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# ISOMORPHISM INVARIANTS FOR ABELIAN GROUPS

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ABSTRACT. Let $A = (A_1, \dots, A_n)$ be an $n$-tuple of subgroups of the additive group, $Q$, of rational numbers and let $G(A)$ be the kernel of the summation map $A_1 \oplus \cdots \oplus A_n \to \sum A_i$ and $G[A]$ the cokernel of the diagonal embedding $\bigcap A_i \to A_1 \oplus \cdots \oplus A_n$. A complete set of isomorphism invariants for all strongly indecomposable abelian groups of the form $G(A)$, respectively, $G[A]$, is given. These invariants are then extended to complete sets of isomorphism invariants for direct sums of such groups and for a class of mixed abelian groups properly containing the class of Warfield groups.

A central theme in the study of abelian groups has been the search for complete sets of numerical isomorphism invariants for nontrivial classes of groups. This quest has met with limited success in the case of torsion-free abelian groups. A promising start was achieved by R. Baer [Ba] in 1937 with the classification of completely decomposable groups, up to isomorphism, in terms of types of rank-1 summands. However, notwithstanding the important breakthroughs of the 1960s and 1970s, one could argue that, in terms of numerical isomorphism invariants for torsion-free abelian groups, no significant extension of Baer's results occurred until 1985, when F. Richman [Ri-1] classified the "doubly incomparable groups."

The groups studied by Richman are a class of *Butler groups* (pure subgroups of completely decomposable groups) defined as follows. Let $A = (A_1, \dots, A_n)$ be an $n$-tuple of subgroups of $Q$, the additive rationals, and $G = G(A)$ be the kernel of the map $A_1 \oplus \cdots \oplus A_n \to Q$ given by $\bigoplus a_i \to \sum a_i$. Define $G_{ij} = G \cap (A_i \oplus A_j) = \{(a, -a) | a \in A_i \cap A_j\}$, a pure rank-1 subgroup of $G$ isomorphic to $A_i \cap A_j$. The group $G = G(A)$ is called *doubly incomparable* provided $\text{type}(G_{ij})$ and $\text{type}(G_{kl})$ are incomparable whenever $\{i, j\}$ and $\{k, l\}$ are distinct two-element subsets of $\{1, \dots, n\}$. Each doubly incomparable group is strongly indecomposable.

In this paper we obtain a complete set of isomorphism invariants for all strongly indecomposable groups of the form $G(A)$ (Theorem 1.6). The isomorphism invariants for doubly incomparable groups occur as a special case. Our characterization has several other consequences. First, a duality is applied to obtain a complete set of isomorphism invariants for all strongly indecomposable groups of the form $G[A]$, the cokernel of the diagonal embedding

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$A_1 \cap \cdots \cap A_n \to A_1 \oplus \cdots \oplus A_n$ (Theorem 2.1). Second, the global Azumaya theorems in [AHR] are used to extend these invariants to complete sets of isomorphism invariants for arbitrary direct sums of strongly indecomposable groups of the form $G(A)$ or $G[A]$ (Theorem 3.4). Finally, valuated group machinery developed by Richman in [Ri-2] is used to find complete sets of isomorphism invariants for a class of mixed groups containing the Warfield groups (Theorem 4.4). Invariants for the last two cases are developed to reflect criteria given in [AHR] for isomorphism of direct sums of objects in a category where all indecomposable objects have endomorphism rings isomorphic to subrings of $Q$.

A special case of Theorem 4.4 is a complete set of isomorphism invariants for mixed groups $H$ containing a nice subgroup $K$, such that $K$ is a direct sum of strongly indecomposable groups of the form $G(A)$ (alternately, $G[A]$) and $H/K$ is a totally projective torsion group.

Our notation and terminology follow [Ar] for torsion-free groups of finite rank, and [AV-5] for Butler groups. A *type* is an isomorphism class of a subgroup of $Q$. If $\tau$ is a type and $G$ is a torsion-free abelian group, the *$\tau$-radical* of $G$ is the pure subgroup $G[\tau] = \bigcap\{\text{Kernel } f \mid f: G \to C_\tau\}$, where $C_\tau$ is a subgroup of $Q$ of type $\tau$, while the *$\tau$-socle* of $G$ is the pure subgroup $G(\tau) = \sum\{\text{Image } f \mid f: C_\tau \to G\}$. If $M$ is a set of types, then $G[M] = \bigcap\{G[\tau] \mid \tau \in M\}$ and $G(M) = \sum\{G(\tau) \mid \tau \in M\}$. Define $r_G[M] = \operatorname{rank} G[M]$ and $r_G(M) = \operatorname{rank} G(M)$. An epimorphism $G \to H$ of torsion-free abelian groups is *balanced* if the induced homomorphism $G(\tau) \to H(\tau)$ is onto for each type $\tau$.

Two finite-rank torsion-free abelian groups $G$ and $H$ are *quasi-isomorphic* if $G$ is isomorphic to a subgroup of finite index in $H$ and *quasi-equal* if $QG$, the divisible hull of $G$, is equal to $QH$, and there is a nonzero integer $n$ with $nG \subseteq H$ and $nH \subseteq G$. The group $G$ is *strongly indecomposable* if $G$ quasi-equal to $H \oplus K$ implies that $H = 0$ or $K = 0$.

## 1. Strongly indecomposable groups of the form $G(A)$

An $n$-tuple $A = (A_1, \dots, A_n)$ of subgroups of $Q$ is *trimmed* if for each $i$, $A_i = \sum\{A_i \cap A_j \mid j \neq i\}$. As noted in [Ri-1], the assumption that $A$ is trimmed may be made without changing $G(A)$, since if $A' = (A'_1, \dots, A'_n)$ is the trimmed $n$-tuple defined by $A'_i = \sum\{A_i \cap A_j \mid j \neq i\}$, then $G(A') = G(A)$. In §1, most $n$-tuples are assumed to be trimmed. It is easy to verify that $A$ is trimmed if and only if each projection $G(A) \to A_i$ is onto.

**Lemma 1.1.** *Let $A = (A_1, \dots, A_n)$ be a trimmed $n$-tuple of subgroups of $Q$ and $G = G(A)$.*

(a) [AV-1, Corollary 1.2] *Inclusion of each $G_{ij}$ into $G$ induces a balanced epimorphism $\bigoplus\{G_{ij} \mid 1 \leq i < j \leq n\} \to G \to 0$.*

(b) [AV-3, Theorem 3] *The group $G$ is strongly indecomposable if and only if for each type $\sigma_i = \text{type}(A_i)$, $G/G[\sigma_i] \simeq A_i$. In this case, the endomorphism ring of $G$ is a subring of $Q$.*

(c) [AV-3, Theorem 6] *Suppose that $A$ and $B$ are trimmed $n$-tuples of subgroups of $Q$ and that $G = G(A)$ and $H = G(B)$ are strongly indecomposable. Then $G$ and $H$ are quasi-isomorphic if and only if $r_G[M] = r_H[M]$ for each set $M$ of types (equivalently, for each subset $M$ of $\{\text{type}(A_i) \mid 1 \leq i \leq n\}$ and each subset $M$ of $\{\text{type}(B_j) \mid 1 \leq j \leq n\}$).*

The last equivalence in part (c) is a stronger statement of the theorem which appears in [AV-3], but is a direct consequence of the proof given there.

