THE GREEN CORRESPONDENCE FOR THE REPRESENTATIONS OF HECKE ALGEBRAS OF TYPE A_{r-1}

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ABSTRACT. We first prove the conjecture mentioned by Leonard K. Jones in his thesis. By applying this conjecture, we obtain that the vertex of an indecomposable $\mathcal{H}_F$ -module is an l -parabolic subgroup. Finally, we establish the Green correspondence for the representations of Hecke algebras of type A_{r-1} .

INTRODUCTION

Let R be a $\mathbb{Q}[u^{1/2}]$ -algebra in which $u^{1/2}$ is invertible. Let (W, S) be the symmetric group on r letters where S is the set of basic transpositions. Then the Hecke algebra $\mathcal{H}_R$ corresponding to W is a free R -module with basis $\{\tilde{T}_w; w \in W\}$ which obey the following multiplication rules (see [Du]):

$$\tilde{T}_w \tilde{T}_s = \begin{cases} \tilde{T}_{ws}, & \text{if } w < ws, \\ (u^{-1/2} - u^{1/2})\tilde{T}_w + \tilde{T}_{ws}, & \text{otherwise,} \end{cases}$$

where “ $<$ ” is the Bruhat order and $w \in W$, $s \in S$.

The study of the representations of the Hecke algebra $\mathcal{H}_R$ has turned out many remarkable q -analogues of the representations of the symmetric groups (see [DJ1 and 2, Ho and Jo]). In this paper we shall generalize some basic results of Green along the lines of the work of L. Jones (see [Jo]). We organize the paper as follows: After recalling some basic results, we shall prove the conjecture (Theorem 2.7) which has been mentioned in [Jo, 5.3]. The Brauer homomorphism constructed by Jones will play a key role in proving that conjecture. With the aid of this conjecture, we obtain that the vertex of an indecomposable $\mathcal{H}_R$ module is an l -parabolic subgroup. In the last section, we shall establish the Green correspondence for the representations of Hecke algebra $\mathcal{H}_R$.

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1. INDUCED AND INDECOMPOSABLE MODULES

Let l be a positive integer, $l \leq r$, and $\Phi_l(u^{1/2})$ is the l th cyclotomic polynomial in $u^{1/2}$. Let R_l be the completion of the polynomial ring in the indeterminate $u^{1/2}$ over $\mathbf{Q}$ localized at the maximal ideal generated by $\Phi_l(u^{1/2})$. Let K be the quotient field of R_l and F the residue class field $R_l/\eta R_l$ where η is the generator of the maximal ideal of R_l .

We call that (K, R_l, F) is a characteristic 0 modular system. Let $R \in \{K, R_l, F\}$.

Let λ be a composition of r . (A composition λ of r , denoted $\lambda \models r$, is a finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of nonnegative integers whose sum is r .) Then the standard Young (or the parabolic) subgroups W_λ of W consists of those permutations of $\{1, 2, \dots, r\}$ which leave invariant the following sets of integers $\{1, 2, \dots, \lambda_1\}$, $\{\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2\}$, $\{\lambda_1 + \lambda_2 + 1, \dots\}$, $\dots$. A parabolic subgroup H is called l -parabolic if, for each proper parabolic subgroup H' of H ,

$$\Phi_l(u) \mid \frac{d_H}{d_{H'}}$$

where $d_{W'}$ denotes the Poincaré polynomial of W' for any parabolic subgroup W' of W .

If W' is a parabolic subgroup of W , we denote by $\mathcal{D}_{W'}$ the set of all distinguished coset representatives of right cosets of W' in W and set $\mathcal{D}_\lambda = \mathcal{D}_{W_\lambda}$ if $W' = W_\lambda$. Let $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$. Then $\mathcal{D}_{\lambda\mu}$ is the set of distinguished $W_\lambda - W_\mu$ double coset representatives. Also, the R -module $\sum_{w \in W'} RT_w$ is a subalgebra of $\mathcal{H}_R$, which is called the parabolic subalgebra of $\mathcal{H}_R$, denoted by $\mathcal{H}_{W'}$. We will use the abbreviation $\mathcal{H}_\lambda$ instead of $\mathcal{H}_{W_\lambda}$.

If M is an $\mathcal{H}_R$ -module and N is an $\mathcal{H}_\lambda$ -module, then we denote by $M_{\mathcal{H}_\lambda}$ the restriction of M from $\mathcal{H}_R$ -module and denote by $N^{\mathcal{H}_R} = N \otimes_{\mathcal{H}_\lambda} \mathcal{H}_R$ the induced module of N . The q -analogue of Mackey's decomposition theorem for finite groups holds (see [DJ1, Jo]).

1.1 Theorem (Mackey's decomposition). *Let $\lambda, \mu \models r$ and let N be an $\mathcal{H}_R$ - $\mathcal{H}_\lambda$ bimodule then*

$$(N^{\mathcal{H}_R})_{\mathcal{H}_\mu} \cong \sum_{d \in \mathcal{D}_{\lambda\mu}} [(N \otimes_{\mathcal{H}_\lambda} \tilde{T}_d) \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_\mu],$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\lambda^d \cap W_\mu$ for all $d \in \mathcal{D}_{\lambda\mu}$. $\square$

Let M be a finitely generated indecomposable right $\mathcal{H}_R$ -module. Then, by [Jo, 3.35], there exists a parabolic subgroup W_λ of W unique up to conjugation such that M is relatively $\mathcal{H}_\lambda$ -projective (see the definition in [Jo, Chapter 3]) and such that W_λ is W -conjugate to a parabolic subgroup of any parabolic subgroup W_μ of W for which M is relatively $\mathcal{H}_\mu$ -projective.

We call W_λ the vertex of M . In the next section we shall see that the vertex of M must be an l -parabolic subgroup of W .

Mackey's theorem can be used to prove the following result.

1.2 Proposition. *Let W_τ be the vertex of the indecomposable $\mathcal{H}_R$ -module M . Then*

(a) *There is an indecomposable $\mathcal{H}_\tau$ -module N such that $M|N^{\mathcal{H}_R}$.*

(b) If N' is another indecomposable $\mathcal{H}_\tau$ -module with the property (a), then there is an element

$$d \in N_W(W_\tau) \cap \mathcal{D}_{\tau\tau}$$

such that

$$N \cong N' \otimes_{\mathcal{H}_\tau} \tilde{T}_d$$

as $\mathcal{H}_\tau$ -modules.

