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CENTRAL LIMIT THEOREMS FOR SUMS OF WICK PRODUCTS OF STATIONARY SEQUENCES

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ABSTRACT. We show, by the method of cumulants, that checking whether the central limit theorem for sums of Wick powers of a stationary sequence holds can be reduced to the study of an associated graph problem (see Corollary 1). We obtain thus central limit theorems under various integrability conditions on the cumulant spectral functions (Theorems 2, 3).

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A. Introduction. Let X_j be a zero mean stationary sequence with all moments finite. We consider the central limit theorem for

$$Y_n = \sum_{j=1}^{n} : X_j^{(m)} :,$$

where $: X_j^{(m)} :$ denotes the mth Wick power of X_j. See [GS] for a definition of Wick powers; in this case it is a certain polynomial of degree m.

We will study the asymptotic behavior of the cumulants of Y_n using the diagram formula, a combinatorial expansion for the cumulants of Wick powers which has been widely used in proving central limit theorems [BM, CS, FT, G1]. We have obtained in [AB and A] a formula relating the order of magnitude of the cumulants of Wick powers to a certain graph-theoretic quantity (see 1.11 below). This formula led to a short proof of a result of Breuer and Major [BM], as well as to a new central limit theorem in the case when X_j is Gaussian (see [A]).

In this paper, we show in Theorem 1 that the same methods may be used to estimate cumulants for more general stationary sequences X_j, under a certain assumption on their cumulant spectral functions (see 1.7). As an application, in Theorem 2 we provide conditions for Y_n to satisfy a central limit theorem, which applies in particular when X_j is given by

$$X_j = \sum_r c_{j-r}\xi_r \tag{1.1.a}$$

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or more generally

$$X_j = \sum_r c_{j-r}\eta_r, \tag{1.1.b}$$

where ξ_r is an i.i.d. sequence, $\sum(c_r)^2 < \infty$, and $\eta_r = h(\xi_r, \xi_{r+1}, \ldots, \xi_{r+d-1})$, with $h(x_1, \ldots, x_d)$ being chosen so that η_R has mean 0 and all moments finite. Similar results, in the case (1.1.a), were recently announced by Giraitis [G2].

B. The diagram expansion. We recall now the diagram expansion for the cumulants of Y_n, where X_j is an arbitrary 0 mean stationary sequence with all moments finite. By multilinearity, the Rth cumulant of Y_n is given by

$$\operatorname{cum}_R(Y_n) = \sum_{j_1=1}^{n} \cdots \sum_{j_R=1}^{n} \operatorname{cum}\left(: X_{j_1}^{(m)} :, \ldots, : X_{j_R}^{(m)} :\right). \tag{1.2}$$

Let P denote a partition of the entries of the $R \times m$ table

$$\begin{array}{c} X_{j_1}, \ldots, X_{j_1} \\ \vdots \\ X_{j_R}, \ldots, X_{j_R} \end{array}$$

satisfying the conditions

(1.3.a) No set $t \in P$ is contained in a single row of the table.

(1.3.b) For each partition of the rows of the table into two disjoint sets, there is a set $t \in P$ containing an element from each of the two sets.

The diagram formula states that the cumulant on the right-hand side of (1.2) is given by $\sum_P \prod_{t\in P} \operatorname{cum}(t)$ where we have summed over all partitions satisfying (1.3), and $\operatorname{cum}(t) = \operatorname{cum}(t, j_1, \ldots, j_R)$ denotes the cumulant of the collection of random variables in t. Thus we obtain

$$\operatorname{cum}_R(Y_n) = \sum_P \sum_{j_1=1}^{n} \cdots \sum_{j_R=1}^{n} \prod_{t\in P} \operatorname{cum}(t) = \sum_P S_n(P). \tag{1.4}$$

Our main result, Theorem 1, provides a method of computing the order of magnitude of $S_n(P)$ under integrability conditions on the cumulant spectral functions. Recall that the kth cumulant spectral function of the sequence X_j is a function $f^{(k)}(x_1, \ldots, x_{k-1})$ satisfying

$$\begin{aligned} \operatorname{cum}(X_{j_1}, \ldots, X_{j_k}) = \int & f^{(k)}(x_1, \ldots, x_{k-1}) \\ & \cdot \exp\{2\pi i[x_1(j_1 - j_k) + \cdots + x_{k-1}(j_{k+1} - j_k)]\} dx_1 \cdots dx_{k-1}. \end{aligned} \tag{1.5}$$

(In all integrals in this paper, each variable is to be integrated from 0 to 1.)

Consider now the case where X_j is a linear sequence given by (1.1.a). In this case $\operatorname{cum}(X_{j_1}, \ldots, X_{j_k}) = d_k \sum_i c_{j_1-i} \cdots c_{j_k-i}$, where d_k denotes the kth cumulant of ξ_i.