E. L. Lady [La] has observed that if $G$ is a Butler group, then $G[\sigma]$ is the pure subgroup of $G$ generated by $\{G(\tau)|\tau \not\leq \sigma\}$. Therefore, in view of Lemma 1.1(a), if $A$ is an $n$-tuple of subgroups of $Q$ and $G = G(A)$, then $G[\sigma]$ is the pure subgroup of $G$ generated by $\{G_{ij}|\text{type}(G_{ij}) \not\leq \sigma\}$.

For each type $\sigma$ and $n$-tuple $A = (A_1, \dots, A_n)$ of subgroups of $Q$ there is an equivalence relation on $\{1, \dots, n\}$ given by $i$ is *$\sigma$-equivalent* to $j$ if $i = j$ or $G_{ij} \cap G[\sigma] \neq 0$, where $G = G(A)$. As noted above, $G[\sigma]$ is purely generated by $\{G_{kl}|\text{type}\, G_{kl} \not\leq \sigma\}$. Thus, $i$ and $j$ are $\sigma$-equivalent if and only if $G_{ij} \cap (\sum G_{kl}) \neq 0$, where the sum is over a set of indices $kl$ such that type $G_{kl} \not\leq \sigma$. A simple induction argument then shows that $i$ and $j$ are $\sigma$-equivalent if and only if there is a sequence of indices $i = i_1, i_2, \dots, i_m = j$ from $\{1, \dots, n\}$ such that $\text{type}(G_{kl}) \not\leq \sigma$ for each $(k, l) = (i_r, i_{r+1})$, $1 \not\leq r < m$. We employ both notions of $\sigma$-equivalence in the sequel. Because $\sigma$-equivalence depends directly on the entries in the $n$-tuple $A$, it is convenient to refer to *$\sigma$-equivalence classes in $A$* and to say $A_i$ is $\sigma$-equivalent to $A_j$ rather than $i$ is $\sigma$-equivalent to $j$. The history of $\sigma$-equivalence may be found in [AV-1, 2, 3].

If $A$ is trimmed, $G = G(A)$ is strongly indecomposable and $\sigma = \text{type}(A_i)$ for some $i$, then there are precisely two $\sigma$-equivalence classes in $A$, $\{A_i\}$ and $\{A_j|j \neq i\}$ (see Lemma 1.1(b)). More generally, if $\sigma$ is any type such that $\text{rank}\, G/G[\sigma] = 1$, then there are precisely two $\sigma$-equivalence classes partitioning $\{A_1, \dots, A_n\}$ [AV-1, Corollary 2.1.e]. In particular, this is the case if $G$ is quasi- isomorphic to a strongly indecomposable group of the form $G(B_1, \dots, B_n)$ and $\sigma = \text{type}(B_i)$ for some $i$.

The basic strategy for deriving isomorphism invariants from the quasi-isomorphism invariants given in Lemma 1.1(c) is to embed $G$ in

$$\bigoplus\{G/G[\sigma]|\text{rank}\, G/G[\sigma] = 1,\ \text{type}\, G/G[\sigma] = \sigma\},$$

identify each $G/G[\sigma]$ with a subgroup of $Q$ and show that a quasi-isomorphism of $G$ induces a "uniform" quasi-isomorphism on this direct sum of subgroups of $Q$. The precise meaning of "uniform" will take some time to develop.

There is a canonical way to produce a rank-1 factor of $G = G(A)$ from a subset of $\{A_1, \dots, A_n\}$. If $X$ is a proper nonempty subset of $\{1, \dots, n\}$, let $D_X[A]$ be the subgroup $\sum\{A_i \cap A_j|i \in X, j \notin X\}$ of $Q$. There is an epimorphism $\pi_X: G(A) \to D_X[A]$ given by $\pi_X(a_1, \dots, a_n) = \sum\{a_i|i \in X\}$, where $(a_1, \dots, a_n) \in G \subset A_1 \oplus \cdots \oplus A_n$ so that $\sum\{a_i|1 \leq i \leq n\} = 0$. The fact that $\pi_X$ is a well-defined epimorphism follows from Lemma 1.1(a) and the observation that $\pi_X(G \cap (A_i \oplus A_j)) \neq 0$ if and only if exactly one of $i, j$ belongs to $X$. If $Y$ is the complement of $X$ in $\{1, \dots, n\}$, then $\pi_Y = -\pi_X$.

If $G = G(A)$ and $\sigma$ is a type such that $\text{rank}\, G/G[\sigma] = 1$ and $\text{type}\, G/G[\sigma] = \sigma$, we may choose $X$ so that $\{A_i|i \in X\}$ is one of the two $\sigma$-equivalence classes in $A$. In this case we will write $\pi_\sigma = \pi_X$ and $D_\sigma[A] = D_X[A]$. Plainly, $D_\sigma[A]$ is unique and $\pi_\sigma$ is unique up to sign. Moreover, kernel $\pi_\sigma = G[\sigma]$ by the definition of $\sigma$-equivalence and the fact that image $\pi_\sigma = D_\sigma[A]$ has type $\sigma$. (See also, [AV-1], Theorem 1.10 and Corollary 1.11(c).)

Given $A$, a trimmed $n$-tuple of subgroups of $Q$, and $G = G(A)$, write $\Delta[A] = \{\sigma|\text{rank}\, G/G[\sigma] = 1 \text{ and } \text{type}\, G/G[\sigma] = \sigma\}$; and $\delta[A] = (D_\sigma[A]\|\sigma \in$

$\Delta[A]$), a vector of subgroups of $Q$. The vector $\delta[A]$ of subgroups of $Q$ can be multiplied by a rational number $q$ by multiplying each group in the vector by $q$. We then say that two such vectors $\delta$ and $\delta'$ are *equivalent* and write $\delta \equiv \delta'$ if $\delta = q\delta'$ for some nonzero rational $q$. The statement $\delta[A] \equiv \delta[B]$ subsumes $\Delta[A] = \Delta[B]$, since $\Delta[A]$ is the index set for $\delta[A]$. The idea of equivalent vectors of subgroups of $Q$ goes back to Richman in [Ri-1].

If $G = G(A)$, $H = G(B)$, and $f: G \to H$ is a quasi-isomorphism, then for each $\sigma \in \Delta[A] = \Delta[B]$ there is a commutative diagram

$$(d_\sigma(f)) \qquad \begin{array}{ccc} G & \xrightarrow{\pi_\sigma} & D_\sigma[A] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d_\sigma(f)} \\ H & \xrightarrow[\pi_\sigma]{} & D_\sigma[B] \end{array}$$

where $d_\sigma(f) \in Q$ is chosen to make the diagram commute. Note that since $-\pi_\sigma$ can be substituted for $\pi_\sigma$, it is possible to take $d_\sigma(f) > 0$. We will call the set of positive rationals, $\{d_\sigma(f) | \sigma \in \Delta[A]\}$, the *scalars associated with* $f$.

The fundamental step in our development of isomorphism invariants is to show that all the $d_\sigma(f)$'s are equal (uniformity).

**Lemma 1.2.** *Let $A = (A_1, A_2, A_3)$ and $B = (B_1, B_2, B_3)$ be trimmed 3-tuples of subgroups of $Q$. If $G(A)$ and $G(B)$ are quasi-isomorphic, then there is a permutation $\rho$ of $\{1, 2, 3\}$ such that $A_i \simeq B_{\rho(i)}$ for each $i$. Moreover, if $G(A)$ and $G(B)$ are strongly indecomposable, then $\Delta[A] = \Delta[B] = \{\text{type}(A_i) | 1 \leq i \leq 3\}$.*

*Proof.* By Lemma 1.1(a), $G = G(A)$ is the rank two homomorphic image of $G_{12} \oplus G_{23} \oplus G_{13} \simeq (A_1 \cap A_2) \oplus (A_2 \cap A_3) \oplus (A_1 \cap A_3)$ under the map induced by inclusion of $G_{ij}$ in $G$. It follows, using the fact that $A$ is trimmed, that the rank-1 factors of $G$ are, up to isomorphism, $A_1 = (A_1 \cap A_2) + (A_1 \cap A_3)$, $A_2$, $A_3$ and $A_1 + A_2 + A_3$. Similarly, the rank-1 factors of $H = G(B)$ are $B_1$, $B_2$, $B_3$ and $B_1 + B_2 + B_3$. It is an easy consequence that, after rearranging, $A_i \simeq B_i$ for $i = 1, 2, 3$. The last statement of the lemma follows from Lemma 1.1(b).

