TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 330, Number 2, April 1992

ANALYTIC OPERATOR VALUED FUNCTION SPACE INTEGRALS AS AN $\mathscr{L}(L_p, L_{p'})$ THEORY

KUN SOO CHANG AND KUN SIK RYU

ABSTRACT. The existence of an analytic operator-valued function space integral as an $\mathscr{L}(L_p, L_{p'})$ theory $(1 \le p \le 2)$ has been established for certain functionals involving the Lebesgue measure. Recently, Johnson and Lapidus proved the existence of the integral as an operator on L_2 for certain functionals involving any Borel measure. We establish the existence of the integral as an operator from L_p to $L_{p'}$ $(1 < p < 2)$ for certain functionals involving some Borel measures.

1. NOTATIONS AND PRELIMINARIES

In this section we present some necessary notations and lemmas which are needed in our subsequent section. Insofar as possible, we adopt the definitions and notations of [6 and 8].

A. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{R}$ be the set of all real numbers. Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ be the set of all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively. Let ρ be a function on the set of all nonnegative integers such that $\rho(0) = 0$ and $\rho(n) = 1$ for $n \ge 1$.

B. Given a number d such that $1 \le d \le \infty$, d and d' will be always related by $1/d + 1/d' = 1$. If $1 < p < 2$ is given, let α in $(1, \infty)$ be such that $\alpha = p/(2-p)$. In our theorems, N will be a positive integer restricted so that $N < 2\alpha$. For $1 < p < 2$, let r be a real number such that $2\alpha/(2\alpha - N) < r < \infty$. The number $N/2\alpha$ will occur often and so it is worthwhile introducing a symbol for it; $\delta \equiv N/2\alpha$. Note that $0 < r'\delta < 1$ where r and r' are conjugate indices.

C. For $1 \le p < \infty$, $L_p(\mathbb{R}^N)$ is the space of $\mathbb{C}$-valued Borel measurable functions ψ on $\mathbb{R}^N$ such that $|\psi|^p$ is integrable with respect to Lebesgue measure m_L on $\mathbb{R}^N$. $L_\infty(\mathbb{R}^N)$ is the space of $\mathbb{C}$-valued Borel measurable functions ψ on $\mathbb{R}^N$ such that ψ is essentially bounded with respect to m_L. Let $\mathscr{L}(L_p, L_{p'})$ be the space of bounded linear operators from $L_p(\mathbb{R}^N)$ into $L_{p'}(\mathbb{R}^N)$.

The notation $\|\cdot\|$ will be used both for the norm of vectors and for the norm of operators; the meaning will be clear from context.

Received by the editors January 20, 1989 and, in revised form, January 18, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28C20; Secondary 28A33, 47D45, 81C30, 81C35.

Key words and phrases. Bochner integral, Feynman integral, Wiener integral, function space integral, Lebesgue decomposition, linear operator, strongly continuous, strongly measurable.

Research partially supported by the Korea Science and Engineering Foundation and by the Ministry of Education.

0002-9947/92 \$1.00 + \$.25 per page

D. Let $1 \leq p \leq 2$ be given. For λ in $\mathbb{C}_+^\sim$, ψ in $L_p(\mathbb{R}^N)$, ξ in $\mathbb{R}^N$ and a positive real number s, let

$$(C_{\lambda/s}\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{N/2} \int_{\mathbb{R}^N} \psi(u) \exp\left(-\frac{\lambda\|u-\xi\|^2}{2s}\right) dm_L(u)$$

where if N is odd we always choose $\lambda^{-1/2}$ with nonnegative real part and if $\operatorname{Re}\lambda = 0$ the integral in the above should be interpreted in the mean just as in the theory of the L_p Fourier transform. If $p = 1$, from [3] $\mathbb{C}_{\lambda/s}$ is in $\mathscr{L}(L_1, L_\infty)$ and $\|C_{\lambda/s}\| \leq (|\lambda|/2\pi s)^{N/2}$. And as a function of λ, $\mathbb{C}_{\lambda/s}$ is analytic in $\mathbb{C}_+$ and weakly continuous in $\mathbb{C}_+^\sim$. If $1 < p \leq 2$ from [1 and 8] $\mathbb{C}_{\lambda/s}$ is in $\mathscr{L}(L_p, L_{p'})$ and $\|C_{\lambda/s}\| \leq (|\lambda|/2\pi s)^\delta$. And as a function of λ, $\mathbb{C}_{\lambda/s}$ is analytic in $\mathbb{C}_+$ and strongly continuous in $\mathbb{C}_+^\sim$.

E. Let $t > 0$ be given. $M(0, t)$ will denote the space of complex Borel measures η on the interval $(0, t)$. Every measure η in $M(0, t)$ has a unique decomposition, $\eta = \mu + \nu$ into a continuous part μ and a discrete part $\nu \equiv \sum_{p=1}^\infty \omega_p \delta_{\tau_p}$ where $\langle \omega_p \rangle$ is a summable sequence in $\mathbb{C}$ and δ_{τ_p} is the Dirac measure [9]. In fact, this is the Lebesgue decomposition of η. And $M(0, t)^*$ will denote the subset of $M(0, t)$ which satisfies the following conditions;

(a) If μ is the continuous part of η in $M(0, t)^*$, then the Radon-Nikodym derivative $d|\mu|/dm$ exists and is essentially bounded where m is the Lebesgue measure on $(0, t)$.

(b) If $\nu = \sum_{p=1}^\infty \omega_p \delta_{\tau_p}$ is the discrete part of η in $M(0, t)^*$, then $\sum_{p=1}^\infty |\omega_p| \tau_p^{-r'\delta}$ converges.

F. Let $C_0[0, t] \equiv C_0$ be the space of $\mathbb{R}^N$-valued continuous functions x on $[0, t]$ such that $x(0) = 0$. We consider C_0 as equipped with N-dimensional Wiener measure m_w. Let $C[0, t] \equiv C$ be the space of $\mathbb{R}^N$-valued continuous functions y on $[0, t]$.

G. For $1 < p \leq 2$ and η in $M(0, t)$, let $L_{\alpha r:\,\eta}([0, t] \times \mathbb{R}^N) \equiv L_{\alpha r:\,\eta}$ be the space of all $\mathbb{C}$-valued Borel measurable functionals θ on $[0, t] \times \mathbb{R}^N$ such that

$$\|\theta\|_{\alpha r:\,\eta} \equiv \left\{ \int_{(0, t)} \|\theta(s, \cdot)\|_\alpha^r \, d|\eta|(s) \right\}^{1/r} < \infty.$$

Note that $L_{\alpha r:\,\eta} \subset L_{\alpha s:\,\eta}$ if $1 \leq s \leq r \leq \infty$. If θ is in $L_{\alpha r:\,\eta}$ and if $\eta = \mu + \nu$ is the Lebesgue decomposition, it is not difficult to show that θ is in $L_{\alpha r:\,\mu} \cap L_{\alpha r:\,\nu}$ and $\|\theta\|_{\alpha r:\,\eta} = \|\theta\|_{\alpha r:\,\mu} + \|\theta\|_{\alpha r:\,\nu}$.

