SZEGŐ'S THEOREM ON A BIDISC

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ABSTRACT. G. Szegő showed that

$$\inf \int_0^{2\pi} |1 - f|^2 w \, d\theta / 2\pi = \exp \int_0^{2\pi} \log w \, d\theta / 2\pi$$

where f ranges over analytic polynomials with mean value zeros. We study extensions of the Szegő's theorem on the disc to the bidisc. We show that the quantity is a mixed form of an arithmetic mean and a geometric one of w in some special cases.

1. INTRODUCTION

Let m be the Haar measure of the torus T^2 , the distinguished boundary of the unit bidisc U^2 in the space of 2-complex variables (z_1, z_2) . Let Z be the set of all integers, Z_+ the set of all nonnegative integers, Z^2 the set of all $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in Z$ and Z_+^2 the set of all $\alpha \in Z^2$ with $\alpha_i \in Z_+$ for $i = 1, 2$. For $1 \leq p \leq \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p; \hat{f}(\alpha) = 0 \text{ if } \alpha \notin Z_+^2\}$, that is, H^p denotes the usual Hardy spaces on the bidisc. Let $H_0^p = \{f \in H^p; \int f \, dm = 0\}$ and $K_0^p = \{f \in L^p; \hat{f}(\alpha) = 0 \text{ if } -\alpha \in Z_+^2\}$.

Let $\mathcal{P}$ be a set of all analytic polynomials z_1, z_2 and $\mathcal{P}_0 = \{f \in \mathcal{P}; \int f \, dm = 0\}$. For each nonnegative function $w \in L^1$ we study the following quantity:

$$S(w) = \inf_{f \in \mathcal{P}_0} \int_0 |1 - f|^2 w \, dm.$$

In the case of one complex variable, G. Szegő [6] showed that

$$S(w) = \exp \int \log w \, dm.$$

In the case of two complex variables, this quantity has been studied by A. G. Miamee [1] under some strong condition and then $S(w) = \exp \int \log w \, dm$.

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However it is easy to see that there exists a nonzero function $w \in L^1$ such that $S(w) \neq \exp \int \log w \, dm$. Even if w is zero on some positive measure on T^2 it is possible that $S(w) > 0$.

In §2 several means are defined to estimate $S(w)$ in the latter sections. In §3 we give expressions in terms of w of the $L^2(w \, dm)$ -distances between 1 and subalgebras of L^∞ which contain $\mathcal{P}_0$ properly. This follows from the theory of an abstract Hardy space [2]. In §4 we estimate $S(w)$ from the above and the below by means which are defined using conditional expectations. In §5 for special weights we give an expression in terms of w of $S(w)$. In §6 we study the $L^2(w \, dm)$ -distance between 1 and K_0^∞ which is a dual version of $S(w)$. In §7 we study relations between $S(w)$ and an invariant subspace defined by w .

2. VARIOUS MEANS

Three typical means of w are the following:

$$\int w \, dm \geq \exp \int \log w \, dm \geq \left(\int w^{-1} \, dm \right)^{-1}.$$

$\int w \, dm$, $\exp \int \log w \, dm$ or $(\int w^{-1} \, dm)^{-1}$ is called an arithmetic mean, a geometric mean or a harmonic mean of w , respectively. We would like to define new means in which the means above are mixed. Put $|\alpha|_r = \alpha_1 - r\alpha_2$ where r is a real number. For $1 \leq p \leq \infty$ $\mathcal{L}_r^p$ denotes the space of all $f \in L^p$ whose Fourier coefficients $\hat{f}(\alpha) = 0$, $\alpha \in \mathbb{Z}^2$ with $|\alpha|_r = 0$. If r is irrational then $\mathcal{L}_r^p$ is trivial but if r is rational then $\mathcal{L}_r^p$ is nontrivial. Moreover $\mathcal{L}_{-\infty} = \mathcal{L}_{\infty}$. $\mathcal{L}_r^\infty$ is a commutative von Neumann algebra and hence $\mathcal{L}_r^p = L^p(T^2, \mathcal{B}_r, m)$ where $\mathcal{B}_r$ is the σ -algebra of subsets E of T^2 for which the characteristic function χ_E lies in $\mathcal{L}_r^\infty$. Let $\mathcal{A}$ be the σ -algebra of measurable sets with respect to m . Then $\mathcal{B}_r$ is a sub- σ -algebra of $\mathcal{A}$. Let $\mathcal{E}^r$ denote the conditional expectation for sub- σ -algebra $\mathcal{B}_r$ of $\mathcal{A}$. Define $\mathcal{E}^r(\log w)$ by $\lim_{\varepsilon \rightarrow 0} \mathcal{E}^r\{\log(w + \varepsilon)\}$. We consider the following three new means for each r ,

