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THE DEFEKTSATZ FOR CENTRAL SIMPLE ALGEBRAS

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ABSTRACT. Let Q be a central simple algebra finite-dimensional over its center F and let V be a valuation ring of F. Then V has an extension to Q, i.e., there exists a Dubrovin valuation ring B of Q satisfying $V = F \cap B$. Generally, the number of extensions of V to Q is not finite and therefore the so-called intersection property of Dubrovin valuation rings $B_1, \dots, B_n$ is introduced. This property is defined in terms of the prime ideals and the valuation overrings of the intersection $B_1 \cap \cdots \cap B_n$. It is shown that there exists a uniquely determined natural number n depending only on V and having the following property: If $B_1, \dots, B_k$ are extensions of V having the intersection property then $k \leq n$ and $k = n$ holds if and only if $B_1 \cap \cdots \cap B_k$ is integral over V. Let n be the extension number of V to Q. There exist extensions $B_1, \dots, B_n$ of V having the intersection property and if $R_1, \dots, R_n$ are also extensions of V having the intersection property then $B_1 \cap \cdots \cap B_n$ and $R_1 \cap \cdots \cap R_n$ are conjugate. The main result regarding the extension number is the Defektsatz: $[Q : F] = f_B(Q/F)e_B(Q/F)n^2p^d$, where $f_B(Q/F)$ is the residue degree, $e_B(Q/F)$ the ramification index, n the extension number, $p = \operatorname{char}(V/J(V))$, and d a natural number.

1. INTRODUCTION

Valuation theory has a long tradition for commutative fields but a couple of years ago only little was known about noncommutative valuation rings. Since around 1980 valuation rings, especially of finite-dimensional division algebras, have been investigated extensively and the results which have already been obtained seem to signify a rich and promising theory.

A first systematic approach to a general noncommutative valuation theory was attempted by Schilling in 1945. Since he was rather engaged in valuations than valuation rings, his rings which arose from his valuations were necessarily invariant (under all inner automorphisms of the division algebra). Although they do not exist very often (indeed, there are finite-dimensional division algebras which do not have any valuation) they turned out to be a useful tool in studying division algebras and many authors could use them to get significant information about finite-dimensional division algebras having special kinds of valuations. A subring B of a division algebra D is called total valuation ring if $x \in B$ or $x^{-1} \in B$ holds for each nonzero x in D and with this notation Schilling's valuation rings are exactly the invariant total valuation rings

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(or invariant valuation rings for short). Originally, total valuation rings were constructed for geometrical reasons but they also provide examples as well as counterexamples in topological algebra and it should be emphasized that non-invariant total valuation rings also exist in finite-dimensional division algebras. From another point of view, the investigation of noncommutative valuation rings can be understood as an investigation of extensions of given commutative valuation rings and this seems to be the right standpoint, especially for finite-dimensional division algebras, since their valuation rings are closely related to their centers. Unfortunately, a general extension theorem does not exist and valuation rings of the center are not necessarily extendible to the entire division algebra. With a view to improving this situation Dubrovin introduced in [D1] a new type of valuation ring for division algebras as well as matrices over division algebras (i.e. simple Artinian rings) and they have many properties which signify that Dubrovin valuation rings are the right object to study in central simple algebras. For example, if Q denotes a central simple algebra finite-dimensional over its center F then each valuation ring V of F can be extended to a Dubrovin valuation ring of Q and all extensions are conjugate. Furthermore, if V is discrete these extensions are exactly the maximal orders over V. Finally, if Q is in addition a division algebra then all extensions are total valuation rings if there is at least one total valuation ring of Q extending V, i.e., general Dubrovin valuation rings appear if no total valuation rings are available.

This paper deals with Dubrovin valuation rings of a central simple algebra Q finite-dimensional over its center F. In [W2] Wadsworth investigates what happens to a Dubrovin valuation ring B of Q with passage to the Henselization of its center $V = F \cap B$ and one of his results is the Ostrowski Theorem or Defektsatz for Dubrovin valuation rings:

$$[Q : F] = f_B(Q/F)e_B(Q/F)(n_B/t_B)^2 p^d ,$$

where $f_B(Q/F)$ is the residue degree, $e_B(Q/F)$ the ramification index, $p = \operatorname{char}(V/J(V))$, and d a nonnegative integer. Finally, n_B/t_B is a positive integer which appears by passing to the Henselization. If Q is a division algebra and B a total valuation ring then n_B/t_B is equal to the number of all extensions of V to Q but generally this number is infinite and so n_B/t_B cannot have this meaning. The purpose of this paper is to state a new Defektsatz where n_B/t_B is replaced by a number n which indicates the relation between the extensions of V. In §6 we define the intersection property of Dubrovin valuation rings $B_1, \dots, B_n$ of Q and roughly speaking $B_1, \dots, B_n$ have this property if each prime ideal P of R belongs to a Dubrovin valuation ring B of Q containing R such that $J(B) \cap R = P$ where R denotes the intersection of $B_1, \dots, B_n$. It is proved that B can be obtained by localization at P (i.e., $B = R_P$) and this localization induces a bijection between the prime ideals of R and the valuation overrings of R. For example, total valuation rings always have the intersection property. With regard to the extensions of V the following statements are proved: First of all, there exist Dubrovin valuation rings $B_1, \dots, B_k$ of Q extending V such that $B_1, \dots, B_k$ have the intersection property. If $B_1, \dots, B_k$ are given this way we can find $B_{k+1}, \dots, B_n$ such that $B_1, \dots, B_n$ still have the intersection property and $R = B_1 \cap \cdots \cap B_n$ is integral over V. This number n is uniquely determined, i.e., if $B'_1, \dots, B'_m$ are also extensions of V having the intersection property such that $R' = B'_1 \cap \cdots \cap B'_m$ is integral over V then $n = m$. Furthermore, R and R' are conjugate. Now, we call n the extension number of V to Q and we prove:

$$[Q : F] = f_B(Q/F)e_B(Q/F)n^2p^d.$$

If Q is a division algebra and B a total valuation ring then n is exactly the number of all extensions of V to Q.

The investigations in this paper require two theorems which have been proved in [W2]. The first theorem describes the center of the residue ring of a Dubrovin valuation ring and the other states that two extensions of a central valuation ring are conjugate. Both proofs use Henselization techniques and they are involved in the large network of proofs given in [W2]. Therefore, in §§3 and 4 we give new direct proofs of these results without using Henselizations and also the other theorems are proved independently from [W2]. The proofs given in this paper explain how Dubrovin valuation rings are built of total valuation rings, matrix rings, maximal valuation rings as well as Azumaya algebras over valuation rings and they also show how to use commutative valuation theory after passing to the residue ring. These results are a further step forward making the structure of Dubrovin valuation rings more understandable.

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2. Preliminaries

In this section we give all information about total valuation rings as well as Dubrovin valuation rings which are needed in this paper and we restrict ourselves to the case where Q is a central simple algebra finite-dimensional over its center F. First of all a few words on notation. If R is any ring then we write $Z(R)$ for the center of R, $J(R)$ for the Jacobson radical of R, and $M_k(R)$ for the $k \times k$-matrix ring of R.

Definition. A subring B of Q is called Dubrovin valuation ring of Q if there exists an ideal M of B such that the following hold.
(i) B/M is simple Artinian.
(ii) For each a in $Q \backslash B$ there exist b, b' in B with $ab, b'a$ in $B \backslash M$.

It turns out that M is the only maximal ideal of B and therefore M is actually the Jacobson radical $J(B)$ of B. Furthermore, $V = F \cap B$ is a valuation ring of F and V is the center of B. If Q is a division algebra then each total valuation ring of Q is a Dubrovin valuation ring.

Theorem 2.1. *Let V be a valuation ring of F. There exists a Dubrovin valuation ring B of Q extending V, i.e., $V = F \cap B$.*

Proof. Cf. [D2, §3, Theorem 2] or [BG2, Theorem 3.8].

The following list is a collection of essential properties of Dubrovin valuation rings which have been proved by Dubrovin in [D1, D2] and which are used in this paper.

Let B be a Dubrovin valuation ring of Q with center $V = F \cap B$, then

(D1) The two-sided ideals of B are totally ordered by inclusion where a two-sided ideal is a B-bimodule of Q.

(D2) Each finitely generated left (resp. right) ideal of B is principal.
(D3) Each overring B' of B in Q is a Dubrovin valuation ring of Q and $J(B')$ is a prime ideal of B. Furthermore, the overrings of B in Q are totally ordered by inclusion.
(D4) If P is a prime ideal of B then the classical localization B_P can be formed and B_P is a Dubrovin valuation ring of Q with $J(B_P) = P$.
(D5) If P is a prime ideal of B and $P' = P \cap F$ then $B_P = B_{P'}$, where $B_{P'} = B(V \backslash P)^{-1}$.
(D6) $B \cap F = B' \cap F$ implies $B = B'$ for each overring B' of B in Q.
(D7) $M_k(B)$ is a Dubrovin valuation ring of $M_k(Q)$ and if B' is a Dubrovin valuation ring of $M_k(Q)$ then $xB'x^{-1} = M_k(B'')$ for some Dubrovin valuation ring B'' of Q and some regular x in $M_k(Q)$.

