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# LINEAR SERIES WITH AN $N$-FOLD POINT ON A GENERAL CURVE

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ABSTRACT. A linear series $(V, \mathscr{L})$ on a curve $X$ has an $N$-fold point along a divisor $D$ of degree $N$ if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geq \dim V - 1$. The dimensions of the families of linear series with an $N$-fold point are determined for general curves.

## 1

We work over the field of complex numbers $\mathbf{C}$.

Let $X$ be a smooth projective curve. A $g_d^r$ on $X$ is a linear series of dimension $r$ and degree $d$ on $X$, i.e., a pair $(V, \mathscr{L})$ consisting of a line bundle $\mathscr{L}$ of degree $d$ and an $r+1$ dimensional subspace $V \subset H^0(X, \mathscr{L})$. The $g_d^r$'s on $X$ are parameterized by a projective scheme $G_d^r(X)$. If $X$ is general in moduli, then $\dim G_d^r(X) = \rho(g, r, d) = g - (r+1)(g+r-d)$ [ACGH].

**Definition.** We say that a $g_d^r(V, \mathscr{L})$ has an $N$-fold point along a divisor $D$ of degree $N \geq 2$ in $X$ if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geq r$.

If $\pi : X \to V$ is a flat proper irreducible family of smooth curves, there exists a scheme $G_d^r(X/V)$ which is projective over $V$ whose fiber over each $v \in V$ is $G_d^r(X_v)$ where $X_v = \pi^{-1}(v)$ [EH-2]. It is easy to see that, by considering the construction of $G_d^r(X \times_V X^N/X^N)$ as the degeneracy locus of a vector bundle map, there is a closed subscheme $N(X/V) \subset G_d^r(X \times_V X^N/X^N)$ such that the fiber over each point $(P_1, \dots, P_N) \in X_V^N$ consists of the $g_d^r$'s on $X_V$ with an $N$-fold point along $\sum P_j$. We will write $N(X)$ if $V$ is a point.

Let $\mathscr{M}_g$ denote the moduli space of smooth curves of genus $g \geq 3$, and let $U$ be the open subset of $\mathscr{M}_g$ corresponding to curves without nontrivial automorphisms. Let $Z \to U$ be the universal curve over $U$. We will prove the following theorem.

**Theorem.** *With $g \geq N \geq 2$, $g \geq 3$, $r \geq 2$ and the notation as above, $\dim N(Z_u) \leq \rho(g, r, d) - r(N-1) + N$ when $u$ is a sufficiently general point of $U$.*

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This extends a result of Marc Coppens who proved the case when $N = 2$ [C]. It is easy to see that $\rho(g, r, d) - r(N-1) + N$ is a lower bound for the dimension of $N(Z_u)$ when $N(Z_u) \neq \emptyset$. If $\rho(g, r, d) - r(N-1) + N \geq 0$ and $\rho(g, r-1, d-N) \geq 0$ then $N(Z_u) \neq \emptyset$ [S].

Let $X$ be a connected complete curve. We say that $X$ is of compact type if and only singularities of $X$ are ordinary double points, and the dual graph is a tree. We say a connected closed subcurve $Y \subset X$ is a tail if it meets $\overline{X-Y}$ at at most one point. We say that a curve $X$ of genus $g$ is of special type if: it is of compact type; each irreducible component is a nonsingular rational or nonsingular elliptic curve; and each irreducible elliptic component is a tail.

