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STABLE AND UNIFORMLY STABLE UNIT BALLS IN BANACH SPACES

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ABSTRACT. Let X be a Banach space with closed unit ball B_X and, for $x \in X$, $r \geq 0$, put $B(x; r) = \{u \in X: \|u - x\| \leq r\}$ and $V(x, r) = B_X \cap B(x; r)$. We say that B_X (or in general a convex set) is *stable* if the midpoint map $\Phi_{1/2}: B_X \times B_X \to B_X$, with $\Phi_{1/2}(u, v) = \frac{1}{2}(u + v)$, is open. We say that B_X is *uniformly stable* (US) if there is a map $\alpha: (0, 2] \to (0, 2]$, called a *modulus of uniform stability*, such that, for each $x, y \in B_X$ and $r \in (0, 2]$, $V(\frac{1}{2}(x + y); \alpha(r)) \subseteq \frac{1}{2}(V(x; r) + V(y; r))$. Among other things, we see: (i) if $\dim X \geq 3$, then X admits an equivalent norm such that B_X is not stable; (ii) if $\dim X < \infty$, B_X is stable iff B_X is US; (iii) if X is rotund, X is uniformly rotund iff B_X is US; (iv) if X is 3.2.I.P, B_X is US and $\alpha(r) = r/2$ is a modulus of US; (v) B_X is US iff $B_{X^{**}}$ is US and X, X^{**} have (almost) the same modulus of US; (vi) B_X is stable (resp. US) iff $B_{C(K, X)}$ is stable (resp. US) for each compact K iff $B_{A(K, X)}$ is stable (resp. US) for each Choquet simplex K; (vii) B_X is stable iff $B_{L_p(\mu, X)}$ is stable for each measure μ and $1 \leq p < \infty$.

0. INTRODUCTION

Let X be a normed space and B_X and S_X the closed unit ball and unit sphere of X, respectively. If A, B are subsets of X, define the distance $d(A, B) = \sup_{x \in A} \inf_{y \in B}\{\|x - y\|\}$. If $x \in X$ and $\varepsilon > 0$, we write $B(x; \varepsilon) = \{y \in X: \|x - y\| \leq \varepsilon\}$ and $V(x; \varepsilon) = B_X \cap B(x; \varepsilon)$. We denote by $\operatorname{Ext}(C)$ the set of extreme points of a set C. A convex set C is said to be *stable* if the midpoint map $\Phi_{1/2}: C \times C \to C$, $\Phi_{1/2}(u, v) = \frac{1}{2}(u + v)$, is open. Stable convex sets have been studied in [4, 12, 3]. Many Banach spaces have stable unit ball, namely: strictly convex or rotund Banach spaces, Banach spaces with 3.2.I.P. [4, p. 195], finite dimensional Musielak-Orlicz spaces [5], etc. In case of stable unit balls, the characterization of some extreme elements is very easy. For instance, if K is a compact space, B_X stable and $f \in C(K, X)$, then $f \in \operatorname{Ext}(B_{C(K, X)})$ iff $f(K) \subseteq \operatorname{Ext}(B_X)$. If K is a Choquet simplex, B_X stable and $f \in A(K, X)$ (= affine continuous functions $g: K \to X$), then $f \in \operatorname{Ext}(B_{A(K, X)})$ iff $f(\operatorname{Ext}(K)) \subseteq \operatorname{Ext}(B_X)$ (see [4, 2.1. Theorem]). Let $\mathscr{K}(X, C(K))$ be the space of compact operators $T: X \to C(K)$. It is known that $\mathscr{K}(X, C(K))$ is isometrically isomorphic to the space $C(K, X^*)$. Follow-

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ing Morris and Phelps [11], an operator $T \in \mathscr{K}(X, C(K))$ is said to be a *nice* operator, if its adjoint T^* satisfies $T^*(K) \subseteq \operatorname{Ext}(B_{X^*})$. Let $\mathscr{N}\mathscr{K}(X, C(K))$ be the set of nice operators. Then clearly $\mathscr{N}\mathscr{K}(X, C(K)) \subseteq \operatorname{Ext}(B_{\mathscr{K}(X, C(K))})$ and, if B_{X^*} is stable, we have $\mathscr{N}\mathscr{K}(X, C(K)) = \operatorname{Ext}(B_{\mathscr{K}(X, C(K))})$ (see [2, 15, 16]).

If $x, y \in B_X$ and $s \geq 0$, denote $D(x, y; s) = \frac{1}{2}[V(x; s) + V(y; s)]$. Obviously, $D(x, y; s)$ is a convex and, in general, not norm closed set; but if X is a dual space, then $D(x, y; s)$ is w^*-compact and so norm closed. A map $\alpha: [0, 2] \to [0, 2]$ is a *modulus of stability* of B_X if $V(\frac{1}{2}(x+y); \alpha(r)) \subseteq D(x, y; r)$ for each $x, y \in B_X$ and $r \in [0, 2]$. The *maximum modulus of stability* α_m of B_X is

$$r \in [0, 2], \qquad \alpha_m(r) = \sup\{s \in [0, 2]\colon \text{ for each } x, y \in B_X,\ V(\tfrac{1}{2}(x+y); s) \subseteq D(x, y; r)\}.$$

Clearly α_m is nondecreasing, null in $[0, r_0)$ and positive in $(r_0, 2]$, for some $r_0 \geq 0$, and $\alpha_m(2) = 2$. The closed unit ball B_X is said to be *uniformly stable* (US) if there exists a map $\alpha: (0, 2] \to (0, 2]$ (called a *modulus of US*) such that for each $x, y \in B_X$ and $r \in (0, 2]$ we have: $V(\frac{1}{2}(x+y); \alpha(r)) \subseteq D(x, y; r)$. Of course, B_X is US iff $\alpha_m(r) > 0$ for each $r \in (0, 2]$.

In this paper we study *stable* and *uniformly stable* (US) unit balls in Banach spaces. In §§1 and 2 we see some elementary facts as: (i) if $\operatorname{Dim} X \geq 3$, X admits an equivalent norm such that B_X is not stable; (ii) if X is a dual space, α_m is right continuous; (iii) if $\operatorname{Dim} X < \infty$, B_X is stable iff B_X is US; (iv) if X is rotund, X is uniformly rotund iff B_X is US; (v) B_X is US if X is 3.2.I.P. In §3 we prove that B_X is US iff $B_{X^{**}}$ is US and the moduli of US are (almost) the same. §§4 and 5 are devoted to study the unit balls $B_{C(K, X)}$ and $B_{A(K, X)}$, where $C(K, X)$ (resp. $A(K, X)$) is the space of continuous (resp. affine continuous) functions $f\colon K \to X$ on the compact K (resp. convex compact K). It is proved that B_X is stable (resp. US) iff $B_{C(K, X)}$ is stable (resp. US) for each compact K iff $B_{A(K, X)}$ is stable (resp. US) for each Choquet simplex K. In §6 we prove that B_X is stable iff $B_{L_p(\mu, X)}$ is stable, $1 \leq p < \infty$. Finally, it is an open problem if $B_{L_p(\mu, X)}$, $1 \leq p < \infty$, is US when B_X is US.

1. Preliminary results

We begin with some elementary remarks: (a) Let C be a convex subset of some locally convex space E, $0 < \lambda < 1$, and define $\Phi_\lambda\colon C \times C \to C$ by $\Phi_\lambda(x, y) = \lambda x + (1-\lambda)y$. Then $\Phi_{1/2}$ is open iff Φ_λ is open (see [4, 1.1 Proposition]).

(b) It is easily seen that B_X is stable, if X is a normed space with strictly convex or rotund norm.

(c) Every norm in $\mathbf{R}$ and $\mathbf{R}^2$ produces stable closed unit balls. In $\mathbf{R}$ this result is evident and, concerning $\mathbf{R}^2$, we can apply that, for a compact convex set $K \subseteq \mathbf{R}^3$, the map $\Phi_{1/2}$ is open iff $\operatorname{Ext}(K)$ is closed (see [4]). But if z belongs to $S_{\mathbf{R}^2} \setminus \operatorname{Ext}(B_{\mathbf{R}^2})$, there exist $x, y \in S_{\mathbf{R}^2} \setminus \{z\}$ such that $z = \frac{1}{2}(x+y)$. Let $\varepsilon = \|z - x\|$. Then $B(z; \varepsilon/2) \cap S_{\mathbf{R}^2} \subseteq S_{\mathbf{R}^2} \setminus \operatorname{Ext}(B_{\mathbf{R}^2})$. Therefore $S_{\mathbf{R}^2} \setminus \operatorname{Ext}(B_{\mathbf{R}^2})$ is open in $S_{\mathbf{R}^2}$ and hence $\operatorname{Ext}(B_{\mathbf{R}^2})$ is closed.

(d) In $\mathbf{R}^3$ it is easy to give a norm such that B_X is not stable. Take

$$C = \{(x, y, 0)\colon x^2 + y^2 = 1\}, \qquad K = \{(0, y, z)\colon \max\{|y|, |z|\} \leq 1\}.$$

The new closed unit ball will be $B = \mathrm{co}(C \cup K)$, that is, the convex hull of the set $C \cup K$. Observe that

$$\mathrm{Ext}(B) = \{C \backslash \{(0, 1, 0), (0, -1, 0)\}\} \cup \{(0, \pm 1, \pm 1)\}.$$

As this set is not closed, by the result of [4] aforementioned in (c), B is not stable.

If $\dim X \geq 3$, the situation is similar. We need the following lemma:

1.1. **Lemma.** *Let X be a normed space. Then*: (A) *If B_X is stable,* $\mathrm{Ext}(B_X)$ *is closed.*

(B) *Let $Y \subseteq X$ be a* 1*-complemented subspace. If B_X is stable, then B_Y is stable.*

Proof. (A) Let $\Delta = \{(x, x) : x \in B_X\}$. As Δ is closed in $B_X \times B_X$, $\Phi_{1/2}$ open and $B_X \backslash \mathrm{Ext}(B_X) = \Phi_{1/2}(B_X \times B_X \backslash \Delta)$, then $B_X \backslash \mathrm{Ext}(B_X)$ is open and $\mathrm{Ext}(B_X)$ is closed in B_X.

