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ON TOPOLOGICAL CLASSIFICATION OF FUNCTION SPACES $C_p(X)$ OF LOW BOREL COMPLEXITY

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ABSTRACT. We prove that if X is a countable nondiscrete completely regular space such that the function space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set, then $C_p(X)$ is homeomorphic to σ^∞, where $\sigma = \{(x_i) \in \mathbf{R}^\infty : x_i = 0$ for all but finitely many $i\}$. As an application we answer in the negative some problems of A. V. Arhangel′skiĭ by giving examples of countable completely regular spaces X and Y such that X fails to be a b_R-space and a k-space (and hence X is not a k_ω-space and not a sequential space) and Y fails to be an $\aleph_0$-space while the function spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic to $C_p(\mathfrak{X})$ for the compact metric space $\mathfrak{X} = \{0\} \cup \{n^{-1} : n = 1, 2, \dots\}$.

1. INTRODUCTION

For a space X we define $C_p(X)$ to be the set of all continuous real valued functions on X endowed with the topology of pointwise convergence. The subspace of $C_p(X)$ consisting of all bounded functions is denoted by $C_p^*(X)$. This paper is devoted to the topological classification of $C_p(X)$ and $C_p^*(X)$ for countable completely regular spaces X. Let us note that if X is nondiscrete, then $C_p(X)$ is a dense linear subspace of the countable cartesian product of real lines $\mathbf{R}^x$ (identified with $\mathbf{R}^\infty$), otherwise $C_p(X) = \mathbf{R}^\infty$ or $\mathbf{R}^k$. In [DGM] it was proved that for every countable metrizable nondiscrete space X the spaces $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ^∞, where $\sigma = \{(x_i) \in \mathbf{R}^\infty : x_i = 0$ for all but finitely many $i\}$ (cf. [vM, BGvM, BGvMP]). Extending the work of [DGM] we focus on the case when $C_p(X)$ is an absolute Borel set. The main result of this paper is the following

1.1. **Theorem.** *Let X be a countable nondiscrete completely regular space such that the function space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set. Then $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ^∞.*

Since, for a countable metrizable space X, the space $C_p(X)$ is an absolute $F_{\sigma\delta}$-set Theorem 1.1 generalizes the result of [DGM]. According to [DGLvM], $C_p(X)$ cannot be an absolute $G_{\delta\sigma}$-set, provided that X is nondiscrete. Thus

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Theorem 1.1 gives a complete topological classification of spaces $C_p(X)$ which are absolute Borel sets of the class not higher than 2. To the best of our knowledge there is no classification result for spaces $C_p(X)$ of higher Borel complexity. Let us mention that all multiplicative classes of Borel sets $\mathfrak{M}_\alpha$, where $\alpha \geq 1$, are represented among spaces $C_p(X)$ (see [LvMP, Ca_1]). We conjecture that the Borel class determines the topological type of a space $C_p(X)$.

The essential step in classifying spaces $C_p(X)$ is the case of countable spaces X which have exactly one nonisolated point. Such X are precisely the spaces $\mathbf{N}_F = \{\infty\} \cup \{0, 1, 2, \dots\}$ topologized by isolating the points of $\mathbf{N} = \{0, 1, 2, \dots\}$ and by using the family $\{A \cup \{\infty\} : A \in F\}$ as a neighborhood base at ∞, where F is a filter on $\mathbf{N}$. We recall that a family $F \subset 2^Y$ is a filter on a set Y if $\varnothing \notin F$, $A \cap B \in F$ provided $A, B \in F$ and $A \subseteq C \subseteq Y$, $A \in F$ implies $C \in F$. We say that filters F on a set Y and G on a set Z are isomorphic if there exists a bijection $\alpha : Y \to Z$ such that $A \in F$ iff $\alpha(A) \in G$. By $\mathfrak{F}$ we denote the filter on $\mathbf{N}$ consisting of all cofinite subsets of $\mathbf{N}$. Obviously, the space $\mathbf{N}_\mathfrak{F}$ is homeomorphic to the space $\mathfrak{X} = \{0\} \cup \{n^{-1} : n = 1, 2, \dots\} \subseteq \mathbf{R}$. The spaces $C_p(\mathbf{N}_F)$ can have arbitrarily high Borel complexity; furthermore they may not be Borelian (see [LvMP]). Corollary 3.6 and Theorem 8.8 seem to be useful in classifying general spaces $C_p(\mathbf{N}_F)$ and they are motivated by the $F_{\sigma\delta}$-case.

Here, similarly as in [DGM], we employ the method of absorbing sets but we do it in a more implicit way. We also explain basic facts on first category filters.

We also discuss two examples of filters F and G such that $\mathbf{N}_F$ fails to be a b_R-space and a k-space (and hence $\mathbf{N}_F$ is not a k_ω-space and not a sequential space) and $\mathbf{N}_G$ fails to be an $\aleph_0$-space while the function spaces $C_p(\mathbf{N}_F)$ and $C_p(\mathbf{N}_G)$ are absolute $F_{\sigma\delta}$-sets and according to Theorem 1.1 are homeomorphic to $C_p(\mathbf{N}_\mathfrak{F})$, for the compact metric space $\mathbf{N}_\mathfrak{F}$. This solves in the negative several problems of A. V. Arhangel′skiĭ from $[\mathrm{Ar}_{1,2}]$.

2. First category filters

As usual $2^\mathbf{N}$ denotes the set of all subsets of $\mathbf{N}$. If $A, S \in 2^\mathbf{N}$, then we write $V(A, S) = \{C \in 2^\mathbf{N} : C \cap S = A \cap S\}$. We will consider the space $2^\mathbf{N}$ endowed with the topology generated by all sets $V(A, S)$ for finite S. Obviously, $2^\mathbf{N}$ can be identified with the Cantor set $\{0, 1\}^\mathbf{N}$. In this notation $\mathbf{N}$ can be replaced by any infinite countable set.

The following two lemmas are inspired by [LM, Theorems 4.6, 5.1, and 6.3]. Their proofs presented below are slight modifications of the reasoning from [LM] and they do not use the language of the game theory employed there.

2.1. **Lemma.** *Let $\{G_n\}$ be a decreasing sequence of open dense subsets of $2^\mathbf{N}$. Then for each finite tuple $(i_1, i_2, \dots, i_k)$ of elements of $\mathbf{N}$ we can assign a finite subset $S(i_1, i_2, \dots, i_k)$ of $\mathbf{N}$ such that*

(1) *the families* $\{S(j)\}_{j=1}^{\infty}$ *and* $\{S(i_1, i_2, \dots, i_k, j)\}_{j=1}^{\infty}$ *are pairwise disjoint,*

(2) *for every sequence* $\{i_k\}_{k=1}^{\infty}$ *there exists* $C \in \bigcap_{n=1}^{\infty} G_n$ *such that* $C \supseteq \mathbf{N} \setminus \bigcup_{k=1}^{\infty} S(i_1, i_2, \dots, i_k)$.

Proof. We will construct the required sequence of finite sets $\{S(i_1, i_2, \dots, i_k)\}$ inductively on k. Simultaneously, we will construct a sequence of subsets of $\mathbf{N}$, $\{C(i_1, i_2, \dots, i_k)\}$. We will use the following notation:

$$R(i_1, i_2, \dots, i_k) = \bigcup_{j=0}^{i_k} S(i_1, i_2, \dots, i_{k-1}, j)$$

and

$$T(i_1, i_2, \dots, i_k) = R(i_1) \cup R(i_1, i_2) \cup \cdots \cup R(i_1, i_2, \dots, i_k).$$

We require that the sequences $\{S(i_1, i_2, \dots, i_k)\}$, $\{R(i_1, i_2, \dots, i_k)\}$, and $\{T(i_1, i_2, \dots, i_k)\}$ together with $\{C(i_1, i_2, \dots, i_k)\}$ satisfy the following conditions indexed by k:

(a_1) $S(0) = \varnothing$ and $C(0) = \mathbf{N}$,

(b_1) $V(C(i_1), R(i_1)) \subseteq G_1 \cap V(C(0), R(i_1 - 1))$ and $S(i_1) \cap R(i_1 - 1) = \varnothing$ for $k = 1$ and $i_1 = 1, 2, \dots$, and

(a_k) the family $S(0), S(1), \dots, S(i_1)$, $S(i_1, 0), S(i_1, 1), \dots, S(i_1, i_2)$, $\dots, S(i_1, i_2, \dots, i_{k-1}, 0)$, $S(i_1, i_2, \dots, i_{k-1}, 1), \dots, S(i_1, i_2, \dots, i_k)$ is pairwise disjoint,

(b_k)

$$\begin{aligned} &V(C(i_1, i_2, \dots, i_{k-1}, 0), T(i_1, i_2, \dots, i_{k-1}, 0)) \\ &\quad \subseteq G_k \cap V(C(i_1, i_2, \dots, i_{k-1}), T(i_1, i_2, \dots, i_{k-1})), \end{aligned}$$

(c_k)

$$\begin{aligned} &V(C(i_1, i_2, \dots, i_{k-1}, i_k), T(i_1, i_2, \dots, i_{k-1}, i_k)) \\ &\quad \subseteq G_k \cap V(D, T(i_1, i_2, \dots, i_{k-1}, i_k - 1)), \end{aligned}$$

where $D = C(i_1, i_2, \dots, i_{k-1}) \cup R(i_1, i_2, \dots, i_{k-1}, i_k - 1)$ if $i_k > 0$ for $k > 1$ and for every k-tuple $(i_1, i_2, \dots, i_k)$.