$$\int \exp \mathcal{E}^r(\log w) \, dm, \quad \left(\int \exp \mathcal{E}^r(\log w^{-1}) \, dm \right)^{-1},$$

and

$$\exp \int \log \mathcal{E}^r(w) \, dm.$$

Lemma 1. *For any r and any nonnegative w in L^1 , the following inequalities are valid.*

$$\begin{aligned} \int w \, dm &\geq \int \exp \mathcal{E}^r(\log w) \, dm \geq \exp \int \log w \, dm \\ &\geq \left(\int \exp \mathcal{E}^r(\log w^{-1}) \, dm \right)^{-1} \geq \left(\int w^{-1} \, dm \right)^{-1} \end{aligned}$$

and

$$\int w \, dm \geq \exp \int \log \mathcal{E}^r(w) \, dm \geq \exp \int \log w \, dm.$$

Proof. We can show the inequality of arithmetic and geometric means for conditional expectation. That is, if v is a real function in L^1 and $\exp v \in L^1$, then $\exp \mathcal{E}^r(v) \leq \mathcal{E}^r(\exp v)$ a.e. Hence

$$\mathcal{E}^r(w) \geq \exp \mathcal{E}^r(\log w) \quad \text{a.e.}$$

and

$$\mathcal{E}^r(w^{-1}) \geq \exp \mathcal{E}^r(\log w^{-1}) \quad \text{a.e.}$$

This implies the first part of the lemma. For the second part, apply to $\mathcal{E}^r(w)$ the classical inequality of arithmetic and geometric means.

For $1 \leq j \leq n < \infty$, let λ_j be a nonnegative number with $\sum_{j=1}^n \lambda_j = 1$ and put $\mathcal{E}^j = \mathcal{E}^{r_j}$ where r_j is a real number. The following lemma gives the inequality for $\sum_{j=1}^n \lambda_j \mathcal{E}^j$.

Lemma 2. *If w is a nonnegative function in L^1 , then*

$$\sum_{j=1}^n \lambda_j \mathcal{E}^j(w) \geq \prod_{j=1}^n \mathcal{E}^j(w)^{\lambda_j} \geq \exp \sum_{j=1}^n \lambda_j \mathcal{E}^j(\log w)$$

and hence

$$\begin{aligned} \int w \, dm &\geq \exp \int \log \prod_{j=1}^n \mathcal{E}^j(w)^{\lambda_j} \, dm \\ &\geq \int \exp \sum_{j=1}^n \lambda_j \mathcal{E}^j(\log w) \, dm \geq \exp \int \log w \, dm \\ &\geq \left(\int \exp \sum_{j=1}^n \lambda_j \mathcal{E}^j(\log w^{-1}) \, dm \right)^{-1} \geq \left(\int w^{-1} \, dm \right)^{-1}. \end{aligned}$$

Let $L_+^p = \{f \in L^p; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2) \text{ and } \alpha_1 \alpha_2 < 0\}$ and $L_-^p = \{f \in L^p; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2) \neq (0, 0) \text{ and } \alpha_1 \alpha_2 \geq 0\}$. $L_+^1 + L_-^1$ is a dense subspace in L^1 and $L_+^1 \cap L_-^1$ consists of constant functions. Let P be a projection from $L_+^1 + L_-^1$ to L_-^1 . We can define a mean using P ,

$$\int \exp P(\log w) \, dm.$$

This mean will be used in §5.

