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A SHARP INEQUALITY FOR MARTINGALE TRANSFORMS AND THE UNCONDITIONAL BASIS CONSTANT OF A MONOTONE BASIS IN $L^p(0, 1)$

K. P. CHOI

ABSTRACT. Let $1 < p < \infty$. Let $d = (d_1, d_2, \dots)$ be a real-valued martingale difference sequence, $\theta = (\theta_1, \theta_2, \dots)$ is a predictable sequence taking values in $[0, 1]$. We show that the best constant of the inequality,

$$\left\|\sum_{k=1}^{n} \theta_k d_k\right\|_p \leq c_p \left\|\sum_{k=1}^{n} d_k\right\|_p, \qquad n \geq 1,$$

satisfies

$$c_p = \frac{p}{2} + \frac{1}{2}\log\left(\frac{1+\gamma}{2}\right) + \frac{\alpha_2}{p} + \cdots,$$

where $\gamma = e^{-2}$ and $\alpha_2 = [\frac{1}{2}\log\frac{1+\gamma}{2}]^2 + \frac{1}{2}\log\frac{1+\gamma}{2} - 2(\frac{\gamma}{1+\gamma})^2$. The best constant equals the unconditional basis constant of a monotone basis of $L^p(0, 1)$.

1. INTRODUCTION

More than fifty years ago, Paley [10] proved the following inequality for the Walsh system of functions ψ_n on the Lebesgue unit interval. If $1 < p < \infty$, there is a positive real number c_p with the property that if $b_1, b_2, \dots$ are real numbers and

$$e_n = \sum_{2^n \leq m < 2^{n+1}} b_m \psi_m,$$

then

$$c_p^{-1}\left\|\sum_{k=1}^{n} e_k\right\|_p \leq \left\|\sum_{k=1}^{n} \varepsilon_k e_k\right\|_p \leq c_p \left\|\sum_{k=1}^{n} e_k\right\|_p, \tag{1.1}$$

for all signs $\varepsilon_k \in \{1, -1\}$ and all positive integer n. Notice that the left-hand side of this inequality follows at once from the right-hand side.

It was then observed by Marcinkiewicz [8] that Paley's inequality can be given an equivalent formulation in terms of the Haar system of functions h_n:

$$c_p^{-1}\left\|\sum_{k=1}^{n} a_k h_k\right\|_p \leq \left\|\sum_{k=1}^{n} \varepsilon_k a_k h_k\right\|_p \leq c_p \left\|\sum_{k=1}^{n} a_k h_k\right\|_p, \qquad 1 < p < \infty. \tag{1.2}$$

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Here $a_1, a_2, \dots$ are real numbers and the constant c_p is the same as in Paley's inequality.

Throughout this work, we adopt the following convention. The constant c_p may change from one use to the next; however, if it is necessary to be more specific, the best constant in an inequality, say (1.1), is denoted by $c_p(1.1)$. With this notation, Marcinkiewicz's result can be stated as follows:

$$c_p(1.1) = c_p(1.2).$$

In 1966, Burkholder [1] extended the result of Paley and Marcinkiewicz to martingales:

$$\left\|\sum_{k=1}^{n} v_k d_k\right\|_p \le c_p \left\|\sum_{k=1}^{n} d_k\right\|_p, \qquad 1 < p < \infty, \tag{1.3}$$

where $v = (v_1, v_2, \dots)$ is a predictable sequence uniformly bounded in absolute value by 1 and $d = (d_1, d_2, \dots)$ is a martingale difference sequence. Here the constant c_p is independent of both v and d.

An important special case of (1.3) is

$$\left\|\sum_{k=1}^{n} \varepsilon_k d_k\right\|_p \le c_p \left\|\sum_{k=1}^{n} d_k\right\|_p, \qquad 1 < p < \infty, \tag{1.4}$$

where, again, $\varepsilon_k \in \{1, -1\}$. Clearly, $c_p(1.4) \le c_p(1.3)$. In 1981, Burkholder [2] showed that $c_p(1.3) \le c_p(1.4)$, so equality holds. The Haar system $h = (h_1, h_2, \dots)$ is a martingale difference sequence, as is $d = (a_1h_1, a_2h_2, a_3h_3, \dots)$ for real numbers a_k. Therefore $c_p(1.2) \le c_p(1.4)$. Maurey [9] proved the reverse inequality. Therefore

$$c_p(1.1) = c_p(1.2) = c_p(1.3) = c_p(1.4).$$

In 1984, Burkholder [3] derived the value of this best constant. It is

$$p^* - 1$$

where p^* is the maximum of p and its conjugate $q = p/(p-1)$. The proof rests on solving a system of nonlinear partial differential equations and inequalities. See his paper [4] for a shorter proof.

Inequality (1.3) carries over to stochastic integrals with no change in the value of the best constant (see [3]). It has applications not only in probability theory but also in Fourier analysis and the theory of singular integrals. It carries over to B-valued martingales for a large class of Banach spaces B where the constant depends both on p and B. A geometrical characterization of this class is given in [2]. For a discussion of some of this, see [5].

The main contribution of this paper is a set of equations (see Theorem 3.3 and (3.11), (3.12) in §3) that determine the best constant in the inequality,

$$\left\|\sum_{k=1}^{n} \theta_k d_k\right\|_p \le c_p \left\|\sum_{k=1}^{n} d_k\right\|_p. \tag{1.5}$$

Here, as before, $1 < p < \infty$ and $d = (d_1, d_2, \dots)$ is a martingale difference sequence, but $\theta = (\theta_1, \theta_2, \dots)$ is a predictable sequence taking values 0 or 1.

The gambling interpretation of (1.5) is obvious: as long as the gambler cannot look into the future and the game is fair in the sense that $d = (d_1, d_2, \dots)$ forms a martingale difference sequence then his fortune $\sum_{k=1}^n \theta_k d_k$ is controlled by the fortune $\sum_{k=1}^n d_k$ that would have been achieved without skipping bets. This holds for any p in $(1, \infty)$, but does not hold for $p = 1$ or $p = \infty$ in general. Inequality (1.5) can be extended to θ_k taking values in $[0, 1]$ with the same optimal constant $c_p(1.5)$.

Using a discretization argument (see §16 of [3]), we can extend the inequality (1.5) to stochastic integrals with $c_p(1.5)$ as the best constant.

The inequality (1.5) has another important connection, a connection with the unconditional basis constant. Let $1 < p < \infty$ and $e = (e_1, e_2, \dots)$ be a basis of real $L^p(0, 1)$. The unconditional basis constant, denoted by $K_p(e)$, is the extended real number (see [7], for example)

$$\sup\left\{\left\|\sum_{k=1}^n \theta_k a_k e_k\right\|_p : \text{ where } \left\|\sum_{k=1}^n a_k e_k\right\|_p = 1, \text{ for } n \geq 1, \right.$$
$$\left. a_1, \dots, a_n \text{ are real numbers and } \theta_k \in \{0, 1\}\right\}.$$

Clearly, the unconditional basis constant of the Haar system satisfies

$$K_p(h) \leq c_p(1.5)$$

and by the method of Maurey [9] the reverse inequality is true. Therefore, we have the following theorem.

Theorem A. *If $K_p(h)$ is the unconditional basis constant of the Haar system, then*

$$K_p(h) = c_p(1.5), \qquad 1 < p < \infty.$$

Let $(\Omega, \mathscr{A}, \mu)$ be a positive measure space and $(P_1, P_2, \dots)$ be a nondecreasing sequence of contractive projections in $L^p(\Omega, \mathscr{A}, \mu)$: for every $n, m \geq 1$,

$$P_m P_n = P_n P_m = P_{\min\{m,n\}} \quad \text{and} \quad \|P_n\| \leq 1.$$

Theorem B. *Let $P = (P_1, P_2, \dots)$ be any nondecreasing sequence of contractive projections in $L^p(\Omega, \mathscr{A}, \mu)$ and let $P_0 = 0$. If $f \in L^p(\Omega, \mathscr{A}, \mu)$, then*

$$\left\|\sum_{k=1}^n a_k (P_k - P_{k-1}) f\right\|_p \leq c_p(1.5)\|f\|_p, \qquad 1 < p < \infty,$$

for all integers $n \geq 1$ and all numbers $a_k \in [0, 1]$. And this inequality is sharp.