(B) We say that Y is 1-complemented in X if there exists a projection $P : X \to Y$ such that $\|P\| = 1$. Let $\Phi^X_{1/2}$, $\Phi^Y_{1/2}$ be the midpoint maps in B_X and B_Y respectively. Since $P \circ \Phi^X_{1/2} = \Phi^Y_{1/2} \circ (P \times P)$ and P is open, we get that $\Phi^Y_{1/2}$ is open if $\Phi^X_{1/2}$ is open. □

1.2. **Proposition.** *Let X be a normed space with* $\dim X \geq 3$. *Then X admits an equivalent norm such that the new closed unit ball is not stable.*

Proof. We write X as a direct topological sum $X = X_1 \oplus X_2$ with $\dim X_1 = 3$. Take in X_1 the norm $\|\cdot\|_1$ used in (d) above, an arbitrary equivalent norm $\|\cdot\|_2$ in X_2 and in X the norm $\|\cdot\|_3$ defined by

$$\text{if } x = (x_1, x_2) \in X_1 \oplus X_2 = X, \text{ then } \|x\|_3 = \sup\{\|x_1\|_1, \|x_2\|_2\},$$

Now it is enough to consider (d) and Lemma 1.1. □

1.3. **Proposition.** *Let $X = Y^*$ be a dual Banach space. Then α_m is right-continuous.*

Proof. Let $s_0 \in [0, 2)$, $\varepsilon > 0$ and suppose that $\lim_{s \to s_0^+} \alpha_m(s) = \alpha_m(s_0) + \varepsilon$. Choose $x, y \in B_X$ such that, if $z = \frac{1}{2}(x+y)$, $V(z; \alpha_m(s_0) + \frac{\varepsilon}{4}) \not\subseteq D(x, y; s_0)$. By hypothesis, if $s > s_0$, we have $V(z; \alpha_m(s_0) + \varepsilon) \subseteq D(x, y; s)$. Clearly the distance $d(D(x, y; s), D(x, y; s_0)) \to 0$ when $s \to s_0^+$. Since $D(x, y; s_0)$ is norm closed (is w^*-compact), we conclude that $\bigcap_{s > s_0} D(x, y; s) = D(x, y; s_0)$. Therefore $V(z; \alpha_m(s_0) + \varepsilon) \subseteq D(x, y; s_0)$, a contradiction. □

1.4. **Proposition.** *Let X be a finite dimensional normed space. The following are equivalent*: (a) *B_X is stable*; (b) *B_X is US.*

Proof. As $b \Rightarrow a$ is clear, we prove that $a \Rightarrow b$. If B_X is not US, there exist $r > 0$ and sequences $\{x_n\}_{n \geq 1}$, $\{y_n\}_{n \geq 1}$ in B_X such that, if $z_n = \frac{1}{2}(x_n + y_n)$, then $V(z_n; 1/n) \not\subseteq D(x_n, y_n; r)$. Since B_X is compact, we can suppose the existence of $\lim_{n \to \infty} x_n = x_0$, $\lim_{n \to \infty} y_n = y_0$ and $\lim_{n \to \infty} z_n = z_0 = \frac{1}{2}(x_0 + y_0)$. Since B_X is stable, there exists $\varepsilon > 0$ such that $V(z_0; \varepsilon) \subseteq D(x_0, y_0; r/2)$. Taking limits in $V(z_n; 1/n) \not\subseteq D(x_n, y_n; r)$, we get a contradiction. □

1.5. **Proposition.** *Let X be a rotund normed space. The following are equivalent:*

(a) *X is uniformly rotund;*
(b) *B_X is US.*

Proof. (a) $\Rightarrow$ (b). Let $\delta: (0, 2] \to (0, 1]$ be a nondecreasing modulus of uniform rotundity of X and take $x, y \in B_X$, $z = (x+y)/2$ and $r \in (0, 2]$. Suppose firstly that $r < \|x-y\| = \varepsilon$. By hypothesis, $B(z; \delta(\varepsilon)) \subseteq B_X$. If $\lambda = r/(\delta(\varepsilon) + \frac{1}{2}\varepsilon)$, the homothecy $h(y; \lambda)$, with centre y and ratio λ, satisfies $h(y; \lambda)[B(z; \delta(\varepsilon))] = B_1 \subseteq V(y; r)$. Also $h(x; \lambda)[B(z; \delta(\varepsilon))] = B_2 \subseteq V(x; r)$. Observe that the radius of B_i, $i = 1, 2$, is $\rho(\varepsilon) = [r\cdot\delta(\varepsilon)]/[\delta(\varepsilon) + \frac{1}{2}\varepsilon]$. In short, if $\varepsilon = \|x-y\| > r$, then $V(z; \rho(\varepsilon)) = B(z; \rho(\varepsilon)) = \frac{1}{2}(B_1 + B_2) \subseteq D(x, y; r)$. Suppose finally that $\|x-y\| \le r$. Then clearly $V(z; r/2) \subseteq D(x, y; r)$. So a modulus of US of B_X is

$$\alpha(r) = \min\left\{\frac{\delta(r)\cdot r}{\delta(r) + r/2}, \frac{r}{2}\right\}, \qquad r \in (0, 2].$$

(b) $\Rightarrow$ (a). Let α be a modulus of US of B_X and $\varepsilon > 0$. We prove that there exists $\delta > 0$ such that $\|(x+y)/2\| \le 1 - \delta$, for each $x, y \in S_X$ with $\|x-y\| \ge \varepsilon$. By hypothesis $V((x+y)/2; \alpha(\varepsilon/2)) \subseteq D(x, y; \varepsilon/2)$. Since X is rotund, $V(x; \varepsilon/2) \cap V(y; \varepsilon/2) \subseteq \{(x+y)/2\}$. But $(x+y)/2 \notin S_X$. Hence $V((x+y)/2; \alpha(\varepsilon/2)) \cap S_X = \varnothing$, because $V((x+y)/2; \alpha(\varepsilon/2)) \cap S_X \subseteq D(x, y; \varepsilon/2) \cap S_X \subseteq V(x; \varepsilon/2) \cap V(y; \varepsilon/2) \cap S_X = \varnothing$. So the distance $d((x+y)/2; S_X) \ge \alpha(\varepsilon/2)$, that is, $\|(x+y)/2\| \le 1 - \alpha(\varepsilon/2)$. □

1.6. **Proposition.** *Let X be a normed space, α a modulus of US of B_X and $Y \subseteq X$ a 1-complemented subspace of X. Then α is a modulus of US of B_Y.*

Proof. Take $\varepsilon > 0$, $x, y \in B_Y$, $z = (x+y)/2$ and $P: X \to Y$ a projection with $\|P\| = 1$. If the subindex Y indicates we work in B_Y, we have the following:

$$V_Y(z; \alpha(\varepsilon)) = P(V(z; \alpha(\varepsilon))) \subseteq P(D(x, y; \varepsilon)) = D_Y(x, y; \varepsilon). \quad \square$$

1.7. **Example.** Let us see some examples of Banach spaces X such that B_X is stable but not US. Indeed, take $X = (\sum_{n\ge1} \bigoplus l_{1+1/n})_p$ or $X = (\sum_{n\ge2} \bigoplus l_n)_p$, $1 < p < \infty$. The unit ball B_X is rotund but not US, because if B_X is US, from Proposition 1.6 we would deduce that the family $\{l_n\}_{n\ge2}$ or $\{l_{1+1/n}\}_{n\ge1}$ is uniformly US (that is, there is a common modulus of US). Now, from the proof of Proposition 1.5, we conclude that these families are uniformly uniformly rotund, which is not true. □

1.8. **Proposition.** *Let X be a Banach space and $Y \subseteq X$ a dense US subspace. Then X is US.*

Proof. Let α be a left-continuous and nondecreasing modulus of US of Y (if α is not left-continuous, take $\tilde{\alpha}(r) = \lim_{\varepsilon\to0^+} \alpha(r-\varepsilon)$). We prove that $m\alpha$ is a modulus of US of B_X for each $0 < m < 1$. Pick $x, y \in B_X$, $r \in (0, 2]$ and $z \in V((x+y)/2; m\alpha(r))$. Since α is left-continuous, we can choose $x' \in V(x; r) \cap Y$, $y' \in V(y; r) \cap Y$ and $0 < \varepsilon$, δ such that

$$z \in \overline{D}_Y(x', y'; r - \varepsilon), \qquad V(x'; r-\varepsilon) \subseteq V(x; r-\delta)$$

and $V(y'; r-\varepsilon) \subseteq V(y; r-\delta)$. Now take $x_1 \in V_Y(x'; r-\varepsilon)$ and $y_1 \in V_Y(y'; r-\varepsilon)$ such that $z \in \overline{D}_Y(x_1, y_1; \delta/2)$. Next take $x_2 \in V_Y(x_1; \delta/2)$ and $y_2 \in V_Y(x_2; \delta/2)$ such that $z \in \overline{D}_Y(x_2, y_2; \delta/4)$. By reiteration we get Cauchy sequences $\{x_n\}_{n\geq 1} \subseteq V(x; r)$, $\{y_n\}_{n\geq 1} \subseteq V(y; r)$ satisfying $\|z-(x_n+y_n)/2\| \leq \delta/2^n$. Then, if $x_0 = \lim_{n\to\infty} x_n$ and $y_0 = \lim_{n\to\infty} y_n$, clearly $x_0 \in V(x; r)$, $y_0 \in V(y; r)$ and $z = (x_0+y_0)/2$. □

The proof of Proposition 1.9 following is straightforward and left to the reader.

