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TRACE FUNCTIONS IN THE RING OF FRACTIONS OF POLYCYCLIC GROUP RINGS

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Dedicated to the memory of I. N. Herstein

ABSTRACT. Let KG be the group ring of a polycyclic-by-finite group G over a field K of characteristic zero, R be the Goldie ring of fractions of KG, S be an arbitrary subring of $R_{n\times n}$. We prove that the intersection of the commutator subring $[S, S]$ with the center $Z(S)$ is nilpotent. This implies the existence of a nontrivial trace function in $R_{n\times n}$.

1

Let G be a polycyclic-by-finite group, K be a commutative field of characteristic zero. (Throughout this paper the term "field" is used in the sense of "skew field.") It is well known that the group ring KG is semiprime Noetherian and hence has a Goldie ring of fractions which we denote by R. Let S be a subring of the matrix ring $R_{n\times n}$, $Z(S)$ be its center and $[S, S]$ be the K-subalgebra of $R_{n\times n}$ generated by all the commutators $[x, y] = xy - yx$, $x, y \in S$. Our first main result is the following theorem which is motivated by R. Snider's article [1].

The *intersection*

$$[S, S] \cap Z(S) \tag{1.1}$$

is a *nilpotent ring* (see Theorem 3). (It is known that (1.1) is a subring; the proof of this fact is easy.)

We obtain immediately from Theorem 3 an affirmative answer to the question, posed by R. Snider in [1]: *Let G be a poly-Z-group, K be a commutative field of characteristic zero, D be the field of fractions of KG. Does*

$$[D, D] \neq D? \tag{1.2}$$

In particular, does

$$1 \notin [D, D]. \tag{1.3}$$

We see thus that the relations (1.2) and (1.3) do hold in D. Furthermore, this result implies that there exists a nontrivial trace function $t: D \to D/[D, D]$, defined by

$$t(d) = d + [D, D]$$

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and this function can be extended to a function $T\colon D_{n\times n} \to D/[D, D]$ by

$$t(d_{ij}) = \sum_i t(d_{ii}),$$

where (d_{ij}) is an arbitrary matrix from $D_{n\times n}$ (see [1–3]). Snider proved in [1] the relation (1.3) and hence the existence of nontrivial trace functions in the case when G is abelian-by-{infinite cyclic}.

The proof of Theorem 3 will be based on the following result (see Theorem 2):

Let K be an arbitrary commutative field and R be the ring of fractions of KG and

$$x_j \qquad (j = 1, 2, \dots, m) \tag{1.4}$$

be given nonzero elements of KG. Then there exists an ideal $C \subseteq KG$ such that the quotient ring $(KG)/C$ is a finite-dimensional K-algebra $K[\widetilde{G}]$, generated by a finite group $\widetilde{G}$ which is the image of G in $(KG)/C$. The homomorphism $\alpha\colon KG \to K[\widetilde{G}]$ is extended to a specialization $\theta\colon R \to K[\widetilde{G}]$, whose domain R_0 contains the elements (1.4). Furthermore the elements $\tilde{x}_j = \theta(x_j)$ $(j = 1, 2, \dots, m)$ are nonzero elements of $K[\widetilde{G}]$.

We will obtain one more result on specializations from R to algebras finite-dimensional over their central subfields; this is Theorem 1 and its corollary. Let H be a torsion-free normal subgroup of finite index in G such that H/H_1 is free abelian, where H_1 is the Fitting radical of H. Then Theorem 1 essentially states that *there exists a G-invariant ideal $A \subseteq KH_1$ and an ideal $B = (A)(KG)$ such that the quotient algebra $(KG)/B \simeq K[\overline{H}]$, where the group $\overline{H}$ is abelian-by-finite; the images $\overline{x}_j$ $(j = 1, 2, \dots, m)$ of the elements (1.4) are nonzero in $K[\overline{G}]$ and a given element x_j is regular in R iff its image $\overline{x}_j$ is regular in $K[\overline{G}]$*. Roseblade's Theorem 11.2.9 in [4] implies that the ideal B is localizable in KG.

It is worth remarking that Theorems 1 and 2 provide a method for construction of specializations from R into finite-dimensional algebras over the *same* field K; they should be compared with the Reduction Theorem (see [5, Theorem 4.1], [6], or [7, 4.2.1]) which gives specializations into algebras over fields of finite characteristic (see a discussion on this in the book [7, p. 137]).

2

Throughout this section let D be a field, generated by a polycyclic-by-finite group G. Thus, D is the field of fractions of its subring generated by the group G; we denote this subring by T. Thus, $T = Z[G]$ or $T = Z_p[G]$, depending on the characteristic of D.