The residue ring. $B/J(B)$ is called residue ring of B and $B/J(B)$ is a central simple algebra finite-dimensional over its center. $V/J(V)$ can be comprehended as a subfield of the center of the residue ring $B/J(B)$ and

$$f_B(Q/F) = [B/J(B) : V/J(V)]$$

is called residue degree where $f_B(Q/F) \leq [Q : F]$ can be proved like the commutative case. If B' is a Dubrovin valuation ring of Q containing B then $B/J(B')$ is a Dubrovin valuation ring of $B'/J(B')$ and $B/J(B')$ is an extension of $V/J(W)$ where $W = F \cap B'$.

The value group. The set of all two-sided ideals of B is an ordered monoid with respect to the usual multiplication and the inclusion and B is the unit element. A two-sided ideal I is called invertible if $IJ = JI = B$ for some two-sided ideal J of B and in this situation J is denoted by I^{-1} since J is uniquely determined. The set Γ_B of all invertible two-sided ideals is an ordered group and is called value group of B, $\widehat{\Gamma}_B = \Gamma_B \cup \{0\}$. It is easily checked that a two-sided ideal I is invertible if and only if there exists a regular q in Q such that $I = qB = Bq$. This shows that our definition is equivalent to Wadsworth's given in [W2] but it is different from the definition given in [Ma] (cf., [G3, 3.4. Theorem]). Let Γ_V be the value group of $V = F \cap B$. Then Γ_V can be identified with its image in Γ_B under the canonical inclusion, i.e., Γ_V is a subgroup of the ordered group Γ_B and $e_B(Q/F) = [\Gamma_B : \Gamma_V]$ is called ramification index. Like the commutative case, $e_B(Q/F)$ is finite since $e_B(Q/F) \leq [Q : F]$ and therefore Γ_B/Γ_V is abelian.

With respect to the overrings there are two types of Dubrovin valuation rings. B belongs to the first type if there exists a maximal Dubrovin valuation ring B' of Q containing B, i.e. $B' \neq Q$ and Q is the only overring of B' in Q. Then $V' = F \cap B'$ is a maximal valuation ring of F and $\Gamma_{V'}$ as well as $\Gamma_{B'}$ are archimedean ordered groups. B belongs to the other type, i.e., B has no maximal overring in Q, if and only if $V = Z(B)$ has no maximal overring in F.

Theorem 2.2. *Let B be a Dubrovin valuation ring of Q having no maximal overring in Q and let $\{a_1, \dots, a_n\}$ be an F-basis of Q. There exists a Dubrovin valuation ring B' of Q, $B' \neq Q$, containing B with center $V' = F \cap B'$ such that following hold.*

(i) $B' = a_1 V' + \cdots + a_n V'$.
(ii) $J(B') = a_1 J(V') + \cdots + a_n J(V')$, *i.e.,* $J(B') = J(V')B'$.
(iii) $f_{B'}(Q/F) = n$.
(iv) $V'/J(V')$ *is the center of* $B'/J(B')$.
(v) B' *is an Azumaya algebra over* V'.

Theorem 2.2 can be obtained by a result of Azumaya (cf. [A, Theorem 12]).

We now turn to the case of total valuation rings, and a Dubrovin valuation ring B is total (and Q is a division algebra) if and only if $J(B)$ is a completely prime ideal of B. Let B be a total valuation ring of Q with center $V = F \cap B$. Then B is integral over V if and only if B is invariant, i.e., $B = qBq^{-1}$ for all nonzero q in Q.

Theorem 2.3. *Let B be an invariant valuation ring of Q with center V. Then $Z(B/J(B))$ is a normal extension of $V/J(V)$ and each $(V/J(V))$-automorphism of $Z(B/J(B))$ is induced by an inner automorphism of Q.*

Proof. Cf. [JW, Proposition 1.7].

If B is a total valuation ring which is not invariant then the following theorem is very useful.

Theorem 2.4. *Let B be a total valuation ring which is not invariant and let $V = F \cap B$.*

(i) *There exists an invariant overring B' of B in Q.*
(ii) *If B' is the minimal invariant overring of B and $V' = F \cap B'$ then $Z(B'/J(B')) \neq V'/J(V')$.*
(iii) *All extensions of V to Q are conjugate.*

Proof. Cf. [BG1, Lemma 5, Theorem 2 and BG2, Theorem 4.3].

3. All extensions are conjugate

In this section Q denotes a central simple algebra finite-dimensional over its center F having a Dubrovin valuation ring B with center $V = B \cap F$.

Proposition 3.1. *Let $\{a_1, \ldots, a_n\}$ be an F-basis of Q such that $B = a_1 V + \cdots + a_n V$. If B' is a Dubrovin valuation ring of Q extending V, then B and B' are conjugate, i.e., $B = qB'q^{-1}$ for some regular q in Q.*

Proof. There exists some q in Q such that $a_1 B' + \cdots + a_n B' = qB'$. Since the identity 1 is in $B \subseteq a_1 B' + \cdots + a_n B' = qB'$, we have that q is regular. It remains to prove $B = qB'q^{-1}$. Let $b_1, \ldots, b_n$ be in B' satisfying $a_1 b_1 q^{-1} + \cdots + a_n b_n q^{-1} = 1$. Since $a_i a_j$ is in $B = a_1 V + \cdots + a_n V \subseteq a_1 B' + \cdots + a_n B'$, we obtain that $a_i a_j b_j q^{-1}$ is in $(a_1 B' + \cdots + a_n B')q^{-1} = qB'q^{-1}$ for all $1 \leq i$, $j \leq n$. Therefore, $a_i = a_i a_1 b_1 q^{-1} + \cdots + a_i a_n b_n q^{-1}$ is in $qB'q^{-1}$ for all $1 \leq i \leq n$ and $B \subseteq qB'q^{-1}$ is shown. Finally, $B \cap F = qB'q^{-1} \cap F$ implies $B = qB'q^{-1}$. Q.E.D.

Proposition 3.2. *Let Q be a division algebra and let B' be a Dubrovin valuation ring of Q extending V. If P (resp. P') denotes the maximal completely prime ideal of B (resp. B') then $P \cap F = P' \cap F$ and $qPq^{-1} = P'$ for some nonzero q in Q.*

Proof. Without restrictions we can assume $P' \cap F \subseteq P \cap F$. There exists a prime ideal P'' of B' containing P' such that $P'' \cap F = P \cap F$. Since P is completely prime, B_P is a total valuation ring of Q. Let $B'' \supseteq B'$ be the Dubrovin valuation ring of Q such that $J(B'') = P''$, i.e., $B'' \cap F = B_P \cap F$. By [BG2, Theorem 4.3], B'' is total and we obtain that P'' is completely prime. Thus $P' = P''$ and $P' \cap F = P \cap F$ is shown. Finally, $qB_Pq^{-1} = B''$ by [BG1, Theorem 2] and therefore $qPq^{-1} = P'$. Q.E.D.

Now, we can give a new proof of

Theorem 3.3. *Let V be a valuation ring of F. If B and B' are two Dubrovin valuation rings of Q extending V, then B and B' are conjugate, i.e., $B = qB'q^{-1}$ for some regular q in Q.*

Proof. We prove the theorem by induction on $n = [Q : F]$. Let $n > 1$.

Case 1. $Q = M_l(D)$ where D is a finite-dimensional division algebra with center F and $l > 1$. Then, there exist Dubrovin valuation rings R, R' of D and regular r, s in Q such that $rBr^{-1} = M_l(R)$, $sB's^{-1} = M_l(R')$, and $R \cap F = R' \cap F$. By induction hypothesis, R and R' are conjugate in D.

Case 2. Q is a division algebra and 0 is the only completely prime ideal of B. By Proposition 3.2, 0 is the only completely prime ideal of B'.

Case 2.1. There exists a maximal Dubrovin valuation ring R of Q containing B. Then, there exists a maximal Dubrovin valuation ring R' of Q containing B'. By [BG2, Theorem 5.4], R and R' are conjugate. Thus, we can assume $R = R'$. Define $W = R \cap F$ and $B/J(R)$, $B'/J(R)$ are two Dubrovin valuation rings of $R/J(R)$ such that $(B/J(R)) \cap (W/J(W)) = (B'/J(R)) \cap (W/J(W))$. By [BG2, Theorem 5.3] we can even assume that $(B/J(R)) \cap Z(R/J(R)) = (B'/J(R)) \cap Z(R/J(R))$. Since $J(R)$ is not completely prime, $B/J(R)$ and $B'/J(R)$ are conjugate by Case 1.

Case 2.2. There is no maximal Dubrovin valuation ring of Q containing B or B'. Then, there exists a valuation ring W of F, $W \neq F$, containing V and an F-basis $\{a_1, \dots, a_n\}$ of Q such that $R = a_1W + \cdots + a_nW$ is a Dubrovin valuation ring of Q extending W and containing B. By Proposition 3.1, we assume that R also contains B'. Similar to Case 2.1 we prove that B and B' are conjugate.

Case 3. Q is a division algebra and the maximal completely prime ideal P (resp. P') of B (resp. B') is nonzero. By Proposition 3.2 we can assume $P = P'$, and B_P is a total valuation ring of Q containing B and B'. Let R be the minimal invariant valuation ring of Q containing B_P. Define $W = R \cap F$. Thus, $B/J(R)$ and $B'/J(R)$ are two Dubrovin valuation rings of $R/J(R)$ such that $(B/J(R)) \cap (W/J(W)) = (B'/J(R)) \cap (W/J(R))$. If $W/J(W) = Z(R/J(R))$ then B_P is invariant by [BG1, Lemma 5], i.e., $R = B_P$. Therefore, $(B/J(R)) \cap Z(R/J(R)) = (B'/J(R)) \cap Z(R/J(R))$, and $B/J(R)$, $B'/J(R)$ are conjugate by Case 2. If $W/J(W) \neq Z(R/J(R))$ then we can assume $(B/J(R)) \cap Z(R/J(R)) = (B'/J(R)) \cap Z(R/J(R))$ by Theorem 2.3 and $B/J(R)$, $B'/J(R)$ are conjugate by induction hypothesis. Q.E.D.