We would like to calculate the means for some special functions w . Let w_j be a nonnegative function in L^1 with $\log w_1 \in L_+^1$ and $\log w_2 \in L_-^1$. If $w = w_1 w_2 \in L^1$ then

$$\int \exp P(\log w) \, dm = \exp \int \log w_1 \, dm \int w_2 \, dm.$$

Let w_r (or w_s) be a nonnegative function in $\mathcal{L}_r^1$ (or $\mathcal{L}_s^1$) and $r \neq s$. If $w = w_r w_s \in L^1$ then

$$\begin{aligned} \int \exp \mathcal{E}^r(\log w) dm &= \exp \int \log w_s dm \int w_r dm, \\ \left(\int \exp \mathcal{E}^r(\log w^{-1}) dm \right)^{-1} &= \exp \int \log w_s dm \left(\int w_r^{-1} dm \right)^{-1}, \\ \exp \int \log \mathcal{E}^r(w) dm &= \exp \int \log w_r dm \int w_s dm, \\ \int \exp \frac{\mathcal{E}^r + \mathcal{E}^s}{2}(\log w) dm &= \int w^{1/2} dm \exp \int \log w^{1/2} dm, \end{aligned}$$

and

$$\left(\int \exp \frac{\mathcal{E}^r + \mathcal{E}^s}{2}(\log w^{-1}) dm \right)^{-1} = (w^{-1/2} dm)^{-1} \exp \int \log w^{1/2} dm.$$

The results above shows that the means with respect to three operators: $\mathcal{E}^r$, $\sum \lambda_j \mathcal{E}^j$ and P , are mixed ones of arithmetic and geometric means.

3. EXTENDED WEAK-* DIRICHLET ALGEBRA

Let $\mathbf{H}_r = \{f \in L^\infty; \hat{f}(\alpha) = 0 \text{ if } \alpha \in Z^2 \text{ and } |\alpha|_r < 0\}$, then $\mathbf{H}_r \cap \overline{\mathbf{H}}_r = \mathcal{L}_r^\infty$, $\mathcal{E}^r$ is multiplicative on $\mathbf{H}_r$ and $\mathbf{H}_r + \overline{\mathbf{H}}_r$ is weak-* dense in L^∞ . That is, $\mathbf{H}_r$ is an extended weak-* Dirichlet algebra with respect to $\mathcal{E}^r$ [2]. If r is irrational then $\mathbf{H}_r$ is a weak-* Dirichlet algebra [5]. Let $I_r = \{f \in L^\infty; \hat{f}(\alpha) = 0 \text{ if } \alpha \in Z^2 \text{ and } |\alpha|_r \leq 0\}$ then $I_r = \{f \in \mathbf{H}_r; \mathcal{E}^r(f) = 0\}$. $\mathbf{H}_r = \mathcal{L}_r^\infty + I_r$. Let $\mathcal{H}_r^\infty = \{f \in \mathcal{L}_r^\infty; \hat{f}(\alpha) = 0 \text{ if } \alpha_2 < 0\}$ and $\mathcal{H}_{r,0}^\infty = \{f \in \mathcal{H}_r^\infty; \int f dm = 0\}$. $\mathcal{H}_r^\infty + I_r$ is a weak-* Dirichlet algebra. Putting $\mathbf{H}_{r,0} = \{f \in \mathbf{H}_r; \int f dm = 0\}$, for any $r < 0$ and $r \neq -\infty$,

$$K_0^\infty \supset \mathbf{H}_{r,0} \supset \mathcal{H}_{r,0}^\infty + I_r \supset I_r \supset H_0^\infty.$$

The following lemma and proposition are essentially known [2].