A basis (for definition, see [7]) $e = (e_1, e_2, \dots)$ in a real Banach space B with norm $\|\cdot\|_B$ is said to be monotone if

$$\left\|\sum_{k=1}^n a_k e_k\right\|_B \leq \left\|\sum_{k=1}^{n+1} a_k e_k\right\|_B,$$

for every $n \geq 1$ and all $a_k \in \mathbf{R}$. If B does have a monotone basis and $x = \sum_{k=1}^\infty a_k e_k$, then let $P_0 x = 0$ and $P_n x = \sum_{k=1}^n a_k e_k$ for all $n \geq 1$. Then it is easy to verify that $P = (P_0, P_1, \dots)$ is a nondecreasing sequence of contractive projections so we have the following conclusion.

Corollary C. *The unconditional basis constant of a monotone basis of* $L^p(0,1)$ *is* $c_p(1.5)$, *i.e.*

$$K_p(e) = c_p(1.5).$$

See [3] for the proof of Theorem B. These theorems illustrate the interest in knowing the value of $c_p(1.5)$.

In §4 (see Theorem 4.3), it is shown that

$$K_p(e) = c_p(1.5) \sim \frac{p}{2} + \frac{1}{2}\log\left(\frac{1+e^{-2}}{2}\right).$$

2. Zigzag martingales

Let $f = (f_1, f_2, \ldots)$ be a real-valued martingale on a probability space $(\Omega, \mathscr{A}, P)$ and $d = (d_1, d_2, \ldots)$ be its difference sequence. Suppose $f_1 = x - y$, where $x, y \in \mathbf{R}$ and $\theta = (1, \theta_2, \theta_3, \ldots)$ is a sequence of real numbers taking values in $\{0, 1\}$. A sequence $g = (g_1, g_2, \ldots)$ is the transform of f by $\theta = (1, \theta_2, \theta_3, \ldots)$ if, for every $n \geq 1$, $g_n = \sum_{k=1}^n \theta_k d_k$, where $\theta_1 = 1$. Indeed, g is also a real-valued martingale.

Let $(X_1, Y_1) = (x, y)$ and for all $n \geq 2$,

$$X_n = x + \sum_{k=2}^{n} \theta_k d_k, \tag{2.1}$$

$$Y_n = y + \sum_{k=2}^{n} (\theta_k - 1) d_k. \tag{2.2}$$

Then $Z = (Z_1, Z_2, \ldots)$, where $Z_n = (X_n, Y_n)$, is an $\mathbf{R}^2$-valued martingale starting at (x, y). Since $\theta_k \in \{0, 1\}$, it is obvious that for each $n \geq 2$ either $X_n - X_{n-1} \equiv 0$ or $Y_n - Y_{n-1} \equiv 0$. In other words, if Z moves at all at the nth step $(n \geq 2)$, it moves either horizontally or vertically, which way depending on n only. In the terminology of [3], Z is a zigzag martingale. Furthermore, we can recover f_n and g_n by

$$f_n = X_n - Y_n, \tag{2.3}$$

$$g_n + y = X_n. \tag{2.4}$$

3. Sharp inequalities

By a standard duality argument, it can be proved that

$$c_p(1.5) = c_q(1.5), \qquad 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1. \tag{3.1}$$

Therefore, we will determine $c_p(1.5)$, $2 < p < \infty$.

Let p_0 be the unique solution to the equation

$$p - 2 = \left[\frac{(p-1)(p-2)}{-p^2 + 5p - 5}\right]^{p-1}, \qquad 2 < p < 3, \tag{3.2}$$

$$p_0 \simeq 2.5455458. \tag{3.3}$$

Indeed, putting $x = p-2$ into (3.2) and simplifying, we obtain an equivalent equation

$$(x+1)\log(-x^2+x+1) = x\log x + (x+1)\log(x+1), \qquad 0 < x < 1. \tag{3.4}$$

The left-hand side is concave in x and the right-hand side is convex in x. The existence and uniqueness of solution to (3.4) can then be deduced readily by considering the behavior of the two sides near the endpoints of $(0, 1)$. Indeed, we have

$$(p-2)[-p^2+5p-5]^{p-1} \geq [(p-1)(p-2)]^{p-1}, \qquad 2 < p \leq p_0, \tag{3.5}$$

and,

$$(p-2)[-p^2+5p-5]^{p-1} < [(p-1)(p-2)]^{p-1}, \qquad p_0 < p \leq 3. \tag{3.6}$$

For $t \in [-1, 1]$, we define

$$E(t) = \operatorname{sgn}(t)|t|^{p-1} - (p-1)t + p - 2, \tag{3.7}$$

$$A(t) = (p-1)(1-t)^2 - [(p-2) - pt]E(t), \tag{3.8}$$

$$D(t) = (p-1)(1-t)^2 + tE(t), \quad \text{and} \tag{3.9}$$

$$B(t) = (p-1)(1-t)^2 E(t) - tA(t), \tag{3.10}$$

or equivalently,

$$B(t) = [(p-1) - pt]E(t) - tD(t). \tag{3.10'}$$

For $p_0 < p < \infty$, let $I_p = (0 \vee (p-3)/(p-1), (p-2)/p)$, and for $2 < p \leq p_0$, $I_p = (-(3-p)/2p, 0]$.

Lemma 3.1. *For $2 < p < \infty$, there exists a unique solution, $t_p \in I_p$, to the equation*

$$[p-2-(p-1)t][A(t)]^{p-1} = [B(t)]^{p-1}. \tag{3.11}$$

For $2 < p < \infty$, let

$$k_p = \frac{A(t_p)}{(p-1)(1-t_p)^2}\left\{\frac{D(t_p)}{A(t_p)}\right\}^p, \quad \text{and} \tag{3.12}$$

$$v(x, y) = |x|^p - k_p|x-y|^p, \qquad (x, y) \in \mathbf{R}^2. \tag{3.13}$$

Lemma 3.2. *For $2 < p < \infty$, there exists a biconcave function $u : \mathbf{R}^2 \to \mathbf{R}$ such that $u(0, 0) = 0$, $u(x, y) \geq v(x, y)$ for all $(x, y) \in \mathbf{R}^2$, and u satisfies the following bounds:*

$$|u(x, y)| \leq c_p(|x|^p + |y|^p), \tag{3.14}$$

$$|u_x(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}), \quad \textit{and}, \tag{3.15}$$

$$|u_y(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}). \tag{3.16}$$

Lemmas 3.1 and 3.2 will be proved in §5.

The following theorem is the main result of this paper.

Theorem 3.3. *For* $2 < p < \infty$,

$$c_p(1.5) = (k_p)^{1/p}.$$

Proof. The proof is based on the idea of Burkholder in [4]. The proof consists of two parts. Part I makes use of Lemma 3.2 to show that $c_p^p(1.5) \le k_p$ and Part II shows by an example that $k_p \le c_p^p(1.5)$.

Part I. Let $d = (d_1, d_2, \dots)$ be a martingale difference sequence of an L^p-bounded martingale f and let g be the martingale transform of f by a predictable sequence $\theta = (\theta_1, \theta_2, \dots)$, where $\theta_k \in [0, 1]$. By a reduction argument (see §2 in [3]), we may assume $d_1 \equiv 0$, $\theta_1 \equiv 1$ and that $\theta = (1, \theta_2, \dots)$ is a sequence of real numbers taking values in $\{0, 1\}$. Construct the zigzag martingale $Z = (Z_1, Z_2, \dots)$, $Z_n = (X_n, Y_n)$ by (2.1) and (2.2) where $x = y = 0$. By Lemma 3.2, there exists a biconcave function $u(x, y)$ that majorizes $v(x, y)$, therefore

$$v(X_n, Y_n) = |X_n|^p - k_p|X_n - Y_n|^p \le u(X_n, Y_n).$$

By (2.3) and (2.4), this implies that

$$|g_n|^p - k_p|f_n|^p \le u(X_n, Y_n).$$

Using the bound (3.14), we see that $u(X_n, Y_n)$ is integrable. Taking expectation, we have that

$$\|g_n\|_p^p - k_p\|f_n\|_p^p \le Eu(X_n, Y_n). \tag{3.17}$$

We also observe that

$$\begin{aligned} u(X_n, Y_n) \le u(X_{n-1}, Y_{n-1}) &+ \theta_n u_x(X_{n-1}, Y_{n-1})d_n \\ &+ (\theta_n - 1)u_y(X_{n-1}, Y_{n-1})d_n, \end{aligned} \tag{3.18}$$

since u is biconcave and $\theta_n = 0$ or $1 - \theta_n = 0$.