1.9. **Proposition.** *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces.*

(A) *The following are equivalent* ($\mathcal{TFAE}$):

(1) *If $X = (\sum_{i\in I} \bigoplus X_i)_0$, B_X is stable;* (2) *B_{X_i} is stable for each $i \in I$.*

(B) $\mathcal{TFAE}$:

(1) *If $X = (\sum_{i\in I} \bigoplus X_i)_\infty$, B_X is stable.*

(2) *Each B_{X_i} is stable and for each $r \in (0, 2]$, there exists a finite subset $F(r) \subseteq I$ such that $\inf\{\alpha_{mi}(r) \colon i \in I\backslash F(r)\} > 0$, where α_{mi} is the maximum modulus of stability of X_i.*

(C) *For $p = 0$ or $p = \infty$,* $\mathcal{TFAE}$:

(1) *α is a modulus of US of B_X, with $X = (\sum_{i\in I} \bigoplus X_i)_p$.*

(2) *α is a modulus of US of B_{X_i}, for each $i \in I$.*

(D) *Let I be an infinite set, Y a Banach space, $X_i = Y$ for each $i \in I$ and $X = (\sum_{i\in I} \bigoplus X_i)_\infty$.* $\mathcal{TFAE}$:

(1) *B_X is stable, with $X = (\sum_{i\in I} \bigoplus X_i)_\infty$.*

(2) *B_Y is US and α is a modulus of US of B_Y.*

(3) *B_X is US and α is a modulus of US of B_X.*

2. Uniformly stable unit balls and intersection property of balls

A normed space X has the n.2.I.P. (n.2. intersection property, see [6, p. 207; 8; 9]) if for each set of closed balls $S_1, S_2, \ldots, S_n$ in X (with varying centers and radii) such that $S_i \cap S_j \neq \varnothing$ for all i and j, it follows that $\bigcap_{i=1}^n S_i \neq \varnothing$. It is known (see [4, p. 195]) that if X is 3.2.I.P, then B_X is stable and we see here that $\alpha(r) = r/2$ is a modulus of US of X.

2.1. **Proposition.** *Let X be a Banach space with 3.2.I.P. Then $\alpha(r) = r/2$, $r \in (0, 2]$, is a modulus of US of X.*

Proof. Take $r \in (0, 2]$, $x, y \in B_X$ and $d \in X$ satisfying $\|d\| \leq r/2$ and $\|x+y+2d\| \leq 2$. Let $z = (x+y)/2$ and $u = z+d$. Then $u \in V(z; r/2)$. We must find $x' \in V(x; r)$ and $y' \in V(y; r)$ such that $u = (x'+y')/2$. Let $B_1 = B(0; 1)$, $B_2 = B(x+d; r/2)$ and $B_3 = B(x+y+2d; 1)$. These balls intersect mutually. So there is $x' \in \bigcap_{i=1}^3 B_i$. Now $x' \in V(x; r)$ (since $\|x'\| \leq 1$), $B_2 \subseteq B(x; r)$ and, if $y' = 2u - x'$, also $y' \in V(y; r)$. Finally $u = (x'+y')/2$. □

3. Uniformly stable unit balls and bidual

Let X be a Banach space. We prove that B_X is US iff $B_{X^{**}}$ is US. Let $N \subseteq X$ be a closed subspace, $Y = X/N$ and $Q \colon X \to Y$ the quotient map.

We say that Q is an *open quotient* if $Q(B_X) = B_Y$ and Q restricted to B_X is open. We say that Q is *uniformly open quotient* (u.o.q.) if Q is open and there exist a map $c: (0, 2] \to (0, 2]$, called an u.o.q. modulus, such that, for each $x \in B_X$ and $r \in (0, 2]$, $V(Q(x); c(r)) \subseteq Q(V((x; r)))$.

3.1. **Examples.** (1) Let K, K_1 be compact Hausdorff spaces such that $K_1 \subseteq K$. Then the restriction map $Q: C(K) \to C(K_1)$, defined by $Q(f) = f_{|K_1}$ for each $f \in C(K_1)$, is an u.o.q. with modulus $c(r) = r$.

(2) Let I be a set, $\mathscr{U}$ an ultrafilter on I, $\{X_i\}_{i\in I}$ a family of Banach spaces, $Y = (\sum_{i\in I} \bigoplus X_i)_\infty$ and $(X_i)_\mathscr{U}$ the ultraproduct with respect to $\mathscr{U}$ (see [10, p. 121]), that is, $(X_i)_\mathscr{U} = Y/N$, where $N = \{x = (x_i)_{i\in I} \in Y: \lim_\mathscr{U} x_i = 0\}$. Then the quotient $Q: Y \to Y/N$ is an u.o.q. with modulus $c(r) = r$. Indeed, denote by (x_i) an element of Y and $Q((x_i)) = (x_i)_\mathscr{U}$ the image element in Y/N. Suppose that $\|(x_i)_\mathscr{U}\| \leq 1$. Define $(y_i) \in Y$ as follows: $y_i = x_i$, if $\|x_i\| \leq 1$, and $y_i = x_i/\|x_i\|$, if $\|x_i\| > 1$. Then $(y_i)_\mathscr{U} = (x_i)_\mathscr{U}$ and $(y_i) \in B_Y$. This proves that $Q(B_Y) = B_{Y/N}$. Now take $(x_i) \in B_Y$, $r \in (0, 2]$ and $(y_i)_\mathscr{U} \in V((x_i)_\mathscr{U}; r)$. We want to find $(z_i) \in V((x_i); r)$ such that $(z_i)_\mathscr{U} = (y_i)_\mathscr{U}$. We can suppose that $1 \geq \|(y_i)\| = \sup\{\|y_i\|: i \in I\}$ (if $1 < \|(y_i)\|$ for some $i \in I$, take (y_i') defined by $y_i' = y_i$, if $\|y_i\| \leq 1$, and $y_i' = y_i/\|y_i\|$ if $\|y_i\| > 1$, that satisfies $\|(y_i')\| \leq 1$ and $(y_i)_\mathscr{U} = (y_i')_\mathscr{U}$). Choose (z_i) as follows: $z_i = y_i$ if $\|x_i - y_i\| \leq r$ and $z_i = x_i + r \cdot (y_i - x_i)/\|y_i - x_i\|$ if $\|x_i - y_i\| > r$. This proves that $V((x_i)_\mathscr{U}; r) \subseteq Q(V((x_i); r))$.

3.2. **Proposition.** *Let X be a Banach space, $N \subseteq X$ a closed subspace and $Q: X \to Y = X/N$ the quotient map. Then*

(a) *If B_X is stable and Q is an open quotient, B_Y is stable.*

(b) *If α is a modulus of US of B_X and Q is an u.o.q. with modulus c, then $c \circ \alpha$ is a modulus of US of B_Y.*

Proof. (a) Take $x, y \in B_X$ and $r > 0$. We prove that there exists $s > 0$ such that $V(Q((x+y)/2); s) \subseteq D(Q(x), Q(y); r)$. Since B_X is stable, there exists $s' > 0$ such that $V((x+y)/2; s') \subseteq D(x, y; r)$. As Q is open, there exists $s > 0$ such that $V(Q((x+y)/2); s) \subseteq Q[V((x+y)/2; s')]$. In consequence

$$\begin{aligned} V\left(Q\left(\frac{x+y}{2}\right); s\right) &\subseteq Q\left[V\left(\frac{x+y}{2}; s\right)\right] \\ &\subseteq Q(D(x, y; r)) \subseteq D(Q(x), Q(y); r). \end{aligned}$$

(b) Let $x, y \in B_X$ and $r \in (0, 2]$. By hypothesis $V((x+y)/2; \alpha(r)) \subseteq D(x, y; r)$. Hence:

$$\begin{aligned} V\left(Q\left(\frac{x+y}{2}\right); c\circ\alpha(r)\right) &\subseteq Q\left[V\left(\frac{x+y}{2}; \alpha(r)\right)\right] \subseteq Q(D(x, y; r)) \\ &\subseteq D(Q(x), Q(y); r). \end{aligned}$$

So $c \circ \alpha$ is a modulus of US of Y. □

3.3. **Corollary.** *Let I be a set, $\mathscr{U}$ an ultrafilter on I, α a modulus of US common to all members of the family of Banach spaces $\{X_i\}_{i\in I}$ and $(X_i)_\mathscr{U}$ the ultraproduct with respect to $\mathscr{U}$. Then α is a modulus of US of $B_{(X_i)_\mathscr{U}}$.*

Proof. By (C) of Proposition 1.9, α is a modulus of US of B_Y, being $Y = (\sum_{i\in I} \bigoplus X_i)_\infty$. Now apply Proposition 3.2 and Example 3.1. □

3.4. **Proposition.** *Let X be a Banach space. Then B_X is US iff $B_{X^{**}}$ is US. Moreover, the maximum moduli α_m, α''_m of US of B_X, $B_{X^{**}}$, respectively, satisfy $\alpha''_m(r) = \lim_{\varepsilon\to 0^+} \alpha_m(r+\varepsilon)$, for each $r \in (0, 2)$.*

Proof. (1) Let α be a modulus of US of $B_{X^{**}}$, $x, y \in B_X$, $r \in (0, 2]$, $\varepsilon > 0$ and $d \in X$ such that $\|d\| \le \alpha(r)$ and $x + y + 2d = u \in B(0; 2)$, that is, $u/2 \in V((x+y)/2; \alpha(r))$. We claim that there exist $x' \in V(x; r+\varepsilon)$, $y' \in V(y; r+\varepsilon)$ such that $x', y' \in X$ and $u/2 = (x'+y')/2$. This fact will imply that $V((x+y)/2; \alpha(r)) \subseteq D(x, y; r+\varepsilon)$ in X. By hypothesis, there exist $z \in V''(x; r)$, $v \in V''(y; r)$ (we put V'' when we work in X^{**}) such that $u/2 = (z+v)/2$. This implies that $z \in [B''(0; 1) \cap B''(x; r) \cap B''(x+2d; r) \cap B''(u; 1)]$, where B'' indicates that we take balls in X^{**}. Let A be the subspace of X^{**} spanned by $\{x, x+2d, u, z\}$ and $\theta > 0$ such that

$$(1+\theta)\|z\| \le 1 + \alpha(\varepsilon/16) \ge (1+\theta)\|z-u\|,$$
$$(1+\theta)\|x-z\| \le r + \varepsilon/2 \ge \|x+2d-z\|(1+\theta).$$

By the principle of local reflexivity (see [10, p. 196]), there exists an operator $T\colon A \to X$ such that: (i) $T(w) = w$, $w \in A \cap X$; (ii) $\|T\| \le 1+\theta$. So if $z_1 = T(z)$, we have

$$z_1 \in [B(0; 1+\alpha(\varepsilon/16)) \cap B(x; r+\varepsilon/2) \cap B(x+2d; r+\varepsilon/2) \cap B(u; 1+\alpha(\varepsilon/16))].$$

