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A DEFORMATION OF TORI WITH CONSTANT MEAN CURVATURE IN $\mathbb{R}^3$ TO THOSE IN OTHER SPACE FORMS

MASAAKI UMEHARA AND KOTARO YAMADA

ABSTRACT. It is shown that tori with constant mean curvature in $\mathbb{R}^3$ constructed by Wente [7] can be deformed to tori with constant mean curvature in the hyperbolic 3-space or the 3-sphere.

INTRODUCTION

In this paper, we will construct tori with constant mean curvature in the hyperbolic 3-space. To be more precious, let T^2 be a torus and $f : T^2 \to \mathbb{R}^3$ be an immersion with constant mean curvature constructed by Wente [7]. Let

$$\mathbb{R}^3(k) = \begin{cases} \mathbb{R}^3 & (\text{if } k \geq 0), \\ \{x \in \mathbb{R}^3 : \sum_{i=1}^3 (x^i)^2 < \frac{1}{|k|}\} & (\text{if } k < 0) \end{cases}$$

be the Riemannian 3-manifold with the Riemannian metric

$$g_k = \left(\frac{2}{1 + k\sum_{i=1}^3 (x^i)^2} \right)^2 \sum_{i=1}^3 (dx^i)^2$$

of constant sectional curvature k. We will show that if f is generic, then for a sufficiently small $\varepsilon > 0$ there exists a local 1-parameter family of immersions $\{f_k : T^2 \to \mathbb{R}^3(k)\}_{|k|<\varepsilon}$ $(f_0 = f)$ with the same constant mean curvature. It should be noted that the induced metrics $\{f_k^* g_k\}_{|k|<\varepsilon}$ on T^2 in this case may not be conformally equivalent to each other. Recently Walter [6] gave another construction of tori with constant mean curvature in the hyperbolic 3-space. But our construction is quite different and depends very much on an idea "deformation of Lie groups".

Wente's construction in [7] is based on doubly periodic solutions of the sinh-Gordon equation on $\mathbb{R}^2$. Even if $k \neq 0$, solutions of the sinh-Gordon give rise to immersions $f_k : \mathbb{R}^2 \to \mathbb{R}^3(k)$ with constant mean curvature. Though f_k may not be doubly periodic, it induces a representation $\rho_k : \pi_1(T^2) \to G_k$ such

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that for any $a \in \pi_1(T^2)$, $\rho_k(a)$ preserves the image of f_k, where

$$G_k = \begin{cases} SO(4) & (\text{if } k > 0), \\ SO(3) \ltimes \mathbb{R}^3 & (\text{if } k = 0), \\ SO^+(3, 1) & (\text{if } k < 0), \end{cases}$$

which are the identity components of the isometry groups of the 3-dimensional space forms. The necessary and sufficient condition for the image of f_k to be closed can be described in terms of the representation ρ_k. To construct a family of doubly periodic immersions, one difficulty arises from the fact that the isometry groups G_k for $k > 0$, $k = 0$, and $k > 0$ are quite different from each other.

In §§1–3, we introduce a differentiable structure on the set $I = \{(k, E) : k \in \mathbb{R}, E \in G_k\}$ such that the family of representations $\rho_k : \pi_1(T^2) \to G_k \subset I$ $(k \in \mathbb{R})$ is smooth with respect to k. In §4, a criterion for the image of f_k to be closed can be taken depending smoothly on k, by virtue of the differentiable structure. Using this criterion, the existence of a deformation f_k with the desired properties are shown in the last section.

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1. Decompositions of isometries

Let $M^3(k)$ be a complete simply connected Riemannian 3-manifold of constant sectional curvature k, and G_k the identity component of the isometry group of $M^3(k)$.

First, we suppose $k > 0$. In this case, $M^3(k)$ is the Euclidean sphere defined by

$$M^3(k) = \left\{ {}^t(x^1, x^2, x^3, t) \in \mathbb{R}^4 : \sum_{i=1}^{3}(x^i)^2 + t^2 = \frac{1}{k} \right\} \tag{1.1}$$

and $G_k = SO(4)$. It is well known that for each $E \in SO(4)$, there exists a matrix $P \in SO(4)$ such that

$$P^{-1} \circ E \circ P = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\nu & -\sin\nu \\ 0 & 0 & \sin\nu & \cos\nu \end{pmatrix}, \tag{1.2}$$

where $e^{\pm i\theta}$ and $e^{\pm i\nu}$ are the eigenvalues of the matrix E.