Now d_n is L^p-integrable and by the bounds (3.15) and (3.16), it follows that $u_x(X_{n-1}, Y_{n-1})$ and $u_y(X_{n-1}, Y_{n-1})$ are L^q-integrable. Indeed

$$\begin{aligned} \|u_x(X_{n-1}, Y_{n-1})\|_q^q &\le c_p E(|X_{n-1}|^{p-1} + |Y_{n-1}|^{p-1})^q \\ &\le c_p(E|X_{n-1}|^{(p-1)q} + E|Y_{n-1}|^{(p-1)q}) \\ &= c_p(\|X_{n-1}\|_p^p + \|Y_{n-1}\|_p^p) \\ &= c_p\left(\left\|\sum_{k=1}^n \theta_k d_k\right\|_p^p + \left\|\sum_{k=1}^n (1 - \theta_k)d_k\right\|_p^p\right) \\ &\le c_p \sum_{k=1}^n \|d_k\|_p^p < \infty. \end{aligned}$$

Similarly $\|u_y(X_{n-1}, Y_{n-1})\|_q < \infty$. By Hölder's inequality, $u_x(X_{n-1}, Y_{n-1})d_n$ and $u_y(X_{n-1}, Y_{n-1})d_n$ are integrable. Therefore

$$Eu_x(X_{n-1}Y_{n-1})d_n = E(E(d_n|\mathscr{A}_{n-1})u_x(X_{n-1}, Y_{n-1})) = 0$$

and, similarly, $Eu_y(X_{n-1}, Y_{n-1})d_n = 0$.

Hence, by (3.18) we get $Eu(X_n, Y_n) \le Eu(X_{n-1}, Y_{n-1})$. Working backward, we have

$$Eu(X_n, Y_n) \le Eu(X_1, Y_1) = u(0, 0) = 0.$$

Combining this and (3.17), we have

$$\|g_n\|_p^p \le k_p \|f_n\|_p^p .$$

Therefore $c_p^p(1.5) \le k_p$.

Part II. We will exhibit an example here to show that $c_p^p(1.5) \ge k_p$. We need the following definitions. Let t_p be as in Lemma 3.2,

$$\alpha = [p-2-(p-1)t_p]\frac{E(t_p)}{D(t_p)}, \tag{3.19}$$

$$\beta = \frac{t_p}{[p-1-pt_p]}\frac{D(t_p)}{E(t_p)}, \tag{3.20}$$

$$\lambda = [p-1-pt_p]\frac{E(t_p)}{D(t_p)}, \tag{3.21}$$

$$\omega = 2\frac{[p-1-pt_p]}{(p-1)(1-t_p)^2}E(t_p). \tag{3.22}$$

Note that $\beta > 0$ when $p_0 < p < \infty$; $\beta \le 0$ when $2 < p \le p_0$.

We need the following technical lemma which can be verified by straightforward computation. Its proof is given in §5.

Lemma 3.4. *For* $2 < p < \infty$,

(i) $\frac{\lambda}{\lambda-\alpha} + \frac{|\beta\lambda|^p}{1-\beta\lambda} = k_p[\frac{(1-\alpha)^p\lambda}{\lambda-\alpha} + \frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}]$,

(ii) $\frac{\lambda}{\lambda-\alpha} + \frac{1}{1-\beta\lambda} = p$,

(iii) $0 < \alpha < \lambda < \beta^{-1}$, *for* $p_0 < p < \infty$, *and*

(iii′) $0 < \alpha < \lambda$, $\beta \le 0$, *for* $2 < p \le p_0$.

Returning to the proof of Theorem 3.3, we proceed as follows. For $x > 0$, there exists a unique $\eta \in (\alpha, \lambda)$ such that

$$x^p = \frac{\lambda}{\lambda-\eta} + \frac{1}{1-\beta\lambda} - p. \tag{3.23}$$

We have, by (ii) in Lemma 3.4, that η converges to α as x converges to 0.

Case (1). $p_0 < p < \infty$. Fix $x > 0$ and choose $\delta \in (0, x)$, actually we will eventually let $\delta \to 0$. For all $k \ge 1$, define

$$\xi_k = 1 - \frac{\lambda\delta}{(\lambda-\eta)(x+k\delta)+\eta\delta},$$

$$\gamma_k = 1 - \frac{\delta}{(1-\beta\lambda)(x+k\delta)},$$

$$\pi_0 = 1, \quad \text{and} \quad \pi_k = \prod_{j=1}^{n} \gamma_j\xi_j .$$

Note that $\beta > 0$.

When there is no risk of ambiguity, we use $[a, b)$ to denote either the interval $\{x : a \le x < b\}$ or the indicator function of the set $[a, b)$.

On the probability space $([0, 1), \mathscr{A}, m)$, where $\mathscr{A}$ is the σ-field of all Borel measurable sets in $[0, 1)$ and m is the Lebesgue measure on $[0, 1)$, we define a sequence of functions on $[0, 1)$, $d = (d_1, d_2, \dots)$ as follows:

$$\begin{gathered} d_1 \equiv (1-\lambda)x, \quad \text{and for all } k \geq 1, \\ d_{2k} = -\lambda\delta[0, \pi_{k-1}\xi_k) + (\lambda - \eta)[x + (k-1)\delta][\pi_{k-1}\xi_k, \pi_{k-1}), \\ d_{2k+1} = \delta[0, \pi_k) - [(1-\beta\lambda)(x + k\delta) - \delta][\pi_k, \pi_{k-1}\xi_k). \end{gathered}$$

It is not difficult to see that $d = (d_1, d_2, \dots)$ forms a martingale difference sequence. Let $\theta_{2k} = 0$ and $\theta_{2k-1} = 1$ for all $k \geq 1$. Let f be the martingale with the martingale difference sequence $d = (d_1, d_2, \dots)$ and g the martingale transform of f by $\theta = (\theta_1, \theta_2, \dots)$. Therefore, we have

$$\begin{aligned} \|f\|_p &= \lim_{n\to\infty} \|f_{2n+1}\|_p^p \\ &= \lim_{n\to\infty} \|X_{2n+1} - Y_{2n+1}\|_p^p \\ &= \lim_{n\to\infty} \left\{ (1-\lambda)^p (x + n\delta)^p \pi_n + \sum_{k=1}^{n} (1-\eta)^p [x + (k-1)\delta]^p \pi_{k-1}(1-\xi_k) \right. \\ &\qquad \left. + \sum_{k=1}^{n} (1-\beta)^p \lambda^p (x + k\delta)^p \pi_{k-1}\xi_k(1-\gamma_k) \right\}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|g + \lambda x\|_p^p &= \lim_{n\to\infty} \|X_{2n+1}\|_p^p \\ &= \lim_{n\to\infty} \left\{ (x + n\delta)^p \pi_n + \sum_{k=1}^{n} [x + (k-1)\delta]^p \pi_{k-1}(1-\xi_k) \right. \\ &\qquad \left. + \sum_{k=1}^{n} (\beta\lambda)^p (x + k\delta)^p \pi_{k-1}\xi_k(1-\gamma_k) \right\}. \end{aligned}$$

We make use of the inequality $1 - w\delta \leq (1-\delta)^w$ for all $w > 1$ and $0 < \delta < 1$. Since $\beta\lambda < 1$ by Lemma 3.4(iii), we introduce $s_1 = \lambda/(\lambda - \eta) > 1$ and $s_2 = 1/(1-\beta\lambda) > 1$, then

$$\begin{aligned} \pi_k &= \left(\prod_{j=1}^{k} \gamma_j\right) \left(\prod_{j=1}^{k} \xi_j\right) \leq \prod_{j=1}^{k} \left(1 - \frac{\delta}{x + j\delta}\right)^{s_2} \times \prod_{j=1}^{k} \left(1 - \frac{\delta}{x + j\delta + \frac{\eta\delta}{\lambda-\eta}}\right)^{s_1} \\ &= \left(\frac{x}{x + k\delta}\right)^{s_1+s_2} \left(1 + \frac{\eta\delta}{(\lambda-\eta)x}\right)^{s_1} \left(\frac{x + k\delta}{x + k\delta + \frac{\eta\delta}{\lambda-\eta}}\right)^{s_1} \\ &\leq H_1(\delta) \left(\frac{x}{x + k\delta}\right)^{s_1+s_2}, \end{aligned}$$

where

$$H_1(\delta) = \left(1 + \frac{\eta\delta}{(\lambda-\eta)x}\right)^{s_1} \to 1 \quad \text{as } \delta \to 0.$$