Let

$$(*) \qquad x_1 = \begin{cases} z_1 & \text{if } z_1 \in B(0; 1), \\ z_1/\|z_1\| & \text{if } z_1 \notin B(0; 1), \end{cases}$$
$$y_1 = \begin{cases} u - z_1 & \text{if } z_1 \in B(u; 1), \\ (u-z_1)/\|u-z_1\| & \text{if } z_1 \notin B(u; 1), \end{cases}$$

and $d_1 \in X$ such that $x_1 + y_1 + 2d_1 = u$. As $\|z_1 - x_1\| \le \alpha(\varepsilon/16) \ge \|x_1 + 2d_1 - z_1\|$, we get $\|d_1\| \le \alpha(\varepsilon/16)$. Apply again the principle of local reflexivity: there exists $z_2 \in X$ such that

$$z_2 \in [B(0; 1+\alpha(\varepsilon/32)) \cap B(x_1; \varepsilon/16 + \varepsilon/32) \cap B(x_1 + 2d_1; \varepsilon/16 + \varepsilon/32) \cap B(u; 1+\alpha(\varepsilon/32))].$$

Now $\|z_1 - z_2\| \le \|z_1 - x_1\| + \|x_1 - z_2\| \le \alpha(\varepsilon/16) + \varepsilon(1/16 + 1/32) \le \varepsilon/4$, because always $\alpha(s) \le s$. Define x_2, y_2 as in $(*)$ using now z_2 and take $d_2 \in X$ such that $x_2 + y_2 + 2d_2 = u$. Then $\|d_2\| \le \alpha(\varepsilon/32)$. Apply the principle of local reflexivity: there exists $z_3 \in X$ such that

$$z_3 \in [B(0; 1+\alpha(\varepsilon/64)) \cap B(x_2; \varepsilon(2^{-5} + 2^{-6})) \cap B(x_2 + 2d_2; \varepsilon(2^{-5} + 2^{-6})) \cap B(u; 1+\alpha(\varepsilon\cdot 2^{-6}))]$$

and $\|z_2 - z_3\| \le \|z_2 - x_2\| + \|x_2 - z_3\| \le \alpha(\varepsilon\cdot 2^{-5}) + \varepsilon(2^{-5} + 2^{-6}) \le \varepsilon\cdot 2^{-3}$. By reiteration, we obtain a Cauchy sequence $\{z_n\}_{n\ge 1}$ such that, if $x' = \lim_{n\to\infty} z_n$, then clearly

$$x' \in [B(0; 1) \cap B(x; r+\varepsilon) \cap B(x+2d; r+\varepsilon) \cap B(u; 1)].$$

Thus $x' \in V(x; r+\varepsilon)$ and, if $y' = u - x'$, then $y' \in V(y; r+\varepsilon)$ and $\frac{u}{2} = \frac{1}{2}(x'+y')$. This proves, in particular, that $\alpha_m(r+\varepsilon) \geq \alpha''_m(r)$ for each $r \in (0, 2)$ such that $r+\varepsilon \leq 2$.

(2) Let B_X be US. It follows from the principle of local reflexivity that X^{**} is 1-complemented in $(X)_{\mathscr{U}}$ for some ultrafilter $\mathscr{U}$. Applying Corollary 3.3, we get that $\alpha''_m \geq \alpha_m$. Thus $B_{X^{**}}$ is US.

Finally for $r \in (0, 2)$ and $\varepsilon > 0$ with $r + \varepsilon \leq 2$, we have: $\alpha''_m(r+\varepsilon) \geq \alpha_m(r+\varepsilon) \geq \alpha''_m(r)$. Thus, as α''_m is right-continuous (see Proposition 1.3), we get that $\alpha''_m(r) = \lim_{\varepsilon\to 0^+} \alpha_m(r+\varepsilon)$, for each $r \in (0; 2)$. □

4. The unit ball in $C(K, X)$

If K is a compact Hausdorff space and X a Banach space, we denote by $C(K, X)$ the Banach space of continuous functions $f: K \to X$ with the supremum norm. We prove in this section that $B_{C(K,X)}$ is stable (resp. US) iff B_X is stable (resp. US).

4.1. **Lemma.** *Let X be a Banach space such that B_X is stable, $x, y \in B_X$ and $\varepsilon > 0$. Then there exist $0 < \delta, \eta$ such that, for each $x' \in V(x; \delta)$, $y' \in V(y; \delta)$, we have*

$$V((x'+y')/2; \eta) \subseteq D(x', y'; \varepsilon).$$

Proof. Let η_1 such that

$$V((x+y)/2; \eta_1) \subseteq D(x, y; \varepsilon/2)$$

and

$$\delta = \min\{\varepsilon/2, \eta_1/2\}.$$

Then, if $x' \in V(x; \delta)$ and $y' \in V(y; \delta)$, we have

$$V((x+y)/2; \eta_1) \subseteq D(x, y; \varepsilon/2) \subseteq D(x', y'; \varepsilon).$$

But

$$V((x'+y')/2; \eta_1/2) \subseteq V((x+y)/2; \eta_1).$$

Thus it is enough to take $\eta = \eta_1/2$. □

4.2. **Proposition.** *Let K be a compact Hausdorff space and X a Banach space. The following are equivalent:* (1) *$B_{C(K,X)}$ is stable*; (2) *B_X is stable.*

Proof. (1) ⇒ (2) As X is 1-complemented in $C(K, X)$, it is enough to apply Lemma 1.1.

(2) ⇒ (1) Take $f, g \in B_{C(K,X)}$, $\varepsilon > 0$ and the compact $H = \{(f(k), g(k)): k \in K\} \subseteq B_X \times B_X$. For each $z = (x, y) \in H$ choose $0 < \delta_z, \eta_z$ fulfilling Lemma 4.1 with respect to z and $\frac{\varepsilon}{2}$. If $U(z; \delta_z) = V(x; \delta_z) \times V(y; \delta_z)$, then the interiors of $\{U(z; \delta_z); z \in H\}$ cover the compact H. So there exists a finite subfamily $\{U(z_i; \delta_{z_i}): i = 1, 2, \dots, n\}$ that also cover H. Let $\eta = \min\{\eta_{z_i}: i = 1, 2, \dots, n\}$. If $h = (f+g)/2$, we have

$$(1) \qquad V(h(k); \eta) \subseteq D(f(k), g(k); \varepsilon/2) \text{ in } B_X \quad \text{for each } k \in K.$$

We claim that $V(h; \eta) \subseteq D(f, g; \varepsilon)$ in $B_{C(K,X)}$. Indeed, let $p \in V(h; \eta)$ and consider the functions

(a) $\phi: B_X \to 2^{B_X \times B_X}$ (= family of subsets of $B_X \times B_X$) such that

$$x \in B_X, \qquad \phi(x) = \left\{(u, v) \in B_X \times B_X : \frac{u+v}{2} = x\right\},$$

(b) $\psi: K \to 2^{B_X \times B_X}$ such that $\psi(k) = [\phi \circ p(k)] \cap [V(f(k); \varepsilon) \times V(g(k); \varepsilon)]$, $k \in K$.

We have the following:

(I) For each $k \in K$, $\psi(k)$ is a nonempty closed convex set.
Indeed, as $\phi \circ p(k)$ and $V(f(k); \varepsilon) \times V(g(k); \varepsilon)$ are closed convex, $\psi(k)$ is closed convex. Let see that $\psi(k) \neq \varnothing$. As $p \in V(h; \eta)$, $\|h - p\| \leq \eta$ and by (1), for each $k \in K$, $p(k) \in D(f(k), g(k); \varepsilon/2)$. Thus:

(2) for each $k \in K$ there exists $u \in [\phi \circ p(k)] \cap [V(f(k); \varepsilon/2) \times V(g(k); \varepsilon/2)]$.
So $u \in \psi(k)$ and $\psi(k) \neq \varnothing$.

(II) ψ is lower semicontinuous.

Let $U = U_1 \times U_2$ be an open subset of $B_X \times B_X$ and $k_0 \in K$ such that $\psi(k_0) \cap U \neq \varnothing$. The set $A = \{(x_1, x_2) \in \psi(k_0) : \|x_1 - f(k_0)\| \langle \varepsilon \rangle \|x_2 - g(k_0)\|\}$ is nonempty (by (2)) and dense in $\psi(k_0)$. As $\psi(k_0) \cap U \neq \varnothing$, also $A \cap U \neq \varnothing$. Take $z = (z_1, z_2) \in A \cap U$. The continuity of f and g implies the existence of $\rho > 0$ and a neighbourhood G_1 of k_0 in K such that

(3) for each $k \in G_1$, $V(z_1; \rho) \times V(z_2; \rho) \subseteq U \cap [V(f(k); \varepsilon) \times V(g(k); \varepsilon)]$.
As B_X is stable and $(z_1 + z_2)/2 = p(k_0)$, there exists $\delta > 0$ satisfying $V(p(k_0); \delta) \subseteq D(z_1, z_2; \rho)$. In consequence:

(4) for each $y \in V(p(k_0); \delta)$, $\phi(y) \cap [V(z_1; \rho) \times V(z_2; \rho)] \neq \varnothing$.
Let $G_2 = p^{-1}(V(p(k_0); \delta))$ and $G = G_1 \cap G_2$ a neighbourhood of k_0 in K. Then for each $k \in G$ we have $\psi(k) \cap U \neq \varnothing$. Indeed, take $k \in G$:

(a) As $k \in G_2$, we have, by (4), that $[\phi \circ p(k)] \cap [V(z_1; \rho) \times V(z_2; \rho)] \neq \varnothing$.

(b) As $k \in G_1$, we have, by (3), that $V(z_1; \rho) \times V(z_2; \rho) \subseteq U \cap [V(f(k); \varepsilon) \times V(g(k); \varepsilon)]$. Therefore $U \cap \psi(k) = [\phi \circ p(k)] \cap [V(f(k); \varepsilon) \times V(g(k); \varepsilon)] \cap U \neq \varnothing$. Thus ψ is lower semicontinuous.