Lemma 1. *Let (1.4) be given nonzero elements of T. Then there exists an ideal $A \subseteq T$ such that the quotient ring $T/A \simeq \prod_{r\times r}$, where $\prod$ is a finite field and the images of the elements (1.4) are invertible in T/A.*

Proof. Wehrfritz proved (see [8] or [7, 4.3.12]) that if R is a finitely generated subring of D, then there exists an ideal C of R of finite index with $\bigcap_{n=1}^{\infty} C^n = 0$; furthermore, every quotient ring R/C^n $(n = 1, 2, \dots)$ is finite. We apply this theorem to the subring S of D, generated by the elements x_j, x_j^{-1} $(j =$

$1, 2, \dots, m)$ and find an ideal $B \subsetneq S$ such that the ring $\overline{S} = S/B$ is finite. Since the images of the elements $x_j \in T$ $(j = 1, 2, \dots, m)$ are invertible in the finite ring $\overline{S}$ they must be invertible in the subring $T/(T \cap B)$. We see now that an arbitrary maximal ideal $A \supseteq (T \cap B)$ satisfies the conclusions of the assertion.

Remark. The current proof of Lemma 1 is somewhat shorter than the proof given in the original version of the paper, where Lemma 1 was obtained as one of the corollaries of the Reduction Theorem [5].

Now let $\Pi[G]$ be a domain, generated by a polycyclic-by-finite group G over a finite field Π. We see that $\Pi[G] \simeq Z_p[G_1]$, where G_1 is the subgroup of units of $\Pi[G]$, generated by G and the multiplicative group of Π. We see thus that Lemma 1 is true for this case, when $T = \Pi[G]$. We will use it in this form in the proof of Proposition 1 below.

Proposition 1. *Let K be an arbitrary commutative field, G be a torsion-free polycyclic group and let (1.4) be given nonzero elements of KG. Then there exists a maximal ideal $A \subseteq KG$ such that the quotient algebra $(KG)/A$ is generated over K by a finite group $\overline{G}$, the image of G under the natural homomorphism $(KG) \to (KG)/A$, and the images of the elements (1.4) in the ring $K[\overline{G}]$ are invertible.*

Proof. We reduce first the proof to the case when the field K is finitely generated. Indeed, assume that the theorem is proved for this special case. Let K_1 be the finitely generated subfield of K, such that K_1G contains all the elements (1.4) and $A_1 \subseteq K_1G$ be the ideal, which satisfies all the conclusions of the theorem. Since

$$(KG)/(KA_1) \simeq K \otimes ((K_1G)/A_1),$$

we obtain an ideal $KA_1 \subseteq KG$ such that the algebra $(KG)/(KA_1)$ is generated by a finite group and the images of the elements (1.4) are invertible in it. Since images of the elements (1.4) are invertible in the algebra $(KG)/(KA_1)$ they are invertible in every simple homomorphic image of it; this implies easily that an arbitrary maximal ideal $A \subseteq KG$, which contains KA_1, satisfies the conclusion of the theorem.

We can assume therefore that the field K is finitely generated. Let $K_0 \subseteq K$ be a finitely generated subring such that K is the field of fractions of K_0. We have the following representations for the elements (1.4)

$$x_j = \sum_i c_{ij} g_i \qquad (c_{ij} \in K;\ j = 1, 2, \dots, m). \tag{2.1}$$

An arbitrary coefficient c_{ij} in (2.1) has a representation

$$c_{ij} = a_{ij} b_{ij}^{-1} \qquad (a_{ij}, b_{ij} \in K_0). \tag{2.2}$$

We can find a maximal ideal $\mathscr{P} \subseteq K_0$ which defines a p-adic valuation in K_0 and contains no one of the elements a_{ij}, b_{ij} in (2.2). If $K_{\mathscr{P}}$ is the ring of fractions of K_0 with respect to $\mathscr{P}$ then all the coefficients c_{ij} in (2.1) belong to $K_{\mathscr{P}}$ and hence

$$x_j \in K_{\mathscr{P}}G \qquad (j = 1, 2, \dots, m).$$

Now consider the natural homomorphism

$$\varphi\colon K_{\mathscr{P}}G \to (K_{\mathscr{P}}G)/(\mathscr{P}), \tag{2.3}$$

where $(\mathscr{P})$ is the ideal of $K_{\mathscr{P}}G$, generated by the ideal $\mathscr{P} \subseteq K_{\mathscr{P}}G$. We observe that the ring $(K_{\mathscr{P}}G)/(\mathscr{P})$ is isomorphic to the group ring ΠG, where $\Pi \simeq (K_{\mathscr{P}})/(\mathscr{P})$ is a finite field and the elements $\varphi(x_j)$ $(j = 1, 2, \dots, m)$ are nonzero. Lemma 1 implies that there exists an ideal $B \subseteq \Pi G$ such that $(\Pi G)/B$ is a simple finite ring and the images $\overline{x}_j$ of the elements $\varphi(x_j)$ $(j = 1, 2, \dots, m)$ are invertible in the ring $(\Pi G)/B$. This together with the homomorphism (2.3) implies that there exists a homomorphism