We say that a sequence $a = (a_0, a_1, \dots, a_r)$ is of type $(r, d)$ if $0 \leq a_0 < a_1 < \cdots < a_r \leq d$. If $X$ is a smooth curve containing a point $P$, and $L = (V, \mathscr{L})$ is a $g^r_d$ on $X$, then the orders of vanishing of the sections of $V$ determine a sequence $\mathbf{a}$ of type $(r, d)$. We call $\mathbf{a}$ the vanishing sequence of $L$ at $P$. We denote by $W(\mathbf{a}) = \sum(a_i - i)$ the weight of the sequence $\mathbf{a}$. If $\mathbf{b} = (b_0, \dots, b_r)$ is a sequence of type $(r, d)$, and a $g^r_d$ $L$ has vanishing sequence $\mathbf{a} = (a_0, \dots, a_r)$ at $P$, we say that $L$ satisfies the vanishing condition $\mathbf{b}$ at $P$ if $a_i \geq b_i$ for $i = 0, \dots, r$.

In §2 we show the existence of a family of smooth curves $X_{T-\{0\}} \to T - \{0\}$ which specialize to a curve $X_0$ of special type and a family $A'$ of $g^r_d$'s with $N$-fold points on $X_{T-\{0\}} \to T - \{0\}$. The $N$-fold points of $A'$ specialize to a genus $M$ tail of $X_0$. The $g^r_d$'s of $A'$ also satisfy a vanishing condition $\mathbf{a}$ at a point. The relative dimension of $A'$ over $T - \{0\}$ is 0, and the codimension of $A'$ in $N(X_{T-\{0\}}/T - \{0\})$ is $\leq N - M + W(\mathbf{a})$.

In §3 we use the theory of limit linear series as developed by Eisenbud and Harris [EH-2] to show that the crude limit linear series on $X_0$ induced by $A'$ forces $\rho(g, r, d) - r(N-1) + M - W(\mathbf{a}) \geq 0$.

The products involving $\mathbf{P}^1$ in the proof of Lemma 1 are taken over Spec $\mathbf{C}$. All other products are fibered over $\overline{\mathscr{M}}_g$ unless specified otherwise.

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## 2

We will make use of Knudsen's results concerning stable $n$-pointed curves [K1, K2]. A stable $n$-pointed curve is a connected projective curve $X$ and $n$ distinct nonsingular points $P_1, \dots, P_n$ of $X$ such that: the only singularities of $X$ are ordinary double points; and on every smooth rational component $Y \subset X$, $\#\{P_i \mid P_i \in Y\} + \#\{Y \cap \overline{X-Y}\} \geq 3$. For each $g$ and $n$, there exists a coarse moduli space for $n$-pointed stable curves of genus $g$ which we denote by $\overline{\mathscr{M}}_{g,n}$. For each $g$ and $n$, $\overline{\mathscr{M}}_{g,n}$ is a projective variety.

The functors of relative stable $n$-pointed curves and relative stable $n-1$-pointed curves with an additional section are isomorphic. For each stable $n$-

pointed curve $(X, P_1, \dots, P_n)$ there is a curve $X_c$ and morphism $c: X \to X_c$ such that: $(X_c, c(P_1), \dots, c(P_{n-1}))$ is a stable $n-1$-pointed curve; and either $c$ is an isomorphism, or $P_n$ lies on a rational component $Y \subset X$ whose image in $X_c$ is a point and $c|_{X-Y}$ is an isomorphism of $X - Y$ with $X_c - c(Y)$. When we wish to consider $\overline{\mathscr{M}}_{g,n}$ as coarsely representing the functor of stable $n-1$-pointed curves with an additional section, we will write $\overline{\mathscr{M}}_{g,n} \simeq \overline{Z}_{g,n-1}$. When $n = 0$ we will write $\overline{\mathscr{M}}_g$ instead of $\overline{\mathscr{M}}_{g,0}$ and $\overline{Z}_g$ instead of $\overline{Z}_{g,0}$.

For each $m < n$ and each subset $\{i_1 < \cdots < i_m\}$ of $\{1, \dots, n\}$, we have a *contraction* morphism $\pi: \overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,m}$ obtained by forgetting the points not indexed by $\{i_1 < \cdots < i_m\}$ and collapsing certain rational subcurves, if necessary.