Letting $c(x)$ denote the Fourier transform of the sequence c_j, one finds in this case that the kth cumulant spectral function is

$$f^{(k)}(x_1, \ldots, x_{k-1}) = d_k c(x_1) \cdots c(x_{k-1}) c(-x_1 - \cdots - x_{k-1}). \tag{1.6}$$

Inspired by this example, we assume that for the general stationary sequence X_j there exist functions $g^{(k)}(x_1, \dots, x_k)$ and constants $\infty \geq p_k \geq 1$, $k \geq 2$, such that

$$f^{(k)}(x_1, \dots, x_{k-1}) = g^{(k)}(x_1, \dots, x_{k-1}, -x_1 - \cdots - x_{k-1}), \tag{1.7.a}$$

$$|||g^{(k)}|||_{p_k} < \infty. \tag{1.7.b}$$

Here $|||\cdot|||_p$ denotes the greatest cross-norm on the tensor product space $L_p^{(k)}$ of L_p with itself k times. If f is a finite sum of products of k functions in L_p, the norm is defined

$$|||f|||_p = \inf \sum_{j=1}^{N} ||f_{j,1}||_p \cdots ||f_{j,k}||_p,$$

where the infimum is taken over all decompositions of $f(x_1, \dots, x_k)$ of the form $f = \sum_{j=1}^{N} f_{j,1} \cdots f_{j,k}$. The tensor product space $L_p^{(k)}$ is then obtained by completing the set of finite sums of products under this norm (see [LC]). Assumption (1.7.b) is used since a generalized Hölder inequality with respect to the $|||\ |||_p$ norms holds (see [AB, Theorem 1′]). Throughout the paper, we use the notation

$$L_p = \begin{cases} L_p[0, 1], & 1 < p < \infty, \\ C[0, 1], & p = \infty. \end{cases}$$

Informally, (1.7.a) amounts to replacing (1.5) by

$$\operatorname{cum}(X_{j_1}, \dots, X_{j_k}) = \int g^{(k)}(x_1, \dots, x_k) e^{2\pi i (j_1 x_1 + \cdots + j_k x_k)} \cdot \delta(x_1 + \cdots + x_k)\, dx_1 \cdots dx_k. \tag{1.8}$$

C. The optimal breaking problem. To each partition P satisfying conditions (1.3) we associate a graph G with two types of vertices: R "row" vertices (one for each row of the table) and T "subset" vertices (one for each of the $T = T(P)$ subsets in partition P). Each element of the table is represented by an edge connecting the "row" and "subset" containing that element. With this edge we associate a "cost" z_k, where k is the cardinality of the partition subset containing that element of the table, and z_k is given by

$$z_k = 1 - (p_k)^{-1}. \tag{1.9}$$

Let E denote the edge set of this graph. With each set of edges $A \subset E$ we associate a "profit"

$$\alpha(A) = C(G \backslash A) - \sum_{e \in A} z_e, \tag{1.10}$$

where z_e denotes the cost of edge e and $C(G \backslash A)$ is the number of components left in G after the edges in A have been removed. The "optimal breaking problem" is to find

$$\alpha_G = \max_{A \subset E} \alpha(A). \tag{1.11}$$

We will show that the order of magnitude of $S_n(P)$ is α_G. More precisely, we have

Theorem 1. *Suppose that the cumulant spectral functions of X_j satisfy conditions* (1.7). *Let P be a partition satisfying conditions* (1.3), *and let $S_n(P)$ be the corresponding term in the expansion* (1.4) *of the* Rth *cumulant of Y_n. If α_G denotes the solution of the associated optimal breaking problem given by* (1.9)–(1.11), *then*

(a) $|S_n(P)| \le Cn^{\alpha_G}$, *where C is a constant depending only on the norms* $|||g^{(k)}|||_{p_k}$.

(b) *If $\alpha_G > 1$, then $S_n(P) = o(n^{\alpha_G})$.*

(c) *If $\alpha_G = 1$, then $\lim_{n\to\infty} S_n(P)/n = I_G$, where I_G is an integral defined as follows.*

Let $t = 1, \dots, T$ denote the subsets in the partition P and n_t denote the cardinality of subset t. Let the matrix M^* be an integer representation of the cutset matroid $\mathscr{E}^*(G)$ of the graph G. Then

$$I_G = \int \prod_{t=1}^{T} f^{(n_t)}(x_{t,1}, \dots, x_{t,n_t})\, dy_1 \cdots dy_N$$

where the vectors

$$x = (x_{1,1}, \dots, x_{1,n_1}, \dots, x_{T,1}, \dots, x_{T,n_t})$$

and $y = (y_1, \dots, y_N)$ are related by $x = yM^*$, with N being the number of rows in M^*. See the appendix for a review of some basic facts from matroid theory.