H. Let $1 < p \leq 2$ be given and θ be in $L_\alpha(\mathbb{R}^N)$. From Lemma 1.3 in [8], a function M_θ: $L_{p'}(\mathbb{R}^N) \to L_p(\mathbb{R}^N)$ defined by $M_\theta(f) = f\theta$, is in $\mathscr{L}(L_{p'}, L_p)$ and $\|M_\theta\| \leq \|\theta\|_\alpha$. It will be convenient to let $\theta(s)$ denote $M_{\theta(s, \cdot)}$ for θ in $L_{\alpha r:\,\eta}$.

Let $\theta_1, \theta_2, \ldots, \theta_{m-1}$ be in $L_\alpha(\mathbb{R}^N)$, ψ in $L_p(\mathbb{R}^N)$ and $0 < s_1 < s_2 < \cdots < s_m < t$. From the Wiener integral formula [12],

$$\begin{aligned}
&\int_{C_0} \theta_1(x(s_1))\theta_2(x(s_2)) \cdots \theta_{m-1}(x(s_{m-1}))\psi(x(s_m))\, dm_w(x) \\
&\quad = [\{C_{1/s} \circ \theta_1(s_1) \circ \cdots \circ C_{1/(s_{m-1}-s_{m-2})} \circ \theta_{m-1}(s_{m-1}) \circ C_{1/(s_m - s_{m-1})}\}\psi](0).
\end{aligned}$$

I. Let $0 < k < 1$ be given and m be in $\mathbb{N}$. For $a < s_1 < s_2 < \cdots < s_m < b$,

$$\int_a^b \int_a^{s_m} \cdots \int_a^{s_1} \{(s_1 - a)(s_2 - s_1) \cdots (b - s_m)\}^{-k} ds_1 ds_2 \cdots ds_m$$
$$= \frac{(b-a)^{m-(m+1)k}\{\Gamma(1-k)\}^{m+1}}{\Gamma((m+1)(1-k))} \quad \text{where } \Gamma \text{ is the gamma function.}$$

Throughout this paper, this value is denoted by $E(a, b; m; k)$.

And let $0 < p \leq 2$ be given and let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers. From the Hölder inequality, we have

$$\sum_{i=1}^n a_i^p \leq n^{(2-p)/2}\left(\sum_{i=1}^n a_i^2\right)^{p/2}.$$

J. Let $1 \leq p \leq 2$ be given. Let F be a functional on C. Given $\lambda > 0$, ψ in $L_p(\mathbb{R}^N)$ and ξ in $\mathbb{R}^N$, let

$$[I_\lambda(F)\psi](\xi) = \int_{C_0} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(t) + \xi)\, dm_w(x).$$

If for m_L-a.e. ξ in $\mathbb{R}^N$, $[I_\lambda(F)\psi](\xi)$ exists in $L_{p'}(\mathbb{R}^N)$ and if the correspondence $\psi \to [I_\lambda(F)]\psi$ gives an element of $\mathscr{L}(L_p, L_{p'})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ. Suppose there exists λ_0 $(0 < \lambda_0 \leq \infty)$ such that $I_\lambda(F)$ exists for all $0 < \lambda < \lambda_0$ and there exists an $\mathscr{L}(L_p, L_{p'})$-valued function which is analytic in $\mathbb{C}_{+,\lambda_0} \equiv \mathbb{C}_+ \cap \{z \in \mathbb{C} \mid |z| < \lambda_0\}$ and agrees with $I_\lambda(F)$ on $(0, \lambda_0)$, then this $\mathscr{L}(L_p, L_{p'})$-valued function is called the operator-valued function space integral of F associated with λ and in this case, we say that $I_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$. If $I_\lambda(F)$ exists for λ in $\mathbb{C}_{+,\lambda_0}$ and $I_\lambda(F)$ is strongly continuous in $\mathbb{C}^{\sim}_{+,\lambda_0} \equiv \mathbb{C}^{\sim}_{+} \cap \{z \in \mathbb{C} \mid |z| < \lambda_0\}$, we say that $I_\lambda(F)$ exists for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$. When λ is purely imaginary, $I_\lambda(F)$ is called the (analytic) operator-valued Feynman integral of F.

K. Let X, Y be two Banach spaces, $\mathscr{L}(X, Y)$ a space of bounded linear operators from X into Y and (Ω, m) be a measure space. Let G: $\Omega \to \mathscr{L}(X, Y)$ be a function such that for each x in X, $\{G(s)\}(x)$ is Bochner integrable with respect to m. Then there exists a linear operator J from X into Y such that

$$J(x) \equiv (B) - \int_\Omega \{G(s)\}(x)\, dm(s) \quad \text{for } x \text{ in } X$$

where $(B)\int_\Omega \{G(s)\}(x)\, dm(s)$ refers to the Bochner integral. Here, this linear operator J is denoted by $(BS)\int_\Omega G(s)\, dm(s)$ and it is called the Bochner integral in the strong operator sense. When $X = Y$, J is called the strong integral of G.

We finish this section with two lemmas.

Lemma 1.1. *Let η be in $M(0, t)^*$ and θ be in $L_{\alpha r:\, \eta}$. Let*

$$F(y) = \int_{(0,t)} \theta(s, y(s))\, d\eta(s) \quad \textit{for } y \textit{ in } C$$

for which the integral exists. Then for every $\lambda > 0$, $F(\lambda^{-1/2}x + \xi)$ is defined for $m_w \times m_L$-a.e. in $C_0 \times \mathbb{R}^N$.

Proof. We can easily check that for every $\lambda > 0$ and $m_w \times m_L$-a.e. (x, ξ) in $C_0 \times \mathbb{R}^N$, $\theta(s, \lambda^{-1/2}x(s) + \xi)$ is defined, see [6, Lemma 0.1].