Now

$$0 \leq (x + n\delta)^p \pi_n \leq H_1(\delta) \left(\frac{x}{x + n\delta}\right)^{s_1+s_2} (x + n\delta)^p,$$

the right side goes to zero as n goes to infinity by (3.23). So

$$\begin{aligned}
&\limsup_{\delta\to 0}\|f\|_p^p\\
&\quad\le \limsup_{\delta\to 0}\left\{\sum_{k=1}^{\infty}(1-\eta)^p[x+(k-1)\delta]^p\pi_{k-1}(1-\xi_k)\right.\\
&\qquad\qquad\left.+\sum_{k=1}^{\infty}(1-\beta)^p\lambda^p(x+k\delta)^p\pi_{k-1}\xi_k(1-\gamma_k)\right\}\\
&\quad= \limsup_{\delta\to 0}\left\{(1-\eta)^p x^p(1-\xi_1)+\sum_{k=1}^{\infty}(1-\eta)^p(x+k\delta)^p\pi_k(1-\xi_{k+1})\right.\\
&\qquad\qquad\left.+\sum_{k=1}^{\infty}(1-\beta)^p\lambda^p(x+k\delta)^p\pi_k\frac{1-\gamma_k}{\gamma_k}\right\}\\
&\quad\le \limsup_{\delta\to 0}\left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\delta\sum_{k=1}^{\infty}(x+k\delta)^{p-1}\pi_k\\
&\quad\le \left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\limsup_{\delta\to 0}H_1(\delta)\delta\sum_{k=1}^{\infty}\frac{x^{s_1+s_2}}{(x+k\delta)^{s_1+s_2-p+1}}\\
&\quad\le \left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\limsup_{\delta\to 0}\delta\int_0^{\infty}\frac{x^{s_1+s_2}}{(x+t\delta)^{s_1+s_2-p+1}}\,dt\\
&\quad= \left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\frac{x^p}{s_1+s_2-p}\\
&\quad= \frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\quad\text{by (3.23)}.
\end{aligned}$$

Similarly,

$$\limsup_{\delta\to 0}\|g+\lambda x\|_p^p\le\frac{\lambda}{\lambda-\eta}+\frac{(\beta\lambda)^p}{1-\beta\lambda}.$$

We proceed to a lower estimate of π_k. Note that if $w>1$ and $w_1=w(x-\delta)/(x-w\delta)$, then

$$1-ws\ge(1-s)^{w_1},\qquad s\in\left(0,\frac{\delta}{x}\right).$$

To see this, consider

$$\frac{\log(1-ws)}{\log(1-s)}=\frac{\log(1-ws)-\log 1}{\log(1-s)-\log 1},$$

by mean value theorem, there exists a $c\in(0,s)$,

$$=\frac{-w/(1-wc)}{-1/(1-c)}=\frac{w-1}{1-wc}+1\le\frac{w-1}{1-w\delta/x}+1=w_1.$$

Therefore,

$$\log(1-ws)\ge w_1\log(1-s),$$

i.e.,

$$1-ws\ge(1-s)^{w_1}.$$

To obtain a lower estimate of π_k, let

$$r_1 = \frac{x-\delta}{(1-\beta\lambda)x-\delta}, \qquad r_2 = \frac{\lambda(x-\delta)}{(\lambda-\eta)x-\lambda\delta},$$

$$\begin{aligned}\pi_k &\geq \prod_{j=1}^{k}\left(1-\frac{\delta}{x+j\delta}\right)^{r_1}\left(1-\frac{\delta}{x+j\delta+\frac{\eta\delta}{\lambda-\eta}}\right)^{r_2} \\ &= \frac{x^{r_1+r_2}}{(x+k\delta)^{r_1+r_2}}\left(\frac{x+k\delta}{x+k\delta+\frac{\eta\delta}{\lambda-\eta}}\right)^{r_2}\left(\frac{x+\frac{\eta\delta}{\lambda-\eta}}{x}\right)^{r_2} \\ &\geq \frac{x^{r_1+r_2}}{(x+k\delta)^{r_1+r_2}}H_2(\delta),\end{aligned}$$

where

$$H_2(\delta) = \left[\frac{x+k\delta}{x+k\delta+\frac{\eta\delta}{\lambda-\eta}}\right]^{r_2} \to 1 \quad \text{as } \delta \to 0.$$

Furthermore, as $\delta \to 0$, $r_1 \to 1/(1-\beta\lambda)$ and $r_2 \to \lambda/(\lambda-\eta)$. Therefore,

$$\begin{aligned}\liminf_{\delta\to 0}\|f\|_p^p &\geq \liminf_{\delta\to 0}\left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\sum_{k=1}^{\infty}\frac{\delta x^{r_1+r_2}H_2(\delta)}{(x+k\delta)^{r_1+r_2-p+1}} \\ &= \liminf_{\delta\to 0}\left\{\frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}\right\}\int_0^{\infty}\frac{\delta x^{r_1+r_2}}{(x+t\delta)^{r_1+r_2-p+1}}\,dt \\ &= \frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda},\end{aligned}$$

i.e.,

$$\lim_{\delta\to 0}\|f\|_p^p = \frac{\lambda(1-\eta)^p}{\lambda-\eta}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda}.$$

Similarly

$$\lim_{\delta\to 0}\|g+\lambda x\|_p^p = \frac{\lambda}{\lambda-\eta}+\frac{(\beta\lambda)^p}{1-\beta\lambda},$$

so

$$\begin{aligned}\lim_{x\to 0}\lim_{\delta\to 0}\|f\|_p^p &= \frac{\lambda(1-\alpha)^p}{\lambda-\alpha}+\frac{(1-\beta)^p\lambda^p}{1-\beta\lambda} \\ &= \frac{1}{k_p}\lim_{x\to 0}\lim_{\delta\to 0}\|g+\lambda x\|_p^p\end{aligned}$$

by Lemma 3.4(i) since $\beta > 0$ for $p_0 < p < \infty$

$$= \frac{1}{k_p}\lim_{x\to 0}\lim_{\delta\to 0}\|g\|_p^p,$$

i.e., $c_p^p(1.5) \geq k_p$.

Case (2). $2 < p \leq p_0$. Fix $x > 0$ and choose $\delta > 0$ such that $\delta \ll x$. For all $k \geq 1$, define ξ_k, γ_k, π_k, d_k and θ_k as in Case (1). Similar calculation as in Case (1) shows that $k_p \leq c_p^p(1.5)$. This completes the proof, apart from the lemmas, of Theorem 3.3. □

Remark. The martingale f described in Part II of the proof of Theorem 3.3 is actually constructed from a zigzag martingale by (2.3). Here we provide a brief geometrical description of this zigzag martingale. Let $x > 0$, choose $\eta \in (\alpha, \lambda)$ by (3.23). The zigzag martingale first starts at $(x, \lambda x)$, it then moves vertically either down to the point $(x, \eta x)$ where it will stop or up to the point $(x, \lambda(x+\delta))$. From the point $(x, \lambda(x+\delta))$, the zigzag martingale moves horizontal either to the left to the point $(\beta\lambda(x+\delta), \lambda(x+\delta))$ where it will stop or to the right to the point $(x+\delta, \lambda(x+\delta))$. The pattern of movement is then repeated.

4. Asymptotic results and remarks

We make use of a result of Burkholder in [3] and triangle inequality to obtain

Theorem 4.1. *For* $1 < p < \infty$,

$$\max\left\{1, \frac{p^*}{2} - 1\right\} \le c_p(1.5) \le \frac{1}{2}p^*,$$

where $p^* = \max\{p, p/(p-1)\}$.

Proof. Let $d = (d_1, d_2, \dots)$ be a martingale difference sequence and $\theta_k \in \{0, 1\}$, $k \ge 1$. Let $\varepsilon_k = 2\theta_k - 1$. Then $\varepsilon_k \in \{1, -1\}$ and

$$\begin{aligned}\left\|\sum_{k=1}^{n} \theta_k d_k\right\| &= \frac{1}{2}\left\|\sum_{k=1}^{n} d_k + \sum_{k=1}^{n} \varepsilon_k d_k\right\|_p \\ &\le \frac{1 + c_p(1.4)}{2}\left\|\sum_{k=1}^{n} d_k\right\|_p .\end{aligned}$$

Therefore, $c_p(1.5) \le (1 + c_p(1.4))/2 = \frac{1}{2}p^*$.