D. Central limit theorems. Let $\mathscr{G}_R$ be the family of graphs arising from partitions involved in the expansion of the Rth cumulant of Y_n. An immediate corollary of Theorem 1 is

Corollary 1. *Suppose* (1.7) *holds and that α_G, defined in* (1.1), *satisfies*

$$\alpha_G \le R/2, \quad \text{for every } G \in \mathscr{G}_R, \quad R \ge 2.$$

Then $n^{-1/2}Y_n$ converges in law to the normal distribution with mean 0 and variance $\sigma^2 = \sum_{G\in\mathscr{G}_2} I_G$.

Using Corollary 1 we are able to prove

Theorem 2. *Suppose* (1.7) *holds and that z_k (given by* 1.9) *satisfies*

$$z_k \ge \begin{cases} \frac{k}{2m} & \text{if } k(k-1) > 2m, \ k \le m+1, \\ \frac{k}{2m(k-m)} & \text{if } m+1 \le k < 2m, \\ \frac{1}{k} + \frac{1}{2m} & \text{otherwise.} \end{cases}$$

Then $n^{-1/2}Y_n$ tends in law to the $N(0, \sigma^2)$ distribution. Theorem 2 implies

Corollary 2. *If $z_k \ge \frac{1}{k} + \frac{1}{2}$ for all k, then $\sum_{j=1}^{n} X_j$ satisfies the central limit theorem.*

Corollary 2 is related to a result of Giraitis [G2].

Note that the lower bound for z_k given by Theorem 2 is maximized over k when $k = 2$ or $k = m+1$, achieving the maximum value $\frac{1}{2} + \frac{1}{2m}$. Thus we obtain

Corollary 3. *Suppose* (1.7) *holds and* $z_k \geq \frac{1}{2} + \frac{1}{2m}$ *for all* k. *Then* Y_n *satisfies the central limit theorem.*

Theorem 3. *If* X_j *is given* (1.1.b) *and* $c(x) = \sum_r c_r e^{-2\pi i r x}$ *is in* L_p *with* $\frac{1}{p} \leq \frac{1}{2} - \frac{1}{2m}$, *then* Y_n *satisfies the central limit theorem.*

Proof. We have

$$\operatorname{cum}(X_{j_1}, \ldots, X_{j_k}) = \sum_{r_1, \ldots, r_k} c_{j_1 - r_1} \cdots c_{j_k - r_k} \operatorname{cum}(\eta_{r_1}, \ldots, \eta_{r_k}).$$

Fixing $r_1, \ldots, r_k$, we see that, if $|r_l - r_1| > kd$ for some $2 \leq l \leq k$ then the random variables $\eta_{r_1} \ldots, \eta_{r_k}$ can be partitioned into two sets which are independent of each other, which implies that $\operatorname{cum}(\eta_{r_1}, \ldots, \eta_{r_k}) = 0$. Thus $\operatorname{cum}(X_{j_1}, \ldots, X_{j_k})$ equals

$$\begin{aligned}
&\sum_{r_1=-\infty}^{\infty} \sum_{r_2=-kd}^{kd} \cdots \sum_{r_k=-kd}^{kd} c_{j_1-r_1} c_{j_2-r_2-r_1} \cdots c_{j_k-r_k-r_1} \operatorname{cum}(\eta_{r_1}, \eta_{r_2+r_1}, \ldots, \eta_{r_k+r_1}) \\
&\quad = \sum_{r_2=-kd}^{kd} \cdots \sum_{r_k=-kd}^{kd} \operatorname{cum}(\eta_0, \eta_{r_2}, \ldots, \eta_{r_k}) \\
&\qquad \cdot \sum_{r_1=-\infty}^{\infty} c_{j_1-r_1} c_{j_2-r_2-r_1} \cdots c_{j_k-r_k-r_1}.
\end{aligned}$$

This implies that the spectral cumulant function of X_j is

$$\begin{aligned}
&\sum_{r_2=-kd}^{kd} \cdots \sum_{r_k=-kd}^{kd} \operatorname{cum}(\eta_0, \eta_{r_2}, \ldots, \eta_{r_k}) \\
&\qquad \cdot e^{-2\pi i r_2 x_2} c(x_2) \cdots e^{-2\pi i r_k x_k} c(x_k) c(-x_2 - \cdots - x_k).
\end{aligned}$$

The result of Theorem 3 now follows from Corollary 3.

Applying Theorem 3 with $\eta_r = \xi_r$, we receive a result of Giraitis [G1].