By Michael's selection theorem [13, p. 5], ψ admits a continuous selection $(s_1, s_2) = s: K \to B_X \times B_X$ such that $p = \frac{1}{2}(s_1 + s_2)$ and $s_1 \in V(f; \varepsilon)$, $s_2 \in V(g; \varepsilon)$. This proves that $V(h; \eta) \subseteq D(f, g; \varepsilon)$ and that $B_{C(K,X)}$ is stable. □

4.3. **Proposition.** *Let K be a compact Hausdorff set and X a Banach space. Then B_X is US iff $B_{C(K,X)}$ is US and the maximum moduli α_m^X, $\alpha_m^{C(K,X)}$ of US of B_X and $B_{C(K,X)}$, respectively, satisfy*

$$\textit{for each } r \in (0, 2], \quad \alpha_m^X(r) \geq \alpha_m^{C(K,X)}(r) \geq \lim_{\varepsilon \to 0^+} \alpha_m^X(r - \varepsilon).$$

Proof. As X is 1-complemented in $C(K, X)$, applying Proposition 1.6, we obtain that B_X is US, provided that $B_{C(K,X)}$ is US and that $\alpha_m^X \geq \alpha_m^{C(K,X)}$.

Suppose now that B_X is US and that α is a modulus of US of B_X. Let $f, g \in B_{C(K,X)}$, $h = (f+g)/2$ and $r \in (0, 2]$. We prove that for each $0 < \varepsilon < r$, $V(h; \alpha(r - \varepsilon)) \subseteq D(f, g; r)$ in $B_{C(K,X)}$. Take $p \in V(h; \alpha(r - \varepsilon))$ and consider the functions ϕ, ψ as in Proposition 4.2, that is,

$$x \in B_X, \qquad \phi(x) = \left\{(u, v) \in B_X \times B_X : \frac{u+v}{2} = x\right\};$$

$$k \in K, \qquad \psi(k) = [\phi \circ p(k)] \cap [V(f(k); r) \times V(g(k); r)].$$

As in Proposition 4.2, ψ is lower semicontinuous and $\psi(k)$ is a closed convex nonempty set, for each $k \in K$. Applying Michael's selection theorem [13, p. 5], we obtain that $p \in D(f, g; r)$ in $B_{C(K,X)}$. Thus $B_{C(K,X)}$ is US and, for each $r \in (0, 2]$, $\alpha_m^{C(K,X)}(r) \geq \lim_{\varepsilon \to 0^+} \alpha_m^X(r - \varepsilon)$. □

5. The unit ball in $A(K, X)$

If K is a compact convex set and X a Banach space, $A(K, X)$ will be the Banach space of affine continuous functions $f: K \to X$ with the supremum norm. A compact convex set K is a Choquet simplex iff $A(K, X)$ is a L_1-predual (see [6, p. 183]; [1, p. 84]).

5.1. **Proposition.** *Let K be a Choquet simplex and X a Banach space. $\mathcal{TFAE}$: (1) $B_{A(K,X)}$ is stable; (2) B_X is stable.*

Proof. (1) $\Rightarrow$ (2) As X is 1-complemented in $A(K, X)$, this follows from Lemma 1.1.

(2) $\Rightarrow$ (1) Take $f, g \in B_{A(K,X)}$, $\varepsilon > 0$ and $h = (f+g)/2$. From Lemma 4.1 it follows that there exists $\eta > 0$ such that, for each $k \in K$, $V(h(k); \eta) \subseteq D(f(k), g(k); \varepsilon/2)$ in B_X. We claim that $V(h; \eta) \subseteq D(f, g; \varepsilon)$ in $B_{A(K,X)}$. Indeed, pick $p \in V(h; \eta)$ and consider the functions ϕ, ψ of Proposition 4.2. We know that ψ is lower semicontinuous and that $\psi(k)$ is a convex closed nonempty set, for each $k \in K$. Moreover, it is easily proved that ψ is affine, that is, if $k_1, k_2 \in K$, $0 \leq \lambda \leq 1$, and $k = \lambda k_1 + (1-\lambda)k_2$, then $\lambda\psi(k_1) + (1-\lambda)\psi(k_2) \subseteq \psi(\lambda k_1 + (1-\lambda)k_2)$. So applying Lazar's theorem [7, Theorem 3.1, p. 511], we get an affine continuous selection $s = (s_1, s_2)$: $K \to B_X \times B_X$ such that $p = \frac{1}{2}(s_1 + s_2)$, $s_1 \in V(f; \varepsilon)$ and $s_2 \in V(g; \varepsilon)$. Therefore $V(h; \eta) \subseteq D(f, g; \varepsilon)$. □

5.2. **Proposition.** *Let K be a Choquet simplex and X a Banach space. Then B_X is US iff $B_{A(K,X)}$ is US, and the maximum moduli α_m^X, $\alpha_m^{A(K,X)}$ of US of B_X and $B_{A(K,X)}$, respectively, satisfy*

$$\textit{for each } r \in (0, 2], \quad \alpha_m^X(r) \geq \alpha_m^{A(K,X)}(r) \geq \lim_{\varepsilon \to 0^+} \alpha_m^X(r - \varepsilon).$$

Proof. This proof is similar to the proof of Proposition 4.3, applying Lazar's theorem instead of Michael's theorem. □

Open problem. Let K be a compact convex set satisfying Proposition 5.1 or Proposition 5.2. Is K a Choquet simplex?

6. The norm in $L_p(\mu, X)$

In this section we study the unit ball $B_{L_p(\mu,X)}$ when the unit ball B_X is stable. Let us begin with $L_\infty(\mu, X)$.

6.1. **Proposition.** *Let X be a Banach space and (Ω, Σ, μ) a measure space.*

(A) *Suppose that μ is purely atomic with a finite number of atoms. Then*

(1) *$B_{L_\infty(\mu, X)}$ is stable iff B_X is stable.*

(2) *α is a modulus of US of $B_{L_\infty(\mu, X)}$ iff α is a modulus of US of B_X.*

(B) *Suppose that the measure space (Ω, Σ, μ) does not reduce to a purely atomic measure with a finite number of atoms. Then $\mathcal{TFAE}$:*

(a) *$B_{L_\infty(\mu, X)}$ is stable.*

(b) *B_X is US.*

(c) *$B_{L_\infty(\mu, X)}$ is US.*

Proof. (A) In this case $L_\infty(\mu, X)$ is isometric to $(\sum_{i=1}^n \bigoplus X_i)_\infty$, with $X_i = X$, $i = 1, 2, \dots, n$, and n the number of atoms of (Ω, Σ, μ). Now apply Proposition 1.9.

(B) (a) $\Rightarrow$ (b) Apply that $(\sum_{n\geq 1} \bigoplus X_n)_\infty$, with $X_n = X$, is 1-complemented in $L_\infty(\mu, X)$ and Proposition 1.9.

(b) $\Rightarrow$ (c) Let $Y \subseteq L_\infty(\mu, X)$ be the subspace of bounded countable X-valued functions $f = \sum_{n\geq 1} x_n \cdot \chi_{A_n}$, $A_n \in \Sigma$, $A_n \cap A_m = \varnothing$ if $n \neq m$. As Y is dense in $L_\infty(\mu, X)$, by Proposition 1.8, it is enough to prove that Y is US. But this follows immediately from (D) of Proposition 1.9 and the fact that, if $f_i \in Y$, $i = 1, 2, \dots, m$, there is in Y an isometric copy of $Z = (\sum_{n\geq 1} \bigoplus X_n)_\infty$, $X_n = X$, $n \geq 1$, such that $f_i \in Z$.

(c) $\Rightarrow$ (a) This is immediate. $\square$

In the following we prove that $B_{L_p(\mu, X)}$, $1 \leq p < \infty$, is stable iff B_X is stable. As X is 1-complemented in $L_p(\mu, X)$, it is enough to prove that $B_{L_p(\mu, X)}$ is stable when B_X is stable. We begin with the case $1 < p < \infty$.

6.2. **Lemma.** *Let X be a Banach space with stable unit ball B_X, K a compact Hausdorff space, $f_i \in C(K, X)$, $i = 0, 1, 2$, such that $f_0 = \frac{1}{2}(f_1 + f_2)$ and $\|f_0(k)\| = \|f_1(k)\| = \|f_2(k)\|$ for each $k \in K$, and $\varepsilon > 0$. Then there exists $\eta > 0$ such that for each $g \in B(f_0; \eta)$ we can choose $g_i \in B(f_i; \varepsilon)$, $i = 1, 2$, satisfying $g = \frac{1}{2}(g_1 + g_2)$ and $\|g(k)\| = \|g_1(k)\| = \|g_2(k)\|$ for each $k \in K$.*

Proof. Define $\phi\colon X \to 2^{X\times X}$ by $\phi(x) = \{(u, v) \in X\times X : x = (u+v)/2, \|x\| = \|u\| = \|v\|\}$, $x \in X$.

(1) Let us see that ϕ is lower semicontinuous. Pick $\delta > 0$, $x_0 \in X$ and $(x_1, x_2) \in \phi(x_0)$. We prove that there exists $\rho > 0$ (depending on δ, x_0, x_1, x_2) such that each $y_0 \in B(x_0; \rho)$ satisfies $\phi(y_0) \cap [B(x_1; \delta) \times B(x_2; \delta)] \neq \varnothing$. If $x_0 = 0$, take $\rho = \delta$. Assume that $x_0 \neq 0$ and, without loss of generality, that $\|x_0\| = 1$. Since B_X is stable, there exists $0 < \rho' < 1$ such that, for each $x_0' \in B(x_0; \rho') \cap S_X$, there exists $x_i' \in B(x_i; \delta/2) \cap S_X$, $i = 1, 2$, satisfying $x_0' = \frac{1}{2}(x_1' + x_2')$. Put $\rho = \min\{\rho'/2, \delta/2\}$ and let $y_0 \in B(x_0; \rho)$ and $x_0' = y_0/\|y_0\|$. As $\|x_0' - x_0\| \leq \|x_0' - y_0\| + \|y_0 - x_0\| \leq 2\rho \leq \rho'$, there exist $x_i' \in B(x_i; \delta/2) \cap S_X$, $i = 1, 2$, with $x_0' = \frac{1}{2}(x_1' + x_2')$. Put $y_i = \|y_0\| \cdot x_i'$, $i = 1, 2$. Then $(y_1, y_2) \in \phi(y_0)$ and $\|x_i - y_i\| \leq \|x_i - x_i'\| + \|x_i' - y_i\| \leq \delta/2 + \delta/2 = \delta$, $i = 1, 2$, that is, $\phi(x_0) \cap [B(x_1; \delta) \times B(x_2; \delta)] \neq \varnothing$.