Let us assume that our construction is completed for some k. Now we shall describe the construction for an arbitrary $(k+1)$-tuple. Fix a k-tuple $(i_1, i_2, \dots, i_k)$. First we choose $S(i_1, i_2, \dots, i_k, 0)$ and $C(i_1, i_2, \dots, i_k, 0)$ in $2^{\mathbf{N}}$ such that $S(i_1, i_2, \dots, i_k, 0)$ is finite and the conditions (a_{k+1}) and (b_{k+1}) are satisfied. The $S(i_1, i_2, \dots, i_k, i_{k+1})$ and $C(i_1, i_2, \dots, i_k, i_{k+1})$ can be found to satisfy (a_{k+1}) and (c_{k+1}).

Now, for every sequence $\{i_k\}_{k=1}^{\infty}$ we have

$$C(i_1, i_2, \dots, i_n) \cap T(i_1, i_2, \dots, i_n) = C(i_1, i_2, \dots, i_m) \cap T(i_1, i_2, \dots, i_n)$$

for $n = 1, 2, \dots$ and $m \geq n$. By our construction, if

$$C = \left(\mathbf{N} \setminus \bigcup_{k=1}^{\infty} S(i_1, i_2, \dots, i_k)\right) \cap \bigcup_{k=1}^{\infty} (S(i_1, i_2, \dots, i_k) \cap C(i_1, i_2, \dots, i_k)),$$

then

$$C \cap \left(\bigcup_{k=1}^{\infty} T(i_1, i_2, \dots, i_k) \right) = \bigcup_{k=1}^{\infty} (C(i_1, i_2, \dots, i_k) \cap T(i_1, i_2, \dots, i_k))$$

and hence the set C satisfies condition (2).

2.2. **Lemma.** *Let F be a family of subsets of* $\mathbf{N}$ *with the property that $A \subseteq B$, $A \in F$ implies $B \in F$. Then the following conditions are equivalent:*

(a) *F is a first category subset of $2^{\mathbf{N}}$.*

(b) *There exists a matrix $\{A(n, m): n, m = 1, 2, \dots\}$ of finite subsets of* $\mathbf{N}$ *such that each row $\{A(n, m): m = 1, 2, \dots\}$ is pairwise disjoint and for every sequence $\{m(n): n = 1, 2, \dots\}$ and every $A \in F$ we have $A \cap \bigcup_{n=1}^{\infty} A(n, m(n)) \neq \varnothing$.*

(c) *There exists a matrix $\{A(n, m): n, m = 1, 2, \dots\}$ of pairwise disjoint finite subsets of* $\mathbf{N}$ *such that for every sequence $\{m(n): n = 1, 2, \dots\}$ and every $A \in F$ we have $A \cap \bigcup_{n=1}^{\infty} A(n, m(n)) \neq \varnothing$.*

Proof. (a) $\Rightarrow$ (b). Since F is a first category subset of $2^{\mathbf{N}}$ there exists a decreasing sequence of open dense subsets $\{G_n\}_{n=1}^{\infty}$ of $2^{\mathbf{N}}$ such that $\bigcap_{n=1}^{\infty} G_n \cap F = \varnothing$. Let $\{S(i_1, i_2, \dots, i_k)\}$ be the family of finite sets satisfying (1) and (2) of Lemma 2.1. The entries of a required matrix will be just the sets $S(i_1, i_2, \dots, i_k)$. We set $A(1, m) = S(m)$ and let each family

$$\{S(i_1, i_2, \dots, i_k, m): m = 1, 2, \dots\}$$

form a row $\{A(n, m): m = 1, 2, \dots\}$. Assume that the matrix $\{A(n, m): n, m = 1, 2, \dots\}$ fails to satisfy (b). Then there exists a sequence $\{i_k\}$ and $A \in F$ such that $A \cap \bigcup_{n=1}^{\infty} S(i_1, i_2, \dots, i_k) = \varnothing$. By (2) of Lemma 2.1, there exists $C \in \bigcup_{n=1}^{\infty} G_n$ such that $\mathbf{N} \setminus \bigcup_{n=1}^{\infty} S(i_1, i_2, \dots, i_k) \subseteq C$. Therefore $A \subseteq C$, yielding $C \in F \cap \bigcap_{n=1}^{\infty} G_n$, a contradiction.

(b) $\Rightarrow$ (c). It is enough to find sequences $\{m(n, j): j = 1, 2, \dots\}$ such that the family $\{A(n, m(n, j)): n, j = 1, 2, \dots\}$ is pairwise disjoint. Set $m(1, 1) = 1$ and assume $\{m(n, j): n + j \leq p\}$ has been constructed. For $l \in \mathbf{N}$ such that $1 \leq l \leq p$ we pick $m(l, p + 1 - 1)$ to be the first index such that $A(l, m(l, p + 1 - l))$ is disjoint with the following finite set:

$$\bigcup \{A(n, m(n, j)): n < 1 \text{ and } n + j = p + 1\} \cup \bigcup \{A(n, m(n, j)): n + j \leq p\}.$$

(c) $\Rightarrow$ (a). Condition (c) is equivalent to the following one: for every $A \in F$ there exists n such that $A(n, m) \cap A \neq \varnothing$ for $m = 1, 2, \dots$. Write $X_n = \{C \in 2^{\mathbf{N}}: \forall_m A(n, m) \cap C \neq \varnothing\}$. It follows that $F \subseteq \bigcup_{n=1}^{\infty} X_n$. Moreover, each set X_n is a closed boundary subset of $2^{\mathbf{N}}$.

Let us recall that a filter F on $\mathbf{N}$ is said to be free if $\bigcap_{A \in F} A = \varnothing$. Obviously, a filter F is free iff F is dense in $2^{\mathbf{N}}$, and iff $\mathfrak{F} \subseteq F$.

2.3. **Lemma.** *Let F be a free filter on* $\mathbf{N}$. *Then the following conditions are equivalent:*

(i) *F is an element of the sigma-algebra generated by the open subsets and the first category subsets of $2^{\mathbf{N}}$.*

(ii) *F is a first category subset of $2^{\mathbf{N}}$.*

Proof. It is enough to show the implication (i) $\Rightarrow$ (ii). By the assumption $F = (U \setminus X) \cup Y$, where U is an open subset of $2^{\mathbf{N}}$ and both X and Y are first category subsets of $2^{\mathbf{N}}$. Assume that F is not a first category subset of $2^{\mathbf{N}}$. Then $U \neq \emptyset$, and hence there exists $C \in U$ and $l \in \mathbf{N}$ such that $V(C, S) \subseteq U$, where $S = \{1, 2, \dots, l\}$. Write $\mathbf{N}_0 = \mathbf{N} \setminus S$ and let $F_0 = \{A \cap \mathbf{N}_0 : A \in F\}$. Since F is dense in $2^{\mathbf{N}}$, F_0 is a dense filter on $\mathbf{N}_0$. Moreover, since F contains a dense G_δ subset of U, F_0 contains a dense G_δ subset of $2^{\mathbf{N}_0}$. Let $\xi : 2^{\mathbf{N}_0} \to 2^{\mathbf{N}_0}$ be the homeomorphism assigning to each C the set $\mathbf{N}_0 \setminus C$. By the filter property we get $\xi(F_0) \cap F_0 = \emptyset$. Consequently, $2^{\mathbf{N}_0}$ contains two disjoint dense G_δ subsets which contradicts the Baire category theorem.