The notation $X|Y$ means that X is isomorphic to a direct summand of Y and $N_W(W_\tau)$ denotes the normaliser of W_τ in W .

Proof. Since M is relatively $\mathcal{H}_\tau$ -projective, we have

$$M|M \otimes_{\mathcal{H}_\tau} \mathcal{H}_R.$$

Thus, there is an indecomposable direct summand N of $M_{\mathcal{H}_\tau}$ such that

$$M|N \otimes_{\mathcal{H}_\tau} \mathcal{H}_R.$$

Hence (a) follows.

Assume that V is an indecomposable $\mathcal{H}_\tau$ -module with

$$M|V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R.$$

Since $N|M_{\mathcal{H}_\tau}$ we have

$$N|(V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R)_{\mathcal{H}_\tau}.$$

By Mackey's theorem,

$$(V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R)_{\mathcal{H}_\tau} \cong \bigoplus_{d \in \mathcal{D}_{\tau\tau}} (V \otimes_{\mathcal{H}_\tau} \tilde{T}_d \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_\tau)$$

where $\nu(d)$ is defined by $W_{\nu(d)} = W_\tau^d \cap W_\tau$. Thus by Krull-Schmidt theorem, there is $d \in \mathcal{D}_{\tau\tau}$ such that

$$N|V \otimes_{\mathcal{H}_\tau} \tilde{T}_d \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_\tau.$$

Therefore

$$M|V \otimes_{\mathcal{H}_\tau} \tilde{T}_d \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_R.$$

Since W_τ is the vertex of M it follows from Higman's criterion [Jo, 3.34] that

$$W_\tau \subseteq_W W_{\nu(d)} = W_\tau^d \cap W_\tau.$$

Therefore

$$W_\tau = W_\tau^d \cap W_\tau \quad \text{and} \quad d \in N_W(W_\tau).$$

Thus we have

$$N|V \otimes_{\mathcal{H}_\tau} \tilde{T}_d.$$

Since V is indecomposable, so is $V \otimes_{\mathcal{H}_\tau} \tilde{T}_d$, hence

$$N \cong V \otimes_{\mathcal{H}_\tau} \tilde{T}_d$$

as $\mathcal{H}_\tau$ -modules. $\square$

The module N is called a *source* of M .

To study the representations of Hecke algebras, the relative norm plays a remarkable role (see [Du, Jo]). Here is the definition of relative norms.

1.3 Definition. Let λ, μ be compositions of r such that $W_\lambda \subseteq W_\mu$. Let M be an $\mathcal{H}_\mu$ - $\mathcal{H}_\lambda$ bimodule and $b \in M$. Define the *relative norm*

$$N_{W_\mu, W_\lambda}(b) = \sum_{w \in \mathcal{D}_\lambda \cap W_\mu} \tilde{T}_{w^{-1}} b \tilde{T}_w.$$

There are a number of nice properties of relative norms which will be used freely in the subsequent discussion. These results are mostly due to P. Hoefsmit and L. Scott. One can find a complete proof in [Jo, Chapter 3].

Let M be an $\mathcal{H}_\lambda$ - $\mathcal{H}_\lambda$ bimodule. We define

$$Z_M(\mathcal{H}_\lambda) = \{m \in M \mid hm = mh \text{ for all } h \in \mathcal{H}_\lambda\}.$$

Obviously $Z(\mathcal{H}_\lambda) = Z_{\mathcal{H}_\lambda}(\mathcal{H}_\lambda)$ is the center of $\mathcal{H}_\lambda$ and, for $M = \text{Hom}_R(N, N)$ where N is a right $\mathcal{H}_\lambda$ -module,

$$Z_M(\mathcal{H}_\lambda) = \text{Hom}_{\mathcal{H}_\lambda}(N, N).$$

One can describe a basis of the center $Z(\mathcal{H}_R)$ of $\mathcal{H}_R$ or a basis of the q -Schur algebra $S_R(n, r)$ in terms of relative norms (see [Du, Jo]), which give us much more facilities.

Let $P_k = W_{(l^k, 1^{r-kl})}$ where k satisfies $r = kl + s$, $s < l$. In [Jo, 5.3] there is a conjecture as follows:

1.4 Conjecture. $N_{W, P_k}(1)$ is invertible.

We shall prove this conjecture in the next section.

2. THE INVERTIBILITY OF $N_{W, P_k}(1)$

In [Jo] a Brauer-type homomorphism is constructed and the image of certain basis of $Z(\mathcal{H}_R)$ is discussed modulo a conjecture which has been proved in [Sh, Proposition 11]. The proof of Proposition 2.6 below is motivated by the argument in [Jo].

Let $\gamma = (l, r-l)$. Then $\mathcal{H}_\gamma \cong \mathcal{H}_l \otimes \mathcal{H}_{r-l}$ where $\mathcal{H}_l = \mathcal{H}_{(l, 1^{r-l})}$ and $\mathcal{H}_{r-l} = \mathcal{H}_{(1^l, r-l)}$. Since W is a disjoint union of the subsets W_γ and $W'xW'$ for $x \notin W_\gamma$ and $W' = W_{(l, 1^{r-l})}$, that is,

$$W = W_\gamma \cup \left(\bigcup_{x \notin W_\gamma} W'xW' \right)$$

(see the proof of [Jo, 5.1.5]), we have

$$(2.a) \quad Z(\mathcal{H}_R) \subseteq Z_{\mathcal{H}_R}(\mathcal{H}_\gamma) = Z(\mathcal{H}_\gamma) \oplus Z_{M_r}(\mathcal{H}_\gamma)$$

where $M_r = \bigoplus_{x \notin W_\gamma} \mathcal{H}_l \tilde{T}_x \mathcal{H}_l$.

Let π be the projection of $Z(\mathcal{H}_R)$ onto $Z(\mathcal{H}_\gamma)$ and let σ be the canonical map from $Z(\mathcal{H}_\gamma)$ onto $Z(\mathcal{H}_\gamma)/N_{W', 1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma)$. Then the map

$$f = \sigma \circ \pi: Z(\mathcal{H}_R) \rightarrow Z(\mathcal{H}_\gamma)/N_{W', 1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma)$$

is an algebraic homomorphism which is called the Brauer homomorphism, following Jones.