Similarly, let $\varepsilon_k \in \{1, -1\}$ and define $\theta_k = (\varepsilon_k + 1)/2 \in \{0, 1\}$. Then

$$\begin{aligned}\left\|\sum_{k=1}^{n} \varepsilon_k d_k\right\|_p &= \left\|2\sum_{k=1}^{n} \theta_k d_k - \sum_{k=1}^{n} d_k\right\|_p \\ &\le [2c_p(1.5) + 1]\left\|\sum_{k=1}^{n} d_k\right\|_p .\end{aligned}$$

Therefore, $c_p(1.4) - 1 \le 2c_p(1.5)$. On the other hand, $c_p(1.5) \ge 1$, Theorem 4.1 follows immediately. □

We assume that $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$, $p_0 < p < \infty$. By (3.10) and (3.8), we rewrite (3.11) as

$$\{t_p + [p - 2 - (p-1)t_p]^{1/(p-1)}\}^{-1} = \frac{1}{E(t_p)} - \frac{p - 2 - pt_p}{(p-1)(1-t_p)^2}. \tag{4.1}$$

From Theorem 3.3, we have $a_0 = 1$, $a_1 = -2$. Substituting $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$ into (4.1) and equating coefficients of p^{-k}, $k = 0, 1, 2, \dots$, we obtain

Theorem 4.2. *Assuming* $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$, $p_0 < p < \infty$, *then*

$$t_p = 1 - \frac{2}{p} - \frac{2(1-\gamma)}{(1+\gamma)}\frac{1}{p^2} - \frac{4\gamma(3-\gamma)(1-\gamma)}{(1+\gamma)^3}\frac{1}{p^3} + \frac{4\gamma(-9\gamma^4+56\gamma^3-186\gamma^2+120\gamma-13)}{3(1+\gamma)^5}\frac{1}{p^4} + \cdots,$$

where $\gamma = e^{-2}$.

We substitute these known coefficients into (3.12), we have

Theorem 4.3. *For* $p_0 < p < \infty$,

$$c_p(1.5) = \frac{p}{2} + \frac{1}{2}\log\left(\frac{1+\gamma}{2}\right) + \frac{\alpha_2}{p} + \cdots,$$

where $\gamma = e^{-2}$ *and*

$$\alpha_2 = \left[\frac{1}{2}\log\left(\frac{1+\gamma}{2}\right)\right]^2 + \frac{1}{2}\log\left(\frac{1+\gamma}{2}\right) - 2\left(\frac{\gamma}{1+\gamma}\right)^2.$$

Remark. That $a_2 = -2(1-\gamma)/(1+\gamma)$ can also be derived from Theorem 4.1 and the fact that $t_p = 1 - 2/p + a_2/p^2 + \cdots$. Indeed, Theorem 4.1 implies that $c_p(1.5) = p/2 + 0(1)$. Substitute, $t_p = 1 - 2/p + a_2/p^2 + O(1/p^3)$ into (3.12) and by Theorem 3.3, we have $c_p(1.5) = p(1+\gamma)/(4+(1+\gamma)a_2) + O(1)$. this implies $a_2 = -2(1-\gamma)/(1+\gamma)$.

5. Proof of lemmas

Proof of Lemma 3.1. We will give a sketch of the proof. For details, see [6].

Case 1. $p_0 < p < \infty$. Define $a(t) = A(t)/(1-t)^2$ and $b(t) = B(t)/(1-t)^2$.

Step 1. We shall show that $a(t)$ is increasing on I_p, $b(t)$ is decreasing on I_p and that they are both positive on I_p.

We note that $E(t)$ is positive and decreasing on $(0, 1)$. Convexity of $E(t)$ on $(0, 1)$ and $E(1) = 0$ imply that $E(t)/(1-t)$ is a positive decreasing function on $(0, 1)$. Note that

$$a(t) = p - 1 - \frac{[(p-2) - pt]}{(1-t)} \cdot \frac{E(t)}{1-t}, \quad \text{and} \tag{5.1}$$

$$b(t) = (p-1)E(t) - ta(t). \tag{5.2}$$

Now $a(t)$ is increasing on $(0, (p-2)/p)$ because $[(p-2)-pt]/(1-t)$ and $E(t)/(1-t)$ are both positive and decreasing on $(0, (p-2)/p)$. By direct verification, we have $a((p-2)/p) > 0$ and $b((p-2)/p) > 0$.

(i) When $p_0 < p \leq 3$, $a(0) \geq 0$. Therefore $a(t) > 0$ on $(0, (p-2)/p)$. It follows from (5.2) that $b(t)$ is decreasing on $(0, (p-2)/p)$ and hence positive on I_p.

(ii) When $3 < p < \infty$. There exists a unique $t_1 \in (0, (p-2)/p)$ such that $a(t_1) = 0$. Since $a((p-3)/(p-1)) > 0$, it implies that $a(t) > 0$ on I_p. It follows from (5.2) that $b(t)$ is decreasing on I_p and hence positive on I_p.

Step 2. Existence of a solution to (3.11). Define

$$\Delta(t) = (p-2) - (p-1)t - \left[\frac{B(t)}{A(t)}\right]^{p-1}, \qquad t \in I_p. \tag{5.3}$$

(i) When $p_0 < p \leq 3$, we can verify that

$$\Delta(0) = p - 2 - \left[\frac{(p-2)(p-1)}{-p^2+5p-5}\right]^{p-1} < 0.$$

(This explains equation (3.2).)

(ii) When $3 < p < \infty$, we have $A((p-3)/(p-1)) < B((p-3)/(p-1))$, therefore $\Delta((p-3)/(p-1)) < 0$. Now $\Delta(t)$ is continuous and $\Delta((p-2)/p) > 0$. Existence of a solution to (3.11) follows.

Step 3. Uniqueness of solution to (3.11). For $3 < p < \infty$, we first show that $a(t)$ is convex on $(0, 1)$. Differentiate (5.1) twice and let $M(t)$ be the numerator. When $4 < p < \infty$, $M''(t) < 0$ on $(0, \tilde{s})$, where $\tilde{s} = (p-4)(p-2)/p^2$; $M''(t) > 0$ on $(\tilde{s}, 1)$. Computing $M'(t)$ at 0 and 1, we see that $M'(t) < 0$ on $(0, 1)$. Therefore, $M(t) > M(1) = 0$. When $3 < p \leq 4$, $M''(t) > 0$ on $(0, 1)$, therefore $M'(t) < M'(1) = 0$ and this implies that $M(t) > 0$ on $(0, 1)$. Now we deduce that $[p-2-(p-1)t](a(t))^{p-1}$ is increasing on I_p. Its derivative equals $(a(t))^{p-2}\{[p-2-pt]a'(t)+ta'(t)-a(t)\}$ which is positive because $ta'(t)-a(t)$ is increasing, therefore $ta'(t)-a(t) > t_1 a'(t_1) > 0$ for $t \in (t_1, (p-2)/p)$. Uniqueness of solution to (3.11) follows immediately since $(b(t))^{p-1}$ is decreasing. For $p_0 < p \leq 3$, we show that $b(t)/a(t)$ is convex on $(0, t_2)$. Uniqueness of solution to (3.11) follows because $[p-2-(p-1)t]$ is linear and $(b(t)/a(t))^{p-1}$ is convex. To show that $b(t)/a(t)$ is convex, we differentiate it twice and rewrite the numerator as

$$N(t) = a(t)[a(t)b''(t) - 2a'(t)b'(t)] + (-a''(t))a(t)b(t) + 2b(t)a'(t).$$

Since $a''(t) < 0$ in this case. To show that $N(t)$ is positive, it suffices to show that $\Delta_1(t) = a(t)b''(t) - 2a'(t)b'(t) \geq 0$. Recall $b(t) = (p-1)E(t) - ta(t)$ from (5.2), $\Delta_1(t) = (p-1)a(t)E''(t) + ta(t)[-a''(t)] + 2(p-1)a'(t)(-E'(t)) + t(a'(t))^2$. Each of these four terms is nonnegative on I_p. This completes the proof of Lemma 3.1 for Case 1.

Case 2. $2 < p \leq p_0$. We will be brief in the proof. We are able to show that $A(t)$ and $B(t)$ are positive on I_p, and by (3.5), $\Delta(0) > 0$ (see (5.3)). It can be shown that $\Delta(-(3-p)/2p) < 0$, existence of a solution follows. Lastly we can show that $[p-2-(p-1)t](B(t)/A(t))^{p-1}$ is decreasing and uniqueness is proved.