2.4. **Proposition.** *For every filter F on $\mathbf{N}$ and every decomposition $\mathbf{N} = \bigcup_{i=1}^{\infty} \mathbf{N}_i$ of $\mathbf{N}$ into infinite pairwise disjoint sets $\mathbf{N}_i$ we write $F_i = \{A \cap \mathbf{N}_i : A \in F\}$. Then we have*

(1) *for every $A_i \in F_i$, $i = 1, 2, \dots, l$, $\bigcup_{i=1}^{l} A_i \cup \bigcup_{i>l} \mathbf{N}_i \in F$,*

(2) *F_i embeds as a closed subset of F,*

(3) *if, in addition, F is a free first category filter, then there exists a decomposition $\mathbf{N} = \bigcup_{i=1}^{\infty} \mathbf{N}_i$ such that each F_i is a free filter on $\mathbf{N}_i$.*

Proof. To verify (1), observe that $\bigcap_{i=1}^{l} \widetilde{A}_i \subseteq \bigcup_{i=1}^{l} A_i \cup \bigcup_{i>l} \mathbf{N}_i$, where $\widetilde{A}_i \in F$ and $\widetilde{A}_i \cap \mathbf{N}_i = A_i$. The map which assigns to each $A \in F_i$ the set $A \cup (\mathbf{N} \setminus \mathbf{N}_i)$ is a closed embedding of F_i into F. To prove the last assertion, pick from Lemma 2.2(c) a matrix $\{A(n, m) : n, m = 1, 2, \dots\}$ of pairwise disjoint finite sets such that $\bigcup \{A(n, m) : n = 1, 2, \dots\} \cap A \neq \emptyset$ for $A \in F$ and $m = 1, 2, \dots$. We let $\mathbf{N}_i = \bigcup_{m=1}^{\infty} A(m, i)$ for $i \geq 2$ and $\mathbf{N}_1 = \mathbf{N} \setminus \bigcup_{i=2}^{\infty} \mathbf{N}_i$.

3. Z_σ-PROPERTY OF FUNCTION SPACES $C_p(X)$

We recall that a closed subset X of an absolute neighborhood retract M is a Z-set if every map $f : K \to M$ of a compactum K into M can be approximated by maps $\overline{f} : K \to M \setminus X$. A space which is a countable union of its own Z-sets is called a Z_σ-space. In this section we prove that some spaces $C_p(X)$ (and their subspaces) are Z_σ-spaces. We will need the following well-known fact about Z_σ-spaces.

3.1. **Fact.** *Let X and Y be dense linear subspaces of $\mathbf{R}^\infty$ such that $X \subseteq Y$ and Y is a Z_σ-space. Then X is a Z_σ-space.*

3.2. **Lemma.** *For every completely regular infinite countable space X the function space $C_p^*(X)$ is a Z_σ-space.*

Proof. We identify X with $\mathbf{N}$. Then $C_p^*(X)$ is a dense linear subspace of $\mathbf{R}^\infty$ which is contained in the subspace $\mathbf{R}^\infty_{\mathrm{bd}} = \{(x_n) \in \mathbf{R}^\infty : \sup |x_n| < \infty\}$. The space $\mathbf{R}^\infty_{\mathrm{bd}}$ is a Z_σ-space. By 3.1 the space $C_p^*(X)$ is a Z_σ-space.

Throughout the paper we use the following subspaces of $\mathbf{R}^\infty$. If $F \subseteq 2^{\mathbf{N}}$, then

$c_F = \{(x_n) \in \mathbf{R}^\infty : \forall_{\varepsilon>0} \exists_{A \in F} |x_n| \le \varepsilon \text{ for all } n \in A\}$,
$c_F^* = \{(x_n) \in c_F : \sup_{n\ge1} |x_n| < \infty\}$, and
$B_F(r) = \{(x_n) \in c_F : \sup_{n \ge 1} |x_n| \le r\}$, where $r > 0$.

3.3. **Proposition.** *Let F be a family of subsets of $\mathbf{N}$ such that $\mathfrak{F} \subseteq F$ and $A \subseteq B$, $A \in F$ implies $B \in F$. If F as a subset of $2^{\mathbf{N}}$ is of the first category, then the sequence spaces c_F, c_F^*, and $B_F(r)$ are Z_σ-spaces.*

Proof. By (b) of Lemma 2.2 there exists a matrix $\{A(n, m) : n, m = 1, 2, \ldots\}$ of pairwise disjoint finite subsets of $\mathbf{N}$ so that for every $A \in F$ there exists n such that $A(n, m) \cap A \neq \varnothing$ for all m. Fix $r > 0$. Let $X_n(r) = \{(x_i) \in c_F : \forall_m \exists_{k \in A(n,m)} |x_k| \le r/2\}$ for $n = 1, 2, \ldots$. Clearly, each $X_n(r)$ is a closed subset of c_F and $c_F = \bigcup_{n=1}^\infty X_n(r)$, $c_F^* = \bigcup_{n=1}^\infty X_n(r) \cap c_F^*$ and $B_F(r) = \bigcup_{n=1}^\infty X_n(r) \cap B_F(r)$. Fix $n \in \mathbf{N}$. We shall show that the sets $X_n(r)$, $X_n(r) \cap c_F^*$, and $X_n(r) \cap B_F(r)$ are Z-sets in the suitable spaces. Let $f : K \to c_F$ be a map of a compactum K. Let, for $m = 1, 2, \ldots$, $g_m : \mathbf{R}^\infty \to \mathbf{R}^\infty$ be the map defined by $g_m((x_i)) = (y_i)$, where $y_i = 0$ for $i > \max A(n, m)$, $y_i = r$ for $i \in A(n, m)$, and $y_i = x_i$ otherwise. If m is sufficiently large then the map $\overline{f} = g_m f : K \to \mathbf{R}^\infty$ closely approximates f and satisfies $\overline{f}(K) \cap X_n(r) = \varnothing$. Since $\mathfrak{F} \subseteq F$ we additionally have $\overline{f}(K) \subseteq c_F$; moreover, if $f(K) \subseteq c_F^*$ or $f(K) \subseteq B_F(r)$ then also $\overline{f}(K) \subseteq c_F^*$ or $\overline{f}(K) \subseteq B_F(r)$, respectively. The proof is complete.

3.4. **Corollary.** *For every free Borelian filter on $\mathbf{N}$ the spaces c_F, c_F^*, and $B_F(r)$, $r > 0$, are Z_σ-spaces.*

Proof. Follows immediately from 2.3 and 3.3.

It is standard that for every filter F the spaces $C_p(\mathbf{N}_F)$ and $C_p^*(\mathbf{N}_F)$ are linearly isomorphic to the products $\mathbf{R} \times c_F$ and $\mathbf{R} \times c_F^*$, respectively.

3.5. **Corollary.** *For every free first category filter F on $\mathbf{N}$ the space $C_p(\mathbf{N}_F)$ is a Z_σ-space.*

Proof. By Proposition 3.3 the space c_F and consequently the product $\mathbf{R} \times c_F$ are Z_σ-spaces. Thus $C_p(\mathbf{N}_F)$, being homeomorphic to $\mathbf{R} \times c_F$, is a Z_σ-space.

3.6. **Corollary.** *Let X be a countable nondiscrete completely regular space such that the space $C_p(X)$ is analytic (i.e., a continuous image of the space of irrationals). Then $C_p(X)$ is a Z_σ-space.*

Proof. Let $a \in X$ be an accumulation point, $Y = X \backslash \{a\}$, and $F = \{A \subseteq Y : a$ is an interior point of $A \cup \{a\}\}$. Then F is a free filter on Y. We shall prove that F is an analytic subset of 2^Y. Let us recall that a set which is simultaneously closed and open is called clopen. First observe that the set $G = \{B \subseteq X : B$ is clopen in X and $a \in B\}$ is analytic in 2^X since it can be identified with a closed subset $\{f \in C_p(X) : f(X) \subseteq \{0, 1\}$ and $f(a) = 0\}$ of

$C_p(X)$. The set $\overline{G} = \{(A, B) \in 2^Y \times 2^X : B \in G \text{ and } B \subseteq A \cup \{a\}\}$ is analytic in $2^Y \times 2^X$ and because X is zero-dimensional F is an image of $\overline{G}$ by the projection onto the first axis. Thus F is analytic in 2^Y.