Let $\eta = \mu + \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$ be a Lebesgue decomposition. Then

$$\begin{aligned}
&\int_{(0,t)} \left(\int_{C_0} |\theta(s, \lambda^{-1/2}x(s) + \xi)| \, dm_w(x) \right) d|\eta|(s) \\
&\overset{[1]}{\leq} \left(\frac{\lambda}{2\pi}\right)^{\delta} \alpha'^{-N/2\alpha'} \left\{ \int_{(0,t)} s^{-\delta} ||\theta(s, \cdot)||_\alpha d|\mu|(s) \right. \\
&\qquad\qquad \left. + \sum_{p=1}^{\infty} (|\omega_p| \tau_p^{-\delta} ||\theta(\tau_p, \cdot)||_\alpha) \right\} \\
&\overset{[2]}{\leq} \left(\frac{\lambda}{2\pi}\right)^{\delta} \alpha'^{-N/2\alpha'} \left\{ ||\theta||_{\alpha r;\, \mu} (\operatorname{ess\,sup} d|\mu|/dm)^{1/r'} \left(\int_{(0,t)} s^{-r'\delta} dm(s) \right)^{1/r'} \right. \\
&\qquad\qquad \left. + \left(\sum_{p=1}^{\infty} |\omega_p| \, ||\theta(\tau_p, \cdot)||_\alpha^r \right)^{1/r} \left(\sum_{p=1}^{\infty} |\omega_p| \tau_p^{-r'\delta} \right)^{1/r'} \right\} \\
&< \infty.
\end{aligned}$$

By applying Wiener integral formula, a simple change of variables and the Hölder inequality, we obtain step [1]. Step [2] results from the Hölder inequality and G. We deduce the last inequality directly from the given conditions.

Hence, by the Fubini theorem

$$\begin{aligned}
&\int_{C_0} \left(\int_{(0,t)} |\theta(s, \lambda^{-1/2}x(s) + \xi)| d|\eta|(s) \right) dm_s(x) \\
&= \int_{(0,t)} \left(\int_{C_0} |\theta(s, \lambda^{-1/2}x(s) + \xi)| \, dm_w(x) \right) d|\eta|(s) \\
&< \infty \quad \text{for } \eta \text{ in } M(0, t)^*.
\end{aligned}$$

Thus, for m_w-a.e. x in C_0 and for all ξ in $\mathbb{R}^N$,

$$\int_{(0,t)} |\theta(s, \lambda^{-1/2}x(s) + \xi)| \, d|\eta|(s)$$

exists. Therefore, for $m_w \times m_L$-a.e. (x, ξ) in $C_0 \times \mathbb{R}^N$, $F(\lambda^{-1/2}x + \xi)$ is defined. The lemma is proved.

The following lemma can be proved by techniques similar to those used in the proof of Lemma 0.2 in [6].

Lemma 1.2. *Let E be a complex Banach space, E^* a dual space of E, (A, m) a finite measure space and let T be a metric space. Consider a function g: $T \times A \to \mathscr{L}(E, E^*)$. Assume that for each λ in T and for each ψ in E, $\{g(\lambda, y)\}\psi$ is a strongly measurable function of y in A. Suppose further that there exists h in $L_1(A, m)$ such that $||g(\lambda, y)|| \leq h(y)$ for m-a.e. y in A and λ in T.*

Set $G(\lambda) = (BS) \int_A g(\lambda, y) \, dm(y)$ for all λ in T.

(1) Assume that for m-a.e. y in A, $g(\lambda, y)$ is a strongly continuous function of λ in T. Then G is strongly continuous in T.

(2) Assume that T is open in $\mathbb{C}$ and for m-a.e. y in A, $g(\lambda, y)$ is an analytic function of λ in T. Then G is analytic in T.

2. An analytic operator-valued function space integral as an $\mathscr{L}(L_p, L_{p'})$ theory

The methods of proof and statements of our results for the L_p case ($1 < p < 2$) are very similar to those of [6] for the L_2 case. However, some care is required to determine sufficient conditions under which the theory goes through in the L_p context. Such conditions ensure the validity of Lemma 1.1, the analogue of Lemma 0,1 in [6], and are adopted throughout this paper.

Theorems 2.1, 2.2, 2.3, 2.4, and 2.5 are the analogous results of Examples 3.3, 3.2, 3.1, 3.4, and Corollary 1.1 in [6], respectively.

Throughout this section, let $1 < p < 2$ be given, let η be in $M(0, t)^*$, let θ be in $L_{\alpha r;\, \eta}$ and let $\eta = \mu + \nu$ be the decomposition of η into its continuous and discrete parts. For n in $\mathbb{N}$, let

$$F_n(y) = \left\{ \int_{(0,t)} \theta(s, y(s))\, d\eta(s) \right\}^n \quad \text{for } y \text{ in } C.$$

Here if $n = 0$, from the definition, directly $I_\lambda(F_0) = C_{\lambda/t}$.

We begin by treating simple cases.

Theorem 2.1 (η purely discrete and finitely supported). *Let $\eta = \sum_{p=1}^{h} \omega_p \delta_{\tau_p}$ where we may assume that $0 < \tau_1 < \tau_2 < \cdots < \tau_h < t$. Suppose that $\theta(\tau_p, \cdot)$, $p = 1, 2, \ldots, h$, are essentially bounded. Then the operator $I_\lambda(F_n)$ exists for all λ in $\mathbb{C}_+^{\sim}$ and for all λ in $\mathbb{C}_+^{\sim}$,*

$$(1) \qquad I_\lambda(F_n) = n! \sum_{\substack{q_1+\cdots+q_h=n \\ 0\le q_1, \ldots, q_h \le n}} \frac{\omega_1^{q_1}\omega_2^{q_2}\cdots\omega_h^{q_h}}{q_1! q_2! \cdots q_h!} L(\lambda; \tau_1, \ldots, \tau_h; q_1, \ldots, q_h)$$

where $L(\lambda; \tau_1, \ldots, \tau_h; q_1, \ldots, q_h) = C_{\lambda/\tau_1} \circ [\theta(\tau_1)]^{q_1} C_{\lambda/(\tau_2-\tau_1)} \circ [\theta(\tau_2)]^{q_2} \circ \cdots \circ [\theta(\tau_h)]^{q_h} \circ C_{\lambda/(t-\tau_h)}$.

(We use the convention $C_{\lambda/(\tau_p-\tau_{p-1})} \circ [\theta(\tau_p)]^0 \circ C_{\lambda/(\tau_{p+1}-\tau_p)} = C_{\lambda/(\tau_{p+1}-\tau_{p-1})}$.) Moreover, for all λ in $\mathbb{C}_+^{\sim}$,

$$(2) \qquad \|I_\lambda(F_n)\| \le n! \left(\frac{|\lambda|}{2\pi}\right)^{(h+1)\delta} \sum_{\substack{q_1+\cdots+q_h=n \\ 0\le q_1, \ldots, q_h \le n}} \frac{|\omega_1|^{q_1}\cdots|\omega_h|^{q_h}}{q_1!\cdots q_h!}$$
$$\times \left\{ \prod_{p=1}^{h} (\|\theta(\tau_p, \cdot)\|_\infty^{q_p-1} \|\theta(\tau_p, \cdot)\|_\alpha)^{\rho(q_p)} \right\} \{\tau_1(\tau_2-\tau_1)\cdots(t-\tau_h)\}^{-\delta}.$$