Next we consider the case $k < 0$. In this case, $M^3(k)$ is the hyperboloid in the Minkowski 4-space $\mathbb{L}^4$ with the induced metric. That is,

$$M^3(k) = \left\{ {}^t(x^1, x^2, x^3, t) \in \mathbb{L}^4 : \sum_{i=1}^{3}(x^i)^2 - t^2 = \frac{1}{k},\ t > 0 \right\} \tag{1.3}$$

and $G_k = SO^+(3, 1)$. Unlike the case $SO(4)$, not all matrices in $SO^+(3, 1)$ can be normalized.

Lemma 1.1. *Let*

$$N = \{ A \in SO^+(3, 1) : \textit{all of the eigenvalues of } A \textit{ are } 1 \}. \tag{1.4}$$

Then for any matrix $E \in SO^+(3, 1)\backslash N$, *there exists* $P \in SO^+(3, 1)$ *such that*

$$P^{-1} \circ E \circ P = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\nu & \sinh\nu \\ 0 & 0 & \sinh\nu & \cosh\nu \end{pmatrix}, \tag{1.5}$$

where $e^{\pm i\theta}$ *and* $e^{\pm\nu}$ *are the eigenvalues of the matrix* E.

Proof. Identify a point $X = {}^t(x^1, x^2, x^3, t) \in \mathbb{L}^4$ with a 2×2-matrix

$$X = \begin{pmatrix} x^3 + t & x^1 + ix^2 \\ x^1 - ix^2 & -x^3 + t \end{pmatrix}.$$

Then $SO(3, 1)$ is isomorphic to $PSL(2, \mathbb{C})$ by the 2-fold covering ρ: $SL(2, \mathbb{C}) \to SO^+(3, 1)$ defined by $\rho(a)X = a \circ X \circ {}^t\bar{a}$. It is easy to check that

$$\rho\begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix} = \begin{pmatrix} \cos u & -\sin u & 0 & 0 \\ \sin u & \cos u & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{pmatrix}, \tag{1.6}$$

where $z = u + iv$. On the other hand, $\rho^{-1}(N) \subset SL(2, \mathbb{C})$ consists exactly of matrices which cannot be diagonalized. Combining these two facts, we obtain the lemma. □

Finally we consider the case $k = 0$. The following lemma holds:

Lemma 1.2. *Let* E *be an isometry of* $\mathbb{R}^3(0)$ *written as* $E = A + \mathbf{c}$ $(A \in SO(3), \mathbf{c} \in \mathbb{R}^3)$, *and suppose* $A \neq \mathrm{id}$. *Then there exists an isometry* P *such that*

$$P \circ E \circ P^{-1}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \tau\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.7}$$

for all $x = {}^t(x^1, x^2, x^3) \in \mathbb{R}^3(0)$. *Moreover, if* E *has such a decomposition, then the* $e^{\pm i\theta}$ *are the eigenvalues of the matrix* A *and* $\pm\tau = \langle \mathbf{c}, \mathbf{e}\rangle$, *where* $\mathbf{e}$ *is the unit eigenvector of* A *corresponding to the eigenvalue* 1 *and* $\langle\ ,\ \rangle$ *denotes the canonical inner product of* $\mathbb{R}^3$.

Proof. Let $P = P_0 + \mathbf{p}$ $(P_0 \in SO(3, 1), \mathbf{p} \in \mathbb{R}^3)$. Then E has the expression $P \circ E \circ P^{-1} = R_\theta + \tau\mathbf{e}_3$ of (1.7) $(R_\theta \in SO(3, 1), \mathbf{e}_3 = {}^t(0, 0, 1))$ if and only if

$$P_0^{-1} \circ R_\theta \circ P_0 = A, \tag{1.8}$$

$$R_\theta\mathbf{p} + \tau\mathbf{e}_3 - \mathbf{p} = P_0\mathbf{c}. \tag{1.9}$$

It is obvious that P_0 satisfying (1.8) exists. Hence, it suffices to show that the existence of $\mathbf{p}$ satisfying (1.9). Note that if such a $\mathbf{p}$ exists, then the $e^{\pm i\theta}$ are the eigenvalues of A by (1.8) and, by (1.9),

$$\tau = \langle \tau\mathbf{e}_3, \mathbf{e}_3\rangle = \langle P_0\mathbf{c}, \mathbf{e}_3\rangle = \langle \mathbf{c}, P_0^{-1}\mathbf{e}_3\rangle = \langle \mathbf{c}, \mathbf{e}\rangle.$$

Now we put $P_0\mathbf{c} = {}^t(\alpha^1, \alpha^2, \alpha^3)$. Then (1.9) is equivalent to

$$\tau = \alpha^3 \quad \text{and} \quad \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}\begin{pmatrix} p^1 \\ p^2 \end{pmatrix} = \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix}.$$

Consequently, the desired τ and $\mathbf{p}$ exist if $\theta \notin 2\pi\mathbb{Z}$. □

2. The stereographic projections

Recall that

$$\mathbb{R}^3(k) = \begin{cases} \mathbb{R}^3 & (\text{if } k \geq 0), \\ \{x \in \mathbb{R}^3 : \sum_{i=1}^3 (x^i)^2 < \frac{1}{|k|}\} & (\text{if } k < 0) \end{cases}$$

is the Riemannian 3-manifold with the Riemannian metric

$$g_k = \left(\frac{2}{1 + k\sum_{i=1}^3 (x^i)^2} \right)^2 \sum_{i=1}^3 (dx^i)^2$$

of constant sectional curvature k.