$$\psi\colon K_{\mathscr{P}}G \to (\Pi G)/B$$

such that the elements

$$\overline{x}_j = \psi(x_j) \qquad (j = 1, 2, \dots, m)$$

are invertible in the ring $(\Pi G)/B$; clearly, $(\Pi G)/B$ is generated over Π by the finite group $\overline{G} = \psi(G)$, i.e.,

$$(\Pi G)/B \simeq \Pi[\overline{G}]. \tag{2.4}$$

Now take a minimal left ideal V in the matrix ring $\Pi[\overline{G}]$; this ideal affords a representation ρ of the group $\overline{G}$ and $\rho(\Pi\overline{G}) \simeq \Pi[\overline{G}]$. Let $\widetilde{K}_0$ be the p-adic completion of K_0, (π) be the maximal ideal of $\widetilde{K}_0$. Since G is polycyclic, the group $\overline{G}$ is solvable and Fong-Swan's Theorem implies that there exists a $\widetilde{K}_0\overline{G}$-module $\widetilde{V}$, free over $\widetilde{K}_0$, such that $\widetilde{V}/(\pi)\widetilde{V} \simeq V$. (In fact, this theorem is proven in [9, 22.1] for the case when the group is p-solvable and $\widetilde{K}_0$ contains a primitive root of degree $(G{:}1)$ from 1 but the last condition is unnecessary (see [10]); this can be shown also by a standard argument based on the Galois theory.) If λ is the representation afforded by $\widetilde{V}$ and $\lambda(\widetilde{K}_0\widetilde{G}) \simeq R$ then $R/(\pi R) \simeq \Pi[\overline{G}]$; it is important that the ideal πR is quasiregular in R.

There exists therefore a system of homomorphisms

$$\widetilde{K}_0 G \xrightarrow{\lambda_1} \widetilde{K}_0\overline{G} \xrightarrow{\lambda} R \xrightarrow{\lambda_2} \Pi[\overline{G}] \tag{2.5}$$

where λ_1 and λ_2 are homomorphisms of $\widetilde{K}_0$-algebras.

The homomorphism

$$\lambda_2\lambda\lambda_1\colon \widetilde{K}_0 G \to \Pi[\overline{G}] \tag{2.6}$$

maps the elements (1.4) into invertible elements $\overline{x}_j$ $(j = 1, 2, \dots, m)$. Since the kernel of λ_2 is a quasiregular ideal we conclude easily that the images of the elements (1.4) under the homomorphism

$$\lambda\lambda_1\colon \widetilde{K}_0 G \to R \tag{2.7}$$

are invertible elements of R. Since the field of fractions of K_0 coincides with K we see that the field of fractions of $\widetilde{K}_0$ is isomorphic to the p-adic completion $\widetilde{K}$ of K; homomorphism (2.7) is extended to a homomorphism of $\widetilde{K}$-algebras

$$\mu\colon \widetilde{K}G \to \widetilde{K}R. \tag{2.8}$$

Since the algebra $\widetilde{K}R$ is generated over $\widetilde{K}$ by the finite group $\overline{G}$, we see that the K-algebra $\mu(KG)$ is also generated over K by the group $\overline{G}$, i.e.

$$\mu(KG) \simeq K[\overline{G}]. \tag{2.9}$$

The homomorphism (2.8) carries out the elements (1.4) into invertible elements of $\widetilde{K}R$; we obtain therefore that the images of these elements under the homomorphism (2.9) are invertible elements of $K[\overline{G}]$. We found thus a homomorphism

$$KG \to (KG)/A \simeq K[\overline{G}]$$

which maps the elements (1.4) into invertible elements of $K[\overline{G}]$. We can assume, of course, that $K[\overline{G}]$ is simple, i.e. the ideal A is maximal. The proof is complete.

3

Let G be a polycyclic-by-finite group, $\rho(G)$ be the Fitting radical of G. It is not difficult to verify that G contains a torsion-free normal subgroup H of finite index such that the quotient group $H/\rho(H)$ is free abelian; it is more convenient to denote the subgroup $\rho(H)$ by H_1. We observe that if A is an arbitrary G-invariant ideal of KH_1 then $B = A(KG)$ is an ideal of KG and $(KG)/B \simeq K[\overline{G}]$, where the group $\overline{G}$ is an extension of the normal subgroup $\overline{H}_1$ by the group $\overline{G}/\overline{H}_1 \simeq G/H_1$. Thus, the algebra $K[\overline{G}]$ is isomorphic to an appropriate cross product of the algebra $K[\overline{H}_1]$ and the group G/H_1 and $K[\overline{H}] \simeq K[\overline{H}_1] * (H/H_1)$.

Theorem 1. *Let K be an arbitrary commutative field,* $\operatorname{char} K = p \geq 0$, *and assume that nonzero elements* (1.4) *of KG are given. Then there exists a G-invariant ideal $A \subseteq KH_1$ and an ideal $B = (A)KG$ such that*

(i) *The image $\overline{H}_1$ of the group H_1 under the natural homomorphism*

$$\varphi: KG \to (KG)/B \simeq K[\overline{G}]$$

is a finite p'-group and hence the group $\overline{H}$ is finite-by-free abelian. Furthermore, there exists a free abelian normal subgroup $N \subseteq \overline{G}$ of finite index, which is contained in $\overline{H}$ and central in it, and whose elements are linearly independent over K; hence $K[N]$ is isomorphic to the group ring KN.