Proof. Let $0 = \tau_0$ and $t = \tau_{h+1}$. Let ψ be in $L_p(\mathbb{R}^N)$, ξ in $\mathbb{R}^N$ and $\lambda > 0$ be given. Then

$$
\begin{aligned}
&[I_\lambda(F_n)\psi](\xi) \\
&\overset{[1]}{=} \int_{C_0}\left(\int_{(0,t)} \theta(s, \lambda^{-1/2}x(s)+\xi)\,d\eta(s)\right)^n \psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x) \\
&\overset{[2]}{=} \int_{C_0}\left(\sum_{p=1}^{h} \omega_p\theta(\tau_p, \lambda^{-1/2}x(\tau_p)+\xi)\right)^n \psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x) \\
&\overset{[3]}{=} n! \sum_{\substack{q_1+\cdots+q_h=n \\ 0\le q_1,\dots,q_h}} \frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!}\left(\frac{\lambda}{2\pi}\right)^{N(h+1)/2} \{\tau_1(\tau_2-\tau_1)\cdots(t-\tau_h)\}^{-N/2} \\
&\quad\times \int_{\mathbb{R}^{N(h+1)}} \left\{\prod_{p=1}^{h}\theta(\tau_p, v_p)^{q_p}\right\}\psi(v_{p+1}) \\
&\quad\times \exp\left\{-\frac{\lambda}{2}\sum_{p=1}^{h+1}\frac{\|v_p-v_{p-1}\|^2}{(\tau_p-\tau_{p-1})}\right\} d\prod_{p=1}^{h+1} m_L(v_p) \\
&\overset{[4]}{=} n! \sum_{\substack{q_1+\cdots+q_h=n \\ 0\le q_1,\dots,q_h}} \frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!}\{L(\lambda;\tau_1,\dots,\tau_h;q_1,\dots,q_h)\psi\}(\xi).
\end{aligned}
$$

By the definition and an elementary calculus, steps [1] and [2] are clear. Step [3] results from the multinomial expansion, the Wiener integral formula and a simple change of variables. From H in §1, $\theta(\tau_p)$, $p = 1, 2, \dots, h$, are in $\mathscr{L}(L_{p'}, L_p)$. Since $\theta(\tau_p, \cdot)$, $p = 1, 2, \dots, h$, are essentially bounded, $[\theta(\tau_p)]^n$, $p = 1, 2, \dots, h$, are in $\mathscr{L}(L_{p'}, L_p)$ for $n \geq 1$. Hence $L(\lambda; \tau_1, \dots, \tau_h; q_1, \dots, q_h)$ is well-defined for any nonnegative integers $q_1, \dots, q_h$. Thus we obtain step [4].

Now, let $\mathscr{F}$ be an $\mathscr{L}(L_p, L_{p'})$-valued function on $\mathbb{C}_+^\sim$ given by $\mathscr{F}(\lambda) = L(\lambda; \tau_1, \dots, \tau_h; q_1, \dots, q_h)$. Then for all λ in $\mathbb{C}_+^\sim$,

$$
\begin{aligned}
\text{(3)}\qquad \|\mathscr{F}(\lambda)\| \le \left(\frac{\lambda}{2\pi}\right)^{(h+1)\delta} &\prod_{p=1}^{h+1}(\|\theta(\tau_p, \cdot)\|_\infty^{q_p-1}\|\theta(\tau_p, \cdot)\|_\alpha)^{\rho(q_p)} \\
&\times\{\tau_1(\tau_2-\tau_1)\cdots(t-\tau_h)\}^{-\delta}.
\end{aligned}
$$

It can be shown that $\mathscr{F}(\lambda)$ is an analytic function of λ in $\mathbb{C}_+$ [8, p. 108]. To show that $\mathscr{F}(\lambda)$ is strongly continuous in $\mathbb{C}_+^\sim$, it suffices to show that

$$
\|\mathscr{F}(\lambda)\psi - \mathscr{F}(-iq)\psi\|_{p'} \to 0
$$

as $\lambda \to -iq$ for ψ in $L_p(\mathbb{R}^N)$ and a nonzero real q. For $1 \le l \le h+1$, let

$$
A_l = C_{-iq/\tau_1} \circ \cdots \circ C_{-iq/(\tau_l-\tau_{l-1})} \circ [\theta(\tau_l)]^{q_l} \circ C_{\lambda/(\tau_{l+1}-\tau_l)} \circ \cdots \circ C_{\lambda/(t-\tau_h)}\psi
$$

and

$$
\psi_l = [\theta(\tau_l)]^{q_l} \circ C_{\lambda/(\tau_{l+1}-\tau_l)} \circ \cdots \circ C_{\lambda/(t-\tau_h)}\psi.
$$

Then

$$\|\mathscr{F}(\lambda)\psi - \mathscr{F}(-iq)\psi\|_{p'} \le \sum_{l=1}^{h+1} \|A_l - A_{l-1}\|_{p'} \le \sum_{l=1}^{h+1} \left\{ \prod_{p=1}^{l-1} \left[\frac{|q|}{2\pi(\tau_p - \tau_{p-1})} \right]^{\delta} \right\}$$
$$\times \left\{ \prod_{p=1}^{l-1} (\|\theta(\tau_p\,,\,\cdot)\|_{\infty}^{q_p-1} \|\theta(\tau_p\,,\,\cdot)\|_{\alpha})^{\rho(q_p)} \right\}$$
$$\times \|C_{\lambda/(\tau_l-\tau_{l-1})}\psi_l - C_{-iq/(\tau_l-\tau_{l-1})}\psi_l\|_{p'}.$$

From D in §1, the right-hand side in the above last inequality converges to zero as $\lambda \to -iq$. Hence, $\mathscr{F}(\lambda)$ is strongly continuous in $\mathbb{C}_+^{\sim}$.

By the uniqueness theorem in [5], $I_\lambda(F_n)$ exists for λ in $\mathbb{C}_+^{\sim}$ and it is given by (1) for all λ in $\mathbb{C}_+^{\sim}$. Furthermore, from (3), a norm estimate of $I_\lambda(F_n)$ is given by (2). Thus the proof of this theorem is complete.