2.1 Lemma. *Let π be as above and let*

$$h = \sum_w a_w \tilde{T}_w \in Z_{\mathcal{H}_R}(\mathcal{H}_\gamma).$$

Then $\pi(h) = 0$ if and only if $s_l = (l, l+1) < w$ for all w with $a_w \neq 0$.

Proof. Immediate from the fact that $w \notin W_\gamma$ if and only if $s_l < w$. $\square$

2.2 Lemma. $Z(\mathcal{H}_\gamma)/N_{W',1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma) \cong [Z(\mathcal{H}_l)/N_{W',1}(\mathcal{H}_l)] \otimes Z(\mathcal{H}_{r-l})$.

Proof. We first claim that

$$(2.b) \quad N_{W',1}(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l}) = (N_{W',1}(\mathcal{H}_l) \otimes \mathcal{H}_{r-l}) \cap (Z(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l})).$$

Obviously, the left-hand side of (2.b) is contained in the right-hand side. Let $\{v_i, 1 \leq i \leq s\}$ be a basis of $Z(\mathcal{H}_l)$ such that $\{v_i, 1 \leq i \leq t\}$ is a basis of $N_{W',1}(\mathcal{H}_l)$. If $a = \sum_{i=1}^t v_i \otimes a_i$ is an element of right-hand side of (2.b) then it is easy to see $a_i \in Z(\mathcal{H}_{r-l})$. Hence the claim is proved. Thus by the claim,

$$\begin{aligned} Z(\mathcal{H}_\gamma)/N_{W',1}(\mathcal{H}_\gamma) \cap Z(\mathcal{H}_\gamma) & \\ & \cong [Z(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l})]/[N_{W',1}(\mathcal{H}_l) \otimes \mathcal{H}_{r-l} \cap Z(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l})] \\ & \cong [Z(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l})]/[N_{W',1}(\mathcal{H}_l) \otimes Z(\mathcal{H}_{r-l})] \\ & \cong [Z(\mathcal{H}_l)/N_{W',1}(\mathcal{H}_l)] \otimes Z(\mathcal{H}_{r-l}). \end{aligned}$$

Hence the result. $\square$

Let $Z = Z(\mathcal{H}_l)/N_{W',1}(\mathcal{H}_l)$ and let $r = kl + s$, $s < l$. For each m , $1 \leq m \leq k$ we define

$$f_m = \underbrace{(\text{id}_Z \otimes \cdots \otimes \text{id}_Z)}_{m-1} \otimes f) \circ \cdots \circ (\text{id}_Z \otimes f) \circ f.$$

Then f_m is a homomorphism from $Z(\mathcal{H}_R)$ into

$$\underbrace{Z \otimes \cdots \otimes Z}_m \otimes Z(\mathcal{H}_{r-ml}).$$

If e is a central primitive idempotent of $\mathcal{H}_R$ then we say that the *defect* of e is m if $f_m(e) \neq 0$ and $f_{m+1}(e) = 0$.

2.3 Lemma. *Let e be a central primitive idempotent of $\mathcal{H}_R$. Then $eZ(\mathcal{H}_R)$ is a local ring.*

Proof. Let

$$\mathcal{A} = \mathcal{H}_R^{\text{op}} \otimes \mathcal{H}_R.$$

Then $\mathcal{H}_R$ is an $\mathcal{A}$ -module defined by

$$h(h_1 \otimes h_2) = h_1 h h_2$$

for $h, h_1, h_2 \in \mathcal{H}_R$, and the $\mathcal{A}$ -submodules of $\mathcal{H}_R$ are ideals of $\mathcal{H}_R$. So $e\mathcal{H}_R$ is an indecomposable $\mathcal{A}$ -submodule. Hence $\text{End}_{\mathcal{A}}(e\mathcal{H}_R)$ is a local ring.

Since $\text{End}_{\mathcal{H}_R}(e\mathcal{H}_R) = e\mathcal{H}_R$, it follows that

$$\text{End}_{\mathcal{A}}(e\mathcal{H}_R) = Z(e\mathcal{H}_R) = eZ(\mathcal{H}_R),$$

hence the result. $\square$

2.4 Lemma. Let e be a central primitive idempotent and $f(e) \neq 0$. Then there exist pairwise orthogonal primitive idempotents $\{e_{i1}\}_{1 \leq i \leq s}$, $\{e_{i2}\}_{1 \leq i \leq s}$ of Z , $Z(\mathcal{H}_{r-l})$ respectively such that

$$f(e) = \sum_{i=1}^s e_{i1} \otimes e_{i2}.$$

Moreover, if the defect of e is m then the defect of $e_{i2} \leq m-1$ for all i .

Proof. Let $\{e_i\}$, $\{e'_j\}$ be the pairwise orthogonal primitive idempotents of Z , $Z(\mathcal{H}_{r-l})$ respectively such that

$$1_Z = \sum_i e_i, \quad 1_{Z(\mathcal{H}_{r-l})} = \sum_j e'_j.$$

Then

$$1_{Z \otimes Z(\mathcal{H}_{r-l})} = \sum_{i,j} e_i \otimes e'_j$$

is a decomposition of primitive idempotents of the identity of $Z \otimes Z(\mathcal{H}_{r-l})$. Since $f(e)$ is an idempotent of $Z \otimes Z(\mathcal{H}_{r-l})$ we may find an expression of $f(e)$ as desired.

Suppose that there is t , $1 \leq t \leq s$, such that e_{t2} is of defect $\geq m$. Thus $f_m(e_{t2}) \neq 0$ and therefore

$$f_{m+1}(e) = \sum_{i=1}^s \text{id}_Z(e_{i1}) \otimes f_m(e_{i2}) \neq 0.$$

This is contrary to our assumption. So the defect of $e_{i2} \leq m-1$ for all i . $\square$

Let ι denote the anti-automorphism of $\mathcal{H}_R$ defined by $\iota(\tilde{T}_w) = \tilde{T}_{w^{-1}}$ and $\iota^2 = 1$. Let $P_m = W_{(1^{(k-m)l}, l^m, 1^{r-kl})}$, $0 \leq m \leq k$.

2.5 Lemma. Let e_0 be a central primitive idempotent of $\mathcal{H}_R$ with defect 0. Then $N_{W, P_k}(1)e_0$ is invertible in $e_0 Z(\mathcal{H}_R)$.