**Lemma 1.3.** *Let $A$ be an $n$-tuple of subgroups of $Q$ and $G = G(A)$. For any type $\sigma$, let $X_1, \ldots, X_r$ index the $\sigma$-equivalence classes in $A$ and let $B = (B_1, \ldots, B_r)$ with $B_j = (\sum\{A_i | i \in X_j\}) \cap (\sum\{A_i | i \notin X_j\})$. Then there is an isomorphism, $G/G[\sigma] \simeq G(B)$, induced by the map $G \to G(B)$ given by $(a_1, \ldots, a_n) \to (b_1, \ldots, b_r)$, where $b_j = \sum\{a_i | i \in X_j\}$. Moreover, $\Delta[B] = \{\tau \in \Delta[A] | \tau \leq \sigma\}$.*

*Proof.* The fact that $(a_1, \ldots, a_n) \to (b_1, \ldots, b_r)$ induces an isomorphism $G/G[\sigma] \simeq G(B)$ is a consequence of [AV-1, Theorem 1.10 and Corollary 1.11]. The result may also be shown via Lemma 1.1(a) and the definition of $\sigma$-equivalence. If $\tau \in \Delta[A]$ satisfies $\tau \leq \sigma$, then plainly $G[\sigma] \subseteq G[\tau]$; and $\tau \in \Delta[B]$ since if $H = G(B)$, then $H/H[\tau] \simeq (G/G[\sigma])/(G[\tau]/G[\sigma]) \simeq G/G[\tau]$. On the other hand, if $\tau \in \Delta[B]$, then $\tau \leq \sigma$ since any rank-1 factor of $G/G[\sigma]$ has type less than or equal to $\sigma$. As before, this implies $G/G[\tau] \simeq H/H[\tau]$ and $\tau \in \Delta[A]$.

The idea underlying the next lemma was first communicated to us by C. Megibben (see Theorem 4.7 in [HM]).

**Lemma 1.4.** *Let $A$ be a trimmed $n$-tuple of subgroups of $Q$, $n \geq 3$, and suppose that $X$ and $Y$ are the index sets for a partition of $A$. Denote $\tau = \text{type}\, D_X[A]$,* *where* $D_X[A] = (\sum\{A_i | i \in X\}) \cap (\sum\{A_i | i \in Y\})$, *and let*

$$\sigma = \text{type}\left(\sum\{A_i | i \in Y\}\right).$$

*Then for distinct indices* $i$, $j$ *in* $X$, $\text{type}(A_i \cap A_j) \not\leq \sigma$ *iff* $\text{type}(A_i \cap A_j) \not\leq \tau$.

*Proof*. Let $\sigma' = \text{type}(\sum\{A_i | i \in X\})$ and note that $\inf\{\sigma, \sigma'\} = \tau$. Clearly, $\rho = \text{type}(A_i \cap A_j) \not\leq \sigma$ implies $\rho \not\leq \tau$. On the other hand, since $\rho \leq \sigma'$, $\rho \leq \sigma$ implies $\rho \leq \inf\{\sigma', \sigma\} = \tau$.

**Proposition 1.5.** *Let* $A = (A_1, \ldots, A_n)$ *and* $B = (B_1, \ldots, B_n)$ *be trimmed n-tuples of subgroups of* $Q$. *Suppose* $G = G(A)$ *and* $H = G(B)$ *are strongly indecomposable and* $f: G \to H$ *is a quasi-isomorphism with associated scalars* $d_\sigma(f)$, $\sigma \in \Delta[A]$. *Then* $\Delta[A] = \Delta[B]$ *and there is a positive rational* $d \in Q$ *with* $d = d_\sigma(f)$ *for each* $\sigma \in \Delta[A]$.

*Proof*. The fact that $\Delta[A] = \Delta[B]$ is immediate, since $G(A)$ and $G(B)$ are quasi-isomorphic. For the rest we use induction on the integer $n$. By multiplying $f$ by a nonzero integer $m$ we may assume that $f(G) \subseteq H$, noting that $d_\sigma(mf) = md_\sigma(f)$ for each $\sigma \in \Delta[A]$. We first consider the special case $\Delta[A] = \{\text{type}(A_i) | 1 \leq i \leq n\}$. In this case, after rearranging, $A_i \simeq B_i$ for each $i$ by Lemma 1.1(b). It follows that for each $i \neq j$, $f(G_{ij}) \subseteq H_{ij}$, since $G_{ij} = \bigcap\{G[\text{type}(A_k)] | k \neq i, k \neq j\}$ (see [AV-3, Lemma 4]) and $f$ preserves radicals. If $(a, -a) \in G_{ij}$, then $f(a, -a) = (b, -b) \in H_{ij}$. However, if $\sigma = \text{type}(A_i)$, then by the diagram $(d_\sigma(f))$, $|a| d_\sigma(f) = |b|$. Similarly, if $\tau = \text{type}(A_j)$, then $|a| d_\tau(f) = |b|$. It follows that $d_\sigma(f) = d_\tau(f) = |b/a|$. Since $\sigma$ and $\tau$ were arbitrary elements of $\Delta[A]$, the proof is complete for this case. Included as special cases are $n = 2: A_1 = A_2 \simeq B_1 = B_2$ because $A$ and $B$ are trimmed; and $n = 3: \Delta[A] = \{\text{type}(A_i) | i = 1, 2, 3\} = \{\text{type}(B_i) | i = 1, 2, 3\} = \Delta[B]$ by Lemma 1.2.