(ii) *The images*

$$\overline{x}_j \qquad (j = 1, 2, \ldots, m)$$

of the elements (1.4) *are nonzero elements of $K[\overline{G}]$. Furthermore, a given element x_j in* (1.4) *is regular in KG if and only if its image $\overline{x}_j$ is regular in $K[\overline{G}]$.*

(iii) *The ideal B is localizable in KG.*

Proof. Let $g_1, g_2, \ldots, g_n$ be a transversal for H in G. The group ring KH contains no zero divisors of KG and we can form the ring R of fractions of KG with respect to the set $(KH)\backslash 0$. If D is the field of fractions of KH then $R \simeq D \otimes_{KH} KG$ and the transversal $g_1 = 1$, $g_2, \ldots, g_n$ gives a basis of the left vector space R over D.

We can assume without loss of generality that the set (1.4) contains regular elements and these are the first m_1 elements

$$x_1, x_2, \ldots, x_{m_1}. \tag{3.1}$$

These elements must be invertible in R; this implies easily that there exist nonzero elements x'_j $(j = 1, 2, \dots, m_1)$ in KG such that

$$\begin{aligned} y_j &= x'_j x_j \in (KH)\backslash 0 \quad (j = 1, 2, \dots, m_1), \\ x'_j x_j &= 0 \quad (j = m_1 + 1, \dots, m). \end{aligned} \tag{3.2}$$

Now let

$$x_j = \sum_{\alpha=1}^{n} c_{\alpha j} g_\alpha, \qquad x'_j = \sum_{\alpha=1}^{n} c'_{\alpha j} g_\alpha \tag{3.3}$$
$$(c_{\alpha j}, c'_{\alpha j} \in KH; \alpha = 1, 2, \dots, n; j = 1, 2, \dots, m)$$

be the representations of the elements x_j, x'_j $(j = 1, 2, \dots, m)$. Let

$$c_1, c_2, \dots c_r \tag{3.4}$$

be all the nonzero coefficients c_{ij} in (3.3). Let h_i $(i \in I)$ be a transversal for H_1 in H and

$$c_\beta = \sum_i \lambda_{i\beta} h_i \qquad (\lambda_{i\beta} \in KH_1; \beta = 1, 2, \dots, r). \tag{3.5}$$

Similarly, we have for the elements y_j in (3.2)

$$y_j = \sum_i \mu_{ij} h_i \qquad (\mu_{ij} \in KH_1; j = 1, 2, \dots, m_1). \tag{3.6}$$

Apply now Proposition 1 and find a maximal ideal $A \subsetneq KH$ such that $(KH)/A \simeq K[\widetilde{H}]$, where $\widetilde{H}$ is a finite group and for all the elements $\lambda_{i\beta}$, μ_{ij} from (3.5) and (3.6) the images of the elements

$$g_\alpha^{-1} \lambda_{i\beta} g_\alpha, \; g_\alpha^{-1} \mu_{ij} g_\alpha \qquad (\alpha = 1, 2, \dots, n) \tag{3.7}$$

are invertible in $K[\widetilde{H}]$. Let

$$A_1 = \bigcap_{\alpha=1}^{n} g_\alpha^{-1} A g_\alpha, \qquad A_2 = A_1 \cap KH_1, \qquad B = (A_2)KG. \tag{3.8}$$

Clearly, A_1 is a G-invariant ideal of KH and as a result of this A_2 is a G-invariant ideal of KH_1. Hence B is an ideal in KG. We have already pointed out that the quotient ring $(KG)/B \simeq K[\overline{G}]$, where the group $\overline{G}$ is an extension of the group $\overline{H}_1$ by the group $\overline{G}/\overline{H}_1 \simeq G/H_1$; the group G/H_1 is an extension of the free abelian group H/H_1 by the finite group G/H. On the other hand, we obtain from (3.8),

$$(KH_1)/(KH_1 \cap B) \simeq (KH_1)/A_2 \simeq (KH_1)/(KH_1 \cap A_1). \tag{3.9}$$