Note that when $k > 0$, $\mathbb{R}^3(k)$ can be understood as the image of the stereographic projection of $M^3(k)$ defined in (1.1) into the (x^1, x^2, x^3)-plane from the south pole $(0, 0, 0, -1/\sqrt{k})$. Similarly, when $k < 0$, $\mathbb{R}^3(k)$ is also the image of the stereographic projection of $M^3(k)$ defined in (1.3) into the (x^1, x^2, x^3)-plane from the south pole $(0, 0, 0, -1/\sqrt{|k|})$.

Let ψ_k $(k \neq 0)$ denote these stereographic projections. Then ψ_k and ψ_k^{-1} are given, independently of the sign of k, by

$$\psi_k(x^1, x^2, x^3, t) = \frac{1}{\sqrt{|k|}t + 1}(x^1, x^2, x^3), \tag{2.1}$$

$$\psi_k^{-1}(x^1, x^2, x^3) = \frac{2}{1 + kr^2}\left(x^1, x^2, x^3, \frac{1 - kr^2}{2\sqrt{|k|}} \right), \tag{2.2}$$

where $r^2 = \sum_{i=1}^3 (x^i)^2$. The Riemannian metric g_k of $\mathbb{R}^3(k)$ is nothing but the one induced from the canonical metric of $M^3(k)$ by ψ_k. Therefore, isometries of $M^3(k)$ can be regarded as isometries of $\mathbb{R}^3(k)$.

Now we interpret the normalized isometries (1.2), (1.5), and (1.7) in terms of the canonical coordinate system of $\mathbb{R}^3(k)$. If $k > 0$, then a matrix $E \in SO(4)$ of the form (1.4) is expressed as

$$\psi_k \circ E \circ \psi_k^{-1} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \cos\nu \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \frac{\mu(1 - kr^2)}{2\sqrt{k}} \begin{pmatrix} 0 \\ 0 \\ -\sin\nu \end{pmatrix}, \tag{2.3a}$$

where

$$\mu = 2\{ 2\sqrt{k}x^3 \sin\nu + \cos\nu(1 - kr^2) + (1 + kr^2) \}^{-1}. \tag{2.3b}$$

Note that the singular point of $\psi_k \circ E \circ \psi_k^{-1}$ corresponds to the point in $M^3(k)$ which is mapped to the south pole by E.

On the other hand, if $k < 0$, then a matrix $E \in SO^+(3, 1)$ of the form (1.5) is expressed as

$$\text{(2.4a)} \qquad \psi_k \circ E \circ \psi_k^{-1} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \cosh\nu \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \frac{\mu(1-kr^2)}{2\sqrt{|k|}} \begin{pmatrix} 0 \\ 0 \\ \sinh\nu \end{pmatrix},$$

where

$$\text{(2.4b)} \qquad \mu = 2\{ 2\sqrt{|k|}x^3 \sinh\nu + \cosh\nu(1-kr^2) + (1+kr^2) \}^{-1}.$$

In (2.3) and (2.4), we now put

$$\text{(2.5)} \qquad \tau = \begin{cases} -\frac{\sin\nu}{2\sqrt{k}} & (\text{if } k > 0), \\ \frac{\sinh\nu}{2\sqrt{|k|}} & (\text{if } k < 0), \end{cases}$$

and denote $\psi_k \circ E \circ \psi_k^{-1}$ by $T_k(\theta, \tau)$. To determine ν uniquely from τ, we assume that $|\nu| < \pi/2$ if $k > 0$. Then $T_k(\theta, \tau)$ is expressed, independently of the sign of k, by

$$\text{(2.6a)} \quad T_k(\theta, \tau) \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \tilde{\mu}_k \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & (1-4k\tau^2)^{1/2} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \tilde{\mu}_k(1-kr^2) \begin{pmatrix} 0 \\ 0 \\ \tau \end{pmatrix},$$

where $|\tau| < 1/2\sqrt{|k|}$ for $k > 0$, and

$$\text{(2.6b)} \qquad \tilde{\mu}_k = 2\{ -4k\tau x^3 + (1-4k\tau^2)^{1/2}(1-kr^2) + (1+kr^2) \}^{-1}.$$

We also define $T_k(\theta, \tau)$ and $\tilde{\mu}_k$ by (2.6a) and (2.6b) even when $k = 0$. Then $\tilde{\mu}_0 = 1$ and $T_0(\theta, \tau)$ is identical with normalized isometry given by (1.7). Thus, for each $k \in \mathbb{R}$, we call $T_k(\theta, \tau)$ a normal form of the isometry of $\mathbb{R}^3(k)$.