4. The residue ring and its center

In this section we give a new proof of

Theorem 4.1. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V = B \cap F$.*

(i) *$Z(B/J(B))$ is a normal extension of $V/J(V)$.*

(ii) *Each $V/J(V)$-automorphism of $Z(B/J(B))$ is induced by an inner automorphism i_q of Q such that $i_q(B) = qBq^{-1} = B$.*

Before proving Theorem 4.1 we need two well-known results.

Proposition 4.2. *Let $F \subseteq K \subseteq L$ be commutative fields such that K is a finite normal extension of F and L is a finite normal extension of K. Then L is a finite normal extension of F if each $\sigma \in \mathrm{Gal}(K/F)$ has an extension $\sigma' \in \mathrm{Gal}(L/F)$, i.e., $\sigma'|_K = \sigma$, and in this case $\mathrm{Gal}(L/F) = \{\sigma'\tau | \sigma \in \mathrm{Gal}(K/F)$ and $\tau \in \mathrm{Gal}(L/K)\}$.*

Proposition 4.3. *Let $F \subseteq K$ be commutative fields such that K is a normal extension of F and let B be a valuation ring of K extending $V = B \cap F$. Then, $B/J(B)$ is a normal extension of $V/J(V)$ and each*

$$\overline{\sigma} \in \mathrm{Gal}((B/J(B))/(V/J(V)))$$

is induced by some $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(B) = B$.

The proof of Proposition 4.2 is straightforward and Proposition 4.3 is proved in [E, Theorem 14.5, Theorem 19.6].

Now, we turn to the

Proof (*of Theorem* 4.1). We prove this theorem by induction on $n = [Q : F]$. Let $n > 1$. We show that three cases can happen where in each case a Dubrovin valuation ring $B' \supseteq B$ of Q with center $V' = B' \cap F$ occurs having one of the following properties:

(I) The theorem holds for B' and $Z(B'/J(B')) \neq V'/J(V')$.

(II) $J(B')$ is not completely prime and $Z(B'/J(B')) = V'/J(V')$.

Case 1. Q is not a division algebra. Define $B' = Q$ and B' has property (II).

Case 2. Q is a division algebra and 0 is the maximal completely prime ideal of B. If there is no maximal Dubrovin valuation ring of Q containing B, there exists B' with property (II) by Theorem 2.2. Now, let B' be a maximal Dubrovin valuation ring of Q containing B. If $Z(B'/J(B')) \neq V'/J(V')$ then B' has property (I) by [BG2, Proposition 5.3]. Otherwise, B' has property (II).

Case 3. Q is a division algebra and there exists a completely prime ideal P of B, $P \neq 0$. Let P be maximal and choose $B' = B_P$, i.e., $J(B') = P$. If $Z(B'/J(B')) = V'/J(V')$ the theorem holds by Case 2. Otherwise, B' has property (II) by [G4, Theorem 3.4].

It remains to show that Theorem 4.1 holds if B' occurs satisfying (I) or (II). If B' has property (II) we can assume that $B'/J(B') = M_l(D)$, $l > 1$, and $B/J(B') = M_l(R)$ where R is a Dubrovin valuation ring of the finite-dimensional division algebra D with center F. The theorem follows by induction hypothesis. If B' satisfies property (I) then the theorem can be proved like [G4, Theorem 3.4]: Let

$$Z = Z(B'/J(B')).$$

By the induction hypothesis, $Z(B/J(B'))/(J(B)/J(B'))$ is a normal extension of $(Z\cap(B/J(B')))/(Z\cap(J(B)/J(B')))$ and each automorphism σ of the corresponding Galois group is induced by an inner automorphism i_d of $B'/J(B')$ that maps $B/J(B')$ onto $B/J(B')$. Thus σ is induced by an inner automorphism i_d of Q which maps B onto B. With respect to Proposition 4.2 it remains to show

(a) $(Z\cap(B/J(B')))/(Z\cap(J(B)/J(B')))$ is a normal extension of

$$(V/J(V'))/(J(V)/J(V')),$$

(b) each σ of the corresponding Galois group is induced by an inner automorphism i_d of Q which maps B onto B.

Clearly, Proposition 4.3 and (I) imply (a). Let σ be as in (b). By Proposition 4.3 and (I), σ is induced by an inner automorphism i_d of Q that maps B' onto B' such that $Z\cap(B/J(B'))=Z\cap(dBd^{-1}/J(B'))$. Finally, there exists a unit q of B' such that $qdB(qd)^{-1}=B$ by Theorem 3.3. Since Z is the center of $B'/J(B')$ the inner automorphism i_q induces in Z the identity. Thus, i_{qd} induces σ in $(Z\cap(B/J(B')))/(Z\cap(J(B)/J(B')))$. Q.E.D.

Corollary 4.4. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V=B\cap F$.*

(i) *There exists a surjective group homomorphism*

$$\Theta_B:\Gamma_B/\Gamma_V\to \mathrm{Gal}(Z(B/J(B))/(V/J(V))),$$

where Θ_B is induced by conjugation by regular elements $q\in Q$ with $qBq^{-1}=B$.

(ii) $\mathrm{Gal}(Z(B/J(B))/(V/J(V)))$ *is abelian.*

Corollary 4.5. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V=B\cap F$. For each Dubrovin valuation ring B' of Q containing B:*

(i) *there is an exact sequence*

$$0\to\Gamma_{\overline{B}}/\Gamma_{\overline{V}}\xrightarrow{\alpha}\Gamma_B/\Gamma_V\xrightarrow{\beta}\Gamma_{B'}/\Gamma_{V'}\xrightarrow{\gamma}G/G^Z\longrightarrow 0,$$

(ii) $|\Gamma_{B'}/\Gamma_{V'}|\cdot|\Gamma_{\overline{B}}/\Gamma_{\overline{V}}|=|\Gamma_B/\Gamma_V|\cdot[G:G^z]$, *where* $V'=B'\cap F$, $G=\mathrm{Gal}(Z(B'/J(B'))/(V'/J(V')))$, $\overline{B}=B/J(B')$, $\overline{V}=V/J(V')$, *and* $G^Z=\{\sigma\in G|\sigma(\overline{B}\cap Z(B'/J(B')))=\overline{B}\cap Z(B'/J(B'))\}$ *is the decomposition group of* $Z(B/J(B'))$ *over* $V/J(V')$.

Proof. (ii) is a consequence of (i). Furthermore, γ is induced by $\Theta_{B'}$ (cf. Corollary 4.4) and the definitions of α, β are obvious. The only nontrivial statement in (ii) is $\operatorname{im}\beta=\ker\gamma$. Clearly, $qB' \mod\Gamma_{V'}$ lies in $\ker\gamma$ if $qB=Bq$, i.e. $\operatorname{im}\beta\subseteq\ker\gamma$. Now, let $qB' \mod\Gamma_{V'}$ be in $\ker\gamma$, i.e. $qB'q^{-1}=B'$ and

$$(qBq^{-1}/J(B'))\cap Z(B'/J(B'))=(B/J(B'))\cap Z(B'/J(B')).$$

By Theorem 3.3, there exists a unit b in B' such that

$$B/J(B')=bqB(bq)^{-1}/J(B'),$$

i.e., $B=bqB(bq)^{-1}$. Since $qB'\equiv bqB' \mod\Gamma_{V'}$, we get $\ker\gamma\subseteq\operatorname{im}\beta$. Q.E.D.

Theorem 4.1, Corollary 4.4, and Corollary 4.5 are due to A. R. Wadsworth, whose original proofs were given in [W2, Theorem B, Corollary B, Theorem E].

5. Overrings of Dubrovin valuation rings

In this section Q denotes a central simple algebra finite-dimensional over its center F. In [D2, §3, Proposition 2] it is shown that each maximal valuation ring B of Q is integral over its center $V = B \cap F$. First of all, we give a new proof of this theorem.

Lemma 5.1. *Let B be a maximal Dubrovin valuation ring of Q with center $V = B \cap F$ and let $\{a_1, \dots, a_n\}$ be an F-basis of Q. There exists a nonzero x in B satisfying $xB = Bx \subseteq a_1J(V) + \cdots + a_nJ(V)$.*

Proof. Let $v: Q \to \widehat{\Gamma}_V$, $q \to BqB$ be the norm belonging to B which was introduced in [BG2, §5] and let $U_\varepsilon = \varepsilon$ be the ε-neighborhood for each ε in $\widehat{\Gamma}_V$. In [BG2, §5] it is shown that v can be extended to a norm $\tilde{v}$ in $\widetilde{F} \otimes_F Q$ where $\widetilde{F}$ denotes the completion of F with respect to v. The restriction of $\tilde{v}$ to $\widetilde{F}$ is a valuation of F and let $\widetilde{V}$ be the corresponding valuation ring of $\widetilde{F}$, i.e., $\widetilde{V} \cap F = V$. We can identify Q resp. $\widetilde{F}$ with their images in $\widetilde{F} \otimes_F Q$ under the canonical inclusion. Thus, $\{a_1, \dots, a_n\}$ is an $\widetilde{F}$-basis of $\widetilde{F}Q = \widetilde{F} \otimes_F Q$. By a well-known theorem (cf., [C, §9.2, Proposition 5]), the topology of $\widetilde{F}Q$ which is induced by $\tilde{v}$ is also induced by the cubical norm $|\ |$ with respect to $\{a_1, \dots, a_n\}$, i.e., $|a_1k_1 + \cdots + a_nk_n| = \max\{\tilde{v}(k_1), \dots, \tilde{v}(k_n)\}$ for all $k_1, \dots, k_n$ in $\widetilde{F}$. Therefore, $U_\varepsilon \subseteq a_1J(\widetilde{V}) + \cdots + a_nJ(\widetilde{V})$ for some ε in Γ_V. Since $\{a_1, \dots, a_n\}$ is a basis, $U_\varepsilon \subseteq Q$ implies $U_\varepsilon \subseteq a_1J(V) + \cdots + a_nJ(V)$. Q.E.D.