Now, by [Ku, §39] and Lemma 2.3, F is the first category subset of 2^Y. Obviously, $C_p(X) \subseteq E$, where

$$E = \{f \in \mathbf{R}^X : \forall_{\varepsilon>0} \exists_{A\in F} \forall_{x\in A} |f(a) - f(x)| < \varepsilon\}.$$

The space E can be identified with $C_p(\mathbf{N}_F)$. Hence, by Corollary 3.5, the space E is a Z_σ-space. By 3.1 the space $C_p(X)$ is a Z_σ-space.

4. Borel complexity of function spaces $C_p(X)$

For a countable ordinal α, $\mathfrak{M}_\alpha$ and $\mathfrak{A}_\alpha$ denote the class of all absolute Borel sets of the multiplicative and additive class α, respectively. If $\alpha \geq 2$, then there exists a filter on $\mathbf{N}$ which belongs to $\mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$ (see [LvMP], cf. [Ca$_2$]). The filter $\mathfrak{F}$ is in the class $\mathfrak{A}_1 \backslash \mathfrak{M}_1$. It is an easy observation that there exist no filters in the class $\mathfrak{M}_1 \backslash \mathfrak{A}_1$ (see [Ca$_1$]). For every nonempty subset $A \subseteq \mathbf{N}$ the filter $F(A) = \{B \subseteq \mathbf{N} : A \subseteq B\}$ is a compact set in $2^{\mathbf{N}}$ and hence $F(A) \in \mathfrak{M}_0$. Moreover, every compact filter is of the form $F(A)$ for $\emptyset \neq A \subseteq \mathbf{N}$.

A filter F is an absolute Borel set (shortly a Borelian filter) iff the space c_F is an absolute Borel set. Moreover, Borel complexity of c_F heavily depends on F and vice versa. Namely, we have

4.1. **Lemma.** *For every filter F there exists a closed embedding of F into the space c_F.*

Proof. The map sending each $A \in F$ onto $\kappa_A \in c_F$, where $\kappa_A(i) = 0$ for $i \in A$ and $\kappa_A(i) = 1$ for $i \notin A$, is a required closed embedding.

4.2. **Lemma.** *Let F be a filter on $\mathbf{N}$ and let α be a countable ordinal, $\alpha \geq 1$. Then*:

(1) *if $F \in \mathfrak{M}_\alpha$, then $c_F \in \mathfrak{M}_\alpha$,*
(2) *if $F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$, then $c_F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$,*
(3) *if $F \in \mathfrak{M}_\alpha \cap \mathfrak{A}_\alpha \backslash \bigcup_{\beta<\alpha}(\mathfrak{A}_\beta \cup \mathfrak{M}_\beta)$, then $c_F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$,*
(4) *if $F \in \mathfrak{A}_\alpha \backslash \mathfrak{M}_\alpha$, then $c_F \in \mathfrak{M}_{\alpha+1} \backslash \mathfrak{A}_{\alpha+1}$.*

Proof. The assertions of this lemma (except for (3)) were proved in [Ca$_{1,2}$]. For the sake of completeness we include our proof of Lemma 4.2.

(1) We will present here a slight simplification of the original proof of (1). Suppose $F \in \mathfrak{M}_\alpha$ for some countable ordinal α. Write

$$T = \mathbf{R} \backslash \left(\bigcup_{n=1}^{\infty} \left(\frac{1}{n} - 4^{-n}, \frac{1}{n} + 4^{-n} \right) \cup \bigcup_{-n=1}^{\infty} \left(\frac{1}{n} - 4^{n}, \frac{1}{n} + 4^{n} \right) \right)$$

and let $f_k : T^\infty \to 2^{\mathbf{N}}$ be defined by $f_k((x_i)) = \{i \in \mathbf{N} : |x_i| < 1/k\}$ for $(x_i) \in T^\infty$ and $k = 1, 2, \ldots$. Then the maps f_k are continuous and $T^\infty \cap c_F =$

$\bigcap_{k=1}^{\infty} f_k^{-1}(F)$. Thus $T^{\infty} \cap c_F \in \mathfrak{M}_{\alpha}$. Let $g: T \to \mathbf{R}$ be a linear extension of the map sending $\frac{1}{n} - 4^{-n}$, $\frac{1}{n} + 4^{-n}$ onto $\frac{1}{n}$ for $n = 1, 2, \dots$, and $\frac{1}{n} - 4^{n}$, $\frac{1}{n} + 4^{n}$ onto $\frac{1}{n}$ for $-n = 1, 2, \dots$. Then the map $g^{\infty}: T^{\infty} \to \mathbf{R}^{\infty}$ defined by $g^{\infty}((x_i)) = (g(x_i))$ is a proper surjection with $(g^{\infty})^{-1}(c_F) = T^{\infty} \cap c_F$ (let us recall that a map $f: X \to Y$ is proper if $f^{-1}(K)$ is compact whenever K is a compact subset in Y). Now, the result of [SR] implies that $c_F \in \mathfrak{M}_{\alpha}$.

(2) This is a consequence of (1) and Lemma 4.1.

(3) We shall use Calbrix's argument of $[\mathrm{Ca}_{1,2}]$. Let F be a filter on $\mathbf{N}$ such that $F \in \mathfrak{M}_{\alpha} \cap \mathfrak{A}_{\alpha} \setminus \bigcup_{\beta<\alpha}(\mathfrak{A}_{\beta} \cup \mathfrak{M}_{\beta})$. Express $\mathbf{N}$ as a union of pairwise disjoint families of infinite sets $\mathbf{N}_i = \{n_{i,1}, n_{i,2}, \dots\}$. Define a new filter F^{∞} consisting of all sets of the form $\bigcup_{i=1}^{\infty} A_i$, where $A_i \in F_i$ and F_i is an isomorphic copy of F on $\mathbf{N}_i$. First we show that $F^{\infty} \in \mathfrak{M}_{\alpha} \setminus \mathfrak{A}_{\alpha}$. It is enough to observe that for every $C \subseteq 2^{\mathbf{N}}$ such that $C \in \mathfrak{M}_{\alpha}$ there exists a continuous map $f: 2^{\mathbf{N}} \to \prod_{i=1}^{\infty} 2^{\mathbf{N}_i} = 2^{\mathbf{N}}$ with $f^{-1}(F^{\infty}) = C$. Let $C = \bigcap_{i=1}^{\infty} C_i$, where $C_i \in \bigcup_{\beta<\alpha} \mathfrak{A}_{\beta}$ for $i = 1, 2, \dots$. Fix $i \geq 1$. By the "Wadge lemma" either there exists a continuous map $f_i: 2^{\mathbf{N}} \to 2^{\mathbf{N}_i}$ such that $f_i^{-1}(F_i) = C_i$ or there exists a continuous map $g_i: 2^{\mathbf{N}_i} \to 2^{\mathbf{N}}$ such that $g_i^{-1}(2^{\mathbf{N}} \setminus C_i) = F_i$. Since $F_i \in \mathfrak{M}_{\alpha} \cap \mathfrak{A}_{\alpha} \setminus \bigcup_{\beta<\alpha}(\mathfrak{A}_{\beta} \cup \mathfrak{M}_{\beta})$ and $C_i \in \bigcup_{\beta<\alpha} \mathfrak{A}_{\beta}$ the map g_i cannot exist. Let $f: 2^{\mathbf{N}} \to \prod_{i=1}^{\infty} 2^{\mathbf{N}_i}$ be the map defined by $f(x) = (f_1(x), f_2(x), \dots)$, where $f_i: 2^{\mathbf{N}} \to 2^{\mathbf{N}_i}$ satisfies $f_i^{-1}(F_i) = C_i$ for $i = 1, 2, \dots$. Now, by (2), we get $c_{F^{\infty}} \in \mathfrak{M}_{\alpha} \setminus \mathfrak{A}_{\alpha}$. The map $\phi: \mathbf{R}^{\infty} \to \mathbf{R}^{\infty}$ given by the formula $\phi((x_{n_{i,j}})) = (\sum_{i=1}^{\infty} 2^{-i-1} |x_{n_{i,j}}| (1 + |x_{n_{i,j}}|)^{-1})_{j=1}^{\infty}$ has the property that $\phi^{-1}(c_F) = c_{F^{\infty}}$. Hence $c_F \notin \mathfrak{A}_{\alpha}$. By (1), $c_F \in \mathfrak{M}_{\alpha}$.