3. A differentiable structure of $\mathscr{T}$

Recall that G_k is the identity component of the isometry group of $M^3(k)$, namely

$$G_k = \begin{cases} SO(4) & (k > 0), \\ SO(3) \ltimes \mathbb{R}^3 & (k = 0), \\ SO^+(3, 1) & (k < 0). \end{cases}$$

Let $\mathscr{T} = \{(k, E) : E \in G_k\}$. Then each of the subsets

$$\mathscr{T}^+ = \{(k, E) \in \mathscr{T} : k > 0\} = (0, \infty) \times SO(4),$$
$$\mathscr{T}^- = \{(k, E) \in \mathscr{T} : k < 0\} = (-\infty, 0) \times SO^+(3, 1)$$

has the canonical differentiable structures as a product. In this section we shall prove the following theorem.

Theorem 3.1. *There exists a differentiable structure on $\mathscr{I}$ whose restriction to $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) is compatible with the canonical product structure of $\mathscr{I}^+$ (resp. $\mathscr{I}^-$).*

Let $\widetilde{\mathscr{I}}$ be a subset of $\mathscr{I}$ defined by

$$\widetilde{\mathscr{I}} = \mathscr{I} \backslash \{(k, E) =: k > 0 \text{ and } E \in SO(4) \text{ maps the north pole of } M^3(k) \text{ to the south pole}\}.$$

For each $(k, E) \in \widetilde{\mathscr{I}}$, we put

$$w^i(E) = \psi_k^i \circ E \circ \psi_k^{-1}(0) \qquad (i = 1, 2, 3), \tag{3.1}$$

$$w^{jl}(E) = \left[\frac{\partial}{\partial x^l}(\psi_k^j \circ E \circ \psi_k^{-1})\right](0) \qquad (j, l = 1, 2, 3), \tag{3.2}$$

and define a map $\mathscr{W} : \widetilde{\mathscr{I}} \to \mathbb{R}^{13}$ by

$$\mathscr{W}(k, E) = (k, w^i(E), w^{jl}(E))_{i,j,l=1,2,3} \in \mathbb{R}^{13},$$

where ψ_k $(k \neq 0)$ is the stereographic projection defined in §1 and ψ_0 is the identity map of $M^3(0)$. The map $\mathscr{W}$ is injective, since every isometry $E \in G_k$ is uniquely determined by the data (3.1) and (3.2). Moreover, it is easily verified that the restriction of the map $\mathscr{W}|_{\widetilde{\mathscr{I}} \cap G_k}$ is an embedding for each $k \in \mathbb{R}$.

Now we introduce some terminology. Let $U \subset \mathbb{R}^7$ be an open subset. Then an immersion $\varphi : U \to \mathbb{R}^{13}$ is said to be *admissible* if it satisfies (Image of φ) $\subset$ (Image of $\mathscr{W}$). Then we have the following

Lemma 3.2. *The image of $\mathscr{W}$ has a unique differentiable structure as an embedded submanifold of $\mathbb{R}^{13}$ such that any admissible immersion induces its local coordinate system.*

Theorem 3.1 can now follow easily from Lemma 3.2:

Proof of Theorem 3.1. Since $\mathscr{W}|_{\mathscr{I}^+}$ and $\mathscr{W}|_{\mathscr{I}^-}$ are locally admissible, the differentiable structure on $\widetilde{\mathscr{I}}$ induced by $\mathscr{W}$ is compatible with the canonical product structures of $\mathscr{I}^+$ and $\mathscr{I}^-$. Thus $\mathscr{I} = \widetilde{\mathscr{I}} \cup \mathscr{I}^+ \cup \mathscr{I}^-$ has a differentiable structure stated in the theorem with respect to the topology generated by $\{\widetilde{\mathscr{I}}, \mathscr{I}^+, \mathscr{I}^-\}$. □