There is a natural *clutching* morphism

$$\gamma: \overline{\mathscr{M}}_{g_1, n+1} \times_{\mathbf{C}} \overline{\mathscr{M}}_{g_2, m+1} \to \overline{\mathscr{M}}_{g_1+g_2, n+m}.$$

If $(X, P_1, \dots, P_{n+1})$ and $(Y, Q_1, \dots, Q_{m+1})$ correspond to a point $(x, y) \in \overline{\mathscr{M}}_{g_1, n+1} \times_{\mathbf{C}} \overline{\mathscr{M}}_{g_2, m+1}$, then $\gamma(x, y)$ corresponds to the $n+m$-pointed curve obtained by joining $X$ and $Y$ at the points $P_{n+1}$ and $Q_{m+1}$.

Fix $g \geq 3$, $2 \leq N \leq g$, $r \geq 2$, and $d$. As before, we let $U \subset \overline{\mathscr{M}}_g$ be the open subset corresponding to smooth curves without nontrivial automorphisms, and we let $\pi: Z \to U$ be the universal curve over $U$. Let $H \subset N(Z/U)$ be a component of the scheme of $g^r_d$'s with $N$-fold points whose dimension is maximal with respect to the property that $\pi(H) = U$.

We have a natural morphism $\alpha: H \to (\overline{Z}_g)^N$. Let $B$ be the closure of $\alpha(H)$ in $(\overline{Z}_g)^N$, and let $\beta: B \to \overline{\mathscr{M}}_g$ be the natural morphism. Note that $\beta(B) = \overline{\mathscr{M}}_g$, because $\pi(H) = U$. Let $M = N - \min\{\dim \beta^{-1}(x) \mid x \in \overline{\mathscr{M}}_g\}$.

Note that $0 \leq M \leq N$. If $M = 0$, then the theorem holds, because the $g^r_d$'s on a general curve with an $N$-fold point along a divisor whose support is a general point have codimension $r(N-1)$ by Theorem 4.5 of [EH-2]. We will henceforth assume $M \geq 1$.

**Lemma 1.** *There exists a point $b \in B$ corresponding to a curve $X_b$ which is of special type and points $P_1, \dots, P_N \in X_b$ which lie on a tail of genus $\leq M$ or $P_1 = P_2 = \cdots = P_N$ is a nonsingular point on a rational component of $X_b$.*

*Proof.* All products involving $\mathbf{P}^1$ in this proof are fibered over $\operatorname{Spec} \mathbf{C}$. All other products are fibered over $\mathscr{M}_g$.

There is a set-theoretic map on closed points $\delta: \overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N \to (\overline{Z}_{0,g})^N$ which we will now describe. A point $w \in \overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N$ corresponds to a $g$-pointed rational curve $(X, Q_1, \dots, Q_g)$ and $N$ points $P_1, \dots, P_N$ on the stable genus $g$ curve $\widetilde{X}_{Q_1, \dots, Q_g}$ obtained by attaching a fixed elliptic curve $E$ to $X$ at each point $Q_1, \dots, Q_g$. Let $\eta: \widetilde{X}_{Q_1, \dots, Q_g} \to X$ be the natural map which collapses the elliptic tails. The point $\delta(w) \in (\overline{Z}_{0,g})^N$ corresponds to $(X, Q_1, \dots, Q_g)$ and the $N$ points $\eta(P_1), \dots, \eta(P_N)$.

For each $I = (i_1, \dots, i_N)$ such that $0 \le i_j \le g$ for $j = 1, \dots, N$ there is a scheme $D_I$ which parameterizes $g$-pointed rational curves $(X, Q_1, \dots, Q_g)$ and points $P_1, \dots, P_N$ on $\widetilde{X}_{Q_1, \dots, Q_g}$ such that each $P_j$ lies on the elliptic tail attached to $X$ at the point $Q_{i_j}$ if $i_j > 0$, or $P_{i_j}$ lies on $X$ if $i_j = 0$. Each $D_I$ is easily shown to exist by considering contraction and clutching morphisms.