The first relation in (3.8) shows that the image of KH under the natural homomorphism $(KH) \to (KH)/A_1$ is a subdirect sum of the rings $(KH)/(g_\alpha^{-1} A g_\alpha)$ $(\alpha = 1, 2, \dots, n)$ which are isomorphic to the simple artinian ring $(KH)/A \simeq K[\widetilde{H}]$; a routine argument (see [5, Lemma 2.9]) implies that in fact $(KH)/A_1$ is a direct sum of rings isomorphic to $K[\widetilde{H}]$. This, together with the relation (3.9), implies first of all that the group $\overline{H}_1$ which is the image of H_1 under the homomorphism $KG \to (KG)/B$, is finite. Furthermore, the images of the elements (3.7) under the homomorphism $KH \to (KH)/A$ are invertible. This implies that the elements

$$\lambda_{i\beta}, \quad \mu_{ij} \tag{3.10}$$

become invertible modulo the ideals $g_\alpha^{-1}Ag_\alpha$ $(\alpha = 1, 2, \dots, n)$ and hence they are invertible modulo the ideal $A_1 = \bigcap_{\alpha=1}^n g_\alpha^{-1}Ag_\alpha$. Since the elements (3.10) belong to KH_1 the second and the third relations in (3.8) imply that they are invertible modulo the ideal B. We have already observed that the image of KH in $(KG)/B$ is isomorphic to

$$K[\overline{H}] \simeq K[\overline{H}_1] * (H/H_1). \tag{3.11}$$

Since the group H/H_1 is free abelian and all the elements

$$\overline{\lambda}_{i\beta}, \quad \overline{\mu}_{ij} \qquad (i = 1, 2, \dots, n)$$

are invertible in $K[\overline{H}_1]$ we conclude easily that the elements

$$\overline{c}_\beta = \sum_i \overline{\lambda}_{i\beta} h_i \qquad (\beta = 1, 2, \dots, r) \tag{3.5$'$}$$

and

$$\overline{y}_j = \sum_i \overline{\mu}_{ij} h_i \qquad (j = 1, 2, \dots, n_1) \tag{3.6$'$}$$

are regular in $K[\overline{H}]$. Since $K[\overline{G}]$ is a free $K[\overline{H}]$-module a routine argument shows that these elements are also regular in $K[\overline{G}]$. We obtain from (3.3)

$$\overline{x}_j = \sum_{\alpha=1}^n \overline{c}_{\alpha j} g_\alpha, \qquad \overline{x'_j} = \sum_{\alpha=1}^n \overline{c'_{\alpha j}} g_\alpha \tag{3.3$'$}$$
$$(\overline{c}_{\alpha j}, \overline{c'_{\alpha j}} \in K[\overline{H}], \ \alpha = 1, 2, \dots, n;\ j = 1, 2, \dots, m).$$

Since the elements (3.5′) are nonzero we obtain from (3.3′) that $\overline{x}_j \neq 0$ $(j = 1, 2, \dots, m)$. The relations

$$\overline{y}_j = \overline{x'_j}\overline{x}_j \qquad (j = 1, 2, \dots, m_1) \tag{3.2$'$}$$

imply, via the regularity of the elements (3.6′), that the elements $\overline{x}_j$ $(j = 1, 2, \dots, m_1)$ are regular in $K[\overline{G}]$. Similarly, the relations $\overline{x}_j\overline{x'_j} = 0$ $(j = m_1 + 1, \dots, m)$ imply that the elements $\overline{x}_j$ $(j = m_1 + 1, \dots, m)$ are zero divisors. We completed thus the proof of statement (ii).

To prove statement (iii) we observe that the ideal $B = (A_2)KG$, where A_2 is an ideal in the group ring of the nilpotent group H_1. Since G is polycyclic-by-finite Roseblade's Theorem 11.2.9 in [5] implies that B is localizable and (iii) is proved.

We have already shown that $(KH)/A_1$ is a direct sum of rings isomorphic to $(KH)/A_1 \simeq K[\widetilde{H}]$, where $\widetilde{H}$ is a finite group and A is a maximal ideal of KH. Hence the ring $(KH)/A_1$ is semisimple artinian. Furthermore, we have a homomorphism

$$K[\overline{H}] \to (K[\overline{H}])/\overline{A}_1 \simeq K[H]/A_1 \tag{3.12}$$

and the second relation (3.9) implies that

$$\overline{A}_1 \cap K[\overline{H}_1] = \overline{0}. \tag{3.13}$$

We have already shown that the group $\overline{H}_1$ is finite. Assume now that $\operatorname{char} K = p$ and prove that $p \nmid ([\overline{H}_1] : 1)$. Indeed, we observe first of all that the group $\overline{H}_1$ is nilpotent since H_1 is. Assume now that $p | (\overline{H}_1 : 1)$, let P be the Sylow p-subgroup of $\overline{H}_1$ and let $\overline{H}_1 \simeq P \times Q$. The elements $h - 1$ $(h \in P)$ generate a nonzero nilpotent ideal in $K[\overline{H}_1]$ because P is a normal subgroup of $\overline{H}_1$. Since $K[H]/A_1$ is semisimple we obtain from (3.12) that $(h - 1) \in \overline{A}_1$ $(h \in P)$ which contradicts (3.13). We proved thus that $\overline{H}_1$ is a finite p'-group.

To complete the proof we need the following assertion which is part of Lemma 3.2 in [5].