Proof. Let χ be the irreducible character of $\mathcal{H}_R$ over $R = \mathbf{Q}(u)$ associated with $e' = \iota(e_0)$. d_χ denotes the generic degree of χ . Then

$$\frac{d_W}{d_\chi} \not\equiv 0 \pmod{(\Phi_l)}$$

(see [Jo, 5.2.29]). By the proof of [Jo, 3.5] we have

$$N_{W, 1}(1)e' = \frac{d_W}{d_\chi} \chi(1)e' \not\equiv 0 \pmod{(\Phi_l)}.$$

Therefore,

$$\text{Tr}(N_{W, 1}(1)e') = \frac{d_W}{d_\chi} \chi(1)^2 \not\equiv 0 \pmod{(\Phi_l)}.$$

On the other hand, we have

$$\begin{aligned} \text{Tr}(N_{W, 1}(1)e') &= \text{Tr}(N_{W, P_k}(N_{P_k, 1}(1))e') \\ &= \sum_{w \in \mathcal{D}_{P_k}} \text{Tr}(\tilde{T}_{w^{-1}} N_{P_k, 1}(1) \tilde{T}_w e') \\ &= \sum_{w \in \mathcal{D}_{P_k}} \text{Tr}(\tilde{T}_w \tilde{T}_{w^{-1}} e' N_{P_k, 1}(1)) \\ &= \text{Tr}(\iota(N_{W, P_k}(1))e' N_{P_k, 1}(1)). \end{aligned}$$

Since $\iota(N_{W, P_k}(1))e'$ commutes with $N_{P_k, 1}(1)$ we have $\iota(N_{W, P_k}(1)e_0)$ is not nilpotent, hence $N_{W, P_k}(1)e_0$ is not nilpotent. Hence $N_{W, P_k}(1)e_0$ is invertible in $e_0Z(\mathcal{H}_R)$ since $e_0Z(\mathcal{H}_R)$ is a local ring. $\square$

We now fix some notation. Let

$$d_0 = 1, \quad d_m = (l, l+m)(l-1, l+m-1) \cdots (l-m+1, l+1)$$

for $1 \leq m \leq m(l) = \min\{l, r-l\}$. Then by [Jo, (5.2.2)]

$$\mathcal{D}_{\gamma\gamma} = \{d_m \mid 0 \leq m \leq m(l)\}$$

and

$$G_m = W_\gamma^{d_m} \cap W_\gamma = W_{(l-m, m, m, r-l-m)}.$$

2.6 Proposition. Let F be the Brauer homomorphism,

$$f: Z(\mathcal{H}_R) \rightarrow [Z(\mathcal{H}_l)/N_{W', 1}(\mathcal{H}_l)] \otimes Z(\mathcal{H}_{r-l}).$$

Then

$$f(N_{W, P_k}(1)) = kN_{W_\gamma, P_k}(1).$$

Proof. By the transitivity of relative norm we have

$$(2.c) \quad \begin{aligned} N_{W, P_k}(1) &= N_{W, W_\gamma}(N_{W_\gamma, P_k}(1)) \\ &= \sum_{m=0}^{m(l)} N_{W_\gamma, G_m}(\tilde{T}_{d_m} N_{W_\gamma, P_k}(1) \tilde{T}_{d_m}). \end{aligned}$$

Let $W'' = W_{(l', r-l)}$. Then $G_m = G_{m1} \times G_{m2}$ where G_{m1}, G_{m2} are the intersections of G_m with W', W'' respectively. Since

$$N_{W_\gamma, P_k}(1) = N_{W'', P_{k-1}}(1) \in Z_{\mathcal{H}_{W''}}(\mathcal{H}_{G_{m2}})$$

we may write by (2.a)

$$N_{W_\gamma, P_k}(1) = N_m + T_m$$

where $N_m \in Z(\mathcal{H}_{G_{m2}})$ and

$$T_m = \sum_{\substack{z \in W'' \\ z \notin G_{m2}}} b_z \tilde{T}_z.$$

Thus, if $d_m z d_m \in W_\gamma$ with $\tilde{T}_z$ involved in T_m , then

$$d_m z d_m \in W_\gamma^{d_m} \cap W_\gamma = G_m$$

hence

$$z \in G_m^{d_m} = G_m.$$

It follows that $(l+m, l+m+1) \in G_m$ since $z \in W''$ and $z \notin G_{m2}$. This is impossible. Therefore $d_m z d_m \notin W_\gamma$. Thus each term $\tilde{T}_w$ involved in $\tilde{T}_{d_m} \tilde{T}_z \tilde{T}_{d_m}$ satisfies $s_l < w$. Therefore, by 2.1,

$$\pi(N_{W_\gamma, G_m}(\tilde{T}_{d_m} T_m \tilde{T}_{d_m})) = 0.$$

We now examine N_m . By [Jo, 4.33] we express N_m as a linear combination of the basis of $Z(\mathcal{H}_{G_{m2}})$:

$$N_m = \sum_{\substack{\alpha \vdash (r-l) \\ W_\alpha \subseteq G_{m2}}} a_\alpha N_{G_{m2}, W_\alpha}(\eta_\alpha)$$

where $\eta_\alpha \in Z(\mathcal{H}_{W_\alpha})$.

Let

$$\eta_\alpha = \sum_{w \in W_\alpha} a_w \tilde{T}_w.$$

Then by [Jo, 5.2.21] we have

$$\begin{aligned} \tilde{T}_{d_m} \eta_\alpha \tilde{T}_{d_m} &= \sum_{w \in G_{m2}} a_w \tilde{T}_{d_m} \tilde{T}_w \tilde{T}_{d_m} \\ &= \sum_{w \in G_{m2}} a_w \left(\tilde{T}_{d_m w d_m} + \sum_{\substack{x \in W \\ s_l < x}} b_{wx} \tilde{T}_x \right) \\ &= \eta_{d_m \alpha d_m} + \sum_{s_l < z} c_z \tilde{T}_z \end{aligned}$$

where $\eta_{d_m \alpha d_m} \in Z(\mathcal{H}_{W_\alpha^{d_m}})$. Thus if we denote $S_\alpha = G_{m1} \times W_\alpha$ then by [Jo, 5.2.10] we have

$$\begin{aligned} \tilde{T}_{d_m} N_{G_{m2}, W_\alpha}(\eta_\alpha) \tilde{T}_{d_m} &= \tilde{T}_{d_m} N_{G_m, S_\alpha}(\eta_\alpha) \tilde{T}_{d_m} \\ &= \sum_{x \in \mathcal{D}_{S_\alpha} \cap G_m} \tilde{T}_{d_m x^{-1}} \eta_\alpha \tilde{T}_x d_m \\ &= \sum_{\hat{x} \in \mathcal{D}_{S_\alpha^{d_m}} \cap G_m} \tilde{T}_{\hat{x}^{-1}} (\tilde{T}_{d_m} \eta_\alpha \tilde{T}_{d_m}) \tilde{T}_{\hat{x}} \\ &= N_{G_m, S_\alpha^{d_m}}(\eta_{d_m \alpha d_m}) + \sum_{s_l < w} d_w \tilde{T}_w. \end{aligned}$$