Furthermore, there are morphisms $\phi_I \colon D_I \to (\overline{Z}_{0,g})^N$ and $\Psi_I \colon D_I \to \overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N$ such that $\delta \circ \Psi_I = \phi_I$. It follows that $\delta(p_Z^{-1}(B))$ is a closed subset of $(\overline{Z}_{0,g})^N$ where $p_Z$ is the projection of $\overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N$ to $(\overline{Z}_g)^N$.

There are morphisms

$$\zeta_N \colon (\overline{Z}_{0,g})^N \to (\overline{Z}_{0,3})^N \simeq (\mathbf{P}^1)^N \quad \text{and} \quad \zeta_g \colon (\overline{Z}_{0,g})^g \to (\overline{Z}_{0,3})^g \simeq (\mathbf{P}^1)^g$$

which correspond to forgetting about the last $g - 3$ points on a $g$-pointed rational curve and designating the first three points as 0, 1 and $\infty$. The morphisms $\zeta_N$ and $\zeta_g$ are products of contraction morphisms. There is a morphism $\varepsilon \colon \overline{\mathscr{M}}_{0,g} \to (\overline{Z}_{0,g})^g$ which corresponds to associating the $g$-pointed rational curve $(X, Q_1, \dots, Q_g)$ to itself and the points $Q_1, \dots, Q_g$ on $X$. The morphism $\varepsilon$ exists because of the functorial properties $\overline{\mathscr{M}}_{0,g}$ and $\overline{Z}_{0,g}$. We get a set-theoretic map

$$f = (\zeta_g \circ \varepsilon \circ p_M) \times (\zeta_N \circ \delta) \colon \overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N \to (\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$$

where $p_M$ is the projection of $\overline{\mathscr{M}}_{0,g} \times (\overline{Z}_g)^N$ onto $\overline{\mathscr{M}}_{0,g}$. Furthermore, $f(p_Z^{-1}(B))$ is closed in $(\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$.

We will prove the lemma by showing that there exists a point

$$(Q_1, \dots, Q_g, P_1, \dots, P_N) \in f(p_Z^{-1}(B)) \subset (\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$$

such that either: (i) $P_1 = \cdots = P_N$ and at most $M$ of the $Q_i$ are equal to $P_1$; or (ii) there exists a point $Q \in \mathbf{P}^1$ such that all of the $P_i$ are distinct from $Q$ and at least $g - M$ of the $Q_i$ are equal to $Q$. Suppose that $(X, Q_1, \dots, Q_g)$ is a stable $g$-pointed rational curve. Let $\widetilde{X}_{Q_1, \dots, Q_g}$ be the curve obtained by attaching the elliptic curve $E$ at each $Q_i$, and let $\eta \colon \widetilde{X}_{Q_1, \dots, Q_g} \to X$ be the map which collapses the elliptic tails as before. The map $\overline{Z}_{0,g} \to \overline{Z}_{0,3} \simeq \mathbf{P}^1$ corresponding to forgetting about the last $g - 3$ points of a $g$-pointed rational curve and designating the first three points as $0$, $1$ and $\infty$ induces a morphism $X \to \mathbf{P}^1$ and hence a morphism $\widetilde{X}_{Q_1, \dots, Q_g} \to \mathbf{P}^1$. Note that if $P$ is a point in $\mathbf{P}^1$ and $k$ of the points $Q_1, \dots, Q_g$ map to $P$, then the pre-image of $P$ in $\widetilde{X}_{Q_1, \dots, Q_g}$ is a tail of genus $k$. hence condition (i) above implies that there exists a point in $B$ corresponding to a curve and $N$ points which lie on a tail of genus $\le M$. Note that the preimage of $\mathbf{P}^1 - \{P\}$ in $\widetilde{X}_{Q_1, \dots, Q_g}$ is a tail of genus $g - k$. Thus condition (ii) will also imply the lemma.