Having treated the above special cases, we may assume that $n \geq 4$ and that there exists a type $\sigma_0 \in \Delta[A] = \Delta[B]$ such that $\sigma_0 \neq \text{type}(A_i)$ for $1 \leq i \leq n$ with $n \geq 4$. Without loss of generality, let $X = \{1, \ldots, k\}$ and $Y = \{k+1, \ldots, n\}$ index the two $\sigma_0$-equivalence classes in $A$, with $2 \leq k \leq n-2$, since $\sigma_0 \neq \text{type}(A_i)$ for each $i$. If $\sigma_X = \text{type}(\sum\{A_i | i \in X\})$, then the $\sigma_X$-equivalence classes in $A$ are the singleton sets $\{A_i\}$ for $i \in X$ and the set $\{A_i | i \in Y\}$. For $i \in X$, the singleton $\{A_i\}$ is clearly a $\sigma_X$-equivalence class, since $\text{type}(A_i \cap A_j) \leq \text{type}(A_i) \leq \sigma_X$ for all $j$. Moreover, by Lemma 1.4, the set $\{A_i | i \in Y\}$ is a $\sigma_X$-equivalence class since it is a $\sigma_0$-equivalence class. If $D = D_{\sigma_0}[A] = (\sum_{i \in X} A_i) \cap (\sum_{i \in Y} A_i)$, then by Lemma 1.3, $G' = G/G[\sigma_X] \simeq G(A_1, \ldots, A_k, D)$. Also by Lemma 1.3, if $A' = (A_1, \ldots, A_k, D)$, then $A'$ is trimmed and $\{\text{type}(D), \text{type}(A_i) | i \in X\} \subseteq \Delta[A'] \subseteq \Delta[A]$. Thus, $G'$ is strongly indecomposable by Lemma 1.1(b). Since the radical $G[\sigma_X]$ is preserved under quasi-isomorphism, $f$ induces a quasi-isomorphism from $G/G[\sigma_X]$ to $H/H[\sigma_X]$. By Lemma 1.3, $H/H[\sigma_X]$ is isomorphic to $G(B')$ for some (trimmed) tuple $B'$ of subgroups of $Q$; and $B'$ must contain exactly $k+1$ entries because $\text{rank}\, G/G[\sigma_X] = \text{rank}\, H/H[\sigma_X]$. As a consequence of the various quasi-isomorphisms discussed above, $f$ induces a quasi-isomorphism, which we denote by $f'$, from $G(A')$ to $G(B')$. Moreover, by Lemma 1.3, $\Delta[A'] = \Delta[B'] = \{\sigma \in \Delta[A] | \sigma \leq \sigma_X\}$. The induction hypothesis allows us to conclude that all the scalars associated with $f'$ are equal. However, by the definition of the isomorphisms $G' \simeq G(A')$ and $H' \simeq G(B')$ (Lemma 1.3), there is a commutative diagram,

$$\begin{array}{ccccccc} G(A) & \to & G(A') & \to & D_\sigma[A] & = & D_\sigma[A'] \\ \downarrow f & & \downarrow f' & & \downarrow d_\sigma(f) & & \downarrow d_\sigma(f') \\ G(B) & \to & G(B') & \to & D_\sigma[B] & = & D_\sigma[B'] \end{array}$$

whenever $\sigma \in \Delta[A'] = \Delta[B']$. This implies that the scalars $d_\sigma(f)$ are equal for $\sigma \in \{\text{type}(A_1), \dots, \text{type}(A_k), \text{type}(D)\}$. By replacing $\sigma_X$ with $\sigma_Y$ in the above argument, we can conclude that the scalars $d_\sigma$ are equal for

$$\sigma \in \{\text{type}(A_{k+1}), \dots, \text{type}(A_n), \text{type}(D)\}.$$

Since $\text{type}(D) = \sigma_0$ was an arbitrary element of $\Delta[A] \backslash \{\text{type}(A_1), \dots, \text{type}(A_n)\}$, we may conclude that all the scalars $d_\sigma(f)$ associated with $f$ are equal.

We call the rational $d$ in Proposition 1.5 the *scalar associated with* $f$.

**Theorem 1.6.** *Let $A$ and $B$ be trimmed $n$-tuples of subgroups of $Q$ such that $G = G(A)$ and $H = G(B)$ are strongly indecomposable. Then $G \simeq H$ if and only if $\delta[A] \equiv \delta[B]$ and $r_G[M] = r_H[M]$ for all sets of types $M \subset \Delta[A] = \Delta[B]$.*

*Proof.* Let $f: G(A) \to G(B)$ be an isomorphism, and let $d$ be the scalar associated with $f$. Then, since $f$ is an isomorphism, diagram $(d_\sigma(f))$ implies that $dD_\sigma[A] = D_\sigma[B]$ for all $\sigma \in \Delta[A]$. Thus, $d\delta[A] = \delta[B]$ and $\delta[A] \equiv \delta[B]$.

For the converse, note that by Lemma 1.1(c), the equalities $r_G[M] = r_H[M]$ imply that $G$ and $H$ are quasi-isomorphic. Let $\psi: G \to H$ be a quasi-isomorphism and let $d$ be the scalar associated with $\psi$ (Proposition 1.5). Let $c \neq 0$ in $Q$ be chosen so that $c\delta[A] = \delta[B]$. We will show that the quasi-isomorphism $\varphi: G \to H$ given by $\varphi = d^{-1}c\psi$ is an isomorphism. By the choice of $c$, for each $\sigma \in \Delta[A]$, there is a commutative diagram,

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & D_\sigma[A] \\ \varphi \downarrow & & \downarrow c \\ H & \xrightarrow[\pi_\sigma]{} & D_\sigma[B] \end{array}$$

with $c$ an isomorphism. If we take $\sigma = \text{type}(B_i)$, then $D_\sigma[B] = B_i$, since $B$ is trimmed. From the embedding $H \xrightarrow{\oplus\pi_\sigma} \bigoplus D_\sigma[B]$, it follows that $\varphi(G) \subseteq B_1 \oplus \cdots \oplus B_n$. However, since $\varphi(G)$ is quasi-equal to $H$, $m\varphi(G) \subseteq H$ for some positive integer $m$. Thus, $m\varphi(G) \subset H \cap m(B_1 \oplus \cdots \oplus B_n) = mH$ by purity of $H$ in $B_1 \oplus \cdots \oplus B_n$. Therefore, $\varphi(G) \subset H$. By symmetry, $\varphi^{-1}(H) = dc^{-1}\psi^{-1}(H) \subset G$ and $\varphi$ is an isomorphism.

*Remark.* The results above hold (with exactly the same proofs) if we substitute $\Delta[A] = \{\sigma = \text{type}(B_1) | B_1$ is a subgroup of $Q$ such that $G = G(A)$ is quasi-isomorphic to $G(B_1, B_2, \dots, B_n)$ for some $B_2, \dots, B_n$ subgroups of $Q\}$. Consequently, our invariants for strongly indecomposable $G(A)$'s are a direct generalization of those given by Richman for doubly incomparable groups (see [Ri-1, p. 176, conditions (1), (2) and (3)]).

## 2. Strongly indecomposable groups of the form $G[A]$

In this section we apply a quasi-isomorphism duality that sends groups of the form $G(A)$ to groups of the form $G[A]$, as described in [AV-4].

If $L$ is a finite distributive lattice of types, then an *L-group* is a pure subgroup (equivalently, homomorphic image) of a completely decomposable group with typeset contained in $L$. Define $B_L$ to be the category with quasi-equality classes of $L$-groups as objects and with morphism sets $Q \otimes_Z \mathrm{Hom}_Z(G, H)$. Isomorphism in this category is quasi-isomorphism and (quasi-equality classes of) strongly indecomposable $L$-groups are the indecomposable objects in $B_L$. For convenience, we often identify an $L$-group with its quasi-equality class in $B_L$.

Let $\alpha: L \to L^*$ be a lattice anti-isomorphism of finite lattices of types and write $\alpha(\tau) = \tau^*$. Such anti-isomorphisms are easy to find. Indeed, any finite distributive lattice is isomorphic to a lattice of subrings of $Q$ [Ri-3] and any subring of $Q$ is equal to a localization $Z_P$ of $Z$. Then the map $Z_P \to Z_{P^*}$, where $P^*$ is the complement of $P$ in the set of primes, provides a lattice anti-isomorphism.