Lemma 2. *Let K be an arbitrary commutative field and $K[U]$ be a ring, generated by a group U, which is an extension of a finite group V be a polycyclic-by-finite group U/V. Assume also that $K[U] \simeq K[V] * (U/V)$. Then there exists a characteristic poly{infinite cyclic} subgroup $F \subseteq U$ of finite index such that the elements of F are linearly independent over K and, hence, $K[F] \simeq KF$.*

Proof. Let F be a poly-infinite cyclic characteristic subgroup of finite index in U. Then $F \cap V = 1$ and it is not difficult to verify that the elements of F are linearly independent over $K[V]$ and hence over K.

We complete now the proof of Theorem 1. Since $\overline{H}_1$ is finite, $\overline{H}/\overline{H}_1$ is free abelian, and H is finitely generated we conclude that $\overline{H}/Z$ is finite, where Z is the center of $\overline{H}$. The relation (3.11) implies, via Lemma 2, the existence of a characteristic subgroup $F \subseteq \overline{H}$ of finite index such that $K[F] \simeq KF$. Take now $N = F \cap Z$ and statement (iii) follows. The proof is completed.

Let R and $\overline{R}$ be the ring of fractions of KG and $K[\overline{G}]$ correspondingly. The ring $\overline{R}$ is isomorphic to the ring of fractions of $K[\overline{G}]$ with respect to the subring KN; since $(\overline{G} : N)$ is finite we conclude easily that $\overline{R}$ has a finite left dimension over the subfield $T = (KN)(KN)^{-1}$ and as a result of it is finite-dimensional over a central subfield $Z \subseteq T$. Furthermore, $\overline{R}$ is a homomorphic image of a suitable cross product $T * \overline{G}/N$; when $\operatorname{char} K = 0$ this cross product is semisimple artinian and so is $\overline{R}$.

If now nonzero elements (1.4) in R are given then

$$(3.14) \qquad x_j = a_j b_j^{-1} \qquad (a_j \in KG;\ b_j \in (KG)\backslash 0;\ j = 1, 2, \ldots, m).$$

We apply Theorem 1 to the set of elements $a_j, b_j \in KG$ $(j = 1, 2, \ldots, n)$ and obtain via well-known facts of the localization theory the following corollary.

Corollary. *Let nonzero elements* (3.14) *in R be given. Then there exists a localizable ideal $B \subseteq KG$ such that the elements* (3.14) *belong to the subring $S \subseteq R$, obtained by the localization of the ideal B, and $S/BS \simeq \overline{R}$, where $\overline{R}$ is the ring of fractions of $K[\overline{G}]$; the ring $\overline{R}$ has a finite dimension over its central subfield Z. Clearly, the ideal BS of S is quasiregular,*

Let Q be an arbitrary ring. We recall (see Cohn [11] and Passman [12]) that a specialization from Q on ring $\overline{Q}$ is a homomorphism $\alpha : Q_0 \to \overline{Q}$ such that $\ker \alpha$ is a quasiregular ideal of Q_0; Q_0 is the domain of α. Theorem 1 thus gives a method for constructing specializations from the K-algebra R to algebras finite-dimensional over their central subfields. Another system of specializations to algebras finite-dimensional over K is obtained from the following theorem.

Theorem 2. *Let* R *be the ring of fractions of* KG *and* (3.14) *be given nonzero elements of* R. *Then there exists an ideal* $C \subseteq KG$ *such that the quotient ring* $(KG)/C$ *is a finite-dimensional* K*-algebra, generated by a finite group* $\widetilde{G}$, *which is the image of* G *in* $(KG)/C$. *The homomorphism* $\alpha \colon KG \to K[\widetilde{G}]$ *is extended to a specialization* $\theta \colon R \to K[\widetilde{G}]$, *whose domain* R_0 *contains the elements* (3.14). *Furthermore,* $\tilde{x}_j = \theta(x_j)$ $(j = 1, 2, \dots, m)$ *are nonzero elements of* $K[\widetilde{G}]$.

Proof. Apply first Theorem 1 and its Corollary and obtain a homomorphism

$$\beta \colon KG \to (KG)/B \simeq K[\overline{G}]$$

such that the elements (3.14) belong to the subring $S \subseteq R$, the domain of the specialization $\pi \colon R \to \overline{R}$ which extend β, and

$$\overline{x}_j = \pi(x_j) \neq 0 \qquad (j = 1, 2, \dots, m). \tag{3.16}$$

We recall that $\overline{G}$ contains a free abelian normal subgroup N of finite index such that $K[N] \simeq KN$. Let T be the field of fractions of KN and $g_1, g_2, \dots, g_r$ be a system of elements of $\overline{G}$ which form a basis of the left vector space $\overline{R}$ over T. Let

$$x_j = \sum_{i=1}^{r} a_{ij} g_i \qquad (a_{ij} \in T;\ j = 1, 2, \dots, m). \tag{3.17}$$

Let $a_1, a_2, \dots, a_s$ be all the elements of KN which occur in the numerators and denominators of the nonzero elements a_{ij} in (3.17); clearly, every element a_k $(k = 1, 2, \dots, s)$ has a finite number of $\overline{G}$-conjugates. Then apply Proposition 1 and find an ideal $A \subseteq KN$ such that the quotient algebra $(KN)/A \simeq K[\widetilde{N}]$ where $\widetilde{N}$ is a finite group and

$$g^{-1} a_k g \notin A \qquad (k = 1, 2, \dots, s;\ g \in \overline{G}).$$

Let $A_1 = \bigcap_{g \in \overline{G}} A$ and $\overline{C} = A_1(K\overline{G})$. The same argument as in the proof of Theorem 1 shows that $\overline{C}$ contains no one of the elements (3.17) and $K[\overline{G}]/\overline{C} \simeq K[\widetilde{G}]$, where $\widetilde{G}$ is a finite group.