Let $D_k = \{(P_1, \dots, P_N) \in (\mathbf{P}^1)^N \mid$ at least $k$ points coincide$\}$. We will use the fact that if $Y$ is a closed subset of $(\mathbf{P}^1)^N$ and $\dim(Y) \geq k-1$, then $D_k \cap Y \neq \varnothing$. This fact follows from: the diagonal is ample in $\mathbf{P}^1 \times \mathbf{P}^1$; thus $\dim(D_k \cap Y) \geq 1$ implies $\dim(D_{k+1} \cap Y) \neq \varnothing$; and $\operatorname{codim} D_k = k-1$.

Let $p_g$ and $p_N$ denote the projections of $(\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$ to $(\mathbf{P}^1)^g$ and $(\mathbf{P}^1)^N$, respectively.

The dimension of the fiber of $f(p_Z^{-1}(B))$ over each point of $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ is $\geq N - M$, because the dimension of the fiber of $B$ over each point of $\overline{\mathscr{M}}_g$ is $\geq N - M$. Hence

$$p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_{N-M+1}) = (0, 1, \infty) \times (\mathbf{P}^1)^{g-3}.$$

Let $\kappa = \max\{k \mid (\mathbf{P}^1)^g \times D_k \cap f(p_Z^{-1}(B)) \neq \varnothing\}$. We will prove the lemma by showing that condition (i) holds if $\kappa = N$ and condition (ii) holds if $\kappa \leq N-1$.

Suppose $\kappa = N$. Then $p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N)$ has codimension $\leq N - (N - M + 1) = M - 1$ in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$. Hence

$$\dim(p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N)) \geq g - 2 - M.$$

Let $E_{M+1} = \{(Q_1, \dots, Q_g) \in (0, 1, \infty) \times (\mathbf{P}^1)^{g-3} \mid$ at least $M+1$ of the points $Q_1, \dots, Q_g$ coincide$\}$. We have

$$\dim(E_{M+1}) = g - 3 - M < \dim p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N).$$

Thus we can find $(Q_1, \dots, Q_g, P_1, \dots, P_N) \in f(P_Z^{-1}(B))$ so that $P_1 = \cdots = P_N$ and at most $M$ of the $Q_i$'s are equal to $P_1$.

Suppose $\kappa \leq N - 1$. Let $W$ be a component of $f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_\kappa$. Then $p_N(W)$ is a point $(P_1, \dots, P_N) \in (\mathbf{P}^1)^N$, because otherwise $(\mathbf{P}^1)^g \times D_{\kappa+1} \cap f(P_Z^{-1}(B)) \neq \varnothing$. Choose a point $R \in \mathbf{P}^1$ so that $R \neq P_i$ for $i = 1, \dots, N$. If $\kappa = N-1$, we choose $R \in \{0, 1, \infty\}$. Now $W$ has codimension $\leq \kappa - (N - M + 1)$ in $f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_{N-M+1}$ so $p_g(W)$ has codimension $\leq \kappa - (N - M + 1)$ in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$. Thus $\dim p_g(W) \geq g - M$ if $\kappa \leq N-2$, and $\dim p_g(W) \geq g - M - 1$ if $\kappa = N - 1$. Since $p_g(W)$ is closed in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ there are $\dim(p_g(W))$ factors of $(\mathbf{P}^1)^{g-3}$ so that the projection of $p_g(W)$ to the product of these factors is onto. Thus there is a point $(Q_1, \dots, Q_g) \in p_g(W) \subset (0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ so that at least $g - M$ of the $Q_i$'s are equal to $R$. Thus condition (ii) holds and the lemma follows.

We can find a smooth curve $T$ containing a point 0 and a morphism $\phi: T \to B \subset (\overline{Z}_g)^N$ such that $\phi(0)$ is the point $b \in B$ described in Lemma 1, and the induced map $T \to \overline{\mathscr{M}}_g$ sends $T - \{0\}$ to the subset of $U$ where the fibers of $\beta: B \to \overline{\mathscr{M}}_g$ have dimension $N - M$.