(4) We can repeat the proof of (3) (cf., $[\mathrm{Ca}_{1,2}]$).

Let X be a countable completely regular space and let a be an accumulation point of X. Then

$$F_a = \{A \in 2^{X \setminus \{a\}} : a \text{ is an interior point of } A \cup \{a\}\}$$

is a free filter on $X \setminus \{a\}$.

4.3. **Lemma.** *Let X be a countable completely regular space such that for every accumulation point $a \in X$ the filter $F_a \in \mathfrak{M}_{\alpha}$. Then $C_p(X) \in \mathfrak{M}_{\alpha}$.*

Proof. Let a be an accumulation point of X and let X_{F_a} denote the space X topologized by isolating points of $X \setminus \{a\}$ and by using the family $\{A \cup \{a\} : A \in F_a\}$ as a neighborhood base at a. The spaces $C_p(X)$ and $C_p(X_{F_a})$ are linear dense subspaces of $\mathbf{R}^X$. By Lemma 4.2 $C_p(X_{F_a}) \in \mathfrak{M}_{\alpha}$. Since

$$C_p(X) = \bigcap \{C_p(X_{F_a}) : a \text{ is an accumulation point of } X\}$$

we have $C_p(X) \in \mathfrak{M}_{\alpha}$.

5. Sequence spaces related to $F_{\sigma\delta}$-filters

In this section we give a complete topological classification of sequence spaces c_F which are absolute $F_{\sigma\delta}$-sets (i.e., $c_F \in \mathfrak{M}_2$). Since for a compact filter $F(C)$, $C \subseteq \mathbf{N}$, $c_{F(C)} = \{(x_i) \in \mathbf{R}^\infty : x_i = 0 \text{ for } i \in C\}$ and for an arbitrary filter F, $c_F \subseteq \{(x_i) \in \mathbf{R}^\infty : x_i = 0 \text{ for } i \in \bigcap_{A\in F} A\}$ we can reduce our classification to the case when the filter F is free. Obviously, a filter F is free iff the space $\mathbf{N}_F$ is completely regular.

5.1. **Proposition.** *Let F be a free filter which is an absolute $F_{\sigma\delta}$-set. Then the sequence space c_F is homeomorphic to σ^∞.*

To prove Proposition 5.1 it is enough to verify that the space c_F satisfies the assumptions of the following lemma:

5.2. **Lemma.** *Let $\{X_i\}$ be a sequence of $F_{\sigma\delta}$-absolute retracts such that, for each i, σ^∞ is embeddable onto a closed subset of X_i. Fix $p_i \in X_i$, $i = 1, 2, \ldots$. Then every $F_{\sigma\delta}$-space X which is a Z_σ-space and satisfies*

$$W(X_i, p_i) = \left\{(x_i) \in \prod_{i=1}^{\infty} X_i : x_i = p_i \text{ for all but finitely many } i\right\} \subseteq X \subseteq \prod_{i=1}^{\infty} X_i$$

is homeomorphic to σ^∞.

Proof. Lemma 5.2 is a slight modification of the characterization of σ^∞, used in [DGM, DM], which follows easily from Lemma 2.3 of [DM] and Theorem 6.5 of [BM].

We will also need the following fact proved in [DM, Corollary 2.5]:

5.3. **Lemma.** *Let X_i, for $i = 1, 2, \ldots$, be an absolute retract which is a Z_σ-space. Then the product $\prod_{i=1}^\infty X_i$ contains a closed copy of σ^∞.*

Below we present a new, more elementary, proof of Lemma 5.3 which is a consequence of the following general observation:

5.4. **Lemma.** *If X is an absolute retract which is a Z_σ-space, then for each σ-compact space A there exists a proper map $f: A \to X$.*

Proof. Assume that A is a subset of the Hilbert cube I^∞ and $A = \bigcup_{n=1}^\infty A_n$, where A_n is compact and $A_n \subseteq A_{n+1}$ for $n = 1, 2, \ldots$. Let Y be a complete absolute retract such that $X \subset Y$ and $Y \backslash X$ is locally homotopy negligible in Y, i.e., for every open family $\mathfrak{U}$ of Y and for every map $f: I^\infty \to Y$ there exists a map $g: I^\infty \to Y$ which is $\mathfrak{U}$-close to f and such that $g(f^{-1}(\bigcup \mathfrak{U})) \subseteq X$, see [T]. We shall construct a map $f: I^\infty \to Y$ with $f^{-1}(X) = A$. Then the restricted map $f|A$ is a proper map of A into X. Since X is a Z_σ-space, we can find Z-sets X_n in Y for $n = 1, 2, \ldots$ such that $X \subset \widetilde{X} = \bigcup_{n=1}^\infty X_n$ and $X_1 \subseteq X_2 \subseteq \cdots$. Fix a complete metric d on Y which is bounded by 1. Let $f_0: I^\infty \to Y \backslash \widetilde{X}$ be a map and let $A_0 = X_0 = \varnothing$. We will inductively construct a sequence of maps $f_n\colon I^\infty \to Y$ satisfying for $n = 1, 2, \dots$ the following conditions:

(i) $f_n(A_n) \subset X$,

(ii) $f_n(I^\infty \setminus A_n) \subset Y \setminus \widetilde{X}$,

(iii) $f_n | A_{n-1} = f_{n-1} | A_{n-1}$,

(iv) $d(f_n(x), f_{n-1}(x)) \le 4^{-n} d(f_{n-1}(x), X_{n-1})$.

Assume that the maps $f_i\colon I^\infty \to Y$ satisfying the conditions (i)–(iv) have been already constructed for $0 \le i \le n$. By the local homotopy negligibility of $Y \setminus X'$ there exists a map $g\colon I^\infty \to X$ such that $g|A_n = f_n|A_n$ and $d(g(x), f_n(x)) \le 4^{-n-2} d(f_n(x), X_n)$. Since $\widetilde{X}$ is a countable union of Z-sets in a complete absolute retract Y there exists a homotopy $h_t\colon Y \to Y$ such that $h_0 = \mathrm{id}_Y$, $h_t | X_n = \mathrm{id}_{X_n}$ for $0 \le t \le 1$, $h_t(Y \setminus X_n) \subset Y \setminus \widetilde{X}$ for $t > 0$, and $\operatorname{diam}\{h_t(y)\colon 0 \le t \le 1\} \le 4^{-n-2} d(y, X_n)$. We let $f_{n+1}(x) = h_{\lambda(x)}(g(x))$, where $\lambda\colon I^\infty \to [0, 1]$ is a continuous function with $\lambda^{-1}(0) = A_{n+1}$. If $x \in A_{n+1}$, then $\lambda(x) = 0$ and consequently $f_{n+1}(x) = h_0(g(x)) = g(x)$; in particular, $f_{n+1}(x) = g(x) = f_n(x)$ for $x \in A_n$. If $x \in I^\infty \setminus A_{n+1}$, then $g(x) \notin X$ and $\lambda_n(x) > 0$; hence $f_{n+1}(x) = h_{\lambda(x)}(g(x)) \in Y \setminus \widetilde{X}$. Thus f_{n+1} satisfies the conditions (i)–(iii). To show (iv) we use the following inequalities:

$$\begin{aligned} d(f_n(x), f_{n+1}(x)) &\le d(f_n(x), g(x)) + d(g(x), h_{\lambda(x)}(g(x))) \\ &\le 4^{-n-2} d(f_n(x), X_n) + 4^{-n-2} d(g(x), X_n) \\ &\le (4^{-n-2} + 4^{-n-2}(1 + 4^{-n-2})) d(f_n(x), X_n) \\ &\le 4^{-n-1} d(f_n(x), X_n). \end{aligned}$$

Now, by (iv), the sequence $\{f_n\}$ uniformly converges to a map $f\colon I^\infty \to Y$. By (i) and (iii) we get $f(A) \subset X$. To show that $f(I^\infty \setminus A) \subset Y \setminus \widetilde{X}$, we first observe that for $n > k$,

$$\begin{aligned} d(f_n(x), X_k) &\ge (1 - 4^{-n}) d(f_{n-1}, X_k) \\ &\ge \cdots \ge (1 - 4^{-n})(1 - 4^{-n+1}) \cdots (1 - 4^{-k-1}) d(f_k(x), X_k). \end{aligned}$$

Hence, we have $d(f(x), X_k) \ge \prod_{n=k+1}^\infty (1 - 4^{-n}) d(f_k(x), X_k)$. If $x \in I^\infty \setminus A$ then $d(f_k(x), X_k) > 0$. Since $\prod_{n=k+1}^\infty (1 - 4^{-n}) > 0$, $k \ge 0$, we obtain $d(f(x), X_k) > 0$ for $x \in I^\infty \setminus A$ and $k = 1, 2, \dots$; consequently $f(x) \notin X_k$ for $k = 1, 2, \dots$.