The next lemma can be proved like [Bo, Chapter V, §1.1, Theorem 1].

Lemma 5.2. *Let B be a Dubrovin valuation ring of Q with center $V = B \cap F$ and let $\{a_1, \dots, a_n\} \subseteq B$ be an F-basis of Q. For each q in Q satisfying $qB \subseteq a_1J(V) + \cdots + a_nJ(V)$ there exist $k_0, \dots, k_{n-1}$ in $J(V)$ such that $q^n + k_{n-1}q^{n-1} + \cdots + k_1q + k_0 = 0$.*

Proof. For all $i = 1, \dots, n$ there exist $k_{i1}, \dots, k_{in}$ in $J(V)$ such that $qa_i = k_{i1}a_1 + \cdots + k_{in}a_n$.

If d is the determinant of the $n \times n$-matrix $(k_{ij} - q\delta_{ij})$ over the commutative ring $V[q]$, then we obtain $da_i = 0$ for all $i = 1, \dots, n$. Thus,

$$d(a_1F + \cdots + a_nF) = dQ = 0$$

is shown, i.e., $d = 0$, and q is a root of the polynomial $\det(k_{ij} - x\delta_{ij})$ with coefficients in V. Up to the sign, this polynomial has the required properties. Q.E.D.

Corollary 5.3. *Let $B \neq Q$ be a Dubrovin valuation ring of Q with center $V = B \cap F$. There exists a Dubrovin valuation ring $B' \neq Q$ of Q with center $V' = B' \cap F$ containing B such that*

(i) *B' is integral over V',*

(ii) *for all q in $J(B')$ there exist $k_0, \dots, k_{n-1}$ in $J(V')$ such that $q^n + k_{n-1}q^{n-1} + \cdots + k_1q + k_0 = 0$.*

Proof. Clearly, (ii) implies (i) since $B'/J(B')$ is algebraic over $V'/J(V')$. Now, if there is no maximal Dubrovin valuation ring of Q containing B, then choose B' as in Theorem 2.2, and (ii) follows by 5.2. Otherwise, choose B' maximal and let $x \neq 0$ be in B' such that $xB' = B'x \subseteq a_1J(V') + \cdots + a_nJ(V') \subseteq J(B')$. Since q^m lies in xB' for some positive integer m, i.e. $q^mB' \subseteq a_1J(V') + \cdots + a_nJ(V')$, (ii) follows by 5.2. Q.E.D.

Theorem 5.4. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V = B \cap F$. Then, the following statements are equivalent:*

(i) *B is integral over V.*

(ii) *For each q in $J(B)$ there exist $a_0, \dots, a_{n-1}$ in $J(V)$ such that $q^n + a_{n-1}q^{n-1} + \cdots + a_1q + a_0 = 0$.*

Proof. (ii) $\Rightarrow$ (i). This is obvious since $B/J(B)$ is algebraic over $V/J(V)$.

(i) $\Rightarrow$ (ii). $B/J(V)B$ is finite-dimensional over $V/J(V)$. Thus, $B/J(V)B$ is Artinian and $J(B)/J(V)B$ is nilpotent; hence $J(B)^n \subseteq J(V)B$, for some positive integer n. Let q be in $J(B)$. Then $q^n = cb$ with $c \in J(V)$ and $b \in B$. As B is integral over V there exists a monic $f(x)$ in $V[x]$ satisfying $f(b) = 0$. Let $k = \deg(f(x))$. Then $c^kf(x/c) = x^k + f_{k-1}x^{k-1} + \cdots + f_1x + f_0$ with $f_0, \dots, f_{k-1} \in J(V)$ and $c^kf(q^n/c) = 0$. Thus q^n, and hence q, is integral over $J(V)$. Q.E.D.

Corollary 5.5. *Let $B \subseteq B'$ be two Dubrovin valuation rings of Q with centers $V = B \cap F$ resp. $V' = B' \cap F$. If B' is integral over V' and $B/J(B')$ integral over $V/J(V')$ then B is integral over V.*

Corollary 5.6. *Let B be a Dubrovin valuation ring of Q with center $V = B \cap F$ such that B is not integral over V. Then there exists a Dubrovin valuation ring B' of Q with center $V' = B' \cap F$ containing B minimal with the property that B' is integral over V', and the following hold:*

(i) *$Z(B'/J(B')) \neq V'/J(V')$.*

(ii) *$V/J(V')$ has at least two extensions to $Z(B'/J(B'))$.*

6. Dubrovin valuation rings with intersection property

In this section we investigate the intersection of a finite number of Dubrovin valuation rings $B_1, \dots, B_n$ of Q where Q denotes a central simple algebra finite-dimensional over its center F. We start with the definition of the so-called intersection property. If Q is a division algebra, then $B_1, \dots, B_n$ have this property whenever $B_1, \dots, B_n$ are total (cf., [G1, 3.4. Korollar]).

Definition. Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q, $R = B_1 \cap \cdots \cap B_n$ and let $\mathscr{B}_i$ be the set of all overrings of B_i in Q $(i = 1, \dots, n)$. Then, $B_1, \dots, B_n$ have the intersection property if

$$\begin{aligned} \Phi : \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_n &\to \operatorname{Spec} R \\ B &\mapsto J(B) \cap R, \end{aligned}$$

is a well-defined anti-order-isomorphism.

The property "Φ is well-defined" signifies that $J(B) \cap R$ is indeed a prime ideal of R whereas "Φ is an anti-order-isomorphism" ensures that $B \subseteq B'$ if and only if $J(B') \cap R \subseteq J(B) \cap R$.

Proposition 6.1. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q and let R be their intersection, $S = R \cap F$.*

(i) *If $T \subset S$, $0 \notin T$, is multiplicative closed then $B_1T^{-1}, \dots, B_nT^{-1}$ have the intersection property if $B_1, \dots, B_n$ have the intersection property.*

(ii) *$B_1, \dots, B_n$ have the intersection property if $(B_1)_M, \dots, (B_n)_M$ have the intersection property for all maximal ideals M of S where*

$$(B_i)_M = (B_i)(S\backslash M)^{-1} \quad \textit{for all } i = 1, \dots, n.$$

Proof. We prove (i); (ii) follows similarly. RT^{-1} is the intersection of all $B_1T^{-1}, \dots, B_nT^{-1}$. The main argument we need is the following statement which can be proved exactly as in the commutative case: $(I \cap R)T^{-1} = I$ for each ideal I of RT^{-1}, and I is prime in RT^{-1} if and only if $I \cap R$ is prime in R.

Now, let B be a Dubrovin valuation ring of Q and let $B_iT^{-1} \subseteq B$ for some i. Then, $J(B) \cap RT^{-1}$ is prime in RT^{-1} since $(J(B) \cap RT^{-1}) \cap R$ is prime in R. If P is a prime ideal of RT^{-1} then $P \cap R$ is prime in R, i.e., $P \cap R = J(B) \cap R$ for some Dubrovin valuation ring B of Q containing at least one B_i. Clearly, $J(B) \cap R$ does not meet T and we obtain $(B_i)T^{-1} \subseteq BT^{-1} = B$ as well as $P = (P \cap R)T^{-1} = (J(B) \cap R)T^{-1} = J(B) \cap RT^{-1}$. Finally, $B \subseteq B'$ if and only if $J(B') \cap R \subseteq J(B) \cap R$, i.e., $B \subseteq B'$ if and only if $J(B') \cap RT^{-1} \subseteq J(B) \cap RT^{-1}$ for all Dubrovin valuation rings B, B' of Q containing some $(B_i)T^{-1}$, $i = 1, \dots, n$. Q.E.D.

Corollary 6.2. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q with centers $V_i = B_i \cap F$, $i = 1, \dots, n$. If $V_1, \dots, V_n$ are pairwise comaximal in F then $B_1, \dots, B_n$ have the intersection property.*

Proposition 6.3. *Let $B, B_1, \dots, B_n$ be Dubrovin valuation rings of Q such that $B_1, \dots, B_n \subseteq B$. Then $B_1, \dots, B_n$ have the intersection property if and only if $B_1/J(B), \dots, B_n/J(B)$ have the intersection property.*

Proposition 6.3 is a simple consequence of

Lemma 6.4. *Let $B, B_1, \dots, B_n$ be Dubrovin valuation rings of Q such that $B_1, \dots, B_n \subseteq B$ and let $R = B_1 \cap \dots \cap B_n$. If I is an ideal of R then $J(B) \subseteq I$ or $I \subseteq J(B)$. Furthermore, if in addition I is prime in R and $I \subseteq J(B)$ then I is a prime ideal of B.*

Proof. Let $V = B \cap F$, $S = R \cap F$ and let $I \nsubseteq J(B)$. We define $J = I + J(B)$ and VJ is an ideal of B containing $J(B)$ properly since $VR = B$. Therefore, $VJ = B$ and there exist $f_1, \dots, f_k$ in V, $j_1, \dots, j_k$ in J such that $f_1j_1 + \dots + f_kj_k = 1$. Let a be in $S\backslash J(V)$ satisfying $af_i \in S$, $i = 1, \dots, k$. We obtain $a = af_1j_1 + \dots + af_kj_k \in J$. Let c be in I, b in $J(B)$ such that $a = c + b$; thus $c = b - a$ is a unit in B. This completes the proof since $J(B) = cJ(B) \subseteq I$.