Lemma 3. Let w be a nonnegative function in L^1 . For any $v \in \mathcal{L}_r^\infty$ and any r ,

$$\begin{aligned} \mathbf{S}(v, r) &= \inf \left\{ \int |v - f|^2 w dm; f \in I_r \right\} \\ &= \int \exp \mathcal{E}^r(\log w) |v| dm. \end{aligned}$$

Hence

$$\inf \left\{ \mathbf{S}(v, r); \int \log |v| dm \geq 0 \right\} = \exp \int \log w dm$$

and

$$\inf \left\{ \mathbf{S}(v, r); \int v dm = 1 \right\} = \left(\int \exp \mathcal{E}^r(\log w^{-1}) dm \right)^{-1}.$$

Proposition 1. *Let w be a nonnegative function in L^1 . Then, for any r*

$$(1) \quad \inf_{f \in \mathcal{H}_{r,0}^\infty} \int |1 - f|^2 w \, dm = \exp \int \log \mathcal{E}^r(w) \, dm,$$

$$(2) \quad \inf_{f \in I_r} \int |1 - f|^2 w \, dm = \int \exp \mathcal{E}^r(\log w) \, dm,$$

$$(3) \quad \inf_{f \in \mathcal{H}_{r,0}^\infty} \int |1 - f|^2 w \, dm = \exp \int \log w \, dm,$$

$$(4) \quad \inf_{f \in \mathbf{H}_{r,0}} \int |1 - f|^2 w \, dm = \left(\int \exp \mathcal{E}^r(\log w^{-1}) \, dm \right)^{-1}.$$

4. $S(w)$ AND MEANS WITH RESPECT TO $\mathcal{E}^r$

In this section we will improve the following known inequality:

$$\int w \, dm \geq S(w) \geq \exp \int \log w \, dm.$$

Theorem 2. *Let w be a nonnegative function in L^1 .*

(1) *If $0 \leq s \leq \infty$ and $-\infty < r < 0$, then*

$$\exp \int \log \mathcal{E}^s(w) \, dm \geq S(w) \geq \int \exp \mathcal{E}^r(\log w) \, dm,$$

(2) *Suppose $0 > r_j > -\infty$, $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$ and $n < \infty$, and $0 \leq s_j \leq \infty$, $\gamma_j \geq 0$, $\sum_{j=1}^l \gamma_j = 1$ and $l < \infty$. Then*

$$\exp \int \log \prod_{j=1}^l \mathcal{E}^{s_j}(w)^{\gamma_j} \, dm \geq S(w) \geq \int \exp \sum_{j=1}^n \lambda_j \mathcal{E}^{r_j}(\log w) \, dm.$$

Proof. (1) Since $-\infty < r < 0$, $H_0^\infty \subset I_r$ and hence by (2) of Proposition 1

$$S(w) \geq \inf_{f \in I_r} \int |1 - f|^2 w \, dm = \int \exp \mathcal{E}^r(\log w) \, dm.$$

Since $0 \leq r \leq \infty$, $\mathcal{H}_{r,0}^\infty \subset H_0^\infty$ and hence by (1) of Proposition 1

$$S(w) \leq \inf_{f \in \mathcal{H}_{r,0}^\infty} \int |1 - f|^2 w \, dm = \exp \int \log \mathcal{E}^r(w) \, dm.$$

(2) Since $-\infty < r_j < 0$, if $f \in H_0^\infty$ then $f \in \bigcap_{j=1}^n I_{r_j}$ and hence by the first part of Lemma 2,

$$\begin{aligned} \int |1 - f|^2 w \, dm &\geq \int \exp \sum_{j=1}^n \lambda_j \mathcal{E}^{r_j}(\log |1 - f|^2 w) \, dm \\ &\geq \int \exp \sum_{j=1}^n \lambda_j \mathcal{E}^{r_j}(\log w) \, dm. \end{aligned}$$

Since $0 \leq s_j \leq \infty$, by (1)

$$\exp \int \log \prod_{j=1}^l \mathcal{E}^j(w)^{\lambda_j} dm = \prod_{j=1}^l \left(\exp \int \log \mathcal{E}^j(w) dm \right)^{\lambda_j} \geq S(w).$$

Corollary 1. *If $w = w_t w_l \in L^1$ where $w_t \in \mathcal{L}_t^1$, $w_l \in \mathcal{L}_l^1$, $-\infty < t < 0$ and $0 \leq l \leq \infty$, then*

$$\exp \int \log \mathcal{E}^l(w) dm = S(w) = \int \exp \mathcal{E}^t(\log w) dm$$

and hence

$$S(w) = \int w_t dm \exp \int \log w_l dm.$$

Proof. It is easy to see that

$$\begin{aligned} \exp \int \log \mathcal{E}^l(w) dm &= \int w_t dm \exp \int \log w_l dm \\ &= \int \exp \mathcal{E}^t(\log w) dm. \end{aligned}$$

Hence (1) of Theorem 2 implies the corollary.