There is a duality $F = F(\alpha): B_L \to B_L^*$ defined by $F(G) = H$ (more precisely, the quasi-equality class of $H$), where $QH = (QG)^* = \mathrm{Hom}_Q(QG, Q)$, $QH[\tau^*] = QG(\tau)^{\perp} = \{f \in (QG)^* | f(G(\tau)) = 0\}$, and $QH(\tau^*) = QG[\tau]^{\perp}$ for each $\tau \in L$. If $f: G \to H$, then $F(f) = f^*: H^* \to G^*$ is the restriction of $f^*: (QH)^* \to (QG)^*$ to $F(H)$. Moreover, $F$ sends a rank-1 group of type $\tau$ to a rank-1 group of type $\tau^*$; $G(A_1, \dots, A_n)$ to $G[F(A_1), \dots, F(A_n)]$ and conversely; and sends balanced sequences in $B_L$ to cobalanced sequences in $B_L^*$ and conversely. An exact sequence $0 \to G \to H \to K \to 0$ in $B_L$ is *balanced* if $0 \to G(\tau) \to H(\tau) \to K(\tau) \to 0$ is exact in $B_L$ for each $\tau \in L$ and *cobalanced* if $0 \to G[\tau] \to H[\tau] \to K[\tau] \to 0$ is exact in $B_L$ for each $\tau \in L$. We will call $F$ the *duality defined by* $\alpha$. A complete discussion of this duality may be found in [AV-4].

The duality $F$, being defined only up to quasi-isomorphism, will not suffice for isomorphism invariants. However, $F$ can be employed to dualize Proposition 1.5, a quasi-isomorphism result which is the key to the proof of Theorem 1.6.

The following notation is dual to that of §1, and the proofs of results corresponding to results in §1 can also be dualized (see [Le] for details). For $G = G[A] = (A_1 \oplus \cdots \oplus A_n)/\mathrm{Diag}(A)$, denote $G^{ij} = G/\mathrm{Image}(\bigoplus A_k | k \neq i, j)$ and let $\pi^{ij}: G \to G^{ij}$ be projection. Note that $G^{ij}$ is a rank-1 factor of $G$ with type equal to $\mathrm{type}(A_i + A_j)$. The $n$-tuple $A$ is *cotrimmed* if the image of each $A_i$ is pure in $G[A]$. If $A$ is an arbitrary $n$-tuple, there is a cotrimmed $n$-tuple $B$ with $G[A] = G[B]$.

For each type $\tau$ there is an equivalence relation, called *$\tau$-equivalence*, on $\{A_1, \dots, A_n\}$ defined by (1) $A_i$ is $\tau$-equivalent to $A_i$ and (2) $A_i$ is $\tau$-equivalent to $A_j$ if there is a sequence $i = i_1, i_2, \dots, i_k = j$ of distinct integers from $\{1, \dots, n\}$ such that $\tau \not\leq \mathrm{type}(G^{i_r i_{r+1}})$ for $1 \leq r < k$. This equivalence relation is distinct from that defined in §1.

Let $A$ be a cotrimmed $n$-tuple of subgroups of $Q$. If $X$ is a proper subset of $\{1, \dots, n\}$, define $D_X(A) = \bigcap\{A_i + A_j | i \in X, j \notin X\} \subseteq Q$. There is a pure embedding $\rho_X: D_X(A) \to G = G[A]$ given by $\rho_X(a) = (a_1, \dots, a_n) + \mathrm{Diag}(A)$, where $a_i = a$ if $i \in X$ and $a_i = 0$ otherwise. It is easy to check that if $Y$ is the complement of $X$ in $\{1, \dots, n\}$, then $D_X(A) = D_Y(A)$ and $\rho_X = -\rho_Y$.

In particular, if $\tau$ is a type such that $\mathrm{rank}(G(\tau)) = 1$, there are precisely two

$\tau$-equivalence classes, indexed, say, by $X$ and $Y$. If, in addition, $\operatorname{type} G(\tau) = \tau$, denote $D_X(A)$ by $D_\tau(A)$, and $\rho_X$ by $\rho_\tau$, noting that $\rho_\tau$ is uniquely defined up to sign. The case $\operatorname{type} G(\tau) = \tau$ occurs, for example, when $\tau = \operatorname{type} A_i$ for some $i$.

Define $\Delta(A) = \{\tau | \operatorname{rank} G(\tau) = 1 \text{ and } \operatorname{type} G(\tau) = \tau\}$ and $\delta(A) = (D_\tau(A) | \tau \in \Delta(A))$. The objective of this section is to prove the following theorem.

**Theorem 2.1.** *Let $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be cotrimmed $n$-tuples of subgroups of $Q$ such that $G = G[A]$ and $H = G[B]$ are strongly indecomposable. Then $G$ and $H$ are isomorphic if and only if $\delta(A) \equiv \delta(B)$ and $r_G(M) = r_H(M)$ for each $M \subseteq \Delta(A) = \Delta(B)$. Moreover, the endomorphism ring of $G$ is a subring of $Q$.*

The fact that the endomorphism ring of a strongly indecomposable group of the form $G[A]$ is a subring of $Q$ follows from Lemma 1.1(a) and the fact that the duality $F$ maps $G[A]$ to $G(F(A))$.

Given a trimmed $n$-tuple $A = (A_1, \ldots, A_n)$ of subgroups of $Q$, let $L(A)$ be the finite lattice of types generated by $\{\operatorname{type}(A_i) | 1 \le i \le n\}$ under sups and infs. As noted above, there is a lattice of types $L^*(A)$ and an anti-isomorphism $\alpha: L(A) \to L^*(A)$ with $\alpha(\tau) = \tau^*$. For each $i$, choose a subgroup $A_i^*$ of $Q$ such that $\operatorname{type}(A_i^*) = \operatorname{type}(A_i)^*$. Denote $A^* = (A_1^*, \ldots, A_n^*)$ and note that $L^*(A) = L(A^*)$ and that $F(A_i) = A_i^*$, where $F = F(\alpha): B_{L(A)} \to B_{L(A^*)}$ is the duality defined by $\alpha$. Let $L_Q(A)$ be the lattice of subgroups of $Q$ generated by the $A_i$'s and extend the mapping $A_i \to A_i^*$ to a correspondence $^*: L_Q(A) \to L_Q(A^*)$ by sending sums to intersections and intersections to sums. In general, the correspondence $^*: L_Q(A) \to L_Q(A^*)$ is onto, but not one-to-one. Clearly, the lattices $L_Q(A)$ and $L_Q(A^*)$ contain representatives for each type in $L(A)$ and $L(A^*)$, respectively. The correspondence $^*$ just described is called a *duality on $A$ induced by $\alpha$*.

**Lemma 2.2.** *Let $A$ be a trimmed $n$-tuple of subgroups of $Q$ and let $\alpha: L(A) \to L^*(A)$ be an anti-isomorphism of lattices of types. Then there is a duality $^*$ on $A$ induced by $\alpha$ such that $A^*$ is cotrimmed. Moreover, if $X$ is a subset of $\{1, \ldots, n\}$, then the maps $\pi_X: G(A) \to D_X[A]$ and $\rho_X: D_X(A^*) \to G[A^*]$ are dual. That is, $F(\pi_X) = \rho_X$, where $F$ is the duality defined by $\alpha$.*

*Proof*. The fact that $^*$ can be chosen so that $A^*$ is cotrimmed is a consequence of the fact that if $A$ is any $n$-tuple, then there is a cotrimmed $n$-tuple $B$ such that $G[A] = G[B]$. The rest of the lemma is a straightforward exercise in vector space duality, recalling that $F(G(A)) = G[A^*]$.

The key to the proof of Theorem 2.1 is the next proposition, which is the dual of Proposition 1.5.