This completes the proof of Lemma 3.1. □

Proof of Lemma 3.4. We note that $(p-1-pt_p)$, $[p-2-(p-1)t_p]$, $D(t_p)$ and $E(t_p)$ are positive, so $\alpha > 0$. Also, $\lambda - \alpha = (1-t_p)E(t_p)/D(t_p) > 0$ and $1 - \beta\lambda = 1 - t_p > 0$. When $p_0 < p < \infty$, $t_p > 0$ so $\beta > 0$. Since $\beta\lambda = t_p < 1$, this implies $\lambda < \beta^{-1}$. When $2 < p \leq p_0$, $t_p \leq 0$ so $\beta \leq 0$. This completes the proof of (iii) and (iii$'$).

It is easy to show (ii) by (3.19) to (3.21).

From (3.7) and (3.9), we see readily that the left-hand side of (i) equals $D(t_p)/(1-t_p)$. By definition of $A(t)$ in (3.8), we have

$$\frac{\lambda(1-\alpha)^p}{\lambda - \alpha} = \frac{[p-1-pt_p]}{(1-t_p)}\left[\frac{A(t_p)}{D(t_p)}\right]^p.$$

From (3.10′),

$$\frac{(1-\beta)^p\lambda^p}{\lambda-\alpha} = \frac{1}{(1-t_p)}\cdot\left[\frac{B(t_p)}{D(t_p)}\right]^p.$$

Since t_p is a solution to (3.11), the right-hand side of (i) can be simplified as

$$\frac{(p-1-pt_p)A(t_p)+[p-2-p-1)t_p]B(t_p)}{(p-1)(1-t_p)^3}$$

which can be further simplified as $D(t_p)/(1-t_p)$ by (3.10′), (3.8) and (3.9). This completes the proof of (i) and hence Lemma 3.4. □

Let

(5.4) $$A = 1-k_p(1-\alpha)^p = v(x,\alpha x)|x|^{-p},$$

(5.5) $$B = |\beta|^p - k_p(1-\beta)^p = v(\beta y, y)|y|^{-p}.$$

To prove Lemma 3.2, we need the following identities and inequalities which are grouped under the following lemma.

Lemma 5.1. *For* $2<p<\infty$, *we have*

(i) $1-\alpha = \dfrac{A(t_p)}{D(t_p)} > 0$,

(ii) $A = \dfrac{[(p-2)-pt_p]}{(p-1)(1-t_p)^2}E(t_p) > 0$,

(iii) $1-k_p(1-\alpha)^{p-2} < 0$,

(iv) $\omega - A = \dfrac{pE(t_p)}{(p-1)(1-t_p)} > 0$,

(v) $\dfrac{\omega-A}{\lambda-\alpha} = pk_p(1-\alpha)^{p-1}$,

(vi) $pA - \dfrac{\alpha(\omega-A)}{\lambda-\alpha} = p[1-k_p(1-\alpha)^{p-1}]$,

(vii) $pA + (p-2)(\omega-A) - \dfrac{2\alpha(\omega-A)}{\lambda-\alpha} = 0$,

(viii) $1-\beta = \dfrac{B(t_p)}{[(p-1)-pt_p]E(t_p)} > 0$,

(ix) $\omega - B\lambda^p = \dfrac{p[(p-1)-pt_p]}{(p-1)(1-t_p)}E(t_p) > 0$,

(x) $p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1-\beta)^{p-1}] = \dfrac{\lambda}{1-\beta\lambda}\left[\dfrac{\omega}{\lambda^p} - B\right]$,

(xi) $-pk_p(1-\beta)^{p-1} = pB - \dfrac{\beta\lambda}{1-\beta\lambda}\left[\dfrac{\omega}{\lambda^p} - B\right]$,

(xii) $pB + (p-2)\left[\dfrac{\omega}{\lambda^p} - B\right] - \dfrac{2\beta\lambda}{1-\beta\lambda}\left[\dfrac{\omega}{\lambda^p} - B\right] = 0$,

(xiii) $\dfrac{\omega - B\lambda^p}{1-\beta\lambda} = pA - \dfrac{\alpha(\omega-A)}{\lambda-\alpha} + (p-1)(\omega-A)$,

(xiv) $pB\lambda^{p-1} + (p-1)\left[\dfrac{\omega}{\lambda^p} - B\right]\lambda^{p-1} - \dfrac{\beta\lambda}{1-\beta\lambda}\left[\dfrac{\omega}{\lambda^p} - B\right]\lambda^{p-1} = \dfrac{\omega-A}{\lambda-\alpha}$,

(xv) $|\beta|^{p-2} - k_p(1-\beta)^{p-2} < 0$.

Proof of Lemma 5.1. From (3.19), (3.8) and (3.9) we get (i). By (i), (3.12) and (3.8), we get (ii). Using (i), (3.12) and (3.9), we obtain

$$1 - k_p(1-\alpha)^{p-1} = 1 - \frac{D(t_p)}{(p-1)(1-t_p)^2} = \frac{-t_p E(t_p)}{(p-1)(1-t_p)^2} < 0.$$

Therefore, $1 < k_p(1-\alpha)^{p-1} < k_p(1-\alpha)^{p-2}$, so (iii) is proved. From (ii) and (3.22), we prove (iv). From (iv), (3.19) and (3.21), both sides of (v) equal $D(t_p)/(p-1)(1-t_p)^2$, so (v) follows.

Now,

$$\begin{aligned} pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} &= pA - p\alpha k_p(1-\alpha)^{p-1} \quad \text{by (v)} \\ &= p[1 - k_p(1-\alpha)^p] - p\alpha k_p(1-\alpha)^{p-1} \quad \text{by (5.4)} \\ &= p[1 - k_p(1-\alpha)^{p-1}], \end{aligned}$$

which is (vi). From (ii), (iv) and (v), we can prove (vii). By (3.20) and (3.10′), we prove (viii).

To show (ix). From (viii), we obtain

$$\begin{aligned} \omega - B\lambda^p &= \frac{2[p-1-pt_p]}{(p-1)(1-t_p)^2}E(t_p) - |t_p^p| + \frac{[B(t_p)]^p}{(p-1)(1-t_p)^2[A(t_p)]^{p-1}} \\ &= \frac{2[p-1-pt_p]}{(p-1)(1-t_p)^2}E(t_p) + [p-1-pt_p] - D(t_p) \\ &\quad + \frac{[p-2-(p-1)t_p]\{[p-1-pt_p]E(t_p) - t_pD(t_p)\}}{(p-1)(1-t_p)^2} \qquad (3.9)\text{–}(3.11) \\ &= \frac{[p-1-pt_p][p-(p-1)t_p]E(t_p)}{(p-1)(1-t_p)^2} - \frac{[p-1-pt_p]}{(p-1)(1-t_p)^2}D(t_p) \\ &\quad + [p-1-pt_p] \\ &= \frac{[p-1-pt_p][p-(p-1)t_p]E(t_p)}{(p-1)(1-t_p)^2} - \frac{[p-1-pt_p]t_pE(t_p)}{(p-1)(1-t_p)^2} \\ &= \frac{p[(p-1)-pt_p]E(t_p)}{(p-1)(1-t_p)} > 0. \end{aligned}$$

To show (x). It is enough to show that

$$\frac{\omega - B\lambda^p}{1 - \beta\lambda} = p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1-\beta)^{p-1}]\lambda^{p-1}.$$

Now,

$$\begin{aligned} \text{R.H.S.} &= p\left\{\operatorname{sgn}(t_p)|t_p|^{p-1} + \frac{D(t_p)[(p-2)-(p-1)t_p]}{(p-1)(1-t_p)^2}\right\} \\ &= p\left\{E(t_p) - [(p-2)-(p-1)t_p] \right. \\ &\qquad\qquad \left. + \frac{D(t_p)[(p-2)-(p-1)t_p]}{(p-1)(1-t_p)^2}\right\} \\ &= p\frac{[(p-1)-pt_p]E(t_p)}{(p-1)(1-t_p)^2} = \text{L.H.S.}, \quad \text{so (x) is verified.} \end{aligned}$$

Now,

$$\begin{aligned} B &= |\beta|^p - k_p(1-\beta)^p \\ &= \beta\{\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1-\beta)^{p-1}\} - k_p(1-\beta)^{p-1}. \end{aligned}$$

Therefore, by (x), we have,

$$B = \frac{\beta\lambda}{p(1-\beta\lambda)}\left[\frac{\omega}{\lambda^p} - B\right] - k_p(1-\beta)^{p-1}.$$