Now, let $I \subseteq J(B)$ and let I be prime in R. We show $aBb \nsubseteq I$ for all a, b in $B\backslash I$. This statement is trivial if a, b are in R or if a, b are in $B\backslash J(B)$. Thus, let a be in $B\backslash R$ and let b be in $J(B)$. There exists k in $S\backslash J(V)$ such that $ak \in R\backslash J(B)$. Since $k^{-1}b$ lies in $J(B)\backslash I$ we conclude $aRb = akRk^{-1}b \nsubseteq I$ because I is prime in R.

It remains to show that I is indeed an ideal of B. Let a be in B. If a lies in R then $aI, Ia \subseteq I$ is trivial. Otherwise, there exists k in $S\backslash J(V)$ such that

ka is in $R\backslash J(B)$ and we are done if $k^{-1}I \subseteq I$ is shown. Since k is in $R\backslash I$, we obtain that $k^{-1}x$ lies in I for all x in I because $kBk^{-1}x \subseteq I$. Q.E.D.

Remark. The second half of Lemma 6.4 follows immediately by [AS, Theorem 2.5]: $J(B)$ is a common ideal of R and B $(\supseteq R)$. If $I \subset J(B)$ is a prime ideal of R then there exists a prime ideal P' of B such that $P = P' \cap R$, i.e., $P = P'$ since $P' \subseteq J(B) \subset R$.

Lemma 6.5. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q having the intersection property such that $V = B_1 \cap F = \cdots = B_n \cap F$. If each B_i is integral over V then $B_1 = \cdots = B_n$.*

Proof. Without loss of generality we an assume $B_1, \dots, B_n \neq Q$. First of all we prove that there exists a nonzero ideal I of B_1 which lies in each $J(B_i)$, $i = 1, \dots, n$. By Theorem 3.3 there is a regular q in Q such that $B_2 = qB_1q^{-1}$ and we can even suppose that q is in $J(B_1) \cap J(B_2)$. Let I_2 be a nonzero ideal of B_1 satisfying $I_2 \subseteq qB_1 = B_2q$ (as q is regular, there exists a nonzero a in $qB_1 \cap V$; set $I_2 = aB_1$). It follows that $I_2 \subseteq J(B_1) \cap J(B_2)$. Similarly, a nonzero ideal I_i of B_1 exists such that $I_i \subseteq J(B_1) \cap J(B_i)$, $i = 2, \dots, n$. $I = I_2 \cap \cdots \cap I_n$. Now, let I be maximal with this property, let P be the minimal prime ideal of B_1 containing I, and let $B \supseteq B_1$ be the corresponding Dubrovin valuation ring of Q, i.e., $J(B) = P$. We show $P \cap R \subseteq J(B_i) \cap R$, $i = 1, \dots, n$. Assume $P \cap R \not\subseteq J(B_i) \cap R$ for at least one i. By $B_1 \cap F = \cdots = B_n \cap F$ and the intersection property, $J(B_i) \cap R$ is a maximal ideal of R. Let a be in $P \cap R$ but not in $J(B_i) \cap R$ and we obtain $RaR + J(B_i) \cap R = R$. Thus, there exists b in $RaR \subseteq P \cap R$ as well as c in $J(B_i) \cap R$ such that $b + c = 1$, i.e., $b = 1 - c$. Since P is minimal relative to $I \subseteq P$, there exists a positive integer n satisfying $b^n \in I$. But $b^n = (1-c)^n \notin J(B_i) \cap R$ is a contradiction and $P \cap R \subseteq J(B_i) \cap R$ is shown for all $i = 1, \dots, n$. By the intersection property, this implies $B_i \subseteq B$ and $P \subseteq J(B_i)$ for all $i = 1, \dots, n$, i.e., $I = P$. If $P \cap F = J(V)$ then $P = J(B_1) = \cdots = J(B_n)$ follows and we are done. Thus, let $P \cap F \neq J(V)$ and $B_1/P, \dots, B_n/P \neq B/P$ are Dubrovin valuation rings of B/P which are integral over $(B_1/P) \cap Z(B/P) = \cdots = (B_n/P) \cap Z(B/P)$. Since $B_1/P, \dots, B_n/P$ have the intersection property there exists an ideal I' of B_1 satisfying $P \subset I \subseteq J(B_1) \cap \cdots \cap J(B_n)$ (consider the Dubrovin valuation rings $B_1/P, \dots, B_n/P$ in B/P). But this is a contradiction because $P = I$ is maximal with this property. Q.E.D.

Corollary 6.6. *Let $B_1, \dots, B_n$ be incomparable Dubrovin valuation rings of Q having the intersection property such that $V = F \cap B_1 \cap \cdots \cap B_n$ is a valuation ring of F. If R_i denotes the minimal Dubrovin valuation ring of Q containing B_i such that R_i is integral over its center $W_i = R_i \cap F$ $(i = 1, \dots, n)$ then $R_1 = \cdots = R_n$.*

Proof. Let $n \neq 1$ and let W be the minimal valuation ring of F containing V such that each extension of W to Q is integral over W. Clearly, $W \subseteq W_i$ for all $i = 1, \dots, n$. Assume $W \neq W_i$ for at least one i and let W_i be maximal among $W_1, \dots, W_n$. Then $R_i = B_i$ since R_i is integral over W_i and minimal relative to $B_i \subseteq R_i$. Let $P = J(W_i)$ and $(B_1)_P \cap F = \cdots = (B_n)_P \cap F = W_i$. By Proposition 6.1, $(B_1)_P, \dots, (B_n)_P$ have the intersection property and $(B_1)_P = \cdots = (B_n)_P = B_i$ follows by Lemma 6.5. This leads to the contradiction

$B_1, \dots, B_n \subseteq B_i$. Therefore, $W_1 = \cdots = W_n$ and $R_i = (B_i)_P$, $i = 1, \dots, n$. Thus, $R_1, \dots, R_n$ have the intersection property and $R_1 = \cdots = R_n$ follows by Lemma 6.5. Q.E.D.

Corollary 6.6 has a conclusion which is crucial for the proofs of the following theorems. Let R be the intersection of Dubrovin valuation rings $B_1, \dots, B_n$ of Q having the intersection property and let $V = R \cap F$ be a valuation ring of F. Without restriction we can assume that $B_1, \dots, B_n$ are pairwise incomparable and two cases can be distinguished:

Case 1. R is a Dubrovin valuation ring, i.e., $n = 1$.

Case 2. R is not a Dubrovin valuation ring, i.e., $n \neq 1$. Let B be the Dubrovin valuation ring of Q containing B_1 minimal relative to the property that B is integral over its center $W = B \cap F$. By Corollary 6.6, $B_1, \dots, B_n \subseteq B$ and $B_1 \neq B$. Thus, $Z(B/J(B)) \neq W/J(W)$ by Corollary 5.6. In many inductive proofs this construction is the crucial point and we will call it the reduction step since $[B/J(B) : Z(B/J(B))] < [Q : F]$.

Corollary 6.7. *Let B_1, B_2 be two Dubrovin valuation rings of Q having the intersection property. If B_1 and B_2 are comaximal then $B_1 \cap F$ and $B_2 \cap F$ are comaximal in F.*

Proof. Let $P \neq 0$ be a common prime ideal of $B_1 \cap F$ and $B_2 \cap F$. Then $(B_1)_P, (B_2)_P \neq Q$ have the intersection property and $(B_1)_P \cap F = (B_2)_P \cap F$. Thus, there exists a Dubrovin valuation ring $B \neq Q$ of Q containing B_1 and B_2 (see above). Q.E.D.

Theorem 6.8. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q.*

(i) *Let B, B' be Dubrovin valuation rings of Q such that $B_i \subseteq B$, $B_j \subseteq B'$ for some $i, j = 1, \dots, n$. If $B_1, \dots, B_n$ have the intersection property then B and B' have the intersection property.*

(ii) *If B_i and B_j have the intersection property for all $i, j = 1, \dots, n$, then $B_1, \dots, B_n$ have the intersection property.*

(iii) *Let $B \subseteq B_1$ be a Dubrovin valuation ring of Q and let B_1, B_i be incomparable $(i > 1)$. If $B_1, \dots, B_n$ have the intersection property then $B, B_2, \dots, B_n$ have the intersection property.*

Proof. We prove the theorem by induction on $[Q : F]$. Let $[Q : F] > 1$.

(i) Let $S = F \cap B \cap B'$. By Proposition 6.1 we have to show that B_M and $(B')_M$ have the intersection property for all maximal ideals M of S. Let $P = M \cap F \cap (B_1 \cap \cdots \cap B_n)$. Obviously, $(B_1)_P, \dots, (B_n)_P$ have the intersection property and $F \cap (B_1)_P \cap \cdots \cap (B_n)_P = V$ is a valuation ring of F. Furthermore, $(B_i)_P \subseteq B_M$ and $(B_j)_P \subseteq (B')_m$. By the reduction step for B_M, $(B')_M$ and the induction hypothesis, (i) is shown.