Proof of 5.3. We write $\mathbf{N} = \bigcup_{i=0}^\infty \mathbf{N}_i$, where the $\mathbf{N}_i$ are infinite and pairwise disjoint for $i = 1, 2, \dots$. Since

$$\prod_{i=1}^\infty X_i = \prod_{i=1}^\infty \left(\prod_{n \in \mathbf{N}_{2i-1}} X_n \times \prod_{n \in \mathbf{N}_{2i}} X_n \right)$$

it is enough to show that for $i = 1, 2, \ldots$ there exists a closed embedding $v_i\colon \sigma \to \prod_{n\in\mathbf{N}_{2i-1}} X_n \times \prod_{n\in\mathbf{N}_{2i}} X_n$. First let us observe that each nontrivial absolute retract contains the interval $[0, 1]$ and the infinite product of such absolute retracts contains the Hilbert cube I^∞. To obtain v_i we choose any embedding $u_i\colon \sigma \to \prod_{n\in\mathbf{N}_{2i-1}} X_n$ and a proper map from Lemma 5.4 $f_i\colon \sigma \to \prod_{n\in\mathbf{N}_{2i}} X_n$ and set $v_i = u_i \times f_i$.

We will also employ the following

5.6. **Lemma.** *For any filter F on $\mathbf{N}$, any decomposition $\mathbf{N} = \bigcup_{i=1}^\infty \mathbf{N}_i$ into pairwise disjoint infinite sets $\mathbf{N}_i$, and for the natural isomorphism $h\colon \mathbf{R}^\mathbf{N} \to \prod_{i=1}^\infty \mathbf{R}^{\mathbf{N}_i}$ we have*

(1) $W(c_{F_i}, 0) \subset h(c_F) \subset \prod_{i=1}^\infty c_{F_i}$,

(2) $W(B_{F_i}(r), 0) \subset h(B_F(r)) \subset \prod_{i=1}^\infty b_{F_i}(r)$, *for* $r > 0$, *and*

(3) $\prod_{i=1}^\infty B_{F_i}(\frac{1}{i}) \subset h(B_F(1))$,

where $F_i = \{A \cap \mathbf{N}_i\colon A \in F\}$.

Proof. The inclusions are easy consequences of the observation that $A_1 \cup A_2 \cup \cdots \cup A_k \cup \mathbf{N}_{k+1} \cup \mathbf{N}_{k+2} \cup \cdots$ belongs to F for every $A_i \in F_i$ and for arbitrary k; cf. Proposition 2.4.

Proof of Proposition 5.1. By Lemma 4.2 the space c_F is an absolute $F_{\sigma\delta}$-set. Corollary 3.4 implies that c_F is a Z_σ-space. Since F is free on $\mathbf{N}$, by (3) of Proposition 2.4, there exists a decomposition $\mathbf{N} = \bigcup_{i=1}^\infty \mathbf{N}_i$ such that each $F_i = \{A \cap \mathbf{N}_i\colon A \in F\}$ is a free $F_{\sigma\delta}$-filter on $\mathbf{N}_i$. Now, by (1) of 5.6 and 5.2, it is enough to show that c_{F_i} (equivalently, c_F) contains σ^∞ as a closed subset. The last follows from (3) of Lemma 5.6, Lemma 5.3, and Corollary 3.4.

Similarly we can prove the following

5.7. **Proposition.** *For every noncompact $F_{\sigma\delta}$-filter F on $\mathbf{N}$ the spaces c_F^* and $B_F(r)$, $r > 0$, are homeomorphic to σ^∞.*

6. Topological classification of function spaces $C_p(X)$ of type $F_{\sigma\delta}$

In this section we prove Theorem 1.1. We start with the following general fact.

6.1. **Proposition.** *Let X be a countable nondiscrete completely regular space. Then one of the following conditions holds:*

(i) *there exists a clopen subset Y of X with exactly one accumulation point.*

(ii) *there exists a decomposition $X = \bigcup_{n=1}^\infty X_n$, where $\{X_n\}_{n=1}^\infty$ is a pairwise disjoint sequence of nondiscrete clopen sets.*

Proof. Suppose that (i) does not hold. Then by induction, we construct the decomposition $X = \bigcup_{n=1}^\infty X_n$ of (ii). Let $X = \{x_1, x_2, \ldots\}$ and $X_0 = \varnothing$.

Assume that we have constructed pairwise disjoint nondiscrete clopen subsets $X_1, X_2, \dots, X_n$ of X such that $Y_n = X \setminus \bigcup_{i=1}^n X_i$ is also nondiscrete and $\{x_1, x_2, \dots, x_n\} \subseteq \bigcup_{i=1}^n X_i$. By our assumption Y_n contains at least two accumulation points. Using the fact that X is zero-dimensional, we can divide Y_n into two nondiscrete clopen sets. We choose one of them as X_{n+1} in such a way that $\{x_1, x_2, \dots, x_{n+1}\}$ is contained in $\bigcup_{i=1}^{n+1} X_i$.

The next proposition summarizes the results of the previous section.

6.2. **Proposition.** *Let F be a filter on $\mathbf{N}$. Then the following conditions are equivalent:*

(a) *$C_p(\mathbf{N}_F)$ is homeomorphic to σ^∞,*
(b) *$C_p(\mathbf{N}_F)$ is an absolute $F_{\sigma\delta}$-set and not a G_δ-set,*
(c) *The filter F is a noncompact $F_{\sigma\delta}$-subset of $2^{\mathbf{N}}$.*

The same is true for the space $C_p^*(\mathbf{N}_F)$.

Proof. The implication (a) $\Rightarrow$ (b) is well known. (b) $\Rightarrow$ (c) follows from Lemma 4.1. Finally (c) $\Rightarrow$ (a) follows from Proposition 5.1 and the fact that $C_p(\mathbf{N}_F)$ is linearly isomorphic to $\mathbf{R} \times c_F$.

Proof of Theorem 1.1. We only present a proof for the space $C_p(X)$ (the proof for the space $C_p^*(X)$ is the same). We shall consider two cases:

(1) The space X satisfies (i) of Proposition 6.1. Let Y be a clopen subset of X with exactly one accumulation point. The space Y is homeomorphic to $\mathbf{N}_F$, where F is a noncompact filter on $\mathbf{N}$. Moreover, the space $C_p(X)$ is linearly homeomorphic to $C_p(Y) \times C_p(X \setminus Y)$. By Proposition 6.2, $C_p(Y)$ is homeomorphic to σ^∞. Hence, by Corollary 5.4 of [BM], it follows that $C_p(X)$ is homeomorphic to σ^∞.

(2) The space X satisfies (ii) of Proposition 6.1. Let $X = \bigcup_{n=1}^\infty X_n$ be a decomposition of X into pairwise disjoint nondiscrete clopen sets. Now, the space $C_p(X)$ is homeomorphic to the product $\prod_{n=1}^\infty C_p(X_n)$, where all spaces $C_p(X_n)$ and $C_p(X)$ are $F_{\sigma\delta}$-absolute retracts which, according to Corollary 3.6, are Z_σ-spaces. From Lemmas 5.2 and 5.3 it follows that $C_p(X)$ is homeomorphic to σ^∞.

7. Examples of special $F_{\sigma\delta}$-filters

In this section we apply Theorem 1.1 to answer in the negative several questions posed by A. V. Arhangel′skiĭ and related to the following general problem: how close do the properties of the spaces X and Y have to be if $C_p(X)$ and $C_p(Y)$ are homeomorphic? We will discuss the properties of the spaces X and Y listed below. A Hausdorff space X is a k-space if for each $A \subseteq X$, the set A is closed in X provided that the intersection of A with any compact subspace K of X is closed in K. A k-space X is a k_ω-space if there exists a countable family $\mathfrak{K}$ of compact subsets of X such that $\bigcup \mathfrak{K} = X$ and for every compact subspace K of X there exists $L \in \mathfrak{K}$ such that $K \subseteq L$. A topological space X is called a sequential space if a set $A \subseteq X$ is closed iff together with any sequence it contains all its limits. A space X is called an $\aleph_0$-space if there exists a countable family $\mathfrak{S}$ of subsets of X such that for every compact subset $K \subseteq X$ and for every neighborhood V of K in X one can find $P \in \mathfrak{S}$ with $K \subseteq P \subseteq V$. A subset A of a space X will be called R-bounded in X, if every function $f \in C_p(X)$ is bounded on A. A function $f\colon X \to \mathbf{R}$ is called strictly b-continuous if for every R-bounded subset $A \subseteq X$ there exists a map $g \in C_p(X)$ such that $f|A = g|A$. A space X is said to be a b_R-space if every strictly b-continuous function $f\colon X \to \mathbf{R}$ is continuous.