Therefore, by (2.1),

$$\pi \left(N_{W_\gamma, G_m} \left(\sum_{s_l < w} d_w \tilde{T}_w \right) \right) = 0$$

and if $0 < m < l$ then $N_{G_{m1}^{d_m}, 1}(1)$ is invertible by [Jo], thus

$$\begin{aligned} N_{W_\gamma, G_m}(N_{G_m, S_\alpha^{d_m}}(\eta_{d_m \alpha d_m})) &= N_{W', G_{m1}^{d_m}}(1) N_{W'', W_\alpha^{d_m}}(\eta_{d_m \alpha d_m}) \\ &= N_{W', 1} \left(\frac{1}{N_{G_{m1}^{d_m}, 1}(1)} N_{W'', W_\alpha^{d_m}}(\eta_{d_m \alpha d_m}) \right) \end{aligned}$$

which lies in the kernel of σ .

By observing (2.c) and the above arguments we obtain that

$$f(N_{W, P_k}(1)) = \begin{cases} N_{W_\gamma, P_k}(1), & \text{if } m(l) < l, \\ \sum_{m=0, l} f(N_{W_\gamma, G_m}(\tilde{T}_{d_m} N_{W_\gamma, P_k}(1) \tilde{T}_{d_m})), & \text{if } m(l) = l. \end{cases}$$

In particular, if $k = 1$ then $m(l) < l$ and

$$f(N_{W, P_k}(1)) = N_{W_\gamma, P_k}(1).$$

So the assertion is true for $k = 1$. Assume now that $k > 1$. Then $m(l) = l$ and we have

$$f(N_{W, P_k}(1)) = N_{W_\gamma, P_k}(1) + f(N_{W_\gamma, G_l}(\tilde{T}_{d_l} N_{W_\gamma, P_k}(1) \tilde{T}_{d_l})).$$

By induction we get

$$N_{W_\gamma, P_k}(1) = N_{W'', P_{k-1}}(1) = (k-1)N_{W_{\gamma'}, P_{k-1}}(1) + \sum_{\substack{w \in W'' \\ (2l, 2l+1) < w}} f_w \tilde{T}_w$$

where $W_{\gamma'} = W_{(1^l, l, r-2l)}$. Thus similar reason as before shows that

$$\pi \left(N_{W_\gamma, G_l} \left(\tilde{T}_{d_l} \sum_{\substack{w \in W'' \\ (2l, 2l+1) < w}} f_w \tilde{T}_w \tilde{T}_{d_l} \right) \right) = 0.$$

So we eventually obtain that

$$\begin{aligned} f(N_{W, P_k}(1)) &= N_{W_\gamma, P_k}(1) + f(N_{W_\gamma, G_l}(\tilde{T}_{d_l}(k-1)N_{W_{\gamma'}, P_{k-1}}(1)\tilde{T}_{d_l})) \\ &= N_{W_\gamma, P_k}(1) + (k-1)f(N_{W_\gamma, G_l}(\tilde{T}_{d_l}N_{G_l, P_k}(1)\tilde{T}_{d_l})) \\ &= N_{W_\gamma, P_k}(1) + (k-1)f(N_{W_\gamma, P_k}(\tilde{T}_{d_l}^2)) \\ &= N_{W_\gamma, P_k}(1) + (k-1)N_{W_\gamma, P_k}(1) \\ &= kN_{W_\gamma, P_k}(1) \end{aligned}$$

since $P_k^{d_l} = P_k$ and

$$\tilde{T}_{d_l}^2 = \tilde{T}_1 + \sum_{\substack{w \in W \\ s_l < w}} g_w \tilde{T}_w$$

by [Jo, 5.2.20]. $\square$

2.7 Theorem. *Let e be a central primitive idempotent of $\mathcal{H}_R$. Then $N_{W, P_k}(1)e$ is invertible in $eZ(\mathcal{H}_R)$. Therefore $N_{W, P_k}(1)$ is invertible in $\mathcal{H}_R$.*

Proof. The first statement is true if e is of defect 0 by (2.5). Assume that e is of defect $m > 0$. By (2.4)

$$f(e) = \sum_{i=1}^s e_{i1} \otimes e_{i2}$$

where e_{i1}, e_{i2} are the central primitive idempotents of $Z, Z(\mathcal{H}_{r-l})$ respectively, and the defect of $e_{i2} < m$. Thus by the previous proposition,

$$\begin{aligned} f(N_{W, P_k}(1)e) &= f(N_{W, P_k}(1))f(e) \\ &= kN_{W_\gamma, P_k}(1) \sum_i e_{i1} \otimes e_{i2} \\ &= k \sum_i e_{i1} \otimes N_{W'', P_{k-1}}(1)e_{i2}. \end{aligned}$$

By induction we have that $N_{W'', P_{k-1}}(1)e_{i2}$ is invertible in $e_{i2}Z(\mathcal{H}_{r-l})$ for all i , and $\{e_{i1} \otimes e_{i2}\}_{1 \leq i \leq s}$ is orthogonal pairwise. So $f(N_{W, P_k}(1)e)$ is not nilpotent. Therefore, $N_{W, P_k}(1)e$ is not nilpotent. So it is invertible in $eZ(\mathcal{H}_R)$ since $eZ(\mathcal{H}_R)$ is a local ring.