After replacing $T$ with a base extension, if necessary, there is a family of 0-pointed stable curves $X \to T$ which corresponds to the morphism $T \to \overline{\mathscr{M}}_g$.

Note that if $T$ is replaced with a base extension and the singularities of $X$ are resolved by blowing up, the curve $X_0$ will change by inserting chains of rational curves at the nodes of $X_0$. Thus we may assume (by replacing $T$ with a base extension and blowing up the singularities of $X$ if necessary) that there is a family of curves $X \to T$ which extends $Z \times_U (T - \{0\})$ and has the following properties: (1) $X_0$ is of special type; (2) there is a tail $Y$ of $X_0$ of genus $\leq M$ so that the sections $s_i\colon T \to X$ induces by the map $\phi\colon T \to B \subset (\overline{Z}_g)^N$ are such that each $s_i(0)$ is smooth point of $X_0$ which lies in $Y$ (if $P_1 = P_2 = \cdots = P_N$ is a point on a rational component of $X_b$ then, after blowing up, there will exist $s_i$ such that the $s_i(0)$ lie on a tail of genus 0 in $X_0$); (3) there is a section $s\colon T \to X$ such that $s(0)$ is a smooth point of $X_0$ which lies in a rational component of $X_0 - Y$; and (4) $X$ is smooth. Note that these properties are unchanged if $T$ is replaced with a base extension which sends one point to 0 and the singularities of the new $X$ are resolved by blowing up.

**Theorem 2.** *There is a sequence* $\mathbf{a}$ *of type* $(r, d)$ *and a closed subscheme* $A$ *of* $H \times_B (T - \{0\})$ *such that:* $A$ *consists of all linear series in* $H \times_B (T - \{0\})$ *which satisfy vanishing condition* $\mathbf{a}$ *along* $s(T - \{0\})$; *the fibers* $A_t$ *are nonempty for each* $t \in T - \{0\}$; *and* $\dim A_t = 0$ *for all but finitely many* $t \in T - \{0\}$.

*Proof*. Note that $H \times_B (T - \{0\})$ is proper over $T - \{0\}$, and the subset of linear series satisfying a particular vanishing condition along $s(T - \{0\})$ is closed. Since there are only finitely many sequences of type $(r, d)$, the lemma is a consequence of the following.

**Lemma 2a.** *Let* $X$ *be a smooth curve. Let* $A$ *be a closed subset of* $G^r_d(X)$, *and let* $\mathbf{a}$ *be a sequence of type* $(r, d)$. *If* $\dim A \geq 1$ *and every linear series in* $A$ *satisfies vanishing condition* $\mathbf{a}$ *at a point* $P \in X$, *then there exists a sequence* $\mathbf{a}'$ *with* $a'_i \geq a_i$ *for* $i = 0, \dots, r$ *and* $a'_k > a_k$ *for some* $k$ *such that* $A$ *contains a linear series which satisfies vanishing condition* $\mathbf{a}'$ *at* $P$.

*Proof*. We may assume $A$ is irreducible. Consider the natural map $\Phi\colon A \to \operatorname{Pic}^d(X)$. Suppose for some $x \in \operatorname{Pic}^d(X)$, the fiber $\Phi^{-1}(x)$ has dimension $\geq 1$, and let $\mathscr{L}$ be the line bundle corresponding to $x$. The set of $(r+1)$ dimensional vector spaces of $H^0(X, \mathscr{L})$ which have vanishing sequence $\mathbf{a}$ at $P$ is a Schubert variety, and hence is affine. It follows that $A$ contains a $g^r_d$ which does not have vanishing sequence $\mathbf{a}$ at $P$, because $\Phi^{-1}(x)$ is closed and $\dim \Phi^{-1}(x) \geq 1$. So we may assume $\Phi(A)$ contains a closed curve in $\operatorname{Pic}^d(x)$.