(2) Let $f_i \in C(K, X)$, $i = 0, 1, 2$, and $\varepsilon > 0$ satisfy the statement of Lemma 6.2. We prove that there exists $\eta > 0$ such that $\phi(x) \cap [B(f_1(k); \varepsilon/2) \times B(f_2(k); \varepsilon/2)] \neq \varnothing$, provided that $k \in K$ and $x \in B(f_0(k); \eta)$. By (1),

for each $k \in K$, there exists $\eta_k > 0$ satisfying, for $x \in B(f_0(k); \eta_k)$, that $\phi(x) \cap [B(f_1(k); \varepsilon/4) \times B(f_2(k); \varepsilon/4)] \neq \varnothing$. For $k \in K$ put

$$G_k = f_0^{-1}[B(f_0(k); \eta_k/2)] \cap f_1^{-1}[B(f_1(k); \varepsilon/4)] \cap f_2^{-1}[B(f_2(k); \varepsilon/4)].$$

The family of neighbourhoods $\{G_k\}_{k\in K}$ cover K. So there is a finite family $\{k_1, \dots, k_n\} \subseteq K$ such that $\{G_{k_i}: i = 1, 2, \dots, n\}$ also cover K. Let $\eta = \min\{\eta_k/2: k = 1, \dots, n\}$. If $k \in K$ and $x \in B(f_0(k); \eta)$, there exists $i_0 \in \{1, \dots, n\}$, for example, $i_0 = 1$, such that $k \in G_{k_1}$ and $f_0(k) \in B(f_0(k_1); \frac{1}{2}\eta_{k_1})$. Thus $x \in B(f_0(k_1); \eta_{k_1})$ and there exists $(x_1, x_2) \in \phi(x) \cap [B(f_1(k_1); \varepsilon/4) \times B(f_2(k_1); \varepsilon/4)]$ satisfying

$$\|x_i - f_i(k)\| \le \|x_i - f_i(k_1)\| + \|f_i(k_1) - f_i(k)\| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \qquad i = 1, 2,$$

that is, $(x_1, x_2) \in \phi(x) \cap [B(f_1(k); \varepsilon/2) \times B(f_2(k); \varepsilon/2)]$.

(3) Pick $g \in C(K, X)$ such that $\|g - f_0\| \le \eta$ and define $\psi: K \to 2^{X\times X}$ as follows:

$$\psi(k) = [\phi \circ g(k)] \cap [B(f_1(k); \varepsilon) \times B(f_2(k); \varepsilon)], \qquad k \in K.$$

As in Proposition 4.2, it is proved that ψ is lower semicontinuous and that $\psi(k)$ is a nonempty closed convex set for each $k \in K$. Applying Michael's theorem [13, p. 5], there exists a continuous selection $s = (s_1, s_2): K \to X \times X$ such that $s(k) \in \psi(k)$ for each $k \in K$. Now take $g_i = s_i$, $i = 1, 2$. □

6.3. **Proposition.** *Let X be a Banach space with B_X stable, K a compact Hausdorff space, μ a positive Radon measure on K and $1 < p < \infty$. Then $B_{L_p(\mu, X)}$ is stable.*

Proof. Let $0 < \varepsilon \le 1$ and $f_i \in B_{L_p(\mu, X)}$, $i = 0, 1, 2$, such that $f_0 = \frac{1}{2}(f_1 + f_2)$. We prove that there exists $\rho > 0$ such that $V(f_0; \rho) \subseteq D(f_1, f_2; \varepsilon)$, assuming $\|f_i\| = 1$, $i = 0, 1, 2$ (if $\|f_0\| < 1$, this fact is clearly true). As $1 < p < \infty$, we have $\|f_0(k)\| = \|f_1(k)\| = \|f_2(k)\|$, $k \in K$, almost everywhere (a.e.). Let $G = \{x \in X\backslash\{0\}: x/\|x\| \notin \operatorname{Ext}(B_X)\}$. Note that G is open because B_X is stable and $\operatorname{Ext}(B_X)$ is closed. If $U = f_0^{-1}(G)$, then $f_1(k) = f_0(k) = f_2(k)$, $k \in K\backslash U$ a.e. If $\mu(U) = 0$, there is nothing to prove. Suppose that $\mu(U) > 0$ and choose a compact subset $K_0 \subseteq U$ such that $\mu(K_0) > 0$, f_i continuous on K_0, $i = 0, 1, 2$, and $[\int_{K\backslash K_i} \|f_i - f_0\|^p \cdot d\mu]^{1/p} \le \varepsilon/8$, $i = 1, 2$. Let $\delta = (\varepsilon/4)/[\mu(K_0)]^{1/p}$. By Lemma 6.2, there exists $0 < \eta < 1$ such that, if $h \in C(K_0, X)$ and $\|h - f_0\|_{K_0} \le \eta$ ($\|\cdot\|_{K_0}$ is the supremum norm on K_0), then there exist $h_i \in C(K_0, X)$ satisfying $\|h_i - f_i\|_{K_0} \le \delta$, $i = 1, 2$, $h = \frac{1}{2}(h_1 + h_2)$ and $\|h(k)\| = \|h_1(k)\| = \|h_2(k)\|$ for each $k \in K_0$. Let

$$M = \max\{1, \|f_i\|_{K_0}: i = 0, 1, 2\} \quad \text{and} \quad \rho = (\varepsilon\eta)/(16M).$$

Pick $g \in V(f_0; \rho)$ in $B_{L_p(\mu, X)}$ and let $A = \{k \in K_0: \|f_0(k) - g(k)\| > \eta\}$. Note that

$$\eta \cdot \mu(A)^{1/p} \le \left[\int_A \|f_0(k) - g(k)\|^p \cdot d\mu\right]^{1/p} \le \rho = (\varepsilon\eta)/(16M)$$

that is, $\mu(A) \le [\varepsilon/(16M)]^p$. Now choose a compact subset $K_1 \subseteq K_0\backslash A$ such that g is continuous on K_1 and $[\int_{K_0\backslash K_1} \|f_i(k) - f_0(K)\|^p \cdot d\mu]^{1/p} \le \varepsilon/4$, $i = 1, 2$. Note that this choice is possible because

$$\left[\int_A \|f_i(k) - f_0(k)\|^p \cdot d\mu\right]^{1/p} \leq 2M \cdot \mu(A)^{1/p} \leq \varepsilon/8, \qquad i = 1, 2.$$

Let $\hat{g}: K_0 \to X$ be a continuous extension of $g_{|K_1}$ to K_0 satisfying $\|f_0 - \hat{g}\|_{K_0} \leq \eta$. By Lemma 6.2, there exists $\hat{g}_i \in C(K_0, X)$ such that $\|\hat{g}_i - f_i\|_{K_0} \leq \delta$, $i = 1, 2$, and for each $k \in K_0$, $\hat{g}(k) = \frac{1}{2}(\hat{g}_1(k) + \hat{g}_2(k))$ and $\|\hat{g}_1(k)\| = \|\hat{g}(k)\| = \|\hat{g}_2(k)\|$. Define

$$k \in K, \qquad g_i(k) = \begin{cases} \hat{g}_i(k), & k \in K_1, \\ g(k), & k \in K \setminus K_1, \end{cases} \qquad i = 1, 2.$$

Then $g = \frac{1}{2}(g_1 + g_2)$, $\|g_i\|_p = \|g\|_p \leq 1$ and $\|f_i - g_i\|_p \leq \varepsilon$, $i = 1, 2$, because

$$\begin{aligned} \|f_i - g_i\|_p \leq & \left[\int_{K_1} \|f_i - g_i\|^p \cdot d\mu\right]^{1/p} + \left[\int_{K\setminus K_1} \|f_i - f_0\|^p \cdot d\mu\right]^{1/p} \\ & + \left[\int_{K\setminus K_1} \|f_0 - g\|^p \cdot d\mu\right]^{1/p} \\ & \leq \delta \cdot \mu(K_1)^{1/p} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \rho \leq \varepsilon. \quad \square \end{aligned}$$

6.4. **Proposition.** *Let (Ω, Σ, μ) be a positive measure space, X a Banach space with B_X stable and $1 < p < \infty$. Then $B_{L_p(\mu, X)}$ is stable.*

Proof. Let K be the Stone compact of the measure algebra (Σ, μ) and m the induced Radon measure on the Borel sets $\mathrm{Bo}(K)$ (see [6, p. 119]). If ρ is a lifting on $L_\infty(\mu)$, we can obtain (see [14]) a map $\tilde{\rho}: \Omega \to K$ such that the map $F: L_p(m, X) \to L_p(\mu, X)$, defined by $F(f) = f \circ \tilde{\rho}$, is an isometry. Now apply Proposition 6.3. $\square$

6.5. **Proposition.** *Let $\{X_i\}_{i \in I}$ be a family of Banach spaces with B_{X_i} stable and $1 < p < \infty$. Then, if $Y = (\sum_{i \in I} \bigoplus X_i)_p$, B_Y is stable.*