Theorem 2.2 ($\eta = \mu$ purely continuous). *We suppose that η is purely continuous. The operator $I_\lambda(F_n)$ exists for all λ in $\mathbb{C}_+^{\sim}$ and for all λ in $\mathbb{C}_+^{\sim}$,*

$$I_\lambda(F_n) = n!(BS)\int_{\Delta_n} L(\lambda\,;\, s_1\,,\,\dots\,,\, s_n)d\prod_{u=1}^{n}\eta(s_u) \tag{4}$$

where $\Delta_n = \{(s_1\,,\,\dots\,,\,s_n) \in (0\,,\,t)^n \mid 0 < s_1 < s_2 < \cdots < s_n < t\}$ and for $(s_1\,,\,\dots\,,\,s_n)$ in Δ_n, $L(\lambda\,;\,s_1\,,\,\dots\,,\,s_n) = C_{\lambda/s_1} \circ \theta(s_1) \circ C_{\lambda/(s_2-s_1)} \circ \cdots \circ \theta(s_n) \circ C_{\lambda/(t-s_n)}$. Moreover, for all λ in $\mathbb{C}_+^{\sim}$

$$\|I_\lambda(F_n)\| \le (n!)^{1/r'} \left(\frac{|\lambda|}{2\pi}\right)^{(n+1)\delta} (\|\theta\|_{\alpha r:\,\eta})^n (\operatorname{ess\,sup} d|\eta|/dm)^{n/r'} \tag{5}$$
$$\times E(0\,,\,t\,;\,n\,;\,r'\delta)^{1/r'}.$$

Proof. Let ψ be in $L_p(\mathbb{R}^N)$, ξ be in $\mathbb{R}^N$ and $\lambda > 0$ be given. Then

$$[I_\lambda(F_n)\psi](\xi)$$
$$\overset{[1]}{=} \int_{C_0}\left\{\int_{(0\,,\,t)^n}\prod_{u=1}^{n}\theta(s_u\,,\,\lambda^{-1/2}x(s_u)+\xi)d\prod_{u=1}^{n}\eta(s_u)\right\}\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)$$
$$\overset{[2]}{=} \int_{C_0}\left(\sum_{\sigma\in p_n}\int_{\Delta_{\sigma(n)}}\prod_{u=1}^{n}\theta(s_u\,,\,\lambda^{-1/2}x(s_u)+\xi)d\prod_{u=1}^{n}\eta(s_u)\right)\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)$$
$$\overset{[3]}{=} n!\int_{C_0}\left(\int_{\Delta_n}\prod_{u=1}^{n}\theta(s_u\,,\,\lambda^{-1/2}x(s_u)+\xi)d\prod_{u=1}^{n}\eta(s_u)\right)\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)$$
$$\overset{[4]}{=} n!\int_{\Delta_n}\left\{\int_{C_0}\left(\prod_{u=1}^{n}\theta(s_u\,,\,\lambda^{-1/2}x(s_u)+\xi)\right)\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)\right\}d\prod_{u=1}^{n}\eta(s_u)$$
$$\overset{[5]}{=} n!\int_{\Delta_n}[L(\lambda\,;\,s_1\,,\,\dots\,,\,s_n)\psi](\xi)d\prod_{u=1}^{n}\eta(s_u).$$

Step [1] follows from the definition, the Fubini theorem and Lemma 1.1. Let $D_{i,j} = \{(s_1\,,\,\dots\,,\,s_n) \in (0\,,\,t)^n \mid s_i = s_j\}$ for $1 \le i \ne j \le n$. Then by the Fubini theorem, $D_{i,j}$ is $\prod_{u=1}^n \eta$-null. Let P_n be a permutation on $\{1\,,\,2\,,\,\dots\,,\,n\}$ and for σ in P_n, let

$$\Delta_{\sigma(n)} = \{(s_1\,,\,\dots\,,\,s_n) \in (0\,,\,t)^n \mid 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t\}.$$

Since $(0, t)^n = \{\bigcup_{\sigma \in P_n} \Delta_{\sigma(n)}\} \cup \{\bigcup_{i \neq j} D_{i,j}\}$, we obtain step [2]. Since the integrand is invariant under permutations of s-variables, the integral over the $n!$ simplexes are equal. Hence, we obtain step [3]. Step [4] follows from the Fubini theorem which will be justified below in conjunction with the proof of the norm estimate. Step [5] is obtained by applying the Wiener integral formula, a simple change of variables, and D and H in §1.

If we use the same techniques as in the proof of Theorem 2.1. in [8, p. 107], it can be shown that $L(\lambda; s_1, \dots, s_n)\psi$ is weakly measurable, so it is strongly measurable since $L_{p'}(\mathbb{R}^N)$ is separable.

$$\begin{aligned} \|I_\lambda(F_n)\psi\|_{p'} &\le n!\|\psi\|_p \int_{\Delta_n} \|L(\lambda; s_1, s_2, \dots, s_n)\| d\prod_{u=1}^{n} |\eta|(s_u) \\ &\le (n!)^{1/r'} \left(\frac{\lambda}{2\pi}\right)^{(n+1)\delta} \|\psi\|_p (\operatorname{ess\,sup} d|\eta|/dm)^{n/r'} \\ &\quad \cdot (\|\theta\|_{\alpha r;\, \eta})^n E(0, t; n; r'\delta)^{1/r'}. \end{aligned} \tag{6}$$

In the above, the first inequality follows from [8], the last inequality results from Hölder inequality and an elementary calculus. This justifies the use of the Fubini theorem in step [4] above. Hence, for $\lambda > 0$,

$$I_\lambda(F_n) = n!(BS) \int_{\Delta_n} L(\lambda; s_1, \dots, s_n) d\prod_{u=1}^{n} \eta(s_u).$$

By the same method as in the proof of Theorem 2.1, $L(\lambda; s_1, \dots, s_n)$ is analytic in $\mathbb{C}_+$ and it is strongly continuous in $\mathbb{C}_+$. Using Lemma 1.2.,

$$n!(BS) \int_{\Delta_n} L(\lambda; s_1, \dots, s_n) d\prod_{u=1}^{n} \eta(s_u)$$

is analytic in $\mathbb{C}_+$ and is strongly continuous in $\mathbb{C}_+^\sim$. By the uniqueness theorem, $I_\lambda(F_n)$ exists for λ in $\mathbb{C}_+^\sim$ and for all λ in $\mathbb{C}_+^\sim$, we obtain (5) as in (6) except with λ replaced by $|\lambda|$. Thus the theorem is proved.