2. Proof of Theorem 1

We begin by establishing three extensions of results in [A]. Let $\Delta_n(x)$ be the Dirichlet kernel

$$\Delta_n(x) = \sum_{k=1}^{n} e^{2\pi i k x}.$$

We consider integrals of the form

$$\begin{aligned}
(2.1)\quad S_n = \int h^{(1)}(u_1^1, \ldots, u_{n_1}^1) \cdots h^{(T)}(u_1^T, \ldots, u_{n_T}^T) \\
\cdot\ \Delta_n(v_1), \ldots, \Delta_n(v_R)\, dx_1 \cdots dx_N,
\end{aligned}$$

where u_k^t, $k = 1, \ldots, n_t$, $t = 1, \ldots, T$, and v_j, $j = 1, \ldots, R$, are linear combinations of the variables $x_1, \ldots, x_N$ with integer coefficients.

We arrange the coefficients of the above linear combinations, taken in that order, into columns, with the first $n_1 + \cdots + n_T$ columns forming the matrix U and the last R columns forming the matrix V. We consider U and V to be sets of columns so that, for example, $a \in V$ is a column in V and $A \subset U$ is a set of columns of U. We assume

$$\text{(2.2)} \qquad \operatorname{rank}(V) = \operatorname{rank}(V \backslash a) \quad \text{for every column } a \in V,$$

where $V \backslash a$ is the matrix obtained by deleting the column a from V.

We consider the matrix $[U, V]$ as a matroid on the columns of U and V, and define $W = [U, V]/V$ to be the matroid obtained by "contracting" the columns of V. (See the appendix for a review of the basic concepts of the matroid theory.) Thus W is a matroid on the columns of U.

Let V_0 be an integer matrix with row space equal to the orthogonal complements of the column space of V. By Proposition A.8 of the appendix, the matroid W is represented by $W_0 = V_0 U$.

We suppose

$$\text{(2.3)} \qquad h^{(t)} \in L_{p_t}^{(n_t)}, \qquad t = 1, \dots, T,$$

where $L_{p_t}^{(n_t)}$ denotes the closure of the tensor product of L_{p_t} with itself n_t times with respect to the greatest cross-norm.

Finally, with each column a of U we associate a number z_a defined so that if $a = u_k^t$ then

$$\text{(2.4)} \qquad z_a = 1 - (p_t)^{-1}.$$

Proposition 1. *If* (2.1)-(2.4) *hold, then*

$$|S_n| \le C \left(\prod_{t=1}^{T} |||h^{(t)}|||_{p_t} \right) n^{\alpha}$$

where C *is a constant depending on* V *only and*

$$\alpha = \operatorname{Cor}(V) + \max_{A \subset U} \left[\sum_{a \in A} (1 - z_a) - r_w(A) \right],$$

with r_W *denoting the rank function of the matroid* W, *and* $\operatorname{cor}(V) = R - \operatorname{rank}(V)$ *denoting the nullity of the linear map* Vx.

Note that in the special case where each function has just one variable and U is an identity matrix, Proposition 1 reduces to Theorem 1 of [A]. To see this, it suffices to show that in this case W is the dual of the matroid generated by the columns of the transpose V^{tr}, i.e. $W = (V^{\text{tr}})^*$. This follows from the equalities

$$W^* = ([I, V]/V)^* = [I, V]^* \backslash V = [V^{\text{tr}}, -I] \backslash V = V^{\text{tr}},$$

where we have used Propositions A.1 and A.6 from the appendix.

Proposition 1 can be proven by following the proof of Theorem 1 of [A], which applies entirely in this more general setting. The only difference between Proposition 1 and Theorem 1 of [A] is that Proposition 1 involves functions of several variables, and hence that a Hölder inequality involving such functions (Theorem 1′ of [AB]) has to be used instead of the univariate Hölder inequality (Theorem 1 of [AB]). Note however that since such an inequality does not hold over the whole $L_p([0, 1]^{(k)})$ spaces (see remark after Theorem 1′ of [AB]), we are forced to assume now that our functions (and the cumulant spectral functions in the C.L.T.'s above) belong to the $L_p^{(k)}$ subspaces (which means that they can be approximated in a strong sense by sums of products of univariate functions).

Theorem 2 of [A] can be extended to

Proposition 2. *Assume that* (2.1)–(2.4) *hold, with* L_p *replacing* L_p, *and that* $\alpha = \operatorname{cor}(V)$ *or equivalently*

$$\sum_{a\in A}(1 - z_a) \le r_W(A) \quad \text{for every } A \subset U. \tag{2.5}$$

Then

$$\lim_{n\to\infty}\left(\frac{S_n}{n^\alpha}\right) = c_{W_0} I(W_0)$$

where C_{W_0} *is a constant which equals* 1 *if* W_0 *is unimodular and*

$$I(W_0) = \int \prod_{t=1}^{T} h^{(t)}(w_1^t, \ldots, w_{n_t}^t)\, dy_1 \cdots dy_d,$$

where $(w_1^1, \ldots, w_{n_1}^1, \ldots, w_1^T, \ldots, w_{n_T}^T) = (y_1, \ldots, y_d)W_0$ *and* d *is the number of rows in* W_0.