Before we prove Lemma 3.2, we define the following transformations of $\mathbb{R}^3(k)$, which may have singular points when $k > 0$:

$$S_1(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu_1 \begin{pmatrix} (1 - 4k\tau^2)^{1/2} & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_1(1 - kr^2)\begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix},$$

$$S_2(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \mu_2 \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & (1-4k\tau^2)^{1/2} & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_2(1-kr^2)\begin{pmatrix} 0 \\ \tau \\ 0 \end{pmatrix},$$

$$S_3(k, \theta, \tau)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = T_k(\theta, \tau)$$
$$= \mu_3 \begin{pmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ (1-4k\tau^2)^{1/2} & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \mu_3(1-kr^2)\begin{pmatrix} 0 \\ \tau \\ 0 \end{pmatrix},$$

where $|\tau| < 1/2\sqrt{k}$ for $k > 0$, and

$$\begin{aligned} \mu_i &= \mu_i(k, \theta, \tau, x^1, x^2, x^3) \\ &= 2\{-4k\tau x^i + (1-4k\tau^2)^{1/2}(1-kr^2) + (1+kr^2)\}^{-1} \qquad (i = 1, 2, 3). \end{aligned}$$

By the same argument as that for $S_3(k, \theta, \tau) = T_k(\theta, \tau)$ in the previous section, we can show that $\psi_k^{-1} \circ S_i(k, \theta, \tau) \circ \psi_k \in G_k$ $(i = 1, 2)$. In fact, if $k < 0$ for instance, the corresponding three matrices in G_k are given by

$$\psi_k^{-1} \circ S_1(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cosh\nu & 0 & 0 & \sinh\nu \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \sinh\nu & 0 & 0 & \cosh\nu \end{pmatrix},$$

$$\psi_k^{-1} \circ S_2(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cosh\nu & 0 & \sinh\nu \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sinh\nu & 0 & \cosh\nu \end{pmatrix},$$

$$\psi_k^{-1} \circ S_3(k, \theta, \tau) \circ \psi_k = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\nu & \sinh\nu \\ 0 & 0 & \sinh\nu & \cosh\nu \end{pmatrix},$$

where $\tau = (\sinh\nu)/2\sqrt{|k|}$ (cf. (2.5)).

Using these, we define a smooth map $h_k : \mathbb{R}^6 \to G_k$ $(k \in \mathbb{R})$ by

$$\begin{aligned} &h_k(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) \\ &\quad = \psi_k^{-1} \circ S_1(k, 0, \tau^1) \circ S_2(k, 0, \tau^2) \circ S_3(k, 0, \tau^3) \\ &\qquad \circ S_1(k, \theta^1, 0) \circ S_2(k, \theta^2, 0) \circ S_3(k, \theta^3, 0) \circ \psi_k. \end{aligned}$$

Then one can easily verify that h_k defines locally a diffeomorphism from a neighborhood of the origin onto a neighborhood of the identity.

Proof of Lemma 3.2. By the implicit function theorem, it is sufficient to show that for each $(k, E) \in \widetilde{\mathcal{F}}$, there exists an admissible immersion $\varphi : U \subset \mathbb{R}^7 \to$

$\mathbb{R}^{13}$ such that $(k, E) \in$ (Image of φ). Since $\mathscr{W}|_{\mathscr{I}^+}$ and $\mathscr{W}|_{\mathscr{I}^-}$ are locally admissible, the existence of such a φ is obvious for $(k, E) \in \widetilde{\mathscr{I}}$ $(k \neq 0)$. Now let $(0, E) \in \widetilde{\mathscr{I}}$. Then, since $h_0 : \mathbb{R}^6 \to G_0$ is surjective, there exists a point $\mathbf{a} \in \mathbb{R}^6$ such that $E \in h_0(\mathbf{a})$. We define a smooth map $\varphi : \mathbb{R}^7 \to \mathbb{R}^{13}$ by

$$\varphi(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) = \mathscr{W}(h_k(\mathbf{a}) \circ h_k(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)).$$

Since $\mathscr{W}|_{G_0}$ is an immersion and h_0 is nonsingular at the origin, it is easy to see that the rank of φ at the origin is 7. (We need not calculate the derivative of φ with respect to k because both the domain and the range of φ have the same parameter k.) Thus φ defines an admissible immersion on some neighborhood of the origin such that $(0, E) \in$ (Image of φ).

These results can be extended to a higher-dimensional case. In fact, let $M^n(k)$ be a complete simply connected Riemannian n-manifold of constant sectional curvature k, and $G_k^{(n)}$ the identity component of its isometry group. Then by the same argument as above a differentiable structure on $\mathscr{I}^{(n)} = \{(k, E): E \in G_k^{(n)}, k \in \mathbb{R}\}$ can also be introduced. Furthermore, Tasaki-Umehara-Yamada [3] developed these results for symmetric spaces. We apply these results to hypersurfaces in $M^n(k)$ as follows. Let M be a compact hypersurface of $M^n(k)$. Then the induced metric g and the second fundamental form h satisfy the Gauss and Codazzi equations:

$$(\mathbf{Ga}_k) \quad R(X, Y, Z, W) = k\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} + h(X, Z)h(Y, W) - h(X, W)h(Y, Z) \qquad (X, Y, Z, W \in TM),$$