(ii) Let $S = F \cap B_1 \cap \cdots \cap B_n$ and let M be a maximal ideal of S. Then, $(B_i)_M$ and $(B_j)_M$ have the intersection property for all $i, j = 1, \dots, n$, and $F \cap (B_1)_M \cap \cdots \cap (B_n)_M = V$ is a valuation ring of F. We can assume that the $(B_i)_M$ are incomparable $(i = 1, \dots, n)$. If R_i denotes the minimal overring of $(B_i)_M$ integral over its center, then $R_i = R_j$ for all $i, j = 1, \dots, n$, i.e., $R_1 = \cdots = R_n$. By the reduction step and the induction hypothesis, (ii) is shown.

(iii) By (ii) we have to show that B and B_i have the intersection property. We define $V_1 = F \cap B_1$, $V_i = F \cap B_i$, and $V = F \cap B$. Let P be the maximal common prime ideal of V_1 and V_i. If $(B_1)_P \neq (B_i)_P$, then (iii) follows by the reduction step since $F \cap (B_1)_P = F \cap (B_i)_P$ is a valuation ring of F. Now, let $R = (B_1)_P = (B_i)_P$ as well as $W = R \cap F$. Since B_1, B_i are incomparable, we conclude that $V_1/J(W)$ and $V_i/J(W)$ are comaximal. Thus $V/J(W)$ and $V_i/J(W)$ are comaximal, and therefore $Z(B/J(R))$ and $Z(B_i/J(R))$ are comaximal. (iii) follows by Corollary 6.2. Q.E.D.

Especially, Theorem 6.8 shows that $B_1, \dots, B_n$ have the intersection property if and only if B_i, B_j have the intersection property for all $i, j = 1, \dots, n$.

Theorem 6.9. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q having the intersection property and let R be their intersection. If P is a prime ideal of R and $\mathscr{C}_R(P) = \{r \in R | r + P$ is regular in $R/P\}$ then the following hold.*

(i) *Each element of $\mathscr{C}_R(P)$ is regular in Q.*

(ii) *$\mathscr{C}_R(P)$ is a (left and right) Ore-set of R such that ${}_PR = R_P$ and R_P is a Dubrovin valuation ring of Q such that $J(R_P) \cap R = P$.*

Proof. We prove the theorem by induction on $[Q : F]$. Let $S = R \cap F$ and $M = P \cap S$ is a prime ideal of S and P_M is a prime ideal of R_M such that $P_M \cap R = P$. It is easy to check that we are done if everything is proved for P_M and R_M instead of P and R. Therefore, we can assume $S = R \cap F = V$ is a valuation ring of F such that $J(V) = V \cap P$ and the statements are valid if R is a Dubrovin valuation ring. Thus, let $B_1, \dots, B_n$ be incomparable having an overring B such that $[B/J(B) : Z(B/J(B))] < [Q : F]$ (reduction step). By $J(B) \cap V \subseteq J(V) = P \cap V$ and Lemma 6.4 we obtain $J(B) \subseteq P$. We can apply the induction hypothesis to the prime ideal $P/J(B)$ of $R/J(B)$ and (i) follows immediately. Now, let r be in R and s in $\mathscr{C}_R(P)$. There exist r' in R and s' in $\mathscr{C}_R(P)$ such that $rs' - sr' = a \in J(B)$. Since $s^{-1}a$ lies in $J(B) \cap R$ we obtain $rs' = s(r' + s^{-1}a)$. Similarly, $\mathscr{C}_R(P)$ is a left Ore-set. Finally, there exists a Dubrovin valuation ring B' of Q, $B' \subseteq B$ such that ${}_PR = R_P = B'$ and $B' \cap R = P$ by induction hypothesis. Q.E.D.

Theorem 6.10. *Let $B_1, \dots, B_n$ be Dubrovin valuation rings of Q having the intersection property and let R be their intersection. If B is a Dubrovin valuation ring of Q containing R then B contains at least one B_i.*

Proof. We prove the theorem by induction on $[Q : F]$. Let $S = R \cap F$ and $P = J(B) \cap S$. We can assume that $S = V$ is a valuation ring of F such that $J(V) = F \cap P$ (otherwise pass to the localization at P) and that $B_1, \dots, B_n$ are incomparable. If R is a Dubrovin valuation ring we are done. Thus, let B' be the minimal Dubrovin valuation ring containing $B_1, \dots, B_n$ such that B' is integral over its center, $P' = J(B') \cap V$. It follows $B' = (B_1)_{P'} = \cdots = (B_n)_{P'} = R_{P'} \subseteq B_{P'}$. If $P \subseteq P'$ then $B_{P'} = B$. Otherwise, $B_{P'} \cap F = B' \cap F$, i.e., $B \subset B_{P'} = B'$. The reduction step and induction hypothesis complete the proof. Q.E.D.

Using the notations of the definition for the intersection property, Theorem 6.10 means that $\mathscr{B}_1 \cup \cdots \cup \mathscr{B}_n$ consists exactly of all Dubrovin valuation rings of Q containing R and Theorem 6.9 signifies $\Phi^{-1}(P) = R_P$ for all prime ideals P of R.

In view of the Defektsatz we now look at the intersection of Dubrovin valuation rings from the standpoint of extending given valuation rings of the center F of Q.

Theorem 6.11. *Let $V_1, \dots, V_n$ be pairwise incomparable valuation rings of F and let S be their intersection.*

(i) *If $B_1, \dots, B_k$ are Dubrovin valuation rings of Q having the intersection property such that $R \cap F = V_1 \cap \cdots \cap V_l$ where $1 \le l \le n$ and $R = B_1 \cap \cdots \cap B_k$ then there exist Dubrovin valuation rings $B_{k+1}, \dots, B_m$ of Q such that $B_1, \dots, B_m$ have the intersection property, $R' \cap F = S$, and R' is integral over S where $R' = B_1 \cap \cdots \cap B_m$.*

(ii) *If $B_1, \dots, B_k$ (resp. $B'_1, \dots, B'_m$) are incomparable Dubrovin valuation rings of Q having the intersection property such that $R \cap F = R' \cap F = S$ and R, R' are both integral over S where $R = B_1 \cap \cdots \cap B_k$, $R' = B'_1 \cap \cdots \cap B'_m$ then $k = m$.*

Proof. We prove the theorem by induction on $[Q : F]$. The case $[Q : F] = 1$ is trivial. Thus, let $[Q : F] > 1$. We prove this part by induction on n.

$n = 1$. (i) $F \cap (B_1 \cap \cdots \cap B_k) = V_1$. First of all we assume $F \cap B_i = V_1$ for all $i = 1, \dots, k$. If R is integral over V_1 we are done. Otherwise, let B be the minimal Dubrovin valuation ring of Q containing $B_1, \dots, B_k$ and $[B/J(B) : Z(B/J(B)] < [Q : F]$. By induction hypothesis there exist Dubrovin valuation rings $B_{k+1}, \dots, B_m \subseteq B$ of Q such that $B_1/J(B), \dots, B_m/J(B)$ have the intersection property, $R'/J(B)$ is integral over $(R'/J(B)) \cap Z(B/J(B))$ where $R' = B_1 \cap \cdots \cap B_m$ and $(R'/J(B)) \cap Z(B/J(B))$ is the integral closure of $V_1/(J(B) \cap F)$ in $Z(B/J(B))$. Now, we consider the general case and we can assume that $B_1, \dots, B_k$ are incomparable. For each $i = 1, \dots, k$ there exists a Dubrovin valuation ring C_i of Q such that $C_i \subseteq B_i$ and $F \cap C_i = V_1$. By Theorem 6.8, $C_1, \dots, C_k$ have the intersection property, and there exist $B_{k+1}, \dots, B_m$ such that $C_1, \dots, C_k, B_{k+1}, \dots, B_m$ have the corresponding properties. Finally, $C_1, \dots, C_k, B_1, \dots, B_m$ are the desired rings.

(ii) If R is a Dubrovin valuation ring, i.e., $k = 1$, then $B'_1, \dots, B'_m$ are integral over their centers and $B'_1 = \cdots = B'_m$ follows by Corollary 6.6, i.e., $m = 1$. Thus, let $k > 1$ and let B (resp. B') be the minimal Dubrovin valuation ring of Q containing $B_1, \dots, B_k$ (resp. $B'_1, \dots, B'_m$) such that B (resp. B') is integral over $B \cap F$ (resp. $B' \cap F$). Therefore, B and B' are conjugate by $B \cap F = B' \cap F$, and we can assume $B = B'$. Since $R/J(B)$ and $R'/J(B)$ are both integral over $V_1/(J(B) \cap F)$ we obtain $(R/J(B)) \cap Z(B/J(B)) = (R'/J(B)) \cap Z(B/J(B))$. Thus, (ii) follows by induction hypothesis considering $B/J(B)$.

$n > 1$. *Case* 1. There is no prime ideal $P \neq 0$ of S such that $P \subseteq J(V_i)$ for all $i = 1, \dots, n$. Then there exists a disjoint union $\{V_1, \dots, V_n\} = \{W_1, \dots, W_t\} \cup \{W'_1, \dots, W'_{t'}\}$ such that W_i and W'_j are comaximal for $i = 1, \dots, t$ and $j = 1, \dots, t'$.