We can ask whether if

$$\inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm = S(w) = \sup_{-\infty < r < 0} \int \exp \mathcal{E}^r(\log w) dm.$$

Unfortunately this equality does not hold for some w . Suppose $w = w_t w_l \in L^1$, and both $w_t \in \mathcal{L}_t^1$ and $w_l \in \mathcal{L}_l^1$ are nonconstant functions.

If $-\infty < t, l < 0$ and $w \in L_-^1$ then

$$\begin{aligned} \inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm &= S(w) \\ &\neq \sup_{-\infty < r < 0} \int \exp \mathcal{E}^r(\log w) dm. \end{aligned}$$

In fact,

$$\inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm = \int w dm$$

and

$$\begin{aligned} &\sup_{-\infty < r < 0} \int \exp \mathcal{E}^r(\log w) dm \\ &= \max \left\{ \exp \int \log w dm, \int w_t dm \exp \int \log w_l dm, \right. \\ &\quad \left. \exp \int \log w_l dm \int w_l dm \right\}. \end{aligned}$$

By (1) of Theorem 4, $S(w) = \int w dm$.

If $0 \leq t, l \leq \infty$ and $w \in L_+^1$ then

$$\inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm \not\geq S(w) \\ = \sup_{-\infty < r < 0} \exp \mathcal{E}^r(\log w) dm.$$

In fact,

$$\inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm \\ = \min \left\{ \int w dm, \exp \int \log w_l dm \int w_l dm, \int w_l dm \exp \int \log w_l dm \right\}$$

and

$$\sup_{-\infty < r < 0} \exp \mathcal{E}^r(\log w) dm = \exp \int \log w dm.$$

While by (2) of Theorem 4 if $w_l \in \mathcal{L}_l^\infty S(w) = \exp \int \log w dm$. Moreover we can ask whether if $S(w) = \inf_{0 \leq s \leq \infty} \exp \int \log \mathcal{E}^s(w) dm$ or $S(w) = \sup_{-\infty < r < 0} \exp \mathcal{E}^r(\log w) dm$. However this is also not true. For such an example, suppose $w = w_l w_l w_k$ where $w_j \in \mathcal{L}_j^\infty$ ($j = t, l, k$), $-\infty < t < 0$ and $0 \leq l, s \leq \infty$.

There does not exist a universal finite constant γ_0 such that

$$S(w) \leq \gamma_0 \exp \int \log w dm$$

for all w in L^∞ with $w^{-1} \in L^\infty$. Let $D_r^\infty = \mathcal{H}_r^\infty + I_r$ and $D_r^2 = \mathcal{H}_r^2 + [I_r]$. For $-\infty \leq r \leq 0$, let γ_r be the norm of the orthogonal projection from K^2 onto D_r^2 in $L^2(w^{-1} dm)$. If both w and w^{-1} are in L^∞ then for each r γ_r is finite.

Theorem 3. Let w be a nonnegative function in L^1 . If $w^{-1} \in L^\infty$ then

$$S(w) \leq (\gamma_0 + \gamma_\infty) \exp \int \log w dm.$$

Proof. By the duality

$$S(w) = \sup \left| \int g w dm \right|$$

where g ranges over the unit ball of $(H_0^\infty)^\perp \cap L^2(w dm)$. If $w^{-1} \in L^\infty$ then

$$(H_0^\infty)^\perp \cap L^2(w dm) = w^{-1} \overline{K}^2.$$

Since

$$K^2 = D_0^2 + D_\infty^2 = D_0^2 \oplus \{D_\infty^2 \ominus [K^2 \ominus D_0^2]\},$$