$$(\mathbf{Co}) \qquad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \qquad (X, Y, Z \in TM),$$

where ∇ is the Levi-Civita connection of g and R denotes its curvature tensor. Now, let g_k $(k \in \mathbb{R})$ be a smooth one-parameter family of Riemannian metrics on a compact $(n-1)$-manifold M and h_k $(k \in \mathbb{R})$ a smooth one-parameter family of symmetric 2-tensors such that g_k and h_k satisfy $(\mathbf{Ga}_k)$ and $(\mathbf{Co})$ for each k. It then follows from the fundamental theorem for hypersurfaces that there exists an immersion $f_k : \widetilde{M} \to M^n(k)$ whose induced metric and second fundamental form coincide with $\pi^* g_k$ and $\pi^* h_k$ respectively, where $\pi : \widetilde{M} \to M$ is the universal covering of M. Note that each deck transformation T of $\widetilde{M}$ preserves $\pi^* g_k$ and $\pi^* h_k$ and hence T extends to an isometry of $M^n(k)$ by the rigidity of f_k. Thus, for each k, we have a representation $\rho_k : \pi_1(M) \to G_k^{(n)}$. Then the following holds.

Proposition 3.3. *The family of the representation*

$$\rho_k : \pi_1(M) \to G_k^{(n)} \subset \mathscr{I}^{(n)} \qquad (k \in \mathbb{R})$$

depends smoothly on the parameter k with respect to the differentiable structure of $\mathscr{I}^{(n)}$.

Proof. We confine our discussion to the case $n = 3$. But the following proof is valid also for the higher-dimensional case. Let $p \in M$ and choose a reference point $q_0 \in \pi^{-1}(p)$. Then each deck transformation T determines uniquely a point $q \in \pi^{-1}(p)$ such that $T(q_0) = q$. If we normalize $\psi_k \circ f_k(q_0) = 0$ and take a frame (e_1, e_2) of (M, g_k) at p, then the isometry E_k corresponding to T satisfies

$$\text{(3.3)}\qquad \begin{gathered} \widetilde{E}_k(0) = \psi_k \circ f_k(q), \qquad d\widetilde{E}_k(\xi_{q_0}) = \xi_q, \\ d(\widetilde{E}_k \circ \psi_k \circ f_k)[(d\pi^{-1})_{q_0}(e_j)] = d(\psi_k \circ f_k)[(d\pi^{-1})_q(e_j)] \quad (j = 1, 2), \end{gathered}$$

where $\widetilde{E}_k = \psi_k \circ E_k \circ \psi_k^{-1} : \mathbb{R}^3(k) \to \mathbb{R}^3(k)$ and ξ is the unit normal vector field of f_0. Since the coefficients of the Frenet equation with respect to the canonical coordinate system of $\mathbb{R}^3(k)$ depends smoothly on k, so does $\psi_k \circ f_k : M \to \mathbb{R}^3(k)$. Thus (3.3) implies that $\mathscr{W}(E_k) \in \mathbb{R}^{13}$ is smooth with respect to k. So, by the definition of our differentiable structure of $\mathscr{I}$, E_k depends smoothly on k. $\square$

4. Smoothness of Normal form

Let

$$\begin{aligned} \mathscr{N}^- &= \{(k, E) \in \mathscr{I} : k < 0 \text{ and all the eigenvalues} \\ &\qquad\qquad \text{of } E \in SO^+(3, 1) \text{ are } 1\}, \\ \mathscr{N}^0 &= \{(0, E) \in \mathscr{I} : E \text{ is the identity or a translation of } \mathbb{R}^3\}, \\ \mathscr{N}^+ &= \{(k, E) \in \mathscr{I} : k > 0 \text{ and } \theta \geq \nu \text{ or } \cos\nu \leq 0 \\ &\qquad\qquad \text{in the decomposition (1.2) of } E \in SO(4)\}, \end{aligned}$$

and define a closed subset in $\mathscr{I}$ by

$$\text{(4.1)}\qquad \mathscr{N} = \mathscr{N}^- \cup \mathscr{N}^0 \cup \mathscr{N}^+.$$

Then for each $(k, E) \in \widetilde{\mathscr{I}} \backslash \mathscr{N}$ there exists $(k, P) \in \mathscr{I}$ such that

$$\text{(4.2)}\qquad P^{-1} \circ E \circ P = T_k(\theta, \tau),$$

where $T_k(\theta, \tau)$ is the normal form defined in §2. In §1 it was proved that θ and τ are determined up to $\mathbb{Z}_2$-ambiguity. In this section, we will see that locally θ and τ are smooth functions on $\widetilde{\mathscr{I}} \backslash \mathscr{N}$ with respect to the differentiable structure defined in §3.