**Proposition 2.3.** *Let $A$ and $B$ be cotrimmed $n$-tuples of subgroups of $Q$ such that $G[A]$ and $G[B]$ are strongly indecomposable and suppose that $f: G[A] \to G[B]$ is a quasi-isomorphism. Then there is a nonzero rational $d$ such that for each $\sigma$ in $\Delta(A) = \Delta(B)$, the following diagram commutes.*

$$\begin{array}{ccc} D_\sigma(A) & \xrightarrow{\rho_\sigma} & G[A] \\ d \downarrow & & \downarrow f \\ D_\sigma(B) & \xrightarrow[\rho_\sigma]{} & G[B] \end{array}$$

*Proof*. Let $\alpha: L(A) \to L(A^*)$ with $\alpha(\tau) = \tau^*$ be an anti-isomorphism and $L_Q(A) \to L_Q(A^*)$ a duality on $A$ induced by $\alpha$, as defined above. Since each $\text{type}(B_i) \in \Delta(B) = \Delta(A)$, $B_i$ is isomorphic to a group in $L_Q(A)$. Thus, we may choose $B_i^*$ in $L_Q(A^*)$ such that $\text{type}(B_i)^* = \text{type}(B_i^*)$ and set $B^* = (B_1^*, \dots, B_n^*)$. Then $F(f) = f^*: F(H) \to F(G)$ is a quasi-isomorphism with $F(G) = G(A^*) = G^*$ and $F(H) = G(B^*) = H^*$. By Proposition 1.5, there is a scalar $d^*$ such that the following diagram commutes.

$$
\begin{array}{ccc}
G^* & \overset{\oplus\pi_\sigma}{\to} & \bigoplus\{D_{\sigma^*}[A^*]: \sigma^* \in \Delta[G^*]\} \\
\uparrow f^* & & \uparrow d^* \\
H^* & \underset{\oplus\pi_\sigma}{\to} & \bigoplus\{D_{\sigma^*}[B^*]: \sigma^* \in \Delta[H^*]\}
\end{array}
\tag{2.3.1}
$$

Note that to make the diagram commute using a single scalar $d^*$, it is necessary to make the proper choice of sign on the various $\pi_\sigma$. Applying $F^{-1} = F(\alpha^{-1})$ and Lemma 2.2 to the diagram (2.3.1), we obtain a commutative diagram

$$
\begin{array}{ccc}
\bigoplus\{D_\sigma(A): \sigma \in \Delta(A)\} & \overset{\oplus\rho_\sigma}{\to} & G \\
\downarrow d & & \downarrow f \\
\bigoplus\{D_\sigma(B): \sigma \in \Delta(B)\} & \underset{\oplus\rho_\sigma}{\to} & H
\end{array}
$$

This completes the proof.

The proof of Theorem 2.1 is now a straightforward dualization of the proof of Theorem 1.6, using the above proposition and the dual of Lemma 1.1(c). The latter result appears as Corollary I in [AV-4].

## 3. Direct sums of strongly indecomposable groups

In this section isomorphism invariants for strongly indecomposable groups of the form $G(A)$ or $G[A]$ are extended to isomorphism invariants for arbitrary direct sums of such groups. This is not automatic, as direct sum decompositions into strongly indecomposable groups of the form $G(A)$ or $G[A]$ need not be unique up to isomorphism and order. There is, however, a "local" uniqueness which is described next.

Let $G$ and $H$ be finite rank torsion-free abelian groups such that $\text{End}(G)$ and $\text{End}(H)$ are (isomorphic to) subrings of $Q$. Then $G$ and $H$ are *isomorphic at a prime* $p$ if there are homomorphisms $f: G \to H$ and $g: H \to G$ such that both $fg$ and $gf$ are integers relatively prime to $p$—in particular $G$ and $H$ are quasi-isomorphic.

Let $C$ be a class of finite rank torsion-free abelian groups with endomorphism rings subrings of $Q$. A finite rank torsion-free abelian group $H$ with $\text{End}(H)$ a subring of $Q$ is *locally in* $C$ if for each prime $p$ in $\text{End}(H)$, there is a $G$ in $C$ such that $H$ is isomorphic to $G$ at $p$. The class $C$ is said to be *locally closed* if $\text{End}(H)$ is a subring of $Q$ and $H$ locally in $C$ implies that $H$ is isomorphic to a group in $C$.

**Lemma 3.1** [AHR]. *Let $C$ be a class of torsion-free abelian groups of finite rank with endomorphism rings subrings of $Q$.*

(a) *If $A \oplus B = \bigoplus\{C_i | i \in I\}$, with each $C_i$ in $C$, then $A = \bigoplus\{A_j | j \in J\}$ for some set $J$ with $\text{End}(A_j)$ a subring of $Q$ and $A_j$ locally in $C$ for each $j \in J$.*

(b) *Two groups $\bigoplus\{C_i | i \in I\}$ and $\bigoplus\{D_j | j \in J\}$, with each $C_i$ and $D_j$ in $C$, are isomorphic if and only if for each $G$ in $C$ and prime $p$, there is a bijection* *from* $\{i \in I | G$ *is isomorphic to* $C_i$ *at* $p\}$ *to* $\{j \in J | G$ *is isomorphic to* $D_j$ *at* $p\}$.

Two classes $C$ are of interest here: the class of strongly indecomposable groups of the form $G(A)$ and the class of strongly indecomposable groups of the form $G[A]$. In either case, the endomorphism rings of groups in $C$ are subrings of $Q$ (Lemma 1.1(b) and Theorem 2.1). Summands of direct sums of groups in $C$ are again direct sums of groups in $C$ provided that $C$ is locally closed, by Lemma 3.1(a). The next lemma shows that this is, in fact, the case for the classes we are considering.

**Lemma 3.2.** *The class of strongly indecomposable groups of the form* $G(A)$ *(respectively,* $G[A]$*) is locally closed.*

*Proof*. The proof for groups of the form $G(A)$ is given in [AV-2, Proposition 3.4]. The proof for groups of the form $G[A]$ is dual and, as such, is omitted.

As a consequence of Lemmas 3.1 and 3.2, isomorphism invariants for direct sums of strongly indecomposable groups of the form $G(A)$, respectively $G[A]$, can be given. First we must translate the isomorphism invariants (given in Theorems 1.6 and 2.1) for the strongly indecomposable $G(A)$'s and $G[A]$'s into a form suitable for this context. Our treatment follows that presented in [Ri-1]. We give an explicit argument for direct sums of strongly indecomposable groups of the form $G(A)$ and leave the analogous definitions and results for direct sums of strongly indecomposable groups of the form $G[A]$ to the reader.

If $A$ is a trimmed $n$-tuple of subgroups of $Q$, define a matrix $E_A$ with rows indexed by types $\sigma \in \Delta[A]$ and columns indexed by primes $p$ as follows. For each such $\sigma$ and $p$, let $e(\sigma, p)$ be the infimum of the set of all integers $m$ such that $p^m a/b \in D_\sigma[A]$ for some integers $a$ and $b$ relatively prime to $p$. Set $E_A = (e_{\sigma p})$, where $e_{\sigma p} = e(\sigma, p) - \min\{e(\tau, p) \neq -\infty | \tau \in \Delta[A]\}$. The $p$th column of $E_A$ specifies the relative positions of the groups $D_\sigma[A]_p$, the localization of $D_\sigma[A]$ at $p$, as subgroups of $Q$.

In contrast to doubly incomparable groups, the strongly indecomposable $G(A)$'s are not known to be classified up to isomorphism by equivalence classes of the vector $\delta[A]$ alone. To utilize Theorem 1.6, the vector $r[A] = (r_G[M] | M \subset \Delta[A])$, where $G = G(A)$ and $r_G[M] = \operatorname{rank} G[M]$, is also needed.