if $F \in K^2$ then

$$w^{-1} F = w^{-1} F_0 + w^{-1} F_\infty$$

for some $F_0 \in D_0^2$ and $F_\infty \in D_\infty^2 \ominus [K^2 \ominus D_0^2]$. By hypothesis if $\int |w^{-1}F|^2 w \, dm \leq 1$ then

$$\int |w^{-1}F_r|^2 w \, dm \leq \gamma_r \quad (r = 0, \infty).$$

Hence

$$S(w) \leq \sum_{r=0, \infty} \gamma_r \sup \left\{ \left| \int h w \, dm \right| ; h \in (D_{r,0}^\infty)^\perp \cap L^2(w \, dm) \right. \\ \left. \text{and } \int |h|^2 w \, dm \leq 1 \right\}.$$

Again by the duality

$$S(w) \leq \sum_{r=0, \infty} \gamma_r \inf \left\{ \int |1 - f|^2 w \, dm ; f \in D_{r,0}^\infty \right\}.$$

Thus by (3) of Proposition 1 the theorem follows.

5. ARITHMETIC MEAN AND GEOMETRIC ONE

The function $g \in H^2$ is called an outer function if

$$\int \log |g| \, dm = \log \left| \int g \, dm \right| > -\infty.$$

The function $g \in H^2$ is called a generator if $[g\mathcal{P}] = H^2$. If $g \in H^2$ is a generator then it is an outer function [4, p. 73]. However there exists an outer function which is not a generator [4, p. 76]. The following lemma is known [4, pp. 73 and 77].

Lemma 4. *Let $w \in L^1_+$ be a nonnegative function. There exists an outer function $g \in H^2$ such that $w = |g|^2$ if and only if $\log w \in L^1_+$.*

Theorem 4. *Let w be a nonnegative function in L^1_+ .*

(1) $w \in L^1_-$ if and only if

$$S(w) = \int w \, dm.$$

(2) $w = |g|^2$ for some generator $g \in H^2$ if and only if

$$S(w) = \exp \int \log w \, dm.$$

Proof. (1) If w is a nonzero function in L^1_- and $\int w \, dm = a$ then $w \, dm/a$ is a representing measure of the evaluation at the origin and hence for any f in H_0^∞ ,

$$\int |1 - f|^2 w \, dm/a \geq 1.$$

Thus $S(w) = \int w \, dm$. Conversely if $S(w) = \int w \, dm$ then for any $f \in H_0^\infty$,

$$\int |f|^2 w \, dm \geq 2 \operatorname{Re} \int f w \, dm.$$

Hence for any $f \in H_0^\infty$ and for any positive number ε ,

$$\varepsilon \int |f|^2 w \, dm \geq 2 \left| \int f w \, dm \right|.$$

As $\varepsilon \rightarrow 0$,

$$\int f w \, dm = 0$$

and hence $w \in L_-^1$.

(2) If $w = |g|^2$ and g is a generator then

$$S(w) = \left| \int g \, dm \right|^2$$

and by the remark above g is an outer function. Thus $S(w) = \exp \int \log w \, dm$. Conversely if $S(w) = \exp \int \log w \, dm$ then by (1) of Theorem 2 for any r with $-\infty < r < 0$,

$$\int \exp \mathcal{E}^r(\log w) \, dm = \exp \int \log w \, dm$$

and hence

$$\mathcal{E}^r(\log w) = \int \log w \, dm.$$

Assuming $\log w \in L^1$ without loss of the generality, $(\log w)^\wedge(\alpha) = 0$ if $|\alpha|_r = 0$ and $-\infty < r < 0$, and so $\log w \in L_+^1$. By Lemma 4 $w = |g|^2$ for some outer $g \in H^2$. Then g has the decomposition $g = g_0 + g_1$ where $g_0 \in [g\mathcal{P}] \ominus [g\mathcal{P}_0]$ and $g_1 \in [g\mathcal{P}_0]$. Hence since $S(w) = \exp \int \log w \, dm$,

$$S(w) = \int |g_0|^2 \, dm = \left| \int g_0 \, dm \right|^2 > 0.$$

Thus g_0 is constant and hence g is a generator.