Multiplying throughout by p and rearranging terms, we obtain (xi). Instead of verifying (xii), we will show that

$$pB\lambda^p = \left[\frac{2\beta\lambda}{1-\beta\lambda} - (p-2)\right][\omega - B\lambda^p],$$

which is equivalent to (xii). Now,

$$\begin{aligned} B\lambda^p &= [B\lambda^p - \omega] + \omega \\ &= \frac{-p[(p-1) - pt_p]E(t_p)}{(p-1)(1-t_p)} + \frac{2[(p-1) - pt_p]E(t_p)}{(p-1)(1-t_p)^2} \quad \text{by (ix)} \\ &= \frac{p[(p-1) - pt_p]E(t_p)}{(p-1)(1-t_p)}\left\{\frac{2}{p(1-t_p)} - 1\right\} \\ &= \frac{p[(p-1) - pt_p]E(t_p)}{(p-1)(1-t_p)}\left\{\frac{2t_p}{p(1-t_p)} - \left(1 - \frac{2}{p}\right)\right\}. \end{aligned}$$

Since $\beta\lambda = t_p$ and by (ix), so

$$B\lambda^p = \left\{\frac{2\beta\lambda}{p(1-\beta\lambda)} - \left(1 - \frac{2}{p}\right)\right\}[\omega - B\lambda^p],$$

this implies (xii). Rewriting (vii) and (xii), they become

$$\frac{\lambda A}{\lambda - \alpha} = \left[\frac{\lambda}{\lambda - \alpha} - \frac{p}{2}\right]\omega \quad \text{and} \quad \frac{B\lambda^p}{1 - \beta\lambda} = \left[\frac{1}{1-\beta\lambda} - \frac{p}{2}\right]\omega.$$

Therefore,

$$\frac{\lambda A}{\lambda - \alpha} + \frac{B\lambda^p}{1 - p\lambda} = \left[\frac{\lambda}{\lambda - \alpha} + \frac{1}{1 - \beta\lambda} - p\right]\omega$$

which is equivalent to (xiii).

$$\begin{aligned} \text{L.H.S. of (xiv)} &= \frac{2\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} - (p-2)\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} \\ &\quad + (p-1)\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} - \frac{\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} \quad \text{by (xii)} \\ &= \left\{1 + \frac{\beta\lambda}{1-\beta\lambda}\right\}\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} \\ &= \frac{p}{(p-1)}\frac{D(t_p)}{(1-t_p)^2} \quad \text{by (ix) and (3.21)}. \end{aligned}$$

By (iv), (3.19) and (3.21), R.H.S. of (xiv) $= pD(t_p)/(p-1)(1-t_p)^2$, so we have proved (xiv).

It remains to show (xv). It is enough to verify $k_p((1-\beta)/|\beta|)^{p-2} > 1$. Now, by (viii), (3.20), (3.12) and (3.11), we have

$$k_p\left(\frac{1-\beta}{|\beta|}\right)^{p-1} - 1 = \frac{[p-2-(p-1)t_p][D(t_p)]^2}{|t_p|^{p-2}(p-1)(1-t_p)^2 B(t_p)} - 1.$$

Therefore, it suffices to show that

$$[(p-2)-(p-1)t_p][D(t_p)]^2 - (p-1)(1-t_p)^2|t_p|^{p-2}B(t_p) > 0. \tag{5.6}$$

Using (3.10′), we can rewrite the

$$\begin{aligned}\text{L.H.S.} = {} & \{[(p-2)-(p-1)t_p]D(t_p) + (p-1)(1-t_p)^2\operatorname{sgn}(t_p)|t_p|^{p-1}\}D(t_p) \\ & - (p-1)(1-t_p)^2|t_p|^{p-2}[(p-1)-pt_p]E(t_p).\end{aligned}$$

We simplify the expression inside { }:

$$\begin{aligned}&[(p-2)-(p-1)t_p]D(t_p) + (p-1)(1-t_p)^2[E(t_p)-(p-2)+(p-1)t_p] \\ &\quad = [(p-2)-(p-1)t_p]\{D(t_p)-(p-1)(1-t_p)^2\} + (p-1)(1-t_p)^2E(t_p) \\ &\quad = [(p-1)-pt_p]E(t_p) \quad \text{by (3.9)}.\end{aligned}$$

So,

$$\begin{aligned}\text{L.H.S.} &= [(p-1)-pt_p]E(t_p)D(t_p) - (p-1)(1-t_p)^2|t_p|^{p-2}[(p-1)-pt_p]E(t_p) \\ &= [(p-1)-pt_p]E(t_p)\{D(t_p)-(p-1)(1-t_p)^2|t_p|^{p-2}\} \\ &> [(p-1)-pt_p]E(t_p)(p-1)(1-t_p)^2(1-|t_p|^{p-2}) \quad \text{by (3.9)} \\ &> 0.\end{aligned}$$

So (xv) is established and this concludes the proof of Lemma 5.1. □

Proof of Lemma 3.2. Define

$$\begin{aligned}\Omega_1 &= \{(x,y)\in\mathbf{R}^2 : y>0,\ y<\alpha x\}, \\ \Omega_2 &= \{(x,y)\in\mathbf{R}^2 : y>0,\ \alpha x<y<\lambda x\}, \\ \Omega_3 &= \{(x,y)\in\mathbf{R}^2 : y>0,\ \beta y<x<\lambda^{-1}y\}, \\ \Omega_4 &= \{(x,y)\in\mathbf{R}^2 : y>0,\ x<\beta y\}.\end{aligned}$$

Let $u(x,y)$ be the continuous function from $\mathbf{R}^2$ to $\mathbf{R}$ satisfying $u(x,y) = u(-x,-y)$, and

$$u(x,y) = \begin{cases} v(x,y), & (x,y)\in\Omega_1\cup\Omega_4, \\ Ax^p + \frac{y-\alpha x}{\lambda-\alpha}(\omega-A)x^{p-1}, & (x,y)\in\Omega_2, \\ By^p + \frac{\lambda(x-\beta y)}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p}-B\right]y^{p-1}, & (x,y)\in\Omega_3.\end{cases}$$

The bounds on u, u_x and u_y, and $u(0,0)=0$ can be verified readily.

Step 1. To show that u is concave in y on $\bigcup_{i=1}^4\Omega_i$. On $\Omega_1\cup\Omega_4$, $u_{yy}(x,y) = -p(p-1)k_p|x-y|^{p-2} < 0$, therefore u is concave in y. On Ω_2, $u_{yy}\equiv 0$, hence u is concave in y. On Ω_3,

$$\begin{aligned}u_{yy}(x,y) = {} & p(p-1)By^{p-2} - \frac{2(p-1)\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p}-B\right]y^{p-2} \\ & + (p-1)(p-2)\frac{\lambda(x-\beta y)}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p}-B\right]y^{p-3}.\end{aligned}$$

Now,

$$u_{yy}\left(\left(\frac{y}{\lambda}\right)-, y\right) = (p-1)\left\{pB + (p-2)\left[\frac{\omega}{\lambda^p} - B\right] - \frac{2\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right]\right\} y^{p-2}$$
$$= 0 \quad \text{by Lemma 5.1(xii)}.$$

So,

$$\begin{aligned} u_{yy}(x, y) &= u_{yy}(x, y) - u_{yy}\left(\left(\frac{y}{\lambda}\right)-, y\right) \\ &= \frac{(p-1)(p-2)(\lambda x - y)}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right] y^{p-3} \\ &< 0 \quad \text{by Lemma 5.1(ix) and that } y > \lambda x \text{ on } \Omega_3. \end{aligned}$$

Step 2. To show that u is concave in x on $\bigcup_{i=1}^4 \Omega_i$. On Ω_1,

$$\begin{aligned} u_{xx}(x, y) &= p(p-1)[x^{p-2} - k_p(x-y)^{p-2}] \\ &< p(p-1)[1 - k_p(1-\alpha)^{p-2}]x^{p-2} < 0 \end{aligned}$$

by Lemma 5.1(iii).