Proof. Introduce vectors $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_d)$. Let q be a column vector $n_1 + \cdots + n_T$ integers. As in the proof of Theorem 2 of [A] it is enough to show that Proposition 2 holds when

$$h^{(t)}(x_{t,1}, \ldots, x_{t,n_t}) = \exp\left\{-2\pi i \sum_{j=1}^{n_t} q_{t,j} x_{t,j}\right\}, \qquad 1 \le t \le T,$$

since the generalized Hölder inequality (Theorem 1′ of [AB]) and Proposition 1 yield then the general case.

Letting $B = \{1, \ldots, n\}^R$ we see that in this case S_n equals

$$\sum_{k\in B} \int e^{-2\pi i x(Uq - Vk) dx_1 \cdots dx_N} = \text{cardinality}\{k \in B : Uq = Vk\},$$

which equals 0 unless Uq is in the column space of V, i.e., unless $0 = V_0 Uq = W_0 q$. As in the proof of Theorem 2 of [A] we conclude

$$\lim_{n\to\infty} \frac{S_n}{n^\alpha} = \begin{cases} c_{W_0} & \text{if } W_0 q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently

$$\lim_{n\to\infty} \frac{S_n}{n^\alpha} = c_{W_0} \int e^{-2\pi i y W_0 q dy_1 \cdots dy_d}.$$

The analogue of Corollary 1 of [A] holds also:

Proposition 3. *If* (2.1)–(2.4) *hold, with* L_p *replacing* L_p *and* $\alpha > \operatorname{cor}(V)$, *then* $S_n = o(n^\alpha)$.

We now refer to the graph $G = G_{R,P}$ introduced in §1. We will also use the graph $\overline{G} = \overline{G}_{R,P}$ formed by adding an extra vertex to G and connecting that vertex by one edge to each of the R row vertices.

Let T denote the number of sets in the partition P and index the edges of G with pairs (t, k) where $t = 1, \dots, T$, and $k = 1, \dots, n_t$, with n_t denoting the cardinality of the subset t. With each edge (t, k) we associate a variable $x_{t,k}$. Let

$$\hat{V}_t = \sum_{k=1}^{n_t} x_{t,k}, \qquad t = 1, \dots, T,$$

$$\overline{V}_j = \sum_{(t,k)\in j} x_{t,k},$$

$$V_j = \sum_{\substack{(t,k)\in j\\ k\neq n_t}} x_{t,k} - \sum_{(t,n_t)\in j} \sum_{k=1}^{n_t-1} x_{t,k}, \qquad j = 1, \dots, R,$$

where we write $(t, k) \in j$ if edge (t, k) is incident to vertex j.

Thus

$$V_j = \overline{V}_j - \sum_{(t,n_t)\in j} \hat{V}_t. \tag{2.6}$$

Lemma 1. *The sum $S_n(P)$ can be written*

$$S_n(P) = \int \prod_{t=1}^{T} g^{(n_t)}(x_{t,1}, \dots, x_{t,n_t-1}, t = 1 - x_{t,1} - \cdots - x_{t,n_t-1}) \cdot \prod_{j=1}^{R} \Delta_n(V_j) \prod_{t=1}^{T} \prod_{k=1}^{t} dx_{t,k}. \tag{2.7}$$

Proof. Relation (2.7) follows by direct substitution of (1.5) and (1.7.a) into the definition of $S_n(P)$. It is easier, however, to plug in the heuristic formula (1.8), leading to

$$S_n(P) = \int \prod_{t=1}^{T} g^{(n_t)}(x_{t,1}, \dots, x_{t,n_t})\delta(\hat{V}_t) \prod_{j=1}^{R} \Delta_n(\overline{V}_j) \prod_{t=1}^{T} \prod_{k=1}^{n_t} dx_{t,k},$$

and then to replace x_{t,n_t} by $-\sum_{k=1}^{n_t-1} x_{t,k}$.

Now let U and V be the matrices obtained by writing as columns the coefficients of the linear combinations

$$\left\{ x_{t,k}, t = 1, \dots, T, k = 1, \dots, n_t - 1, -\sum_{k=1}^{n_t-1} x_{t,k}, t = 1, \dots, T \right\}$$

and $\{V_j, j = 1, \dots, R\}$, respectively.