Theorem 4.1. *Let $(k, E) \in \widetilde{\mathscr{I}} \backslash \mathscr{N}$. Then there exists a neighborhood $U \subset \widetilde{\mathscr{I}} \backslash \mathscr{N}$ of (k, E) such that, by taking suitable branches, θ and τ in (4.2) are defined as smooth functions on U.*

If $k \neq 0$, the theorem is obvious. So we may assume $k = 0$. Since each $E \in G_0 \backslash \mathscr{N}^0$ is equivalent to a normal form $T_0(\alpha, \beta)$ $(\alpha, \beta \in \mathbb{R}, \alpha \notin 2\pi\mathbb{Z})$ by (4.2), it is sufficient to prove the following lemma.

Lemma 4.2. *Let $\mathscr{U} : \mathbb{R}^7 \to \mathscr{I}$ be a map defined by*

$$\begin{aligned} \mathscr{U}(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3) = S_1(k, \theta^1, \tau^1) &\circ S_2(k, \theta^2, \tau^2) \circ T_k(\theta^3, \tau^3) \\ &\circ S_2^{-1}(k, \theta^2, \tau^2) \circ S_1^{-1}(k, \theta^1, \tau^1). \end{aligned}$$

Then the Jacobian of $\mathscr{U}$ does not vanish at the point $(0, 0, 0, \alpha, 0, 0, \beta)$ $(\alpha \notin 2\pi\mathbb{Z})$.

Proof. Consider the map $\mathscr{W} \circ \mathscr{U} : \mathbb{R}^7 \to \mathbb{R}^{13}$ Then

$$\operatorname{rank}(d(\mathscr{W} \circ \mathscr{U})) = \operatorname{rank} \frac{\partial(k, w^i, w^{jl})}{\partial(k, \theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)} = 1 + \operatorname{rank} \left.\frac{\partial(w^i, w^{jl})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right|_{k=0}$$

at the point $(0, 0, 0, \alpha, 0, 0, \beta)$. By a straightforward calculation, the derivatives on the right-hand side are given by

$$\begin{aligned}
dw^1 &= (0, -\beta, 0, 0, 1 - \cos\alpha, \sin\alpha), \\
dw^2 &= (-\beta, 0, 0, 0, -\sin\alpha, 1 - \cos\alpha), \\
dw^3 &= (\beta, \beta, 0, 2\beta, 2\beta, 1), \\
dw^{11} &= (0, 0, -\sin\alpha, 2\cos\alpha, 2\cos\alpha, 0), \\
dw^{21} &= (0, 0, \cos\alpha, 2\sin\alpha, 2\sin\alpha, 0), \\
dw^{13} &= (-\sin\alpha, -1 + \cos\alpha, 0, 0, 0, 0), \\
dw^{23} &= (-1 + \cos\alpha, -\sin\alpha, 0, 0, 0, 0),
\end{aligned}$$

which yield

$$\det\left\{\frac{\partial(w^1, w^2, w^3, w^{11}, w^{13}, w^{23})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right\} = 8\sin\alpha(1 - \cos\alpha)^2,$$

$$\det\left\{\frac{\partial(w^1, w^2, w^3, w^{21}, w^{13}, w^{23})}{\partial(\theta^1, \theta^2, \theta^3, \tau^1, \tau^2, \tau^3)}\right\} = -8\cos\alpha(1 - \cos\alpha)^2.$$

Hence, $d\mathscr{U}$ is nondegenerate at $(0, 0, 0, \alpha, 0, 0, \beta)$, since $\alpha \notin 2\pi\mathbb{Z}$. □

By Theorem 4.1, we may regard θ and τ as globally defined functions $\theta : \widetilde{\mathscr{F}\backslash\mathscr{N}} \to \mathbb{R}/2\pi\mathbb{Z}$ and $\tau : \widetilde{\mathscr{F}\backslash\mathscr{N}} \to \mathbb{R}$.

5. Deformation of the immersion

Let $\Omega(a_0, b_0) = (-a_0, a_0) \times (-b_0, b_0)$ be a rectangular domain of $\mathbb{R}^2$. Then the Dirichlet problem of the sinh-Gordon equation

$$\Delta\omega + \cosh\omega \sinh\omega = 0 \tag{5.1}$$

on $\Omega(a_0, b_0)$ has a unique positive solution ω_0 if $a_0^{-2} + b_0^{-2} > 4\pi^{-2}$ [7, 1, 2]. By the odd reflections about $\partial\Omega(a_0, b_0)$, this solution can be extended to a doubly periodic solution $\widetilde{\omega}_0$ of (5.1), which has a rectangular fundamental domain. To get solutions with twisted fundamental domain, we can perturb $\widetilde{\omega}_0$ in the following fashion.