Let $1 = \sum_{i=1}^s e_i$ be a decomposition of central primitive idempotents of the identity of $\mathcal{H}_R$. Then

$$N_{W, P_k}(1) = \sum_{i=1}^s N_{W, P_k}(1)e_i.$$

Since $N_{W,P_k}(1)e_i$ is invertible in $e_i Z(\mathcal{H}_R)$ for all i we have $N_{W,P_k}(1)$ is invertible. $\square$

3. GREEN CORRESPONDENCE

In this section we shall present a couple of applications of Theorem 2.7.

Recall from §1 that if M is an indecomposable $\mathcal{H}_R$ -module then there is a “minimal” parabolic subgroup W_λ of W such that M is relatively $\mathcal{H}_\lambda$ -projective. Such a group W_λ is called a vertex of M . Now we can say more about the vertex.

3.1 Theorem. *Let M be a finitely generated indecomposable $\mathcal{H}_R$ -module. Then the vertex of M is an l -parabolic subgroup of W .*

Proof. Let H be the vertex of M and P the maximal l -parabolic subgroup of H . Then, by Higman’s criterion,

$$N_{W,H}(\text{Hom}_{\mathcal{H}_H}(M, M)) = \text{Hom}_{\mathcal{H}_R}(M, M).$$

Since $N_{H,P}(1)$ is invertible, we have

$$\begin{aligned} N_{W,H}(\text{Hom}_{\mathcal{H}_H}(M, M)) &= N_{W,P} \left(\frac{1}{N_{H,P}(1)} \text{Hom}_{\mathcal{H}_H}(M, M) \right) \\ &\subseteq N_{W,P}(\text{Hom}_{\mathcal{H}_P}(M, M)) \end{aligned}$$

therefore

$$N_{W,P}(\text{Hom}_{\mathcal{H}_P}(M, M)) = \text{Hom}_{\mathcal{H}_R}(M, M).$$

By Higman’s criterion again we get M is relatively $\mathcal{H}_P$ -projective. This forces $P = H$. $\square$

In the modular representation theory of finite groups, the Green correspondence connects indecomposable modules for the group G with modules for its local subgroups (see [Al, F]). Now, we start to establish a q -analogue of the Green correspondence for the representations of Hecke algebras. Such a generalization becomes apparent as soon as 3.1 is set up. First of all we need a couple of lemmas.

3.2 Lemma. *Let M be an indecomposable $\mathcal{H}_R$ -module with vertex W_τ and W_ρ is a parabolic subgroup containing W_τ . Then there is an indecomposable $\mathcal{H}_\rho$ -module N satisfying any two of the following statements:*

- (a) $N|M_{\mathcal{H}_\rho}$;
- (b) $M|N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R$;
- (c) N has vertex W_τ .

Proof. Since W_τ is a vertex of M we have

$$M|M \otimes_{\mathcal{H}_\tau} \mathcal{H}_R.$$

Thus,

$$M|(M \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho) \otimes_{\mathcal{H}_\rho} \mathcal{H}_R$$

since $W_\tau \subseteq W_\rho$. Hence, M is relatively $\mathcal{H}_\rho$ -projective by Higman’s criterion. Therefore

$$M|M \otimes_{\mathcal{H}_\rho} \mathcal{H}_R.$$

Thus, there is an indecomposable summand N of $M_{\mathcal{H}_\rho}$ such that

$$M|N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R.$$

So (a) and (b) hold.

Let V be a source of M . Then $M|V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R$ and hence, $M|(V^{\mathcal{H}_\rho})^{\mathcal{H}_R}$. Thus there is an indecomposable summand N of $V^{\mathcal{H}_\rho}$ with $M|N^{\mathcal{H}_R}$. We claim that N has vertex W_τ .

Since $N|V^{\mathcal{H}_\rho}$, we have N is relatively $\mathcal{H}_\tau$ -projective, so there is a vertex $W_{\tau'}$ of N with $W_{\tau'} \subseteq W_\tau$. Let V' be a $\mathcal{H}_{\tau'}$ -module with $N|(V')^{\mathcal{H}_\rho}$. Then

$$N^{\mathcal{H}_R}|((V')^{\mathcal{H}_\rho})^{\mathcal{H}_R} \quad \text{and} \quad ((V')^{\mathcal{H}_\rho})^{\mathcal{H}_R} = (V')^{\mathcal{H}_R},$$

hence

$$M|N^{\mathcal{H}_R}|V' \otimes_{\mathcal{H}_{\tau'}} \mathcal{H}_R.$$

That is M is relatively $\mathcal{H}_{\tau'}$ -projective. Thus $W_{\tau'}$ contains a conjugate of W_τ . Since $W_{\tau'} \subseteq W_\tau$ we have $W_{\tau'} = W_\tau$. (b) and (c) are true.

By the proof of 1.2 there is an indecomposable $\mathcal{H}_\tau$ -module V such that $V|M_{\mathcal{H}_\tau}$ and $M|V^{\mathcal{H}_R}$. Hence there is an indecomposable $\mathcal{H}_\rho$ -module N with $N|M_{\mathcal{H}_\rho}$ and $V|N_{\mathcal{H}_\tau}$. We shall prove that N has vertex W_τ .

Since $N|M_{\mathcal{H}_\rho}$ we have $N|(V^{\mathcal{H}_R})_{\mathcal{H}_\rho}$. By Mackey's theorem there exists $d \in \mathcal{D}_{\tau\rho}$ such that

$$N|V \otimes_{\mathcal{H}_\tau} \tilde{T}_d \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_\rho.$$

Hence N has a vertex $W_{\tau'}$ with $W_{\tau'} \subseteq W_\tau^d \cap W_\rho = W_{\nu(d)}$. Assume that V' is a source of N , $N|V' \otimes_{\mathcal{H}_{\tau'}} \mathcal{H}_\rho$. Thus

$$V|(V' \otimes_{\mathcal{H}_{\tau'}} \mathcal{H}_\rho)_{\mathcal{H}_\tau}.$$

By Mackey's theorem we see that V is relatively $\mathcal{H}_{W_\tau^z \cap W_\tau}$ -projective for some z . Thus M is also relatively $\mathcal{H}_{W_\tau^z \cap W_\tau}$ -projective, hence $W_\tau \subseteq_W W_\tau^z \cap W_\tau$. Therefore,

$$W_{\tau'} = W_\tau^d \quad (d \in \mathcal{D}_{\tau\rho})$$

since $W_{\tau'} \subseteq W_\tau^d$. Hence W_τ is a vertex of N . (a) and (c) hold. $\square$

3.3 Lemma. *Let τ, ρ be as in 3.2. If N is a relatively $\mathcal{H}_\tau$ -projective $\mathcal{H}_\rho$ -module, then*

$$(N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R)_{\mathcal{H}_\rho} \cong N \oplus Y$$

where every indecomposable summand of Y is relatively projective for a subgroup of the form

$$W_\tau^d \cap W_\rho, \quad \text{for } d \in \mathcal{D}_{\tau\rho}, \quad d \neq 1.$$

Proof. Since N is $\mathcal{H}_\tau$ -projective we have $N|V \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho$ for some $\mathcal{H}_\tau$ -module V . Thus

$$V \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho = N \oplus T$$

for some $\mathcal{H}_\rho$ -module T .