Lemma 2. (a) *The matrix $[U, V]$ is a representation of $\mathscr{E}^*(\overline{G})$, the cutset matroid of $\overline{G}$.*

(b) *The contraction matroid $[U, V]/V$ is equivalent to $\mathscr{E}^*(G)$, the cutset matroid of G.*

Proof. We show instead that $[U, V]^*$ is a representation of $\mathscr{E}(\overline{G})$, the cycle matroid of $\overline{G}$. To do so, we abuse notation by using $\hat{V}_t$, $\overline{V}_j$, and V_j to denote the row vectors of coefficients of the corresponding linear combinations. Also, $\tilde{V}_t$ will denote the vector of coefficients of $-\sum_{k=1}^{n_t-1} x_{t,k}$. Thus

$$[U, V] = [I_{mR-T}, \tilde{V}_1^{\text{tr}}, \dots, \tilde{V}^{\text{tr}}, V_1^{\text{tr}}, \dots, V_R^{\text{tr}}].$$

By Proposition A.6 of the appendix, $[U, V]^*$ is represented by

$$\left[\begin{array}{c|c} \tilde{V}_1 & \\ \vdots & \\ \tilde{V}_T & -I_{T+R} \\ V_1 & \\ \vdots & \\ V_R & \end{array}\right].$$

Now we take one of the last R rows, starting with a certain V_j, and subtract it from each row t among the first T rows for which $(t, n_t) \in j$. We perform this operation for each of the last R rows. Using (2.6) and letting T columns of $-I_{T+R}$ correspond to x_{t,n_t}, $t = 1, \dots, T$, we see that the matrix we have obtained is

$$\left[\begin{array}{c|c} -\hat{V}_1 & \\ \vdots & 0 \\ -\hat{V}_T & \\ \hline \overline{V}_1 & \\ \vdots & \\ & -I_R \\ \overline{V}_R & \end{array}\right]. \tag{2.8}$$

This matrix has precisely one 1 and one -1 in each of the first mR columns. If we append to it another row equal to minus the sum of the previous rows, we obtain the incidence matrix of $\overline{G}$, the last row corresponding to the "extra" vertex. Part (a) of Lemma 2 follows from Proposition A.5. To prove part (b), note that $([U, V]/V)^* = [U, V]^* \backslash V$ by Proposition A.1, so that $([U, V] \backslash V)^*$ is represented by the first mR columns of (2.8), which is the incidence matrix of G.

Proof of Theorem 1. (a) By Lemma 1, $S_n(P)$ is an integral of the form (2.1). By Lemma 2, in this case $W = \mathscr{E}^*(G)$, so that $r_W(A) = |A| - C(G\backslash A) + 1$ by Proposition A.3. Part (a) follows from Proposition 1.

(b) This follows from Proposition 3.

(c) This follows from Proposition 2.

Proof of Theorem 2. We will show that the conditions of Theorem 2 imply those of Corollary 1. Let G be a fixed graph in the family $\mathscr{G}_R$ of graphs with R row vertices. For each subset vertex t in G, we may identify the elements of the corresponding subset with edges incident to t. Thus $|t|$ denotes the degree of t and we write $e \in t$ if edge e is incident to t.

We begin by assigning to each edge $e \in ta$ cost $z_e = 1/|t| + 1/2m$. These costs are chosen so that the "total breaking" (obtained by deleting all of the edges of G) achieves a profit of $R/2$ for all $G \in \mathcal{G}_R$. To see this, introduce the notation $z_A = \sum_{e \in A} z_e$ for any set of edges A, and note that for each subset vertex t

$$z_t = \sum_{e \in t} z_e = |t| \left(\frac{1}{|t|} + \frac{1}{2m} \right) = 1 + \frac{|t|}{2m}.$$

Since $\sum_t |t| = mR$, the profit associated with the total breaking is

$$R + \sum_t (1 - z_t) = R - \frac{1}{2m} \sum_t |t| = \frac{R}{2}.$$

If the total breaking were optimal at costs z_e for every graph in $\mathcal{G}_R$, Corollary 1 would imply that a central limit theorem holds if z_k (given by (1.9)) satisfies $z_k \geq \frac{1}{k} + \frac{1}{2m}$. However, the total breaking is not optimal, because of a phenomenon we call a "bond:" a set B of b edges connecting the same two vertices, such that $z_B > 1$.

We will show below that a given subset vertex t can contain at most one bond. If t contains a bond B, we modify the costs z_e for $e \in t$ as follows: The cost of each bond is "discounted" to $\frac{1}{b}$, so that the bond becomes "removable." At the same time, we increase the cost of the other edges in t so that the total cost of the edges in t is unchanged, i.e. the total discount $z_B - 1$ is divided equally among the other edges in t. We denote the resulting costs by z'_e. Theorem 2 follows directly from the next two propositions and Corollary 1.

Proposition 4. *At the cost z'_e, the total breaking is optimal and achieves a profit $R/2$ for every graph G in $\mathcal{G}_R$.*

Proposition 5. *Let t be a subset vertex with k edges, containing an edge e.*
(a) $z'_e = \frac{1}{k} + \frac{1}{2m}$ *if* $k(k-1) \leq 2m$ *or* $k \geq 2m$.
(b) $z'_e \leq \frac{k}{2m}$ *if* $k(k-1) > 2m$, $k \leq m+1$.
(c) $z'_e \leq \frac{k}{2m(k-m)}$ *if* $m+1 < k < 2m$.