The ideal $\overline{C}$ is localizable in $K[\overline{G}]$; this can be verified in a straightforward way or obtained from Roseblade's theorem in [5]. We see therefore that the homomorphism $\gamma \colon K[\overline{G}] \to K[\widetilde{G}]$ is extended to a specialization $\tau \colon \overline{R} \to K[\widetilde{G}]$ and

$$\tilde{x}_j = \tau(\overline{x}_j) \neq 0. \tag{3.18}$$

Finally, let C be the inverse image of the ideal $\overline{C}$ in KG. Clearly,

$$(KG)/C \simeq (K[\overline{G}])/\overline{C} \simeq K[\widetilde{G}].$$

Furthermore, the natural homomorphism

$$\alpha \colon KG \to (KG)/C \simeq K[\widetilde{G}]$$

is a composition of two homomorphisms β and γ, which are extended to specializations π and τ correspondingly. We obtain from this (see [8, Chapter 6] or [11]) that α can be extended to a specialization $\tau\pi = \theta \colon \overline{R} \to K[\widetilde{G}]$, whose domain contains the elements (3.14). The assertion follows now from (3.16) and (3.18).

4

We will need in the proof of Theorem 3 the following fact:

Lemma 3. *Let D be a field, x be a given matrix from $D_{n\times n}$. Assume that D has a system of subrings T_i $(i\in I)$ such that $x\in(T_i)_{n\times n}$ for all $i\in I$ and*

(1) given any finite set of elements $M\subseteq D$ there is a T_i with $M\subseteq T_i$,

(2) each T_i has an ideal $U_i\neq T_i$ such that the image of the matrix x in the quotient ring

$(T_i)_{n\times n}/(U_i)_{n\times n}\simeq(T_i/U_i)_{n\times n}$ is nilpotent. Then the matrix x is nilpotent.

Proof. Assume that x is not nilpotent and hence

$$x^n\neq 0. \tag{4.1}$$

The powers of x are linearly dependent over D; there exists therefore elements $0\neq d_j\in D$ $(j=1,2,\dots,r)$ such that

$$\sum_{j=1}^{r} d_j x^{n_j}=0 \qquad (1\leq n_1<n_2<\cdots<n_r\leq n^2+1). \tag{4.2}$$

Find in the system of subrings T_i $(i\in I)$ a subring T and its ideal $U\neq T$ such that T contains all the elements d_j, d_j^{-1} $(j=1,2,\dots,r)$, all the nonzero entries of the matrix x (and x^n) and the inverses of these entries. Since all these elements are invertible in T and $U\neq T$, their images in T/U are nonzero. Let $\overline{X}$ denote the image of a subset $X\subseteq T_{n\times n}$ under the homomorphism $(T)_{n\times n}\to(T/U)_{n\times n}$. We see that the elements $\overline{d}_j$ are invertible in $(T/U)_{n\times n}$, the element $\overline{x}$ is nilpotent but

$$\overline{x}^n\neq\overline{0} \tag{4.3}$$

and

$$\sum_{j=1}^{r}\overline{d}_j\overline{x}^{n_j}=\overline{0}. \tag{4.2$'$}$$

Now let k be the smallest natural number such that $\overline{x}^k=0$. It follows from (4.3) that $k>n$. We multiply (4.2$'$) on the right by $\overline{x}^{k-n_1-1}$ and obtain that $\overline{d}_1\overline{x}^{k-1}=0$. Since $\overline{d}_1$ is invertible we see that $\overline{x}^{k-1}=0$ which contradicts (4.3). Thus assumption (4.1) leads to a contradiction, i.e., x is nilpotent.

The following fact is known (see [13, Lemma II.5.4]).

Lemma 4. *Let U be a finite-dimensional algebra over a field K of characteristic zero, Z be its center. Then the intersection $[U,U]\cap Z$ is a nilpotent ring.*

We can now prove our main result.

Theorem 3. *Let G be a polycyclic-by-finite group, K be a field of characteristic zero and R be the ring of fractions of KG. Let S be a subring of the matrix ring $R_{n\times n}$, Z be its center. Then the intersection $[S,S]\cap Z$ is a nilpotent ring.*

Proof. In order to prove Theorem 3 it is enough to prove that the ring $[S,S]\cap Z$ is nil because a nil subring of a matrix ring over the artinian ring R must be nilpotent.