If the lemma were false, then there would exist a family $F$ of $g^0_{d-a_r}$'s obtained from $A$ by taking $(H^0(Y, \mathscr{L}(-a_r P)) \cap V, \mathscr{L}(-a_r P))$ for every $(V, \mathscr{L})$ of $A$. But a family of $g^0_{d-a_r}$'s is a family of divisors of degree $d - a_r$. Since $\dim(F) \geq 1$, it must contain a divisor with $P$ in its support. But the linear series in $A$ associated with this divisor satisfies vanishing condition $(a_0, \dots, a_{r-1}, a_r + 1)$.

If $T$ is replaced with an appropriate base extension, and the singularities of $X$ blown up, we may also assume that $A \to T - \{0\}$ gives an isomorphism $A' \to T - \{0\}$ for some component $A'$ of $A$.

The codimension of $H \times_B T$ in $H \times_U T$ is $N - M$, and $A'$ has codimension $\le W(\mathbf{a})$ in $H \times_B T$. Thus the theorem will follow when we show that $W(\mathbf{a}) + N - M \le \rho(g, r, d) - r(N-1) + N$.

## 3

The family of curves $X \to T$ and $A'$ determine a crude limit linear series on $X_0$ [EH-2]. This crude limit linear series is a collection consisting of a $g^r_d$ for each of the components of $X_0$. If $C$ is a component of $X_0$, the $g^r_d(V_C, \mathscr{L}_C)$ on $C$ is determined in the following manner. Let $R$ be the local ring of $T$ at 0, and let $\eta$ be the generic point of $\operatorname{Spec} R$. Then $A'$ determines a unique line bundle $\mathscr{L}^C$ on $X_{\operatorname{Spec} R}$ such that the restriction of $\mathscr{L}^C$ to $X_0$ has degree 0 on every component of $X_0$ except $C$. Also, $A'$ determines a subspace $V \subset H^0(X_\eta, \mathscr{L}^C|_{X_\eta})$. Let $\widetilde{V}_C = V \cap H^0(X_{\operatorname{Spec} R}, \mathscr{L}^C)$. Then $V_C = \widetilde{V}_C \otimes k(0) \subset H^0(X_0, \mathscr{L}^C|_{X_0})$ and $\mathscr{L}_C = \mathscr{L}^C|_C$ where $k(0)$ the residue field of $R$.

Let $Q = Y \cap (\overline{X_0 - Y})$ and let $Y'$ be the component of $Y$ containing $Q$. Let $Y_1, \dots, Y_k$ be the remaining components of $Y$. In addition to parameterizing a family of $g^r_d$'s on $X_{T-\{0\}} \to T - \{0\}$, $A'$ also determines a family of $g^{r-1}_{d-N}$'s corresponding to the sections which vanish along $\sum s_i(T - \{0\})$.

Let $(W_{Y'}, \eta_{Y'})$ be the $g^{r-1}_{d-N}$ on $Y'$ of the limit $g^{r-1}_{d-N}$ on $X_0$. Then

$$\eta^{Y'} = \mathscr{L}^{Y'}\left(-\sum s_i(T) - \sum n_i Y_i\right) \quad \text{where } n_i \ge 0,$$

so $\eta_{Y'} = \eta^{Y'}|_{Y'} = \mathscr{L}_{Y'}(-\sum Q_i)$ where the $Q_i$ are points not equal to $Q$. Also,

$$W_{Y'} = (\widetilde{V}_{Y'} \cap H^0(X_{\operatorname{Spec} R}, \eta^{Y'})) \otimes k(0) \subset V_{Y'} \cap H^0(\eta_{Y'}).$$

So the vanishing sequence $\mathbf{b}$ of $(W_{Y'}, \eta_{Y'})$ at $Q$ is a subsequence of the vanishing sequence $\mathbf{c}$ of $(V_{Y'}, \mathscr{L}_{Y'})$ at $Q$.