Before proving Propositions 4 and 5, we establish two lemmas. For any set of edges A, define $z'_A = \sum_{e \in A} z'_e$.

Lemma 3. *Let t be a subset vertex containing a bond B with b edges.*
(a) $|t|/2 < b \leq m$. *No subset can contain two bonds.*
(b) $z'_{t-B} < 1$.

Proof. (a) It is clear that $b \leq m$. If $b \leq |t|/2$, then $z_B = b/|t| + b/2m \leq \frac{1}{2} + \frac{1}{2}$, contradicting the assumption that B is a bond.

(b) Since $z'_t = z_t$ and $z'_B = 1$,

$$z'_{t-B} = z_t - 1 = \sum_{e \in T} \left(\frac{1}{|t|} + \frac{1}{2m} \right) - 1 = \frac{|t|}{2m} < 1,$$

by the result of part (a).

The second part of Lemma 3 implies that when the costs z'_e are adopted no new bonds are introduced, so that there are no bonds with cost z'_e.

Lemma 4. *If G' is a subgraph of G, let X be the set of row vertices in G', and Y the set of subset vertices in G'. Then at costs z'_e the total breaking of G' achieves a profit of at least $|X|/2$.*

Proof. We may write $Y = Y_1 \cup Y_2 \cup Y_3$, where under costs z_e the vertices in Y_1 had no bonds, each vertex in Y_2 had a bond connected to a vertex in X, and each vertex in Y_3 had a bond connected to a vertex not in X. For any $t \in Y$ denote the set of edges from t to X by $t(X)$. Next we show

$$z'_{t(X)} \leq 1 + \frac{|t(X)|}{2m}, \qquad t \in Y. \tag{3.1}$$

Indeed, for $t \in Y_1$,

$$z'_{t(X)} = z_{t(X)} = \frac{|t(X)|}{|t|} + \frac{|t(X)|}{2m},$$

implying (3.1).

If $t \in Y_2$, then $z'_{t(X)} \leq z_{t(X)}$ so (3.1) follows as before. If $t \in Y_3$, then Lemma 3 implies $z'_{t(X)} \leq 1$, which establishes (3.1). The profit from the total breaking of G' is now

$$|X| + \sum_{t \in Y}(1 - z'_{t(X)}) \geq |X| - \frac{1}{2m}\sum_{t \in Y}|t(X)| \geq |X| - \frac{m|X|}{2m} = \frac{|X|}{2}.$$

Proof of Proposition 4. Suppose that there is an optimal breaking other than the total breaking. Let G' be a component left after the removal of the edges in this breaking, and X the set of row vertices in G'. If $|X| = 1$ then, since there are no bonds any more, we may remove all of the edges in G' without decreasing the profit. If $|X| \geq 2$, Lemma 4 implies that we may remove the edges in G' without decreasing the profit. Thus the total breaking is optimal.

Since the profit of the total breaking at costs z_e is $R/2$, and the prices z'_e were chosen to have the same total cost, it follows that the total breaking has profit $R/2$ at costs z'_e.

Proof of Proposition 5. First suppose $k \leq m+1$. In view of (1.3.a), no bond in t can contain more than $k-1$ edges, so t cannot contain a bond (at prices z_e) if $(k-1)\left(\frac{1}{k} + \frac{1}{2m}\right) \leq 1$, i.e. if $k(k-1) \leq 2m$. Thus $z'_e = z_e$ if $k(k-1) \leq 2m$, establishing the first half of part (a). If $k(k-1) > 2m$ and $k \leq m+1$, then bonds are possible. The largest value of z'_e occurs when t contains a bond B with $k-1$ edges and e is the edge not in B, in which case

$$z'_e = \left(\frac{1}{k} + \frac{1}{2m}\right) + (k-1)\left(\frac{1}{k} + \frac{1}{2m}\right) - 1 = \frac{k}{2m},$$

establishing part (b).

Now suppose $k > m+1$. By Lemma 3, no bond can contain more than m edges. Thus t cannot contain a bond if $m(\frac{1}{k} + \frac{1}{2m}) \leq 1$, i.e. if $k \geq 2m$, so $z'_e = z_e$ when $k \geq 2m$, establishing the second half of part (a). If $m+1 < k < 2m$, then bonds are possible. The largest value of z'_e occurs when t contains a bond B with m edges and $e \in t - B$, in which case

$$z'_e = \left(\frac{1}{k} + \frac{1}{2m}\right) + \frac{m\left(\frac{1}{k} + \frac{1}{2m}\right) - 1}{k-m} = \frac{k}{2m(k-m)},$$

establishing part (c).