Proof. Let $x, y, z \in S_Y$ such that $z = \frac{1}{2}(x + y)$. We can suppose that $I = \mathbb{N}$. As $1 < p < \infty$, then $\|x_n\| = \|y_n\| = \|z_n\|$, $n \geq 1$. Let us prove that, for a given $\varepsilon > 0$, there exists $\rho > 0$ such that $V(z; \rho) \subseteq D(x, y; \varepsilon)$. Choose $m \in \mathbb{N}$ such that $[\sum_{n \geq m} \|x_n - z_n\|^p]^{1/p} \leq \varepsilon/4 \geq [\sum_{n \geq m} \|y_n - z_n\|^p]^{1/p}$. From the proof of Lemma 6.2 we know that there exist $\eta > 0$ such that, if $\|z_n - z_n'\| \leq \eta$, $n \leq m$, there exist $x_n', y_n' \in X_n$ fulfilling $\|x_n'\| = \|y_n'\| = \|z_n'\|$, $z_n' = \frac{1}{2}(x_n' + z_n')$, $\|x_n' - x_n\| \leq \varepsilon/(4 \cdot m^{1/p}) \geq \|y_n' - y_n\|$. Take $\rho = \min\{\varepsilon/2, \eta\}$ and $z' \in V(z; \rho)$. Let x', y' such that x_n', y_n', $n \leq m$, are as above and $x_n' = z_n' = y_n'$, $n > m$. Then $z' = \frac{1}{2}(x' + y')$ with $x' \in V(x; \varepsilon)$ and $y' \in V(y; \varepsilon)$. $\square$

6.6. **Lemma.** *Let X be a Banach space with B_X stable, K a compact Hausdorff space and $f_i \in C(K, X)$, $i = 0, 1, 2$, continuous functions such that $f_0 = \frac{1}{2}(f_1 + f_2)$ and $\|f_0(k)\| = \frac{1}{2}(\|f_1(k)\| + \|f_2(k)\|)$, $k \in K$. Then, given $\varepsilon > 0$, there exists $\eta > 0$ such that, for each $g \in B(f_0; \eta)$, there exist $g_i \in B(f_i; \varepsilon)$, $i = 1, 2$, satisfying*

$$g = \tfrac{1}{2}(g_1 + g_2), \qquad \|g(k)\| = \tfrac{1}{2}(\|g_1(k)\| + \|g_2(k)\|), \qquad k \in K.$$

Proof. Define $\phi: X \to 2^{X\times X}$ by $\phi(x) = \{(u, v) \in X \times X : x = (u+v)/2, \|x\| = \frac{1}{2}(\|u\| + \|v\|)\}$, $x \in X$.

(1) Let us see that ϕ is lower semicontinuous. Take $\varepsilon > 0$, $x_0 \in X$ and $(x_1, x_2) \in \phi(x_0)$. We prove that there exists $\eta > 0$ such that, for each $x_0' \in B(x_0; \eta)$, $\phi(x_0') \cap [B(x_1; \varepsilon) \times B(x_2; \varepsilon)] \neq \varnothing$. If $x_i = \lambda_i \cdot x_0$, $\lambda_i \geq 0$, $i = 1, 2$, take $\eta = \varepsilon/2$. Suppose that $x_i \neq \lambda_i \cdot x_0$. Without loss of generality, assume that $\|x_2\| = 1$, $\|x_0\| = 1 - a > 0$ and $\|x_1\| = 1 - 2a > 0$. Let $y_i = x_i/\|x_i\|$, $i = 0, 1, 2$. We claim that $y_0 \in [y_1, y_2]$ $(= \{\mu y_1 + (1-\mu)y_2 : 0 \leq \mu \leq 1\})$. Indeed, if $\lambda = (1-2a)/2(1-a)$, then

$$\begin{aligned} y_0 = \frac{x_0}{1-a} = \frac{1}{2}\frac{x_1}{1-a} + \frac{1}{2}\frac{x_2}{1-a} &= \frac{1}{2}\frac{1-2a}{1-a}\frac{x_1}{1-2a} \\ &+ \left(1 - \frac{1}{2}\frac{1-2a}{1-a}\right)x_2 = \lambda y_1 + (1-\lambda)y_2. \end{aligned}$$

Thus $[y_1, y_2] \subseteq S_X$. Observe that, if $0 < t < 1$ and we define $\phi_t: X \to 2^{X\times X}$ as $\phi_t(x) = \{(u, v) \in X \times X : x = tu + (1-t)v, \|x\| = \|u\| = \|v\|\}$, $x \in X$, then the proof of Lemma 6.2 implies that ϕ_t is lower semicontinuous. Then, for $\lambda = (1-2a)/2(1-a)$, there exists $\eta' > 0$ such that, for each $y_0' \in B(y_0; \eta')$, there exists $y_i' \in B(y_i; \varepsilon)$, $i = 1, 2$, such that

$$(*) \qquad y_0' = \lambda y_1' + (1-\lambda)y_2', \qquad \|y_0'\| = \|y_1'\| = \|y_2'\|.$$

Let $\eta = (1-a)\eta'$ and $x_0' \in B(x_0; \eta)$. Then, if $y_0' = x_0'/(1-a)$, we have $y_0' \in B(y_0; \eta')$. Thus there exist $y_i' \in B(y_i; \varepsilon)$, $i = 1, 2$, satisfying $(*)$. In particular

$$y_0' = \frac{x_0'}{1-a} = \tfrac{1}{2}\frac{1-2a}{1-a}y_1' + \tfrac{1}{2}\frac{1}{1-a}y_2'.$$

Let $x_1' = y_1'(1-2a)$ and $x_2' = y_2'$. Then

$$\begin{aligned} x_0' &= \tfrac{1}{2}x_1' + \tfrac{1}{2}x_2', \\ \|x_0'\| &= (1-a)\|y_0'\| = (1-a)\tfrac{1}{2}(\|y_1'\| + \|y_2'\|) \\ &= \tfrac{1}{2}[(1-2a)\|y_1'\| + (1-a+a)\|y_2'\|] = \tfrac{1}{2}(\|x_1'\| + \|x_2'\|). \end{aligned}$$

Moreover $\|x_i - x_i'\| \leq \varepsilon$, that is, $(x_1', x_2') \in \phi(x_0') \cap [B(x_1; \varepsilon) \times B(x_2; \varepsilon)]$. Thus ϕ is lower semicontinuous.

(2) Let $f_i \in C(K, X)$, $i = 0, 1, 2$, be continuous functions and $\varepsilon > 0$. Then there exists $\eta > 0$ such that, if $x \in B(f_0(k); \eta)$, $k \in K$, we have that $[B(f_1(k); \varepsilon/2) \times B(f_2(k); \varepsilon/2)] \cap \phi(x) \neq \varnothing$ (the proof is analogous to the part (2) of Lemma 6.2). Take $g \in B(f_0; \eta)$ in $C(K, X)$ and define $\psi: K \to 2^{X\times X}$ as $\psi(k) = [\phi \circ g(k)] \cap [B(f_1(k); \varepsilon) \times B(f_2(k); \varepsilon)]$, $k \in K$. As in Proposition 4.2, it is proved that ψ is lower semicontinuous and that $\psi(k)$ is a nonempty closed convex set for each $k \in K$. Applying Michael's theorem [13, p. 5], there exists a continuous selection $s = (s_1, s_2): K \to X \times X$ such that $s(k) \in \psi(k)$ for each $k \in K$. Now take $g_i = s_i$, $i = 1, 2$. $\square$

6.7. **Proposition.** *Let X be a Banach space with B_X stable, K a compact Hausdorff space and μ a positive Radon measure on K. Then $B_{L_1(\mu, X)}$ is stable.*

Proof. Take $f_i \in S_{L_1(\mu,X)}$, $i = 0, 1, 2$, satisfying $f_0 = \frac{1}{2}(f_1 + f_2)$, that implies $\|f_0(k)\| = \frac{1}{2}(\|f_1(k)\| + \|f_2(k)\|)$ a.e. Given $0 < \varepsilon \le 1$, let us prove that there exists $\rho > 0$ such that $V(f_0; \rho) \subseteq D(f_1, f_2; \varepsilon)$ in $B_{L_1(\mu,X)}$. If

$$\int_K | \|f_i(k)\| - \|f_0(k)\| \,| \, d\mu = 0, \qquad i = 1, 2,$$

(what implies that $\|f_0(k)\| = \|f_i(k)\|$, $i = 1, 2$) the conclusion follows from the proof of Proposition 6.3. Assume that $\int_K | \|f_i(k)\| - \|f_0(k)\| \,| \, d\mu > 0$ and denote

$$U_i = \{k \in K : \|f_i(k)\| > \|f_0(k)\|\}, \qquad i = 1, 2.$$

Then

$$\int_{U_1} (\|f_1(k)\| - \|f_0(k)\|) \, d\mu = \int_{U_2} (\|f_2(k)\| - \|f_0(k)\|) \, d\mu$$
$$= \frac{1}{2} \int_K | \|f_i(k)\| - \|f_0(k)\| \,| \, d\mu = H' > 0.$$

Let $H = \min\{1, H'\}$ and, as in Proposition 6.3, choose a compact $K_0 \subseteq K$ satisfying that f_i is continuous on K_0 and $\int_{K \setminus K_0} \|f_i - f_0\| \, d\mu \le \varepsilon H/32$, $i = 0, 1, 2$. Let $\delta = \varepsilon H / 16\mu(K_0)$. By Lemma 6.6, there exists $0 < \eta < 1$ such that, if $h \in C(K_0, X)$ and $\|h - f_0\|_{K_0} \le \eta$, there exist $h_i \in C(K_0, X)$ satisfying

$$\|h_i - f_i\|_{K_0} \le \delta, \qquad h = \tfrac{1}{2}(h_1 + h_2)$$

and

$$\|h(k)\| = \tfrac{1}{2}(\|h_1(k)\| + \|h_2(k)\|), \qquad k \in K_0.$$

Let $M = \max\{1, \|f_i(k)\|_{K_0} : i = 0, 1, 2\}$ and $\rho = H\varepsilon\eta/(32 \cdot M)$. Pick $g_0 \in V(f_0; \rho)$ in $B_{L_1(\mu,X)}$ and put $A = \{k \in K_0 : \|f_0(k) - g_0(k)\| > \eta\}$. Then

$$\eta \cdot \mu(A) \le \int_A \|f_0(k) - g_0(k)\| \, d\mu \le \rho$$

that is, $\mu(A) \le H\varepsilon/(32 \cdot M)$. So we can choose a compact $K_1 \subseteq K_0 \setminus A$ such that g_0 is continuous on K_1 and

$$\int_{K_0 \setminus K_1} \|f_i - f_0\| \, d\mu \le \varepsilon H/8, \qquad i = 1, 2.$$