Let thus $z \in ([S, S] \cap Z)$, where $S \subseteq R_{m\times m}$. There exist therefore elements $u_i, v_i \in S$ $(i = 1, 2, \ldots, r)$ such that

$$\sum_{i=1}^{r}[u_i, v_i] = z. \tag{4.4}$$

Pick in R an arbitrary finite subset which has a form

$$x_1, x_2, \ldots, x_k; \qquad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1} \tag{4.5}$$

and contains all the nonzero entries of the matrices u_i, v_i $(i = 1, 2, \ldots, r)$ (and of z). Apply Theorem 2 and find a subring $T \subseteq R$ and an ideal $U \subseteq T$ such that elements (4.5) belong to T and $T/U \simeq K[\widetilde{G}]$, where $K[\widetilde{G}]$ is a finite-dimensional algebra over K. Relation (4.4) implies the following relation in $(T/U)_{m\times m}$ for the images of the elements u_i, v_i, z $(i = 1, 2, \ldots, r)$:

$$\sum_{i=1}^{r}[\overline{u}_i, \overline{v}_i] = \overline{z}. \tag{4.4$'$}$$

Since the element $\overline{z}$ commutes with all the elements $\overline{u}_i$, $\overline{v}_i$ $(i = 1, 2, \ldots, r)$, we obtain from Lemma 4 that $\overline{z}$ is nilpotent. Lemma 3 now implies that z is nilpotent which completes the proof of Theorem 3.

Corollary 1. *Let the subring S in Theorem 3 be semiprime. Then* $[S, S] \cap Z = 0$.

Now let G be a residually torsion-free nilpotent group, K be a commutative field. Let

$$G = N_1 \supseteq N_2 \supseteq \cdots$$

be a series of normal subgroups in G such that every quotient group G/N_i $(i = 1, 2, \ldots)$ is torsion-free nilpotent and $\bigcap_{i=1}^{\infty} N_i = 1$. It is not difficult to define in G an order such that all the homomorphisms $G \to G/N_i$ are homomorphisms of ordered groups (see [14]). Let $K\langle G\rangle$ be the appropriate Malcev-Neumann power series ring and Δ be its subfield, generated by the group ring. We will give now a sketch of proof of the following result.

Proposition 2. (i) *If* $\operatorname{char} K = 0$ *then the conclusion of Theorem 3 is valid for an arbitrary subring $S \subseteq \Delta_{n\times n}$.*

(ii) *If K has an arbitrary characteristic then*

$$1 \notin [\Delta, \Delta]. \tag{4.6}$$

Proof. Let Δ_i be the field of fractions of the group ring $K(G/N_i)$. The results of [14] imply that for every given i there exists a specialization $\theta_i \colon \Delta \to \Delta_i$, extending the natural homomorphism $G \to G/N_i$ and that for every given elements of D,

$$x_1, x_2, \ldots, x_k; \qquad x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}$$

an index i_0 can be found such that for every $i \geq i_0$ these elements belong to the domain T_i of the specialization θ_i. Since Theorem 3 holds for the subrings of $(\Delta_i)_{n\times n}$ we obtain now easily from Lemma 3 the statement (i).

We prove now (ii). A routine argument reduces the proof to the case when the group G is finitely generated; we can assume also that the field K is algebraically closed. Assume that $1 \in [\Delta, \Delta]$, i.e. there exist nonzero elements $u_j, v_j \in \Delta$ $(j = 1, 2, \dots, s)$ such that

$$1 = \sum_{j=1}^{s} [u_j, v_j]. \tag{4.7}$$

Apply Proposition 2.8 in [15] and find a specialization $\pi: \Delta \to K[\widetilde{G}]$ such that $K[\widetilde{G}]$ is a simple algebra generated by a finite q-group $\widetilde{G}$ where q is an arbitrary prime number unequal to $\operatorname{char} K$ and the domain T of π contains all the elements u_i, v_i from (4.7). The relation (4.7) now yields the following relation in $K[\widetilde{G}]$,

$$\tilde{1} = \sum_{j=1}^{s} [\tilde{u}_j, \tilde{v}_j]. \tag{4.7'}$$

Since $K[\widetilde{G}]$ is a simple algebra over an algebraically closed field K and $q \neq \operatorname{char} K$ we obtain that $K[\widetilde{G}]$ is isomorphic to a matrix algebra of degree q^m over K. The relation (4.7′) however is impossible in the algebra $K_{q^m \times q^m}$ since the trace of the right side is zero whereas $T_r(\tilde{1}) = q^m \neq 0$. This completes the proof.

Since free groups and free soluble groups are residually torsion-free nilpotent, we obtain that Propositon 2 is valid for the universal field of fractions of free group rings or for Ore fields of fractions of group rings of free soluble groups.

The truth of (4.6) for a ring of fractions R of a ring $(KG)/P$, where G is a finitely generated nilpotent group, $\operatorname{char} K = 0$ and P is a prime ideal of KG, was established by M. Lorenz in [16].

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