The following lemma is an immediate consequence of Theorem 4.5 of [EH-2] and Theorem 2.3 of [EH-1].

**Lemma 3.** *If $C$ is a genus $g$ curve of special type and $P$ and $Q$ are two smooth points in rational components of $C$, then the existence of a crude limit $g^r_d$ on $C$, which satisfies vanishing conditions $\mathbf{a}$ and $\mathbf{b}$ at points $P$ and $Q$, respectively, implies that $\rho(g, r, d) - W(\mathbf{a}) - W(\mathbf{b}) \ge 0$.*

Now the crude limit $g^{r-1}_{d-N}$ on $X_0$ determined by $A'$ restricts to a limit $g^{r-1}_{d-N}$ on the genus $M$ tail $Y$ which satisfies vanishing condition $\mathbf{b}$ at $Q$. Hence $\rho(M, r-1, d-N) - W(\mathbf{b}) \ge 0$. Since $\mathbf{b}$ is a subsequence of $\mathbf{c}$ we have $W(\mathbf{c}) \le W(\mathbf{b}) + d - r$. So $W(\mathbf{c}) \le \rho(M, r-1, d-N) + d - r$.

Let $F$ be the component of $\overline{X_0 - Y}$ which contains $Q$. If $\mathbf{c}'$ is the vanishing sequence of $V_F$ at $Q$, then the definition of crude limit linear series requires

$$\begin{aligned} W(c') &\geq (r+1)(d-r) - W(c) \\ &\geq r(d-r) - \rho(M, r-1, d-N). \end{aligned}$$

Now the crude limit $g_d^r$ on $X_0$ determined by $A'$ restricts to a crude limit $g_d^r$ on $\overline{X - Y}$ satisfying vanishing conditions $\mathbf{c}$ at $Q$ and $\mathbf{a}$ at $s(0)$. Thus,

$$\begin{aligned} 0 &\leq \rho(g-M, r, d) - W(\mathbf{a}) - W(\mathbf{c}') \\ &\leq \rho(g-M, r, d) - W(\mathbf{a}) + \rho(M, r-1, d-N) - r(d-r) \\ &= (r+1)(d-r) - r(g-M) - w(\mathbf{a}) \\ &\quad + r(d-N-r+1) - (r-1)M - r(d-r) \\ &= (r+1)(d-r) - rg - W(\mathbf{a}) - r(N-1) + N - (N-M) \\ &= \rho(g, r, d) - W(\mathbf{a}) - r(N-1) + N - (N-M). \end{aligned}$$

Thus $W(\mathbf{a}) + N - M \leq \rho(g, r, d) - r(N-1) + N$, and the theorem follows.

## References


[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves*, Vol. 1, Springer, New York, 1984.

[C] M. Coppens, *A remark on the embedding theorem for general smooth curves*, Preprint No. 405, Univ. of Utrecht, 1986.

[EH-1] D. Eisenbud and J. Harris, *Divisors on general curves and cusptial rational curves*, Invent. Math. **74** (1983), 371–418.

[EH-2] ——, *Limit linear series: basic theory*, Invent. Math. **85** (1986), 337–371.

[K1] F. Knudsen, *The projectivity of the moduli space of stable curves.* II: *The stacks* $M_{g,n}$, Math. Scand. **52** (1983), 161–199.

[K2] ——, *The projectivity of the moduli space of stable curves.* III: *The line bundles on* $M_{g,n}$, *and a proof of the projectivity of* $\overline{M}_{g,n}$ *in characteristic* $0$, Math. Scand. **52** (1983), 200–212.

[S] D. Schubert, *Linear series with cusps and* $n$*-fold points*, Trans. Amer. Math. Soc. **304** (1987), 689–703.



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