**Lemma 3.3.** *Let* $A$ *and* $B$ *be trimmed* $n$*-tuples of subgroups of* $Q$ *such that* $G(A)$ *and* $G(B)$ *are strongly indecomposable. Then the following are equivalent*:

(i) $G(A) \simeq G(B)$,

(ii) $\delta[A] \equiv \delta(B)$ *and* $r[A] = r[B]$,

(iii) $\Delta[A] = \Delta[B]$, $E_A = E_B$ *and* $r[A] = r[B]$.

*Proof*. (i) if and only if (ii) is Theorem 1.6. (ii) if and only if (iii) has the same proof as that of Theorem 2.1 of [Ri-1], noting that $r[A] = r[B]$ appears in both statements and that the $p$th column of $E_A$ specifies the relative position of the groups $D_\sigma[A]_p$ in $Q$.

Denote by $C$ the class of strongly indecomposable groups of the form $G(A)$. A direct sum of groups in $C$ may be specified by a subfamily $\varphi$ of $\{A | A$ is a trimmed $n$-tuple of subgroups of $Q$ with $G(A)$ strongly indecomposable$\}$. Let $G(\phi)$ denote $\bigoplus\{G(A) | A \in \phi\}$. Define $T_A = \{(\sigma, r) | \sigma \in \Delta[A], r = r[A]\}$ and associate with $\phi$ the set $\mathscr{T}[\phi] = \{T_A | A \in \phi\}$. Note that, by Lemma

1.1(c), the set $T_A$ determines $G(A)$ up to quasi-isomorphism. Next associate to each $T$ in $\mathscr{T}[\phi]$ and prime $p$ the multiset $\mathscr{E}_\phi(T, p) = \{\overline{E}_A(-, p) | A \in \phi$, and $T = T_A\}$, viewing $\overline{E}_A(-, p)$ as a function from $T$ to $Z \cup \{-\infty\}$ with $\overline{E}_A((\sigma, r), p) = E_A(\sigma, p) = e_{\sigma p}$. Here $E_A$ is the matrix $(e_{\sigma p} | \sigma \in \Delta[A]$, $p$ prime) defined prior to Lemma 3.3. Finally, regard $\mathscr{E}_\phi$ as a function from the Cartesian product $\mathscr{T}[\phi] \times \{\text{primes}\}$ into multisets of functions. As noted in [Ri-1], the function $\mathscr{E}_\phi$ does not record the fact that the functions $\overline{E}_A(-, p)$ and $\overline{E}_A(-, p')$ come from the same $A$ in $\phi$. Thus the family $\{G(A) | A \in \phi\}$ cannot be recovered from $\mathscr{T}[\phi]$ and $\mathscr{E}_\phi$, a reflection of the fact that nonisomorphic multisets of groups in $C$ can produce isomorphic direct sums.

**Theorem 3.4.** *Let $\phi$ and $\phi'$ be families of trimmed $n$-tuples $A$ of subgroups of $Q$ with $G(A)$ strongly indecomposable. Then $\bigoplus\{G(A) | A \in \phi\}$ is isomorphic to $\bigoplus\{G(A) | A \in \phi'\}$ if and only if $\mathscr{T}[\phi] = \mathscr{T}[\phi']$ and $\mathscr{E}_\phi = \mathscr{E}_{\phi'}$.*

*Proof.* In view of Lemma 3.1(b), $G(\phi) \simeq G(\phi')$ if and only if for each trimmed $n$-tuple $B$ such that $G = G(B)$ is strongly indecomposable and each prime $p$, there is a bijection from $\{A \in \phi | G$ is isomorphic at $p$ to $G(A)\}$ to $\{A' \in \phi' | G$ is isomorphic at $p$ to $G(A')\}$. But $G$ is isomorphic to $G(A)$ at $p$ if and only if $T_A = T_B$ and $\overline{E}_A(-, p) = \overline{E}_B(-, p)$, as can be seen by a proof analogous to that of Lemma 4.3 of [Ri-1]. An appeal to the definitions of $\mathscr{T}[\phi]$ and $\mathscr{E}_\phi$ establishes the theorem.

## 4. Isomorphism invariants for a class of mixed groups

In this section, machinery developed by F. Richman in [Ri-2] is used to extend the isomorphism invariants given in §3 to isomorphism invariants for a large class of mixed abelian groups. As in §3, the interested reader can dualize our results involving strongly indecomposable groups of the form $G(A)$ to the corresponding results for strongly indecomposable groups of the form $G[A]$.

Suppose that $G$ is a finite rank torsion-free abelian group with $\operatorname{End}(G)$ a subring of $Q$. Define $\tau$ to be the *inner type* of $G$, the infimum of the types of pure rank-1 subgroups of $G$. It is part of the folklore of the subject that $G \simeq U \otimes G'$, where $G'$ is a torsion-free group whose inner type is the *divisible part* of $\tau$ (for each $p$ the $p$th entry is 0 or $\infty$, according to whether $\tau$ is finite or infinite at $p$) and $U$ is a subgroup of $Q$ with type equal to the *reduced part* of $\tau$, i.e. there exists $u \in U$ and a height vector $h \in \tau$ such that $p$-height$(u) = h(p)$ whenever $h(p)$ is finite, and $p$-height$(u) = 0$ whenever $h(p)$ is infinite. Also, $G$ is strongly indecomposable if and only if $G'$ is strongly indecomposable. Finally, it is easy to check that $G = G(A)$ for some $n$-tuple $A$ of subgroups of $Q$ if and only if $G' = G(A')$ for some $n$-tuple $A'$ of subgroups of $Q$, where if $A' = (A'_1, \dots, A'_n)$, then $A = (A_1, \dots, A_n)$ with $A_i \simeq U \otimes A'_i$.

Let $C_0$ be the class of strongly indecomposable groups of the form $G(A)$, for $A = (A_1, \dots, A_n)$ a trimmed $n$-tuple of subgroups of $Q$ such that $\bigcap_i A_i$ is isomorphic to a subring of $Q$. This last condition guarantees that the inner type of $G(A)$ is *divisible*, i.e. the type of a subring of $Q$.

Each rank-1 torsion-free abelian group $X$ can be identified with a valuated cyclic group $U = \langle x \rangle$, for $0 \neq x \in X$, where the valuation on $U$ is given by the height of elements computed in $X$. This valuation on $U$ is a special valuation, as it is *gap-free*, $v_p(px) = v_p(x) + 1$. There is a notion of types of valuated torsion-free cyclic groups that extends the notion of types of rank-1 torsion-free abelian groups, as described in [Ri-2, p. 463]. For the remainder of this section, type refers to the type of a valuated torsion-free cyclic group.

Let $U$ be a cyclic torsion-free valuated group and $H$ a finite rank torsion-free group valuated by the height valuation. Then $U \otimes H$ is a valuated group with valuation given by $v_p(x \otimes h) = \infty$ if $v_p(h) = \infty$ and $v_p(x \otimes h) = v_p(p^n x)$ if $v_p(h) = n$ is finite.

Define $TF$ to be the class of valuated direct sums of valuated subgroups of torsion index in groups of the form $U \otimes G$, where

(4.1) $U$ is a cyclic torsion-free valuated group with reduced type, $G \in C_0$, and $\inf\{\text{type}(U), \text{ inner type}(G)\} = \text{type}(Z)$.

It follows from the above remarks that all direct sums of strongly indecomposable groups of the form $G(A)$ are included in $TF$. Furthermore, the endomorphism ring of each $U \otimes G$ is a subring of $Q$, since the endomorphism ring of each $G$ in $C_0$ is a subring of $Q$ by Lemma 1.1(b).