Now,

$$V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R \cong (N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R) \oplus (T \otimes_{\mathcal{H}_\rho} \mathcal{H}_R)$$

and

$$(V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R)_{\mathcal{H}_\rho} \cong N \oplus Y \oplus T \oplus X$$

where

$$(N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R)_{\mathcal{H}_\rho} = N \oplus Y, \quad (T \otimes_{\mathcal{H}_\rho} \mathcal{H}_R)_{\mathcal{H}_\rho} = T \oplus X$$

for suitable $\mathcal{H}_\rho$ -modules X, Y by Mackey's theorem.

On the other hand, by Mackey's theorem again,

$$(V \otimes_{\mathcal{H}_\tau} \mathcal{H}_R)_{\mathcal{H}_\rho} = \bigoplus_{d \in \mathcal{D}_{\tau\rho}} (V \otimes_{\mathcal{H}_\tau} \tilde{T}_d \otimes_{\mathcal{H}_{\nu(d)}} \mathcal{H}_\rho) = V \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho \oplus U$$

where $d = 1$ gives $V \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho$, the U is the direct sum of all terms for $d \neq 1$ and so each indecomposable summand of U is relatively projective for a subgroup of the form $W_\tau^d \cap W_\rho$, $d \in \mathcal{D}_{\tau\rho}$, $d \neq 1$. The Krull-Schmidt theorem implies that $X \oplus Y \cong U$. So Y is as claimed. $\square$

From now on we assume that $W_\rho = W_{(ml, r-ml)}$, $W_\theta = W_{(lm, l'r-ml)}$. Then

$$W_\rho \supseteq N_W(W_\theta) \supseteq W_{(lm, r-ml)} \supseteq W_\theta.$$

Let $\mathcal{P}$ be the collection of all parabolic subgroups of W . If $\mathcal{S}$ is a collection of parabolic subgroups of W , then $P \in_W \mathcal{S}$ for $P \in \mathcal{P}$ means $P = H^x$ for some $H \in \mathcal{S}$, $x \in W$.

We say that $\mathcal{H}_R$ -module M is relatively $\mathcal{S}$ -projective if $M = \bigoplus_i M_i$, and each M_i is relatively projective for a group in $\mathcal{S}$.

Let

$$\begin{aligned} \mathcal{X} &= \{H \in \mathcal{P} \mid H \subseteq W_\theta^d \cap W_\theta \text{ for some } d \in W, d \notin W_\rho\}, \\ \mathcal{Y} &= \{H \in \mathcal{P} \mid H \subseteq W_\theta^d \cap W_\rho \text{ for some } d \in W, d \notin W_\rho\}, \\ \mathcal{Z} &= \{P \in \mathcal{P} \mid P \subseteq W_\theta \text{ is } l\text{-parabolic}, P \notin_W \mathcal{X}\}. \end{aligned}$$

Observe that $W_\rho \supseteq N_W(W_\theta)$, $\mathcal{X}$ consists of proper subgroups of W_θ , but $W_\theta \in \mathcal{Z}$.

3.4 Lemma. *If W_τ is an l -parabolic subgroup of W_θ then the following assertions are equivalent:*

- (a) $W_\tau \in_W \mathcal{X}$;
- (b) $W_\tau \in \mathcal{X}$;
- (c) $W_\tau \in \mathcal{Y}$;
- (d) $W_\tau \in_{W_\rho} \mathcal{Y}$.

Proof. (a) $\Rightarrow$ (b) If (a) holds then there exists $x \in W$ with

$$W_\tau \subseteq (W_\theta^d \cap W_\theta)^x$$

for some $d \in W$, $d \notin W_\rho$. Since either dx or x is not in W_ρ we have $W_\tau \in \mathcal{X}$.

(b) $\Rightarrow$ (c) Obvious, since $\mathcal{X} \subseteq \mathcal{Y}$.

(c) $\Rightarrow$ (d) Obvious.

(d) $\Rightarrow$ (a) Suppose that (d) holds. Then there exist $x \in W_\rho$, $d \in W$, $d \notin W_\rho$ with

$$W_\tau \subseteq (W_\theta^d \cap W_\rho)^x.$$

Thus,

$$W_\tau \subseteq W_\theta^{dx} \cap W_\theta$$

and $dx \notin W_\rho$. Hence, $W_\tau \in_W \mathcal{X}$. $\square$

3.5 Corollary. (a) If M is relatively $\mathcal{X}$ -projective $\mathcal{H}_R$ -module then $M_{\mathcal{H}_\rho}$ is $\mathcal{Y}$ -projective;

(b) If N is relatively $\mathcal{Y}$ -projective $\mathcal{H}_\rho$ -module with a vertex contained in W_θ for each indecomposable summand of N , then $N^{\mathcal{H}_R}$ is relatively $\mathcal{X}$ -projective.

Proof. If L is an indecomposable summand of M then L is relatively projective for a parabolic subgroup W_λ of the group $W_\theta^d \cap W_\theta$, for $d \in W$, $d \notin W_\rho$. Thus we have

$$L|L \otimes_{\mathcal{H}_\lambda} \mathcal{H}_R.$$

By Mackey's theorem, $L_{\mathcal{H}_\rho}$ is relatively projective for the collection $\mathcal{S}$,

$$\mathcal{S} = \{Q \in \mathcal{P} | Q \subseteq W_\lambda^z \cap W_\rho \text{ for some } z \in \mathcal{D}_{\lambda\rho}\}.$$

Since $W_\lambda^z \cap W_\rho \subseteq (W_\theta^d \cap W_\theta)^z \cap W_\rho$ and either dz or z is not in W_ρ we have $\mathcal{S} \subseteq \mathcal{Y}$. Therefore $L_{\mathcal{H}_\rho}$ and $M_{\mathcal{H}_\rho}$ are relatively $\mathcal{Y}$ -projective.