Recall that $\mathfrak{X} = \{0\} \cup \{n^{-1}\colon n = 1, 2, \ldots\}$ and that $\mathfrak{X}$ can be identified with the space $\mathbf{N}_{\mathfrak{F}}$.

7.1. **Example.** There exists a free $F_{\sigma\delta}$-filter F on $\mathbf{N}$ such that:

(a) the space $\mathbf{N}_F$ is countable and completely regular,
(b) the function spaces $C_p(\mathbf{N}_F)$ and $C_p(\mathfrak{X})$ are homeomorphic,
(c) the space $\mathbf{N}_F$ is not a k-space, while $\mathfrak{X}$ is compact metric,
(d) the space $\mathbf{N}_F$ is not a k_ω-space,
(e) the space $\mathbf{N}_F$ is not a sequential space,
(f) the space $\mathbf{N}_F$ is not a b_R-space.

Proof. Let F be a filter of sets of density 1, i.e.,

$$F = \left\{A \subseteq \mathbf{N}\colon \lim_{n\to\infty} n^{-1}\operatorname{card}(A \cap \{1, 2, \ldots, n\}) = 1\right\},$$

where $\operatorname{card}(B)$ denotes the cardinality of a set B (cf. [AU, p. 119; V, p. 98]). Since $F = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{A \subseteq \mathbf{N}\colon k^{-1}\operatorname{card}(A \cap \{1, 2, \ldots, k\}) \geq 1 - n^{-1}\}$, F is a free $F_{\sigma\delta}$-filter on $\mathbf{N}$. The assertion (a) is obvious and (b) follows from Lemma 4.2 and Theorem 1.1. Since for every infinite $M \subseteq \mathbf{N}$ there is $A \in F$ such that $M \setminus A$ is infinite the space $\mathbf{N}_F$ does not contain any nontrivial convergent sequence. Moreover, all compact subspaces of $\mathbf{N}_F$ are finite. Hence $\mathbf{N}_F$ is not a k-space and (c) holds. The assertions (d) and (e) are immediate consequences of (c). One can easily observe that all R-bounded subsets of $\mathbf{N}_F$ are finite. Hence every function from $\mathbf{N}_F$ into $\mathbf{R}$ is strictly b-continuous. Consequently $\mathbf{N}_F$ is not a b_R-space.

Example 7.1 answers the problems 12, 24, 25, 26 of [Ar_1] and 6, 7, 26 of [Ar_2].

7.2. **Example.** There exists a free $F_{\sigma\delta}$-filter G on $\mathbf{N}$ such that:

(a) the space $\mathbf{N}_G$ is countable and completely regular,
(b) the function spaces $C_p(\mathbf{N}_G)$ and $C_p(\mathfrak{X})$ are homeomorphic,
(c) the space $\mathbf{N}_G$ is not an $\aleph_0$-space while $\mathfrak{X}$ is compact metric.

Proof. As a filter G we take one of the filters described in [LvMP]. Let 2^n be the set of all functions from $\{0, 1, \ldots, n-1\}$ into $\{0, 1\}$ for $n = 1, 2, \ldots$. Let us put $T = \bigcup_{n=1}^{\infty} 2^n$. For each function $x: \mathbf{N} \to \{0, 1\}$ we define $B_x = \{x|n \in 2^n: n = 1, 2, \ldots\}$ to be a branch in T, where $x|n$ denotes the restriction of the function x to the set $\{0, 1, \ldots, n-1\}$. The filter G on the countable set T is generated by the family $\{T\backslash(B_{x_1} \cup B_{x_2} \cup \cdots \cup B_{x_n} \cup S): n \geq 1, x_i \in \{0, 1\}^{\mathbf{N}}$ and S is a finite subset of $T\}$. We identify T with $\mathbf{N}$ and consider G as a filter on $\mathbf{N}$. Obviously, the filter G is free. By [Ca$_1$] G is an F_σ-subset of $2^{\mathbf{N}}$. The assertion (b) follows from Lemma 4.2 and Theorem 1.1. Now we shall verify (c). We identify $\mathbf{N}_G$ with $T \cup \{\infty\}$. Let $\mathfrak{S} = \{P_n: n = 1, 2, \ldots\}$ be a family of subsets of $T \cup \{\infty\}$. We will construct a compact set K and an open set V in $T \cup \{\infty\}$ such that $K \subseteq V$ and for every n the set K is not contained in P_n or the set P_n is not contained in V. By induction one can easily define a sequence $\{t_n\}_{n=1}^{\infty}$, where $t_n \in 2^n$ and such that

(*) $t_n|\{0, 1, \ldots, n-2\} = t_{n-1}$, if $n > 1$,

(**) $t_n \in P_n$ or $s_n \notin P_n$, where $s_n \in 2^n$ is defined by $s_n|\{0, 1, \ldots, n-2\} = t_n|\{0, 1, \ldots, n-2\}$, if $n > 1$, and $s_n(n-1) \neq t_n(n-1)$ if $n \geq 1$.

Now, let $x \in \{0, 1\}^{\mathbf{N}}$ be such that $x|\{0, 1, \ldots, n-1\} = t_n$. We set $V = (T \cup \{\infty\})\backslash B_x$ and $K = \{s_n: n = 1, 2, \ldots\} \cup \{\infty\}$. Let us observe that $B_x \cap K = \varnothing$ and for every $y \in \{0, 1\}^{\mathbf{N}}$, $y \neq x$, $B_y \cap K$ is finite (if n is such that $x|n \neq y|n$, then $s_k \notin B_y$ for $k \geq n+1$). Hence the set $K \subseteq V$ is compact and, by (**), the set P_n is not contained in V or the set K is not contained in P_n for every $n = 1, 2, \ldots$.

Example 7.2 answers the problem 34 of [Ar$_1$] (cf. also problem 36 of [Ar$_2$]).

8. Sequence spaces of higher Borel complexity

According to [LvMP, Ca$_2$] and Lemma 4.2 for every countable ordinal $\alpha \geq 2$ there exists a filter F on $\mathbf{N}$ such that $c_F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$. In [BM] it was shown that in each class $\mathfrak{M}_\alpha$ there exists a maximal object Ω_α which can be characterized as follows:

8.1. **Proposition.** *A space X is homeomorphic to Ω_α iff X satisfies the following conditions*:

(1) *X is an absolute retract,*

(2) $X \in \mathfrak{M}_\alpha$,

(3) *X is a Z_σ-space,*

(4) *X is homeomorphic to X^∞.*

(5) *X is $\mathfrak{M}_\alpha$-universal, i.e., each $Y \in \mathfrak{M}_\alpha$ is embeddable onto a closed subset of Ω_α.*

Let us note that Ω_2 is just σ^∞ and in §6 we have proved that if X is a countable completely regular space such that $C_p(X) \in \mathfrak{M}_2 \backslash \mathfrak{A}_2$, then $C_p(X)$ is homeomorphic to σ^∞. It suggests the following

8.2. **Conjecture.** For every countable completely regular space X such that $C_p(X) \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$, $C_p(X)$ is homeomorphic to Ω_α.

Now, we focus on the spaces c_F for Borelian filters F on $\mathbf{N}$. For spaces c_F the condition (1) is clear. According to [Ca$_1$] (see Lemma 4.2) $c_F \in \mathfrak{M}_\alpha$ provided $F \in \mathfrak{M}_\alpha$. The condition (3) is a consequence of Corollary 3.4. The conditions (4) and (5) are the major obstacles in order to confirm Conjecture 8.2 for higher Borelian classes.

8.3. **Problem.** Let F be a filter on $\mathbf{N}$ such that $F \in \mathfrak{A}_\alpha \cup \mathfrak{M}_\alpha$, where $\alpha > 1$. Is c_F homeomorphic to $(c_F)^\infty$?