Define a category $W$ as follows: the objects of $W$ are valuated groups and a morphism in $W$ between valuated groups $A$ and $B$ is a homomorphism from a valuated subgroup of torsion index in $A$ to $B$. The category $W$ is a valuated group extension of a category defined by R. B. Warfield (see [Ri-4]).

Two valuated groups are isomorphic in $W$ if and only if they have isomorphic full-rank valuated subgroups [Ri-4, Theorem 2.1]. In particular, any element of $TF$ is $W$-isomorphic to a direct sum of valuated groups of the form $U \otimes G$ with $U$ and $G$ satisfying (4.1).

Define $\text{Mix}(TF)$ to be the class of mixed groups $H$ with height valuation such that $H$ has a valuated subgroup $K$ in $TF$ satisfying.

(4.2) $K$ is a nice full-rank subgroup of $H$ and $H/K$ has a *nice composition series* (there is a smooth ascending chain $0 \subset N_1 \subset \cdots \subset N_\alpha \subset \cdots$ of nice valuated subgroups of $H/K$ whose union is all of $H/K$ and such that $N_{\alpha+1}/N_\alpha$ is cyclic of prime order.

**Lemma 4.3** [Ri-2]. *Let $H$ be an element of* $\text{Mix}(TF)$ *and $K$ in $TF$ a valuated subgroup of $H$ satisfying the condition* (4.2). *Then the isomorphism class of $H$, as an abelian group, is completely determined by the Ulm invariants of $H$ and the isomorphism class of $K$ in $W$.*

*Proof*. Recall that $TF$ is the class of valuated direct sums of valuated groups taken from the class $D$ of subgroups of torsion index in groups of the form $U \otimes G$ satisfying (4.1). Each group in $D$ is countable, each valuated subgroup of a group in $D$ is nice [Ri-2, Theorem 8.3], and $D$ is closed under full-rank subgroups. Furthermore, the endomorphism ring of each $U \otimes G$ is a subring of $Q$, the endomorphism ring of each such group coincides with its endomorphism ring in $W$ [Ri-2, Theorem 8.1], and the class of $U \otimes G$'s is locally closed as a consequence of Lemma 3.2. An application of the global Azumaya theorem in the category $W$ given in [AHR, Theorem B], shows that $TF$ is closed in $W$ under valuated direct summands. Thus, [Ri-2, Corollary 6.6 and Theorem 5.3] can be applied to $TF$ to show that the group isomorphism class of $H$ is completely determined by its Ulm invariants and its $W$-isomorphism class. Finally, $H$ and $K$ are isomorphic in $W$, which completes the proof.

As remarked above, $K$ is isomorphic in $W$ to a direct sum of valuated groups of the form $U \otimes G$, with $U$ and $G$ satisfying (4.1) and each such $U \otimes G$

has $W$-endomorphism ring a subring of $Q$. The global Azumaya theorem in the category $W$ [ARH, Theorem D] gives a criterion for two direct sums of such groups to be isomorphic in $W$, analogous to that given in Lemma 3.1. Our final task is to specify a collection of isomorphism invariants for such sums, an analog in $W$ of the invariants for direct sums given in §3.

A direct sum of groups of the form $U \otimes G$ satisfying (4.1) is determined by a family $\mathscr{F}$ of pairs $(U, A)$, where $U$ and $G = G(A)$ satisfy (4.1). Denote by $G(\mathscr{F})$ the valuated direct sum of the valuated groups $\{U \otimes G(A) | (U, A) \in \mathscr{F}\}$. To each pair $(U, A) \in \mathscr{F}$, associate the set of triples $T[U, A] = \{(\text{type}(U), \sigma, r[A]) | \sigma \in \Delta[A]\}$, which specifies the quasi-isomorphism class of the abelian group $U \otimes G(A)$. Then to the family $\mathscr{F}$ we associate the set $\mathscr{T}_{\mathscr{F}} = \{T[U, A] | (U, A) \in \mathscr{F}\}$. Finally, to each element $T$ of $\mathscr{T}_{\mathscr{F}}$ and prime $p$, associate the multiset $\mathscr{E}_{\mathscr{F}}(T, p) = \{(\overline{U}, \varphi(A, p)) | (U, A) \in \mathscr{F}, T[U, A] = T\}$, where $\overline{U}$ is the class of all torsion-free cyclic valuated groups which are $W$-isomorphic to $U$ at $p$ (see Theorem 8.2 of [Ri-2]), and $\varphi(A, p)$ is the function from $T$ to $Z \cup \{-\infty\}$ defined by $\varphi(A, p)(\text{type}(U), \sigma, r[A]) = E_A(\sigma, p)$.

Let $H$ be an element of $\text{Mix}(TF)$ and $K, K'$ in $TF$ valuated subgroups of $H$ satisfying (4.2). Write $K = \bigoplus\{U \otimes G(A) | (U, A) \in \mathscr{F}\}$ and $K' = \bigoplus\{U \otimes G(A) | (U, A) \in \mathscr{F}'\}$. Since $K$ and $K'$ are isomorphic in $W$, Theorems 8.2 and 9.3 of [Ri-2] and Theorem 3.4 above imply $\mathscr{T}_{\mathscr{F}} = T_{\mathscr{F}'}$ and $\mathscr{E}_{\mathscr{F}} = \mathscr{E}_{\mathscr{F}'}$. Thus, these sets depend only on $H$ and we may use the notation $\mathscr{E}_H = \mathscr{E}_{\mathscr{F}}$ and $\mathscr{T}_H = \mathscr{T}_{\mathscr{F}}$.

**Theorem 4.4.** *Let $H$ and $H'$ be two elements of* $\text{Mix}(TF)$. *Then $H$ and $H'$ are isomorphic as abelian groups if and only if $\mathscr{T}_H = \mathscr{T}_{H'}$, $\mathscr{E}_H = \mathscr{E}_{H'}$ and $H$ and $H'$ have the same Ulm invariants.*

*Proof*. Let $K \subset H$ and $K' \subset H'$ be nice subgroups belonging to $TF$ and satisfying (4.2). By Lemma 4.3, $H$ and $H'$ are isomorphic as abelian groups if and only if $H$ and $H'$ have the same Ulm invariants and $K$ and $K'$ are isomorphic in $W$. Write $K = \bigoplus\{U \otimes G(A) | (U, A) \in \mathscr{F}\}$ and $K' = \bigoplus\{U \otimes G(A) | (U, A) \in \mathscr{F}'\}$, where $U$ and $G(A)$ satisfy (4.1) for each $(U, A)$ in $\mathscr{F} \cup \mathscr{F}'$. By [AHR, Theorem D], the groups $K$ and $K'$ are isomorphic in $W$ if and only if for each prime $p$ and valuated group $V \otimes G(B)$ satisfying (4.1), there is a bijection from $\{(U, A) \in \mathscr{F} | U \otimes G(A)$ is isomorphic in $W$ to $U \otimes G(B)$ at $p\}$ to $\{(U, A) \in \mathscr{F}' | U \otimes G(A)$ is isomorphic in $W$ to $U \otimes G(B)$ at $p\}$. However, by Theorems 8.2 and 9.3 of [Ri-2] and Theorem 3.4 above, $U \otimes G(A)$ is isomorphic in $W$ to $V \otimes G(B)$ at $p$ if and only if $T[U, A] = T[V, B]$ and $(\overline{U}, \varphi(A, p)) = (\overline{V}, \varphi(B, p))$. This completes the proof.

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Furthermore, we recommend that paper to the reader for an approach to the study of Butler groups via "equivalence theorems".

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