On Ω_4,
$$\begin{aligned} u_{xx}(x, y) &= p(p-1)[|x|^{p-2} - k_p(y-x)^{p-2}] \\ &< p(p-1)[|\beta|^{p-2} - k_p(1-\beta)^{p-2}]y^{p-2} \\ &< 0 \quad \text{by Lemma 5.1(xv)}. \end{aligned}$$

On Ω_2,
$$\begin{aligned} u_{xx}(x, y) &= p(p-1)Ax^{p-2} - \frac{2(p-1)\alpha(\omega - A)}{\lambda - \alpha}x^{p-2} \\ &\quad + (p-1)(p-2)\frac{y - \alpha x}{\lambda - \alpha}(\omega - A)x^{p-3}. \end{aligned}$$

Since

$$u_{xx}(x, \lambda x-) = (p-1)\left\{pA + (p-2)(\omega - A) - \frac{2\alpha(\omega - A)}{\lambda - \alpha}\right\} x^p = 0,$$

so

$$\begin{aligned} u_{xx}(x, y) &= u_{xx}(x, y) - u_{xx}(x, \lambda x-) \\ &= \frac{(p-1)(p-2)(\omega - A)}{\lambda - \alpha}(y - \lambda x)x^{p-3} < 0, \end{aligned}$$

by Lemma 5.1(iv) and the fact that $y < \lambda x$ on Ω_2. The function, u, is affine in x on Ω_3, so u is concave in x on $\bigcup_{i=1}^4 \Omega_i$.

Step 3. To show that the first derivatives match up at the boundaries of the regions. At $y = \alpha x$, we have

$$\begin{aligned} u_x(x, \alpha x+) &= \left[pA - \frac{\alpha}{\lambda - \alpha}(\omega - A)\right] x^{p-1} \\ &= p[1 - k_p(1-\alpha)^{p-1}]x^{p-1} = u_x(x, \alpha x-) \end{aligned}$$

from Lemma 5.1(vi). Also, we have

$$\begin{aligned} u_y(x, \alpha x+) &= \frac{\omega - A}{\lambda - \alpha}x^{p-1} \\ &= pk_p(1-\alpha)^{p-1}x^{p-1} = u_y(x, \alpha x-) \quad \text{by Lemma 5.1(v)}. \end{aligned}$$

At $y = \lambda x$,

$$u_x(x, \lambda x+) = \frac{\omega - B\lambda^p}{1-\beta\lambda} x^{p-1}$$
$$= \left[pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} + (p-1)(\omega - A)\right] x^{p-1} = u_x(x, \lambda x-)$$

from Lemma 5.1(xiii); and

$$u_y(x, \lambda x+) = \left\{pB\lambda^{p-1} + (p-1)\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1} - \frac{\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right]\lambda^{p-1}\right\} x^{p-1}$$
$$= \frac{\omega - A}{\lambda - \alpha} x^{p-1} = u_y(x, \lambda x-)$$

from Lemma 5.1(xiv). At $x = \beta y$, we get

$$u_x(\beta y-, y) = p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1-\beta)^{p-1}]y^{p-1}$$
$$= \frac{\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right] y^{p-1} = u_x(\beta y+, y)$$

from Lemma 5.1(x); and

$$u_y(\beta y-, y) = -pk_p(1-\beta)^{p-1}y^{p-1}$$
$$= \left\{pB - \frac{\beta\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right]\right\} y^{p-1} = u_y(\beta y+, y)$$

from Lemma 5.1(xi).

Step 4. To show that $\lambda < 1$. Since

$$1 - \lambda = \{D(t_p) - [p - 1 - pt_p]E(t_p)\}/D(t_p)$$
$$= (1-t_p)^2\left\{a(t_p) - \frac{E(t_p)}{1-t_p}\right\}/D(t_p),$$

it is enough to show that $a(t_p) - E(t_p)/(1-t_p) > 0$. From Step 1 of the proof of Lemma 3.1, we see that $a(t)$ and $E(t)/(1-t)$ are increasing and decreasing functions respectively. Therefore, $\psi(t) = a(t) - E(t)/(1-t)$ is an increasing function. For $p_0 \le p \le 3$, $\psi(0) = a(0) - E(0) = A(0) - E(0) = (3-p)(p-1) \ge 0$, therefore $\psi(t_p) > 0$; for $3 < p < \infty$,

$$\psi\left(\frac{p-3}{p-1}\right) = -(p-1)\left[\frac{p-3}{p-1}\right]^{p-1} < 0$$

and

$$\psi\left(\frac{p-2}{p}\right) = \frac{p}{2}\left[1 - \left(\frac{p-2}{p}\right)^{p-1}\right] > 0.$$

Therefore, there exists a unique $s_p \in I_p$ such that $\psi(s_p) = 0$ and $\psi(t) > 0$ on $(s_p, (p-2)/p)$. So, $\psi(t_p) > 0$ follows if we can prove that $s_p < t_p$. This can be reduced to proving

$$[p - 2 - (p-1)s_p] - (b(s_p)/a(s_p))^{p-1} < 0. \tag{5.7}$$

As $\psi(s_p) = 0$, it implies $(1-s_p)a(s_p) = E(s_p)$. Using this and (5.2), we get $b(s_p)/a(s_p) = p - 1 - ps_p > 1$. Therefore,

$$\left(\frac{b(s_p)}{a(s_p)}\right)^{p-1} > \frac{b(s_p)}{a(s_p)} = p - 1 - ps_p > (p-2) - (p-1)s_p.$$

So, (5.7) follows and this finishes Step 4.

Step 5. To verify that $u(x, y) \geq v(x, y)$ on $\mathbf{R}^2$. By homogeneity and symmetry, it suffices to show that (i) $u(1, y) \geq v(1, y)$ for $y \in (\alpha, \lambda)$; and (ii) $u(x, 1) \geq v(x, 1)$ for $x \in (\beta, \lambda^{-1})$.

Case (i). We observe that the graph of $u(1, y)$ for $\alpha < y < \lambda$ is part of the line tangent to the graph of the concave function $v(1, y)$ at $y = \alpha$. Therefore, $u(1, y) > v(1, y)$ for $\alpha < y < \lambda$.

Case (ii). Observe that again the graph of $u(x, 1)$ is tangent to the graph of the function $v(x, 1)$ at the point $x = \beta$. Since $u(1, \lambda) > v(1, \lambda)$ from case (i), $u(\lambda^{-1}, 1) > v(\lambda^{-1}, 1)$ by homogeneity. Let m be the slope of the line defined by $u(x, 1)$ on (β, λ^{-1}), i.e., $m = (\lambda/(1-\beta\lambda))[\omega/\lambda^p - B]$. Consider the function $V(x) = v(x, 1)$. We have,

$$V'(x) = p\{\operatorname{sgn}(x)|x|^{p-1} - \operatorname{sgn}(x-1)|x-1|^{p-1}\}$$

and

$$V''(x) = p(p-1)\{|x|^{p-2} - k_p|x-1|^{p-2}\}.$$

There exist ξ_1, ξ_2 such that $0 < \xi_1 < 1 < \xi_2$ and $V''(\xi_i) = 0$, $i = 1, 2$. Furthermore, V is concave on each of the connected components of $\mathbf{R}\backslash[\xi_1, \xi_2]$; and V is convex on (ξ_1, ξ_2). If $\xi_2 \geq \lambda^{-1}$. The affinity of $u(x, 1)$, the concave-convex situations as described above, the fact that $u(x, 1)$ is tangent to $V(x)$ at $x = \beta$ and $u(x, 1) \geq v(x, 1)$ at $x = \beta$ and $x = \lambda^{-1}$ imply that $u(x, 1) \geq v(x, 1)$ on (β, λ^{-1}). If $\xi_2 < \lambda^{-1}$. Now $k_p(\xi_2 - 1)^{p-2} = \xi_2^{p-2}$ and it follows that $V'(\xi_2) = p\xi_2^{p-2} < p\lambda^{-(p-2)}$. Consider,

$$\begin{aligned} m - p\lambda^{-(p-2)} &= \frac{\lambda}{1-\beta\lambda}\left[\frac{\omega}{\lambda^p} - B\right] - p\lambda^{-(p-2)} \\ &= \left\{\frac{\omega - B\lambda^p}{1-\beta\lambda} - p\lambda\right\}\lambda^{-(p-1)} \\ &= \frac{pt_p[p-1-pt_p][E(t_p)]^2}{(p-1)(1-t_p)^2 D(t_p)}\lambda^{-(p-1)} > 0 \end{aligned}$$

from Lemma 5.1(ix), (3.21) and (3.9). Therefore, $m > p\lambda^{-(p-2)} > V'(\xi_2)$. Since $u(x, 1) \geq v(x, 1)$ at $x = \beta$, $x = \xi_2$ and $x = \lambda^{-1}$, it follows that $u(x, 1) \geq v(x, 1)$ on (β, λ^{-1}). This completes the proof of Case (ii) and hence the proof of Lemma 3.2. □

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Department of Mathematics, National University of Singapore, Lower Kent Ridge, Singapore 0511