The equivariant Dehn's lemma and loop theorem*, Comment. Math. Helv. **56** (1981), 225–239.

11. P. Scott, *A new proof of the annulus and torus theorems*, Amer. J. Math **102** (1980), 241–277.

12. J. Tollefson, *Involutions of sufficiently large* 3-*manifolds*, Topology **20** (1981), 323–352.

13. F. Waldhausen, *Über Involutionen der* 3-*Sphäre*, Topology **8** (1969), 81–91.

Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta, T1K 3M4 Canada