Denote $U_{i1} = U_i \cap K_1$, $i = 1, 2$, and observe that

$$\int_{U_i \setminus K_1} (\|f_i(k)\| - \|f_0(k)\|) \, d\mu \le \int_{K \setminus K_1} \|f_i - f_0\| \, d\mu \le \frac{\varepsilon H}{32} + \frac{\varepsilon H}{8}.$$

Thus

$$\int_{U_{i1}} (\|f_i(k)\| - \|f_0(k)\|) \, d\mu \ge H - \frac{\varepsilon H}{32} - \frac{\varepsilon H}{8} \ge \frac{27}{32} H.$$

Let $\hat{g}_0 : K_0 \to X$ a continuous extension of $g_{0|K_1}$ to K_0 satisfying $\|f_0 - \hat{g}_0\|_{K_0} \le \eta$. By Lemma 6.6 there exist continuous functions $\hat{g}_i : K_0 \to X$, $i = 1, 2$, such that

$$\begin{aligned} &k \in K_0, \qquad \|\hat{g}_i(k) - f_i(k)\| \le \delta, \qquad i = 1, 2, \\ &\hat{g}_0(k) = \tfrac{1}{2}(\hat{g}_1(k) + \hat{g}_2(k)), \\ &\|\hat{g}_0(k)\| = \tfrac{1}{2}(\|\hat{g}_1(k)\| + \|\hat{g}_2(k)\|). \end{aligned} \tag{1}$$

Let $b_i = \int_{K_1}(\|\hat{g}_i(k)\| - \|g_0(k)\|)\,d\mu$, $i = 1, 2$, and note that $b_1 + b_2 = 0$ and

$$\begin{aligned}
|b_i| &= \left|\int_{K_1}(\|\hat{g}_i(k)\| - \|g_0(k)\|)\,d\mu\right| \\
&\le \left|\int_{K_1}(\|\hat{g}_i(k)\| - \|f_i(k)\|)\,d\mu\right| \\
&\quad + \left|\int_{K_1}(\|f_i(k)\| - \|f_0(k)\|)\,d\mu\right| + \left|\int_{K_1}(\|f_0(k)\| - \|g_0(k)\|)\,d\mu\right| \\
&\le \int_{K_1}\|\hat{g}_i(k) - f_i(k)\|\,d\mu + \int_{K\setminus K_1}\|f_i(k) - f_0(k)\|\,d\mu \\
&\quad + \int_{K_1}\|f_0(k) - g_0(k)\|\,d\mu \\
&\le \delta\cdot\mu(K_1) + \varepsilon H/32 + \varepsilon H/8 + (\varepsilon\eta H)/(32M) \le \varepsilon H/4.
\end{aligned}$$

Here we use (1), that

$$\int_K [\|f_i\| - \|f_0\|]\,d\mu = 0 \qquad \left(\text{thus } \left|\int_{K_1}[\|f_i\| - \|f_0\|]\,d\mu\right| = \left|\int_{K\setminus K_1}[\|f_i\| - \|f_0\|]\,d\mu\right|\right)$$

and that $g_0 \in V(f_0;\rho)$. Next define measurable functions $g_i'\colon K_1 \to X$, $i = 1, 2$, satisfying

$$k \in K_1, \qquad g_0(k) = \tfrac{1}{2}(g_1'(k) + g_2'(k)), \qquad \|g_0(k)\| = \tfrac{1}{2}(\|g_1'(k)\| + \|g_2'(k)\|)$$
$$\int_{K_1}\|g_i'(k)\|\,d\mu = \int_{K_1}\|g_0(k)\|\,d\mu, \qquad i = 1, 2.$$

If $b_1 = b_2 = 0$, take $g_i' = \hat{g}_i$, $i = 1, 2$. Suppose, for instance, that $b_1 > 0$ (the case $b_2 > 0$ is analogous) and denote $B = \{k \in K_1\colon \|\hat{g}_1(k)\| > \|g_0(k)\|\}$. Then $\int_B \|\hat{g}_1(k)\|\,d\mu \ge \int_B \|g_0(k)\|\,d\mu + b_1$. For $0 \le \lambda \le 1$, let $g_\lambda(k) = \lambda\hat{g}_1(k) + (1-\lambda)g_0(k)$, that satisfies $\|g_\lambda(k)\| = \lambda\|\hat{g}_1(k)\| + (1-\lambda)\|g_0(k)\|$, $k \in K_1$. Thus there exists $0 \le \lambda_0 \le 1$ such that $\int_B \|g_{\lambda_0}(k)\|\,d\mu = \int_B \|\hat{g}_1(k)\|\,d\mu - b_1$. Define g_i', $i = 1, 2$, by

$$k \in K_1, \qquad g_1'(k) = \begin{cases} g_{\lambda_0}(k), & k \in B, \\ \hat{g}_1(k), & k \in K_1\setminus B, \end{cases} \qquad g_2'(k) = 2g_0(k) - g_1'(k).$$

Note that

(1) $g_0(k) = \frac{1}{2}(g_1'(k) + g_2'(k))$, $\|g_0(k)\| = \frac{1}{2}(\|g_1'(k)\| + \|g_2'(k)\|)$, $k \in K_1$.

(2) $\int_{K_1}\|g_i'(k)\|\,d\mu = \int_{K_1}\|g_0(k)\|\,d\mu$, $i = 1, 2$.

(3) We claim that $\int_{K_1}\|g_i'(k) - f_i(k)\|\,d\mu \le 3\varepsilon/4$, $i = 1, 2$. Let us see that

$1-\lambda_0 \le \varepsilon/3$:

$$
\begin{aligned}
\frac{\varepsilon H}{4} \ge b_1 &= (1-\lambda_0)\int_B (\|\hat{g}_1(k)\| - \|g_0(k)\|)\,d\mu \\
&\ge (1-\lambda_0)\int_{U_{11}} (\|\hat{g}_1(k)\| - \|g_0(k)\|)\,d\mu \\
&= (1-\lambda_0)\Big[\int_{U_{11}} (\|\hat{g}_1(k)\| - \|f_1(k)\|)\,d\mu + \int_{U_{11}} (\|f_1(k)\| - \|f_0(k)\|)\,d\mu \\
&\qquad + \int_{U_{11}} (\|f_0(k)\| - \|g_0(k)\|)\,d\mu\Big] \\
&\ge (1-\lambda_0)\left(-\delta\cdot\mu(U_{11}) + \frac{27}{32}\cdot H - \rho\right) \\
&\ge (1-\lambda_0)\left(\frac{27}{32}H - \frac{\varepsilon H}{16} - \frac{\varepsilon\eta H}{32M}\right) \\
&\ge (1-\lambda_0)\left(\frac{27}{32} - \frac{1}{16} - \frac{1}{32}\right)H = (1-\lambda_0)\frac{24}{32}H.
\end{aligned}
$$

So $(1-\lambda_0) \le \varepsilon/3$. Also, as $\|\hat{g}_1(k) - f_i(k)\| \le \delta$, $k \in K_1$, and $\|f_i\|_1 = 1 \ge \|g_0\|_1$, we obtain

$$
\begin{aligned}
&\int_B \|\hat{g}_i(k) - g_0(k)\|\,d\mu \\
&\qquad \le \int_B \|g_0(k)\|\,d\mu + \int_B \|f_i(k)\|\,d\mu + \delta\cdot\mu(B) \le 2 + \frac{\varepsilon H}{16}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\int_{K_1} \|f_i(k) - g_i'(k)\|\,d\mu &\le \int_{K_1} \|f_i(k) - \hat{g}_i(k)\|\,d\mu \\
&\quad + \int_B \|\hat{g}_i(k) - (\lambda_0\hat{g}_i(k) + (1-\lambda_0)g_0(k))\|\,d\mu \\
&\le \delta\cdot\mu(K_1) + (1-\lambda_0)\int_B \|\hat{g}_i(k) - g_0(k)\|\,d\mu \le \frac{\varepsilon H}{16} + \frac{\varepsilon}{3}\left(2 + \frac{\varepsilon H}{16}\right) \le 3\varepsilon/4.
\end{aligned}
$$

Finally define g_i, $i = 1, 2$,

$$
k \in K, \qquad g_i(k) = \begin{cases} g_i'(k), & k \in K_1, \\ g_0(k), & k \in K\backslash K_1. \end{cases}
$$

Then we have

(a) $k \in K$, $g_0(k) = \frac{1}{2}(g_1(k) + g_2(k))$ and $\|g_0(k)\| = \frac{1}{2}(\|g_1(k)\| + \|g_2(k)\|)$.

(b) $\int_K \|g_i(k)\|\,d\mu = \int_K \|g_0(k)\|\,d\mu \le 1$, $i = 1, 2$.

(c) For $i = 1, 2$, we have

$$\begin{aligned}
\int_K &\|f_i(k) - g_i(k)\| \, d\mu \\
&= \int_{K\setminus K_1} \|f_i(k) - g_0(k)\| \, d\mu + \int_{K_1} \|f_i(k) - g_i(k)\| \, d\mu \\
&\leq \int_{K\setminus K_1} \|f_i(k) - f_0(k)\| \, d\mu + \int_{K\setminus K_1} \|f_0(k) - g_0(k)\| \, d\mu \\
&\quad + \int_{K_1} \|f_i(k) - g_i(k)\| \, d\mu \\
&\leq \frac{\varepsilon H}{32} + \frac{\varepsilon H}{8} + \rho + \frac{3\varepsilon}{4} < \varepsilon .
\end{aligned}$$

Thus $V(f_0; \rho) \subseteq D(f_1, f_2; \varepsilon)$. □

6.8. **Proposition.** *Let X be a Banach space with B_X stable and (Ω, Σ, μ) a positive measure space. Then $B_{L_1(\mu, X)}$ is stable.*

Proof. Use the Stone compact corresponding to the measure algebra (Σ, μ) as in Proposition 6.4. □

6.9. **Proposition.** *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces with B_{X_i} stable. Then, if $Y = (\sum_{i\in I} \bigoplus X_i)_1$, B_Y is stable.*

Proof. The proof, analogous to that of Proposition 6.5 (with some changes, as in Proposition 6.7), is left to the reader. □

Open problem. Suppose that B_X is US. Is $B_{L_p(\mu, X)}$ US, $1 \leq p < \infty$?

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