Theorem 2.3 (Finitely supported measure). *Let $\nu = \sum_{p=1}^{h} \omega_p \delta_{\tau_p}$ where we may assume that $0 < \tau_1 < \tau_2 < \cdots \tau_h < t$. Suppose that $\theta(\tau_p, \cdot)$, $p = 1, 2, \dots, h$, are essentially bounded. Then the operator $I_\lambda(F_n)$ exists for all λ in $\mathbb{C}_+^\sim$ and for all λ in $\mathbb{C}_+^\sim$,*

$$\begin{aligned} I_\lambda(F_n) &= n! \sum_{\substack{q_0 + \cdots + q_h = n \\ 0 \le q_0, \dots, q_h}} \frac{\omega_1^{q_1} \cdots \omega_h^{q_h}}{q_1! \cdots q_h!} \\ &\quad \times (BS) \int_{\Delta_{q_0; j_1, \dots, j_h}} L_0 \circ L_1 \circ \cdots \circ L_h d\prod_{u=1}^{q} \mu(s_u) \end{aligned} \tag{7}$$

where for nonnegative integers $q_0, q_1, \dots, q_h$ and $j_1, \dots, j_{h+1}$, $\Delta_{q_0, j_1, \dots, j_{h+1}} = \{(s_1, s_2, \dots, s_{q_0}) \in (0, t)^{q_0} \mid 0 < s_1 < s_2 < \cdots < s_{j_1} < \tau_1 < s_{j_1+1} < \cdots < s_{j_1+j_2} < \tau_2 < \cdots < \tau_h < s_{j_1+\cdots+j_h+1} < \cdots < s_{j_1+\cdots+j_{h+1}} = s_{q_0} < t\}$ and for $(s_1, s_2, \dots, s_{q_0}) \in \Delta_{q_0; j_1, \dots, j_{h+1}}$ and $m \in \{0, 1, \dots, h\}$,

$$L_m = [\theta(\tau_m)]^{q_m} C_{\lambda/(s_{j_1+\cdots+j_m+1}-\tau_m)} \circ \theta(s_{j_1+\cdots+j_m+1}) \circ C_{\lambda/(s_{j_1+\cdots+j_m+2}-s_{j_1+\cdots+j_m+1})} \circ \cdots \circ \theta(s_{j_1+\cdots+j_{m+1}}) \circ C_{\lambda/(\tau_{m+1}-s_{j_1+\cdots+j_{m+1}})}.$$

(*We use the conventions* $\tau_0 = 0$, $\tau_{h+1} = t$ *and* $[\theta(\tau_0)]^{q_0} = 1$, *an identity map on* $L_{p'}(\mathbb{R}^N)$.)

Moreover, for all λ *in* $\mathbb{C}_+^\sim$,

(8)

$$\|I_\lambda(F_n)\| \le n! \sum_{\substack{q_0+\cdots+q_h=n\\ 0\le q_0,\dots,q_h}} \frac{|\omega_1|^{q_1}\cdots|\omega_h|^{q_h}}{q_1!\cdots q_h!}(q_0!)^{-1/r}\left(\frac{|\lambda|}{2\pi}\right)^{(q_0+h+1)\delta}$$
$$\times \left(\frac{(q_0+h)!}{q_0!h!}\right)^{1/2r'}\left[\prod_{l=1}^{h}(\|\theta(\tau_l,\cdot)\|_\infty^{q_l-1}\|\theta(\tau_l,\cdot)\|_\alpha)^{p(q_l)}\right](\text{ess sup}\, d|\mu|/dm)^{1/2r'}$$
$$\times (\|\theta\|_{\alpha r;\,\mu})^{q_0}\left[\sum_{j_1+\cdots+j_{h+1}=q_0}\left\{\prod_{l=0}^{h}E(\tau_l,\tau_{l+1};j_{l+1};r'\delta)\right\}^{2/r'}\right]^{1/2}.$$

Proof. Let $\lambda > 0$ be given, ψ in $L_p(\mathbb{R}^N)$ and ξ in $\mathbb{R}^N$. Then

(9)

$$\begin{aligned}
&[I_\lambda(F_n)\psi](\xi)\\
&\overset{[1]}{=} \int_{C_0}\left\{\int_{(0,t)}\theta(s,\lambda^{-1/2}x(s)+\xi)\,d\mu(s) + \sum_{p=1}^{h}\omega_p\theta(\tau_p,\lambda^{-1/2}x(\tau_p)+\xi)\right\}^n \psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)\\
&\overset{[2]}{=} n!\sum_{\substack{q_0+\cdots+q_h=n\\ 0\le q_0,\dots,q_h}}\frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!}\sum_{j_1+\cdots+j_{h+1}=q_0}\int_{\Delta_{q_0;j_1,\dots,j_{h+1}}}\\
&\quad\cdot\left\{\int_{C_0}\left(\prod_{p=1}^{h}\theta(\tau_p,\lambda^{-1/2}x(\tau_p)+\xi)^{q_p}\right)\right.\\
&\quad\cdot\left.\left(\prod_{u=1}^{q_0}\theta(s_u,\lambda^{-1/2}x(s_u)+\xi)\right)\psi(\lambda^{-1/2}x(t)+\xi)\,dm_w(x)\right\}d\prod_{u=1}^{q_0}\mu(s_u)\\
&\overset{[3]}{=} n!\sum_{\substack{q_0+\cdots+q_h=n\\ 0\le q_0,\dots,q_h}}\frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!}\\
&\quad\cdot\sum_{j_1+\cdots+j_{h+1}=q_0}\left(\int_{\Delta_{q_0;j_1\cdots j_{h+1}}}(L_0\circ L_1\circ\cdots\circ L_h)\psi(\xi)\,d\prod_{u=1}^{q_0}\mu(s_u)\right).
\end{aligned}$$

Step [1] is clear. Step [2] follows from the multinomial expansion, the "simplex trick" and the Fubini theorem which will be justified below in conjunction with the proof of the norm estimate. By D and H in §1, we obtain step [3].

$$(10)\quad \|I_\lambda(F_n)\psi\|_{p'} \le n! \sum_{\substack{q_0+\cdots+q_h=n\\ 0\le q_0,\dots,q_h}} \frac{|\omega_1|^{q_1}\cdots|\omega_h|^{q_h}}{q_1!\cdots q_h!}\|\psi\|_p$$

$$\cdot \sum_{j_1+\cdots+j_{h+1}=q_0} \int_{\Delta_{q_0;j_1,\dots,j_{h+1}}} \|L_0\circ\cdots\circ L_h\| d\prod_{u=1}^{q_0}\mu(s_u)$$

$$\le n!\|\psi\|_p \sum_{\substack{q_0+\cdots+q_h=n\\ 0\le q_0,\dots,q_h}} \frac{|\omega_1|^{q_1}\cdots|\omega_h|^{q_h}}{q_1!\cdots q_h!}(q_0!)^{-1/r}$$

$$\cdot\left(\frac{\lambda}{2\pi}\right)^{(q_0+h+1)\delta}\left(\frac{(q_0+h)!}{q_0!h!)}\right)^{1/2r'}$$

$$\cdot\left[\prod_{l=1}^{h}(\|\theta(\tau_l,\cdot)\|_\infty^{q_l-1}\|\theta(\tau_l,\cdot)\|_\alpha)^{\rho(q_l)}\right]$$

$$\cdot(\operatorname{ess\,sup} d|\mu|/dm)^{1/2r'}(\|\theta\|_{\alpha r:\,\mu})^{q_0}$$

$$\cdot\left[\sum_{j_1+\cdots+j_{h+1}=q_0}\left\{\prod_{l=0}^{h}E(\tau_l,\tau_{l+1};j_{l+1};r'\delta)\right\}^{2/r'}\right]^{1/2}.$$

The first inequality in the above follows from [5, p. 82]. The last inequality results from the Hölder inequality, the Schwarz inequality, and D, H, I in §1.