We want to present a mixed form of (1) and (2) of Theorem 4.

Proposition 5. Let w_j be a nonnegative function in L^1 and $w = w_1 w_2 \in L^1$.

(1) If $\log w_1 \in L_+^1$ and $w_2 \in L_-^1$ then

$$S(w) \geq \exp \int \log w_1 \, dm \int w_2 \, dm.$$

(2) If $w_1 = |g|^2$ for some generator in H^2 and $w_2 \in L_-^\infty$ then

$$S(w) = \exp \int \log w_1 \, dm \int w_2 \, dm.$$

Proof. (1) By Lemma 4 $w_1 = |g|^2$ for some outer function $g \in H^2$ and by the hypothesis $w_2 \in L_-^1$. Hence

$$\begin{aligned} S(w) &= \inf_{f \in \mathcal{P}_0} \int |g - gf|^2 w_2 dm \\ &\geq \inf_{f \in \mathcal{P}_0} \left| \int g w_2 dm - \int g f w_2 dm \right|^2 \left(\int w_2 dm \right)^{-1} \\ &= \left| \int g w_2 dm \right|^2 \left(\int w_2 dm \right)^{-1} \\ &= \left| \int g dm \right|^2 \int w_2 dm \\ &= \exp \int \log w_1 dm \int w_2 dm. \end{aligned}$$

(2) Since $w_2 \in L^\infty$, H_0^2 is in the closure of $g\mathcal{P}_0$ in $L^2(w_2 dm)$ and hence

$$\begin{aligned} \inf_{f \in \mathcal{P}_0} \int |g - gf|^2 w_2 dm &\leq \left| \int g dm \right|^2 \int w_2 dm \\ &= \exp \int \log w_1 dm \int w_2 dm. \end{aligned}$$

Let $L_{-+}^\infty = \{f \in L^\infty; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2), \alpha_1 \leq 0 \text{ and } \alpha_2 \geq 0\}$. $L_{-+}^\infty + \overline{L_{-+}^\infty}$ is weak-* dense in L_-^∞ and L_{-+}^∞ is a weak-* closed algebra. If w satisfies that

$$\log w \in L_+^1 + L_{-+}^\infty + \overline{L_{-+}^\infty}$$

then $w = w_1 w_2$, $\log w_1 \in L_+^1$, $\log w_2 \in L_-^1$ and $w_2 \in L_-^\infty$ and

$$\int \exp P(\log w) dm = \exp \int \log w_1 dm \int w_2 dm.$$

6. THE DUAL VERSION OF SZEGÖ'S THEOREM

$\overline{K}_0^\infty$ is an annihilator of $\mathcal{P}$ (and hence H^∞) in L^∞ . Hence we would like to give an expression in terms of w of the following quantity:

$$S^\perp(w) = \inf_{f \in K_0^\infty} \int |1 - f|^2 w dm.$$

For any r with $-\infty < r < 0$,

$$H_0^\infty \subset I_r \subset \mathcal{H}_{r,0}^\infty + I_r \subset \mathbf{H}_{r,0} \subset K_0^\infty$$

and hence by (3) of Proposition 1

$$\int w dm \geq S(w) \geq \exp \int \log w dm \geq S^\perp(w) \geq \left(\int w^{-1} dm \right)^{-1}$$

and moreover by (4) of Proposition 1

$$\left(\int \exp \mathcal{E}^r(\log w^{-1}) dm \right)^{-1} \geq S^\perp(w).$$

If both w and w^{-1} are in L^∞ , by Theorem 1 in [3]

$$S^\perp(w) = S(w^{-1})^{-1}.$$

Unfortunately we do not know whether if both w and w^{-1} in L^1 then $S^\perp(w) = S(w^{-1})^{-1}$. However by Theorem 1 in [3]

$$\begin{aligned} S(w) &\geq S(w^{-1})^{-1} \geq \exp \int \log w dm \\ &\geq S^\perp(w) \geq S(w^{-1})^{-1} \geq \left(\int w^{-1} dm \right)^{-1}. \end{aligned}$$