8.4. **Problem.** Let for a filter F on $\mathbf{N}$ the space $c_F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$. Can every $X \in \mathfrak{M}_\alpha$ be embeddable onto a closed subset of c_F?

For $\alpha = 2$ the condition (5) is a consequence of the remaining four conditions. For higher α, we ask

8.5. **Problem.** Let X satisfy the following conditions:

(1′) X is an absolute retract,
(2′) $X \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$,
(3′) X is a Z_σ-space,
(4′) X is homeomorphic to X^∞.

Is X homeomorphic to Ω_α?

8.6. *Remark.* We say that a set X in a compact space M is Wadge $\mathfrak{M}_\alpha$-maximal in M if $X \in \mathfrak{M}_\alpha$ and for a subset Y of M, with $Y \in \mathfrak{M}_\alpha$, there exists a map $f\colon M \to M$ such that $f^{-1}(X) = Y$. An inspection of the proof of Lemma 5.3 yields the fact that a space X satisfying the conditions (1)–(4) is Wadge $\mathfrak{M}_2$-maximal in a topological copy of the Hilbert cube I^∞. The last result can be considered as a Hilbert cube counterpart of the fact that in the Cantor set each set $A \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$ is Wadge $\mathfrak{M}_\alpha$-maximal (see [W]). To answer 8.5 in the positive, it is enough to show that a space X satisfying (1′)–(4′) is Wadge $\mathfrak{M}_\alpha$-maximal in a topological copy of the Hilbert cube.

The condition (4) for spaces c_F is closely related to decomposability of filters F described in Proposition 2.4. If a space $c_F \in \mathfrak{M}_2 \backslash \mathfrak{A}_2$, then the restrictions F_i of the filter F of Proposition 2.4 are in the class $(\mathfrak{M}_2 \backslash \mathfrak{A}_2) \cup (\mathfrak{A}_1 \backslash \mathfrak{M}_1)$ and consequently $c_{F_i} \in \mathfrak{M}_2 \backslash \mathfrak{A}_2$. This was the crucial step in verifying the condition (4). For higher α the Borel type of spaces c_{F_i} for restricted filters F_i can be essentially lowered. That is why we introduce the following definition. A filter F on $\mathbf{N}$ is decomposable if there exist infinite, disjoint sets $\mathbf{N}_1$ and $\mathbf{N}_2$ such that $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2$ and $F_i = \{A \cap \mathbf{N}_i\colon a \in F\}$ is a filter on $\mathbf{N}_i$ which is isomorphic to F for $i = 1, 2$. Then we have $F = \{A_1 \cup A_2\colon A_1 \in F_1$ and $A_2 \in F_2\}$ and we write $F = F_1 \times F_2$. By an easy induction we obtain

8.7. **Lemma.** *If F is a decomposable filter on $\mathbf{N}$, then there exists a sequence $\{\mathbf{N}_i\}$ of infinite pairwise disjoint subsets of $\mathbf{N}$ with $\mathbf{N} = \bigcup_{i=1}^\infty \mathbf{N}_i$, and such that each $F_i = \{A \cap \mathbf{N}_i\colon a \in F\}$ is a filter on $\mathbf{N}_i$ which is isomorphic to F.*

The main result of this section is the following:

8.8. **Theorem.** *Let F be a first category filter on $\mathbf{N}$ which is free and decomposable. Then the sequence space c_F is homeomorphic to $(c_F)^\infty$.*

The proof of Theorem 8.8 is based on the following lemma which is a standard fact about absorbing sets (see [BM]):

8.9. **Lemma.** *Let X and Y be absolute retracts which are Z_σ-spaces. Assume that there are noncompact absolute retracts M and N and $p \in M$, $q \in N$ satisfying $W(M, p) \subseteq X \subseteq M^\infty$ and $W(N, q) \subseteq Y \subseteq N^\infty$, where for a space Z and $z \in Z$ we write*

$$W(Z, z) = \{(z_i) \in Z^\infty \colon z_i = z \text{ for all but finitely many } i\}.$$

If $X = \bigcup_{i=1}^\infty X_i$ and $Y = \bigcup_{i=1}^\infty Y_i$, where X_i is closed in X and Y_i is closed in Y for $i = 1, 2, \ldots$ and moreover each X_i embeds onto a closed subset of N and each Y_i embeds onto a closed subset of M, then X and Y are homeomorphic.

Proof of Theorem 8.8. In the proof we will use the spaces c_F, $B_F(1)$, and c_F^* and their products. These spaces are noncompact absolute retracts. Since F is a first category filter, Proposition 3.3 implies that c_F, $B_F(1)$, and c_F^* are Z_σ-spaces. Consider a decomposition $\mathbf{N} = \bigcup_{i=1}^\infty \mathbf{N}_i$ into infinite pairwise disjoint sets $\mathbf{N}_i$ so that the restricted filters F_i are isomorphic to F (see Lemma 8.7). Thus the spaces c_{F_i}, $B_{F_i}(1)$, and $c_{F_i}^*$ are linearly isomorphic to c_F, $B_F(1)$, and c_F^*, respectively. This together with Lemma 5.6 gives a homeomorphism $h\colon \mathbf{R}^\infty \to \mathbf{R}^\infty$ satisfying:

$$W(c_F, 0) \subseteq h(c_F) \subseteq (c_F)^\infty,$$

$$\prod_{i=1}^\infty B_F\left(\frac{1}{i}\right) \cup W(B_F(1), 0) \subseteq h(B_F(1)) \subseteq (B_F(1))^\infty,$$

$$W(c_F^*, 0) \subseteq h(c_F^*) \subseteq (c_F^*)^\infty.$$

Now, Theorem 8.8 follows from Lemma 8.9 applied for $M = N = c_F$, $X = h(c_F)$, and $Y = (c_F)^\infty$ and from the following fact:

(i) $(c_F)^\infty$ embeds onto a closed subset of c_F.

The last is a consequence of (ii)–(iv) below.

(ii) $B_F(1)$ is homeomorphic to $(B_F(1))^\infty$.

By the obvious fact that $B_F(r)$ is homeomorphic to $B_F(1)$, for $r > 0$, and by Lemma 5.6(3), the product $(B_F(1))^\infty$ embeds as a closed subset of $B_F(1)$. Now, we apply Lemma 8.9, with $M = N = B_F(1)$, $X = h(B_F(1))$, and $Y = (B_F(1))^\infty$.

(iii) c_F^* is homeomorphic to $B_F(1)$.

First let us observe that $c_F^* = \bigcup_{n=1}^\infty B_F(n)$. Now, (iii) follows from Lemma 8.9 applied for $M = c_F^*$, $N = B_F(1)$, $X = h(c_F^*)$, and $Y = h(B_F(1))$.

(iv) c_F embeds onto a closed subset of $(c_F^*)^\infty$.

Let $r\colon \mathbf{R} \to [-1, 1]$ be the retraction defined by

$$r(x) = (\operatorname{sgn} x) \min(|x|, 1).$$

Write for $(x_i) \in \mathbf{R}^\infty$,

$$f_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, r(x_{n+1}), r(x_{n+2}), \dots).$$

Then $f = (f_1, f_2, \dots)$ defines a closed embedding of $\mathbf{R}^\infty$ into $(\mathbf{R}^\infty_{\mathrm{bd}})^\infty$ (recall that $\mathbf{R}^\infty_{\mathrm{bd}} = \{(x_n) \in \mathbf{R}^\infty : \sup |x_n| < \infty\}$). Moreover, we have $f^{-1}((c_F^*)^\infty) = c_F$. Thus $f|c_F$ is an embedding of c_F onto a closed subset of $(c_F^*)^\infty$.

8.10. *Remark.* For every countable ordinal $\alpha \geq 2$ there exists a filter F such that $c_F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$ and c_F is homeomorphic to $(c_F)^\infty$.

Proof. Let $F \in \mathfrak{M}_\alpha \backslash \mathfrak{A}_\alpha$ be a filter on $\mathbf{N}$ and let F^∞ be a filter defined in the proof of Lemma 4.2(3). Obviously, the space c_{F^∞} is homeomorphic to $(c_F)^\infty$ and consequently to $(c_{F^\infty})^\infty$.

Added in proof. The authors have just learned that an equivalent version of Lemma 2.2 for filters is contained in Theorem 21 of [Ta].

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