4. Appendix

Here we review some elementary facts about matroids. More detail may be found in [W], [B] or [BP], for example.

Basic definitions. If E is a finite set, a *matroid* on E is a nonempty collection $M(E)$ of subsets of E, called *independent sets*, satisfying

(a) Subsets of independent sets are independent sets.

(b) For any $A \subset E$, every maximal independent subset of A has the same size, denoted by $r(A)$.

The function $r(A)$ is called the *rank function* of the matroid. An independent set of cardinality $r(E)$ is called a *basis* of the matroid.

If M is a matrix over a field, we may view M as a set of columns and define a matroid for which an independent set is a linearly independent set of columns of M. The same letter is often used to denote both the matrix and the associated matroid.

Induced matroids. If $M(E)$ is over E, the *dual matroid* to $M(E)$ is the matroid with independent sets $M^*(E) = \{A : A \subset E - B$ for some basis B of $M(E)\}$. The bases of $M^*(E)$ are the complements of the bases of $M(E)$, so that $(M^*)^* = M$.

Any set $X \subset E$ induces two matroids on $E - X$ with independent sets $M(E)\backslash X = \{A \subset E - X : A \in M(E)\}$ and $M(E)/X = \{A \subset E - X : A \cup B \in M(E)$, where B is a maximal independent subset of $X\}$. We say that $M(E)\backslash X$ is obtained by *deleting* X and $M(E)/X$ is obtained by *contracting* X.

Proposition A.1. *For any* $X \subset E$, $(M(E)/X)^* = M^*(E)\backslash X$.

(This is Theorem (4.3.2) of [W], or Theorem 14 of [B], for example.)

Proposition A.2. *The rank function of a contraction matroid is given by*

$$r_{M(E)/X}(A) = r(A \cup X) - r(X).$$

(See for example [W, p. 61, (4.3.2)], or [BP, Theorem 2.8)].

Graphic matroids. Let G be a connected undirected graph with edge set E. The *cycle matroid* $\mathscr{E}(G)$ is the matroid on E for which an independent set is a collection of edges containing no cycle. The bases of $\mathscr{E}(G)$ are the spanning trees of G.

The dual matroid to $\mathscr{E}(G)$, denoted $\mathscr{E}^*(G)$, is called the *bond*, or *cutset matroid* of G. An independent set in $\mathscr{E}^*(G)$ is a collection of edges whose removal does not disconnect G.

Proposition A.3. *The rank function* $\mathscr{E}^*(G)$ *is* $r(A) = |A| - C(\mathscr{E}\backslash A) + 1$, *where* $C(\mathscr{E}\backslash A)$ *is the number of components in* G *after the edges in* A *are deleted.* (*This follows from* [W, *p.* 35, (2.1.5) *and p.* 29], (1.10.5), *or* [BP, *Exercise* 3.3 *and p.* 125].)

Representable matroids. A matroid $M(E)$ is said to be *represented* by a matrix M if there is a one-to-one correspondence between the elements of E and the columns of M which preserves independence.

Proposition A.4. *If* $M(E)$ *is represented by a matrix* M, *then* $M^*(E)$ *is represented by any matrix* M^* *with row space equal to the orthogonal complement of the row space of* M.

Proposition A.4 is Theorem 17 of [B], or Theorem 9.3.2 of [W]. The next proposition can be found on p. 350 of [B], or in [W, pp. 171–172].

Proposition A.5. *If G is a connected graph, introduce a directed graph G' by assigning an arbitaray orientation to each edge of G. Then the edge-vertex incidence matrix of G' represents $\mathscr{E}(G)$.*

Proposition A.6. *If $M(E)$ is represented by the matrix $[I, M]$, then $M^*(E)$ is represented by $[M^{\mathrm{tr}}, -I]$. (See* [W, *Corollary* 9.3.1], *or Theorem* 18 *of* [B].)

Proposition A.7. *Suppose $M(E)$ is represented by M. Let $X \subset E$. Row reduce M so that the columns corresponding to X are in echelon form, obtaining a matrix M'. Form M'' by deleting from M' the columns corresponding to X, and also deleting any row which has a nonzero entry in one of those columns. Then M'' represents $M(E)/X$.*

Proposition A.7 is Theorem 9 of [B].

Proposition A.8. *Given a matrix of the form $[U, V]$, the contraction $[U, V]/V$ is represented by MU, where M is any matrix with row space equal to the orthogonal complement of the column space of V.*

Proof. Let B be a maximal nonsingular submatrix of V, which we assume for convenience to lie in the upper left corner of V. Reducing $[U, V]$ to put V in echelon form is equivalent to multiplying on the left by

$$\left[\begin{array}{c} B^{-1}, 0 \\ \hline M \end{array}\right].$$

The result now follows from Proposition A.7.

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