Since the right-hand side in the above last inequality is finite, we justify the use of the Fubini theorem in step [2].

By the same method as in the proof of Theorem 2.1 of [8], $(L_0\circ L_1\circ\cdots\circ L_h)\psi$ is Bochner integrable. And by the same method as in the proof of Theorem 2.1,

$$(BS)\int_{\Delta_{q_0;j_1,\dots,j_{h+1}}} L_0\circ L_1\circ\cdots\circ L_h d\prod_{u=1}^{q_0}\mu(s_u)$$

is analytic in $\mathbb{C}_+$ and is strongly continuous in $\mathbb{C}_+^\sim$. Thus $I_\lambda(F_n)$ exists for λ in $\mathbb{C}_+^\sim$ and it is given by (7). Moreover, we obtain (8) as in (10) except with λ replaced by $|\lambda|$. Therefore, the theorem is proved.

Now, we treat the general case. Let $\eta=\mu+\nu$ be in $M(0,t)^*$ with $\nu=\sum_{p=1}^\infty \omega_p\delta_{\tau_p}$. And for h in $\mathbb{N}$, let σ be a permutation on $\{1,2,\dots,h\}$ such that $\tau_{\sigma(1)}<\tau_{\sigma(2)}<\cdots<\tau_{\sigma(h)}$.

Theorem 2.4. *Suppose that $\theta(\tau_p,\cdot)$, $p=1,2,\dots,$ are essentially bounded. Then the operator $I_\lambda(F_n)$ exists for all λ in $\mathbb{C}_+^\sim$ and for all λ in $\mathbb{C}_+^\sim$,*

$$(11)\qquad I_\lambda(F_n)=n!\sum_{h=0}^{\infty}\sum_{\substack{q_0+\cdots+q_h=n\\ q_0\ne 0}}\frac{\omega_1^{q_1}\cdots\omega_h^{q_h}}{q_1!\cdots q_h!}\sum_{j_1+\cdots+j_{h+1}=q_0}$$

$$\times(BS)\int_{\Delta_{q_0;j_1,\dots,j_{h+1}}} L_0\circ L_1\circ\cdots\circ L_h d\prod_{u=1}^{q_0}\mu(s_u)$$

where for each h in $\mathbb{N}$, $\Delta_{q_0;j_1,\dots,j_{h+1}}=\{(s_1,\dots,s_{q_0})\in(0,t)^{q_0}\mid 0<s_1<\cdots<s_{j_1}<\tau_{\sigma(1)}<s_{j_1+1}<\cdots<s_{j_1+j_2}<\tau_{\sigma(2)}<\cdots<s_{q_0}<t\}$ and where for

$(s_1, \ldots, s_{q_0}) \in \Delta_{q_0; j_1, \ldots, j_{h+1}}$ *and* $m \in \{0, 1, 2, \ldots, h\}$,

$$L_m = [\theta(\tau_{\sigma(m)})]^{q_{\sigma(m)}} \circ C_{\lambda/(s_{j_1+\cdots+j_m+1}-\tau_{\sigma(m)})} \circ \theta(s_{j_1+\cdots+j_m+1}) \circ \cdots \circ \theta(s_{j_1+\cdots+j_{m+1}}) \circ C_{\lambda/(\tau_{\sigma(m+1)}-s_{j_1+\cdots+j_{m+1}})}.$$

(We use the convention $\tau_{\sigma(0)} = 0$, $\tau_{\sigma(h+1)} = t$ *and* $[\theta(\tau_{\sigma(0)})]^{q_{\sigma(0)}} = 1$, *an identity map on* $L_{p'}(\mathbb{R}^N)$*.) Moreover,*

$$(12) \quad \|I_\lambda(F_n)\| \le n! \sum_{h=0}^{\infty} \sum_{\substack{q_0+\cdots+q_h=n \\ q_h \ne 0}} \frac{|\omega_1|^{q_1} \cdots |\omega_h|^{q_h}}{q_1! \cdots q_h!} (q_0!)^{-1/r} \left(\frac{(q_0+h)!}{q_0! h!} \right)^{1/2r}$$
$$\times \left(\frac{|\lambda|}{2\pi} \right)^{(q_0+h+1)\delta} \left[\prod_{n=1}^{h} (\|\theta(\tau_{\sigma(n)}, \cdot)\|_\infty^{q_n-1} \|\theta(\tau_{\sigma(n)}, \cdot)\|_\alpha)^{\rho(q_n)} \right]$$
$$\times (\operatorname{ess\,sup} d|\mu|/dm)^{q_0/r'}$$
$$\times (\|\theta\|_{\alpha r;\, \mu})^{q_0} \left[\sum_{j_1+\cdots+j_{h+1}=q_0} \left\{ \prod_{n=0}^{h} E(\tau_{\sigma(n)}, \tau_{\sigma(n+1)}; j_{n+1}; r'\delta) \right\}^{2/r'} \right]^{1/2}.$$

Denote this norm estimate by $B_n(|\lambda|)$.

Proof. It can be proved by the same method as in the proof of Theorem 2.3 by using the dominated convergence theorem and the χ_0-nomial formula [6, p. 41].

Now, we prove the main theorem in this paper. Let $\lambda_0 > 0$ be given. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on $\mathbb{C}_{+,\lambda_0}^{\sim}$ such that $\sum_{n=0}^{\infty} |a_n| B_n(|\lambda|)$ is finite for all λ in $\mathbb{C}_{+,\lambda_0}^{\sim}$. Let η be in $M(0, t)^*$, θ be in $L_{\alpha r;\, \eta}$ and let

$$F(y) = f\left(\int_{(0,t)} \theta(s, y(s))\, d\eta(s) \right) \quad \text{for } y \text{ in } C.$$

Theorem 2.5. *Suppose that* $\theta(\tau_p, \cdot)$, $p = 1, 2, \ldots$, *are essentially bounded. Then* $I_\lambda(F)$ *exists for all* λ *in* $\mathbb{C}_{+,\lambda_0}^{\sim}$ *and is given by*

$$(13) \qquad I_\lambda(F) = \sum_{n=0}^{\infty} a_n I_\lambda(F_n).$$

Moreover, for λ *in* $\mathbb{C}_{+,\lambda_0}^{\sim}$ *the series* $\sum_{n=0}^{\infty} a_n I_\lambda(F_n)$ *converges in operator norm and*

$$(14) \qquad \|I_\lambda(F)\| \le \sum_{n=0}^{\infty} |a_n| B_n(|\lambda|).$$

Proof. Since $\sum_{n=0}^{\infty} \|a_n I_\lambda(F_n)\| \le \sum_{n=0}^{\infty} |a_n| B_n(|\lambda|)$ for all λ in $\mathbb{C}_{+,\lambda_0}^{\sim}$, $\sum_{n=0}^{\infty} a_n I_\lambda(F_n)$ is in $\mathscr{L}(L_p, L_{p'})$. And since $B_n(|\lambda|)$ is increasing as $|\lambda| \uparrow$, $\sum_{n=0}^{\infty} a_n I_\lambda(F_n)$ converges uniformly in $\mathbb{C}_{+,\lambda_1}^{\sim}$ for fixed λ_1 in $(0, \lambda_0)$.