(i) Let $B_1, \dots, B_k \neq Q$ be incomparable. Without restrictions there is a disjoint union $\{B_1, \dots, B_k\} = \{R_1, \dots, R_s\} \cup \{R'_1, \dots, R'_{s'}\}$ such that $W_1 \cap \cdots \cap W_r = F \cap R_1 \cap \cdots \cap R_s$ and $W'_1 \cap \cdots \cap W'_{r'} = F \cap R'_1 \cap \cdots \cap R'_{s'}$, where $1 \le r \le s$, $1 \le r' \le s'$ and $\{W_1, \dots, W_r, W'_1, \dots, W'_{r'}\} = \{V_1, \dots, V_l\}$. By induction hypothesis there exist Dubrovin valuation rings $R_{s+1}, \dots, R_p$ of

Q such that $R_1, \dots, R_p$ have the intersection property, $T \cap F = W_1 \cap \dots \cap W_t$ and T is integral over $W_1 \cap \dots \cap W_t$ where $T = R_1 \cap \dots \cap R_p$. Let $R'_{s'+1}, \dots, R'_{p'}$ and T' be defined similarly. We choose $\{B_{k+1}, \dots, B_m\} = \{R_{s+1}, \dots, R_p, R'_{s'+1}, \dots, R'_{p'}\}$. By Proposition 6.1(ii), $B_1, \dots, B_m$ have the desired property.

(ii) Without restrictions, $\{B_1, \dots, B_k\} = \{B_1, \dots, B_s\} \cup \{R_1, \dots, R_{s'}\}$ is a disjoint union where $W_1 \cap \dots \cap W_t = F \cap B_1 \cap \dots \cap B_s$ and $W'_1 \cap \dots \cap W'_{t'} = F \cap R_1 \cap \dots \cap R_{s'}$ and let $B'_1, \dots, B'_r, R'_1, \dots, R'_{r'}$ be defined similarly. Furthermore, let N be the set of all s in S which does not belong to any $J(W_i) \cap S$ $(i = 1, \dots, t)$ and we obtain $SN^{-1} = W_1 \cap \dots \cap W_t$, $RN^{-1} = B_1 \cap \dots \cap B_s$, $R'N^{-1} = B'_1 \cap \dots \cap B'_r$, $RN^{-1} \cap F = R'N^{-1} \cap F = SN^{-1}$ and RN^{-1}, $R'N^{-1}$ are both integral over SN^{-1}. By induction hypothesis, $s = r$ follows. Similarly, $s' = r'$ and $k = m$.

Case 2. There exists a prime ideal $P \neq 0$ of S such that $P \subseteq J(V_i)$ for all $i = 1, \dots, n$. Let P be maximal relative to this property.

(i) First of all we consider the case that for all $i = 1, \dots, k$ there is $j = 1, \dots, l$ such that $F \cap B_i = V_j$. If $(B_1)_P \cap \dots \cap (B_k)_P = B$ is a Dubrovin valuation ring then $(B_1)_P = \dots = (B_k)_P = B$. We consider $B/J(B)$ as well as all extensions of $V_1/(F \cap J(B)), \dots, V_n/(F \cap J(B))$ to $Z(B/J(B))$ and Case 1 can be applied. If $(B_1)_P \cap \dots \cap (B_k)_P$ is no Dubrovin valuation ring then there exists a Dubrovin valuation ring B of Q containing all $B_1, \dots, B_k$ as well as $V_1, \dots, V_n$ such that $[B/J(B) : Z(B/J(B))] < [Q : F]$. Considering $B/J(B)$ and all extensions of $V_1/(F \cap J(B)), \dots, V_n/(F \cap J(B))$ to $Z(B/J(B))$, the statement follows by induction hypothesis. Now, we investigate the general case and we can assume that $B_1, \dots, B_k$ are incomparable. For each $i = 1, \dots, k$ there exists a Dubrovin valuation ring C_i of Q such that $C_i \subseteq B_i$ and $F \cap C_i = V_j$ for some $j \in \{1, \dots, l\}$. By Theorem 6.8, $C_1, \dots, C_k$ have the intersection property, and there exist $B_{k+1}, \dots, B_m$ such that $C_1, \dots, C_k, B_{k+1}, \dots, B_m$ have the corresponding properties. Finally, $C_1, \dots, C_k, B_1, \dots, B_m$ are the desired rings.

(ii) If $(B_1)_P \cap \dots \cap (B_k)_P = B$ is a Dubrovin valuation ring then $(B_1)_P = \dots = (B_k)_P = B$ and B is integral over $B \cap F$. Therefore, $(B'_1)_P = \dots = (B'_m)_P = B'$ and we can assume $B = B'$ since B and B' are conjugate. The statement follows by Case 1. If $(B_1)_P \cap \dots \cap (B_k)_P$ is no Dubrovin valuation ring the statement follows by the reduction step and the induction hypothesis (see above). Q.E.D.

If S is the intersection of valuation rings $V_1, \dots, V_n$ of F then there exist Dubrovin valuation rings $B_1, \dots, B_m$ of Q having the intersection property such that $S = F \cap R$ and R is integral over S where $R = B_1 \cap \dots \cap B_m$. This follows by Theorem 2.1 and Theorem 6.11.

Theorem 6.12. *Let R (resp. R') be the intersection of a finite number of Dubrovin valuation rings of Q having the intersection property. If $S = F \cap R = F \cap R'$ and R, R' are both integral over S then R and R' are conjugate, i.e., $R = qR'q^{-1}$ for some regular q in Q.*

The proof of this theorem requires the general approximation theorem for Dubrovin valuation rings which was recently stated by P. Morandi (cf., [M2, Theorem 2.3]). He introduces the following condition for Dubrovin valuation rings B, B' of Q: Let R be the least overring in Q of B and B'. Then $(B/J(R)) \cap Z(R/J(R))$ and $(B'/J(R)) \cap Z(R/J(R))$ are comaximal. Morandi proves that the general approximation theorem holds for $B_1 \dots, B_n$ if and only if B_i, B_j satisfy this condition for all $i, j = 1, \dots, n$, $i \neq j$.

First of all, two Dubrovin valuation rings of Q satisfy Morandi's condition if and only if they have the intersection property: By Corollary 6.2 and Proposition 6.3, Morandi's condition implies the intersection property. Furthermore, Proposition 6.3 and Corollary 6.7 show that the intersection property implies Morandi's condition.

Proof (*of Theorem* 6.12). Let $S = V_1 \cap \dots \cap V_n$, $R = B_1 \cap \dots \cap B_k$, $R' = B'_1 \cap \dots \cap B'_k$ where $V_1, \dots, V_n$ are pairwise incomparable valuation rings of F and $B_1, \dots, B_k$ (resp. $B'_1, \dots, B'_k$) are incomparable Dubrovin valuation rings of Q having the intersection property. We prove the theorem by induction on $[Q : F]$ where the case $[Q : F] > 1$ is proved by induction on n.

$n = 1$. The proof is exactly the same as "$n = 1$, (ii)" in the proof of Theorem 6.11.

$n > 1$. *Case* 1. There is no prime ideal $P \neq 0$ of S such that $P \subseteq J(V_i)$ for all $i = 1, \dots, n$. Let $W_1, \dots, W_t, W'_1, \dots, W'_{t'}, B_1, \dots, B_s, R_1, \dots, R_{s'}$, $B'_1, \dots, B'_s, R'_1, \dots, R'_{s'}$ as in "$n > 1$, (ii)" of the proof of Theorem 6.11. By induction hypothesis there exist regular elements a, b in Q such that $a(B_1 \cap \dots \cap B_s)a^{-1} = B'_1 \cap \dots \cap B'_s$ and $b(R_1 \cap \dots \cap R_{s'})b^{-1} = R'_1 \cap \dots \cap R'_{s'}$. Without restrictions we can assume a, b in $R \cap R'$. By construction and the general approximation theorem there exists q in Q such that

$$q - a \in aJ(B_i) \quad \text{for all } i = 1, \dots, s,$$
$$q - b \in bJ(R_j) \quad \text{for all } j = 1, \dots, s'.$$

Thus, $q = a(1 + x_i)$ for all $i = 1, \dots, s$ where x_i is in $J(B_i)$ as well as $q = b(1 + x_j)$ for all $j = 1, \dots, s'$ where x_j is in $J(R_j)$ and q is regular. Finally, $qB_iq^{-1} = aB_ia^{-1}$ for all $i = 1, \dots, s$ and $qR_jq^{-1} = bR_jb^{-1}$ for all $j = 1, \dots, s'$.

Case 2. There is a prime ideal $P \neq 0$ of S such that $P \subseteq J(V_i)$ for all $i = 1, \dots, n$. This case can be treated like Case 2 in "$n > 1$, (ii)" of the proof of Theorem 6.11. Q.E.D.

Definition. Let Q be a central simple algebra finite-dimensional over its center F and let V be a valuation ring of F. If $B_1, \dots, B_n$ are Dubrovin valuation rings of Q extending V and having the intersection property such that $B_1 \cap \dots \cap B_n$ is integral over V then n is called extension number of V to Q.

By Theorem 6.11, the extension number is well-defined. If Q is a division algebra having a total valuation ring which extends V then the extension number of V to Q is exactly the number of all extensions of V.