Hence by (1) of Theorem 2, for any s with $0 \leq s \leq \infty$,

$$S^\perp(w) \geq \left(\exp \int \log \mathcal{E}^s(w^{-1}) dm \right)^{-1}.$$

If $w = |g|^2$ for some generator $g \in H^2$, by (2) of Theorem 4 then $S^\perp(w) = S(w^{-1})^{-1}$. If both w and w^{-1} are in L_-^1 , by (1) of Theorem 4 then $S^\perp(w) = S(w^{-1})^{-1}$.

7. INVARIANT SUBSPACE

A closed subspace M of L^2 is said to be invariant if

$$z_j M \subseteq M \quad \text{for } j = 1, 2.$$

In this section we study invariant subspaces M which are singly generated, that is, $M = [vH^\infty]$ for some $v \in L^2$. Then we can give an expression in terms of w of $S(w)$.

Let w be a nonnegative function in L^1 and $H^2(w)$ the $L^2(w dm)$ -closure of $\mathcal{P}$. It is easy to see that

$$w^{1/2} H^2(w) = [w^{1/2} H^\infty] \subset L^2$$

and

$$\text{dist}(w^{1/2}, [w^{1/2} H_0^\infty]) = S(w)^{1/2}.$$

Let w_2 be a nonnegative function in L_-^1 . Then

$$H^2(w_2) = [1] \oplus H_0^2(w_2)$$

where $H_0^2(w_2)$ is the $L^2(w_2 dm)$ -closure of $\mathcal{P}_0$. We can expect that $w_2^{1/2} H^2(w_2) = [w_2^{1/2} \mathcal{P}]$ has a simple structure.

Conjecture. If M is a singly generated invariant subspace and $S_0 = M \ominus [\mathcal{P}_0 M] \neq [0]$, then S_0 is one dimension and contains a cyclic vector.

Proposition 6. Suppose the conjecture is true for a singly generated invariant subspace M of L^2 with $S_0 \neq [0]$.

(1) $M = qw_2^{1/2}H^2(w_2)$ where q is unimodular and $w_2 \in L_-^1$.

(2) Let g be a nonzero function in L^2 . If M is generated by g then

$$|g|^2 = |h|^2 w_2$$

where $w_2 \in L_-^1$ and h is a generator for $H^2(w_2)$.

(3) Let w be a nonnegative function in L^1 . If M is generated by $w^{1/2}$ then

$$S(w) = \left| \int h w_2 dm \right|^2 \left(\int w_2 dm \right)^{-1}$$

where $w_2 \in L_-^1$, h is a generator for $H^2(w_2)$ and $w = |h|^2 w_2$.

Proof. (1) Suppose $S_0 = [u]$. Since u is orthogonal to $u\mathcal{P}_0$, $|u|^2 \in L_-^1$. Put

$$q(x) = \begin{cases} u(x)/|u(x)| & \text{if } u(x) \neq 0, \\ 1 & \text{if } u(x) = 0, \end{cases}$$

and $w_2 = |u|^2$, then $u = qw_2^{1/2}$. Since M is generated by u ,

$$M = [qw_2^{1/2}\mathcal{P}] = qw_2^{1/2}H^2(w_2).$$

(2) is clear by (1), (3). By (2), putting $g = w^{1/2} w = |h|^2 w_2$. Hence

$$\begin{aligned} S(w) &= \inf \left\{ \int |h - hf|^2 w_2 dm; f \in \mathcal{P}_0 \right\} \\ &= \left| \int h w_2 dm \right|^2 \left(\int w_2 dm \right)^{-1}. \end{aligned}$$

In (3) of Proposition 6 if $w_2^{-1} \in L^\infty$ then $S(w) = \left| \int h dm \right|^2 \int w_2 dm$. If w_2 is in L^∞ then putting $w_1 = |h|^2$

$$S(w) = \exp \int \log w_1 dm \int w_2 dm.$$

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