Let λ be in $(0, \lambda_0)$ and φ, ψ be in $L_p(\mathbb{R}^N)$. Then

$$(15)\quad \int_{\mathbb{R}^N}\left[\int_{C_0}\sum_{n=0}^{\infty}|a_nF_n(\lambda^{-1/2}x+\xi)|)|\varphi(\xi)|\,|\psi(\lambda^{-1/2}x(t)+\xi)|\,dm_w(\xi)\right]dm_L(\xi)$$
$$\le \sum_{n=0}^{\infty}|a_n|\int_{\mathbb{R}^N}|\varphi(\xi)|[I_\lambda(|F_n|)]|\psi|(\xi)\,dm_L(\xi)$$
$$\le \|\varphi\|_p\|\psi\|_p\sum_{n=0}^{\infty}|a_n|B_n(|\lambda|).$$

Hence for $m_\omega\times m_L$-a.e. (x, ξ) in $C_0\times\mathbb{R}^N$,

$$\sum_{n=0}^{\infty}|a_n|\,|F_n(\lambda^{-1/2}x+\xi)|\,|\varphi(\xi)||\psi(\lambda^{-1/2}x(t)+\xi)|$$

is finite. By considering φ and ψ which never vanish one sees that

$$\sum_{n=0}^{\infty}a_nF_n(\lambda^{-1/2}x+\xi)$$

converges absolutely for a.e.- (x, ξ) in $C_0\times\mathbb{R}^N$. Consider a functional Φ given by

$$\Phi(\varphi)=\int_{\mathbb{R}^N}\varphi(\xi)(I_\lambda(F)\psi)(\xi)\,dm_L(\xi)\quad\text{for }\varphi\text{ in }L_p(\mathbb{R}^N).$$

Then Φ is bounded and linear. Hence, by the Riesz representation theorem, $I_\lambda(F)\psi$ is in $L_{p'}(\mathbb{R}^N)$ and

$$\Phi(\varphi)=\int_{\mathbb{R}^N}\varphi(\xi)\sum_{n=0}^{\infty}\left(a_n\int_{C_0}F_n(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(t)+\xi)\,dm(x)\right)dm_L(\xi)$$
$$=\int_{\mathbb{R}^N}\varphi(\xi)\sum_{n=0}^{\infty}a_n[I_\lambda(F_n)\psi](\xi)\,dm_L(\xi).$$

Hence, for λ in $\mathbb{C}^{\sim}_{+,\lambda_1}$, $I_\lambda(F)\psi=\sum_{n=0}^{\infty}a_nI_\lambda(F_n)\psi$ for a.e.-ξ in $\mathbb{R}^N$, which implies that $I_\lambda(F)=\sum_{n=0}^{\infty}a_nI_\lambda(F_n)$ for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$. By Theorem 3.18.1. in [5], $\sum_{n=0}^{\infty}a_nI_\lambda(F_n)$ is analytic in $\mathbb{C}_{+,\lambda_0}$, that is, $I_\lambda(F)$ is analytic in $\mathbb{C}_{+,\lambda_0}$. Now, we claim that $I_\lambda(F)$ is strongly continuous in $\mathbb{C}^{\sim}_{+,\lambda_0}$. Let $0<|q|<\lambda_0$ be given. Then for each ψ in $L_p(\mathbb{R}^N)$ and λ in $\mathbb{C}_{+,(\lambda_0+|q|)/2}$,

$$\lim_{\lambda\to -iq}I_\lambda(F)\psi=\lim_{\lambda\to -iq}\sum_{n=0}^{\infty}a_nI_\lambda(F_n)\psi$$
$$=\lim_{k\to\infty}\lim_{\lambda\to -iq}\sum_{n=0}^{k}a_nI_\lambda(F_n)\psi$$
$$=\lim_{k\to\infty}\sum_{n=0}^{k}a_nI_{-iq}(F_n)\psi=[I_{-iq}(F)]\psi$$

with all the limits in $L_{p'}$-norm. Thus, $I_\lambda(F)$ is strongly continuous in $\mathbb{C}^{\sim}_{+,\lambda_0}$. Therefore, $I_\lambda(F)$ exists for λ in $\mathbb{C}^{\sim}_{+,\lambda_0}$. Clearly, we obtain the norm estimate in (14) and the series $\sum_{n=0}^{\infty}a_nI_\lambda(F_n)$ converges in operator norm. Thus the proof of the theorem is complete.

REFERENCES

1. R. H. Cameron and D. A. Storvick, *An operator valued function space integral and a related integral equation*, J. Math. Mech. **18** (1968), 517–552.
2. ——, *An operator valued function space integral applied to integrals of functions of class* L_2, J. Math. Anal. Appl. **42** (1973), 330–372.
3. ——, *An operator valued function space integral applied to integrals of functions of class* L_1, Proc. London Math. Soc. **27** (1973), 345–360.
4. J. D. Dollard and G. N. Friedman, *Product integration with applications to differential equations*, Encyclopedia of Math. and its Appl., vol. 10, Addison-Wesley, 1979.
5. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
6. G. W. Johnson and M. L. Lapidus, *Generalized Dyson series, generalized Feynman diagrams, the Feynman integral and Feynman's operational calculus*, Mem. Amer. Math. Soc., vol. 62, no. 351, 1986.
7. G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral; the* L_1 *theory*, J. Math. Anal. and Appl. **50** (1975), 647-667.
8. ——, *The Cameron-Storvick function space integral: An* $\mathscr{L}(L_p, L_{p'})$ *theory*, Nagoya Math. J. (1976), 93–137.
9. M. Reed and B. Simon, *Methods of mathematical physics*, Vols. I, II, III, rev. and enlarged ed., Academic Press, New York, 1980.
10. B. Simon, *Functional integration and quantum physics*, Academic Press, New York, 1979.
11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean space*, Princeton Univ. Press, Princeton, N. J., 1971.
12. J. Yeh, *Stochastic processes and the Wiener integral*, Dekker, New York, 1973.

DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SEOUL 120-749, KOREA

DEPARTMENT OF MATHEMATICS, HANNAM UNIVERSITY, DAEJON 300-791, KOREA