Proposition 6.13. *Let Q be a central simple algebra finite-dimensional over its center F and let V be a valuation ring of F. If n is the extension number of V to Q then n is the extension number of V to $M_k(Q)$ for any positive integer k.*

Proof. Let $B_1, \dots, B_n$ be extensions of V to Q having the intersection property. Then $B'_i = M_k(B_i)$, $i = 1, \dots, n$, are Dubrovin valuation rings of

$M_k(Q)$ extending V. It is not hard to check that $B_1', \dots, B_n'$ indeed have the intersection property and it remains to show that $B_1' \cap \cdots \cap B_n'$ is integral over V. But this follows by [Rw, Theorem 4.2.8]. Q.E.D.

Proposition 6.14. *Let $B_1, \dots, B_n$ be incomparable Dubrovin valuation rings of Q having the intersection property and let $R = B_1 \cap \cdots \cap B_n$, $S = R \cap F$. For each maximal ideal M of S let n_M be the extension number of S_M to Q and k_M the number of all maximal ideals M' of R satisfying $M' \cap S = M$. If R is integral over S then the following hold:*

(i) *$M' \cap S$ is a maximal ideal of S for each maximal ideal M' of R.*

(ii) *$n_M = k_M$ for each maximal ideal M of S, i.e., there exist exactly n_M Dubrovin valuation rings among $B_1, \dots, B_n$ extending S_M.*

Proof. (i) follows immediately by [MR, Theorem 13.8.14] or can be proved in the following way: For each B_i there exists a Dubrovin valuation ring C_i of Q such that $S \subseteq C_i \subseteq B_i$ and $J(C_i) \cap S$ is maximal in S. Then $C_1, \dots, C_n$ have the intersection property and $R' = C_1 \cap \cdots \cap C_n$ is integral over $S = F \cap R'$. Finally, $R \cong R'$ by 6.12.

(ii) k_M is the number of maximal ideals of R_M, and R_M is integral over $S_M = F \cap R_M$. Finally, R_M is the intersection of k_M incomparable Dubrovin valuation rings having the intersection property and extending S_M, i.e., $k_M = n_M$. Q.E.D.

7. The "Defektsatz" for central simple algebras

In this section we prove the

Defektsatz. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V = F \cap B$. If n denotes the extension number of V to Q and $p = \operatorname{char}(V/J(V))$ then*

$$(*) \qquad [Q : F] = f_B(Q/F) e_B(Q/F) n^2 p^d,$$

for some nonnegative integer d.

We divide the proof of the Defektsatz into several parts and we say that the Defektsatz holds for Q and V if $(*)$ holds for an extension B of V to Q.

First we observe that the Defektsatz holds for all $M_k(Q)$ and V if it holds for Q and V. Let B be an extension of V to Q and $B' = M_k(B)$. Clearly, $f_{B'}(M_k(Q)/F) = k^2 f_B(Q/F)$. Furthermore, if I is an ideal of B then $M_k(I)$ is invertible if and only if I is invertible, i.e., $e_{B'}(M_k(Q)/F) = e_B(Q/F)$. The final argument is provided by Proposition 6.13. Especially, if V is Henselian then the Defektsatz holds for Q and V by [Dr, Theorem 2].

Proposition 7.1. *The Defektsatz holds for Q and V if V is maximal.*

Proof. Let $\widetilde{F}$ be the completion of F with respect to V. In [BG2, §5] it is shown that there exists a Dubrovin valuation ring $\widetilde{B}$ of $\widetilde{Q} = \widetilde{F} \otimes_F Q$ extending B such that $\widetilde{B}/J(\widetilde{B}) \cong B/J(B)$ relative to the canonical inclusion. Furthermore, since Q is dense in $\widetilde{Q}$ with respect to $\widetilde{B}$ each element of $\Gamma_{\widetilde{B}}$ can be written as $q\widetilde{B}$ where q is a regular element of Q such that $q\widetilde{B}q^{-1} = \widetilde{B}$. Since $\widetilde{F}$ is an immediate extension of F relative to V we obtain $f_{\widetilde{B}}(\widetilde{Q}/\widetilde{F}) = f_B(Q/F)$ as well as $e_{\widetilde{B}}(\widetilde{Q}/\widetilde{F}) = e_B(Q/F)$. Since $\widetilde{V} = \widetilde{F} \cap \widetilde{B}$

is Henselian the Defektsatz holds for $\widetilde{Q}$ and $\widetilde{V}$, i.e., $[Q : F] = [\widetilde{Q} : \widetilde{F}] = f_{\widetilde{B}}(\widetilde{Q}/\widetilde{F})e_{\widetilde{B}}(\widetilde{Q}/\widetilde{F})p^d = f_B(Q/F)e_B(Q/F)p^d$ since $\widetilde{B}$ is integral over $\widetilde{V}$ and B is integral over V. Q.E.D.

Proposition 7.2. *Let $B \subsetneq B'$ be Dubrovin valuation rings of Q being integral over $V = F \cap B$ resp. $V' = F \cap B'$. If the Defektsatz holds for Q and V' as well as for $B'/J(B')$ and $(B/J(B')) \cap Z(B'/J(B'))$ then the Defektsatz holds for Q and B.*

Proof. Notice that the Defektsatz holds for $Z(B'/J(B'))$ and $V/J(V')$ by [E, Theorem 20.21] where $V/J(V')$ has exactly one extension to $Z(B'/J(B'))$. Corollary 4.5(ii) completes the proof. Q.E.D.

Proposition 7.3. *The Defektsatz holds for Q and V if V has an extension B to Q such that B is integral over V.*

Proof. We prove the statement by induction on $[Q : F]$ and let $[Q : F] > 1$.

Case 1. Q is no division algebra. Then the proof is obvious since $Q \cong M_k(D)$ where $[D : F] < [Q : F]$.

Case 2. Q is a division algebra and 0 is the only completely prime ideal of B. If there is a maximal Dubrovin valuation ring of Q containing B then the statement follows by 7.1, 7.2, and Case 1. Otherwise, there exists B' as in Theorem 2.2 and let $V' = F \cap B'$. Each element of Q can be written in the form kq where k is in F and q in $B' \backslash J(B')$, i.e., $e_{B'}(Q/F) = 1$. The proposition follows by 7.2 and Case 1.

Case 3. Q is a division algebra and $P \neq 0$ is the maximal completely prime ideal of Q, i.e., B_P is an invariant valuation ring of Q. The statement follows by [M1, Theorem 3.3], 7.2, and Case 2. Q.E.D.

Now, we are ready to prove the Defektsatz by induction on $[Q : F]$. If B is integral over V we are done. Otherwise, let B' be the minimal Dubrovin valuation ring of Q containing B such that B' is integral over $V' = F \cap B'$. By Corollary 5.6 we can use the induction hypothesis for $B'/J(B')$, and by [E, Theorem 20.21] we obtain

$$f_{B'}(Q/F) = f_{B/J(B')}e_{B/J(B')}(n')^2tp^s,$$

where
$f_{B/J(B')} = f_{B/J(B')}((B'/J(B'))/(V'/J(V'))) = f_B(Q/F)$,
$e_{B/J(B')} = e_{B/J(B')}((B'/J(B'))/(V'/J(V')))$,
n' is the extension number of $(B/J(B')) \cap Z(B'/J(B'))$ to $B'/J(B')$,
t is the number of all extensions of $V/J(V')$ to $Z(B'/J(B'))$,
s is a nonnegative integer.
Finally, Corollary 4.5 shows $e_{B'}(Q/F)e_{B/J(B')} = e_B(Q/F)t$ and we get

$$f_{B'}(Q/F)e_{B'}(Q/F) = f_B(Q/F)e_B(Q/F)(n't)^2p^s.$$

By Proposition 6.14, $n't$ is the extension number of V to Q since B' is integral over V', and Proposition 7.3 completes the proof. Q.E.D.

In [W2], A. R. Wadsworth proved the following version of the Defektsatz (cf., [W2, p. 306]):

$$[Q : F] = f_B(Q/F)e_B(Q/F)(n_B/t_B)^2p^d,$$

where n_B and t_B as well as n_B/t_B are positive integers which are defined as follows:

$t_B =$ matrix size of $B/J(B)$ (i.e., $B/J(B) = M_{t_B}(E)$, where E is a division ring).

$n_B =$ matrix size of $Q \otimes_F F^h$, where F^h denotes the Henselization of F with respect to V.

Proposition 7.4. *Let Q be a central simple algebra finite-dimensional over its center F and let B be a Dubrovin valuation ring of Q with center $V = F \cap B$. Then n_B/t_B is equal to the extension number n of V to Q.*

Proof. We prove the proposition by induction on n.

$n = 1$. B is integral over V, i.e. $n_B/t_B = 1$ by [W2, Theorem F].

$n > 1$. Let B' be the minimal overring of B integral over its center V', and let n' be the extension number of $(B/J(B')) \cap Z(B'/J(B'))$. By Corollary 5.6(ii), we obtain $n' < n$, and therefore $n_{\overline{B}}/t_{\overline{B}} = n'$ by induction hypothesis where $\overline{B} = B/J(B')$. If l denotes the number of all extensions of $V/J(V')$ to $Z(B'/J(B'))$, then we conclude $n = ln'$. Since B' is integral over V', the statement follows by [W2, Theorem E(iii)]. Q.E.D.

Proposition 7.4 together with [W2, Theorem D] can be helpful for computing the extension number (at least if V has finite Krull dimension; in this case [W2, Theorem D] can be applied where $Q_1 \subset \cdots \subset Q_k$ are all prime ideals of V).

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