On the other hand, if L is an indecomposable summand of N and L has a vertex W_τ with $W_\tau \in \mathcal{Y}$, $W_\tau \subseteq W_\theta$, then $L^{\mathcal{H}_R}$ is relatively $\mathcal{H}_\tau$ -projective and $W_\tau \in \mathcal{X}$ by 3.4. So $L^{\mathcal{H}_R}$ and $N^{\mathcal{H}_R}$ is relatively $\mathcal{X}$ -projective. $\square$

We now prove our main result in this section, which is a q -analogue of the Green correspondence in the representation theory of finite groups.

3.6 Theorem. *There is a one to one correspondence between isomorphic classes of indecomposable $\mathcal{H}_R$ -modules with vertex in $\mathcal{X}$ and isomorphic classes of indecomposable $\mathcal{H}_\rho$ -modules with vertex in $\mathcal{X}$, which can be characterized as follows:*

(a) Let M be an indecomposable $\mathcal{H}_R$ -module with vertex W_τ in $\mathcal{X}$. Then $M_{\mathcal{H}_\rho}$ has a unique indecomposable direct summand $f(M)$ with W_τ as vertex.

Furthermore,

$$M_{\mathcal{H}_\rho} \cong f(M) \oplus \left(\bigoplus_i N_i \right)$$

where a vertex N_i lies in $\mathcal{Y}$ for all i .

(b) Let N be an indecomposable $\mathcal{H}_\rho$ -module with vertex W_τ in $\mathcal{X}$. Then $N^{\mathcal{H}_R}$ has a unique indecomposable direct summand $g(N)$ with W_τ as vertex.

Furthermore,

$$N^{\mathcal{H}_R} = g(N) \oplus \left(\bigoplus_j M_j \right)$$

where M_j has a vertex in $\mathcal{X}$ for all j .

(c) In particular, $g(f(M)) = M$ and $f(g(N)) = N$.

Proof. (a) By 3.2 there is an indecomposable $\mathcal{H}_\rho$ -module V with vertex W_τ and

$$(3.a) \quad M|V \otimes_{\mathcal{H}_\rho} \mathcal{H}_R.$$

Applying 3.3 we obtain

$$(V \otimes_{\mathcal{H}_\rho} \mathcal{H}_R)_{\mathcal{H}_\rho} = V \oplus Y_1$$

where Y_1 is $\mathcal{Y}$ -projective. Thus, $M_{\mathcal{H}_\rho}$ is isomorphic to $V \oplus Y$ or Y for some summand Y of Y_1 . However, again by 3.2, $M_{\mathcal{H}_\rho}$ has an indecomposable summand V' with vertex W_τ . Now we claim that V' cannot be isomorphic

to a summand of Y_1 . Otherwise, $W_\tau \in_{W_\rho} \mathcal{Y}$ by [Jo, 3.35], hence $W_\tau \in_W \mathcal{Z}$ by 3.4. This is contrary to $W_\tau \in \mathcal{Z}$. Hence $V' \cong V$ and

$$(3.b) \quad M_{\mathcal{H}_\rho} \cong V \oplus Y$$

just as claimed. The argument also shows V is unique up to isomorphism. Let $f(M) = V$. Then f is well defined.

(b) Let

$$(3.c) \quad N^{\mathcal{H}_R} = M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

be a direct sum of indecomposable $\mathcal{H}_R$ -module. Since, by 3.3,

$$(N^{\mathcal{H}_R})_{\mathcal{H}_\rho} \cong N \oplus Y$$

where Y is relatively $\mathcal{Y}$ -projective, we have, after renumbering, that

$$(3.d) \quad (M_1)_{\mathcal{H}_\rho} \cong N \oplus Y_1, \quad (M_i)_{\mathcal{H}_\rho} \cong Y_i, \quad 2 \leq i \leq t,$$

where the Y_i 's are $\mathcal{H}_\rho$ -modules and

$$Y \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_t.$$

We claim that M_1 has a vertex in $\mathcal{Z}$ and that $M_2, \dots, M_t$ are $\mathcal{Z}$ -projective.

Indeed, since $M_i | N \otimes_{\mathcal{H}_\rho} \mathcal{H}_R$ and $N | N \otimes_{\mathcal{H}_\tau} \mathcal{H}_\rho$ we have

$$M_i | N \otimes_{\mathcal{H}_\tau} \mathcal{H}_R.$$

Hence M_i has a vertex contained in W_θ . Let $W_{\tau'} \subseteq W_\theta$ be a vertex of M_1 . Suppose that M_1 is relatively $\mathcal{Z}$ -projective. Then 3.5 implies that $(M_1)_{\mathcal{H}_\rho} \cong N \oplus Y_1$ is relatively $\mathcal{Y}$ -projective. It follows that $W_\tau \in_{W_\rho} \mathcal{Y}$ since the vertex of N is W_τ , hence $W_\tau \in_W \mathcal{Z}$ by 3.4, a contradiction. So M_1 is not $\mathcal{Z}$ -projective, that is $W_{\tau'} \notin_W \mathcal{Z}$ by 3.4 again. Hence $W_{\tau'} \in \mathcal{Z}$, as claimed.

Moreover, if M_i , ($i > 1$) was not relatively $\mathcal{Z}$ -projective then, by 3.5, $(M_i)_{\mathcal{H}_\rho}$ would not be relatively $\mathcal{Y}$ -projective since M_i has a vertex contained in W_θ , a contradiction. Hence M_i ($i > 1$) is indeed relatively $\mathcal{Z}$ -projective. Let $g(N) = M_1$. We have seen that $g(N)$ is unique up to isomorphism, so g is well defined.

It remains to check (c). By (3.a) and (3.c) we have

$$M | f(M)^{\mathcal{H}_R}, \quad f(M)^{\mathcal{H}_R} \cong g(f(M)) \oplus X$$

where X is relatively $\mathcal{Z}$ -projective. Hence, $g(f(M)) \cong M$. Similarly, (3.b) and (3.d) imply $f(g(N)) \cong N$.

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