Lemma 5.1 [6, Theorem 1]. *For sufficiently small $a_0, b_0 > 0$, there exist a neighborhood U of $(a_0, b_0, 0) \in \mathbb{R}^3$ and a smooth function $\omega(u, v; a, b, c)$ on $\mathbb{R}^2 \times U$ which satisfy the following conditions:*

(1) *For each $(a, b, c) \in U$, $\omega(u, v; a, b, c)$ is a solution of* (5.1) *on $\mathbb{R}^2$.*

(2) *Let $\mathbf{p}_1 = (2a, 0)$ and $\mathbf{p}_2 = (2c, 2b)$. Then*

$$\omega(\mathbf{u} + \mathbf{p}_1; \mathbf{a}) = \omega(\mathbf{u} + \mathbf{p}_2; \mathbf{a}) = \omega(-\mathbf{u}; \mathbf{a}) = -\omega(\mathbf{u}; \mathbf{a}), \tag{5.2}$$

where $\mathbf{u} = (u, v)$ and $\mathbf{a} = (a, b, c)$.

(3) $\omega(u, v; a_0, b_0, 0) = \widetilde{\omega}_0(u, v)$.

Let $\omega(u, v) = \omega(u, v; a, b, c)$ be a doubly periodic solution determined as above. Define the first fundamental form ds^2 by

$$ds^2 = \frac{e^{2\omega}}{4(H^2+k)}(du^2 + dv^2), \tag{5.3}$$

and the second fundamental form $h = h_{11}du^2 + 2h_{12}du\,dv + h_{22}dv^2$ by

$$\begin{aligned} h_{11} &= \frac{He^{2\omega}}{4(H^2+k)} - \frac{\cos 2\beta}{4(H^2+k)^{1/2}}, \\ h_{12} &= -\frac{\sin 2\beta}{4(H^2+k)^{1/2}}, \\ h_{22} &= \frac{He^{2\omega}}{4(H^2+k)} + \frac{\cos 2\beta}{4(H^2+k)^{1/2}}. \end{aligned} \tag{5.4}$$

Then it is not hard to see that for $H \equiv 1/2$ and $k > -1/4$, ds^2 and h satisfy the Gauss and Codazzi equations in $\mathbb{R}^3(k)$. Hence, by the fundamental theorem for surfaces, they determine, up to an isometry of $\mathbb{R}^3(k)$, an isometric immersion $f_k = f_k(a, b, c, \beta)\colon (\mathbb{R}^2, ds^2) \to \mathbb{R}^3(k)$ with constant mean curvature $H \equiv 1/2$. Since the Frenet equation of f_k with respect to the canonical coordinates on $\mathbb{R}^3(k)$ depends smoothly on k, the immersion $f_k(a, b, c, \beta) : \mathbb{R}^2 \to \mathbb{R}^3(k)$ also depends smoothly on the parameters a, b, c, β, and k.

Since ω has the doubly periodicity condition (5.2), there exist motions $E_i = E_i(k, a, b, c, \beta)$ $(i = 1, 2)$ of $\mathbb{R}^3(k)$ such that

$$f_k(\mathbf{u} + 2\mathbf{p}_i; a, b, c, \beta) = E_i \circ f_k(\mathbf{u}; a, b, c, \beta) \qquad (i = 1, 2). \tag{5.5}$$

It follows from Proposition 3.3 that $E_i(k, a, b, c, \beta)$ $(i = 1, 2)$ are smooth with respect to the parameters a, b, c, β, and k.

Properties of the immersions $f_k = f_k(a, b, c, \beta)$ at $k = 0$ are carefully analyzed by Wente [8], in which those of the form $f_0(a, b, 0, 0)$ $(a^{-2} + b^{-2} > 4\pi^{-2})$ whose images are compact are called *symmetric examples.* The existence of symmetric examples has been shown in Wente [7], Abresch [1], and Walter [5]. Now we assume that $f_0(a_0, b_0, 0, 0)$ yields a symmetric example. Then we may put

$$E_1(0, a_0, b_0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_2(0, a_0, b_0, 0, 0) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\pi < \alpha < 2\pi$ and $\alpha \in 2\pi\mathbb{Q}$ (see [1, 8]).

Note that, since $E_1(0, a_0, b_0, 0, 0) \in \mathscr{N}$, Theorem 4.1 cannot apply directly. So we change a generator $\mathbf{p}_1$ of the lattice $\Gamma = \{\mathbf{p}_1, \mathbf{p}_2\}$ for

$$\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2.$$

Let

$$E_3(k, a, b, c, \beta) = E_1(k, a, b, c, \beta) \circ E_2(k, a, b, c, \beta).$$

Then it is obvious that

$$f_k(\mathbf{u}+2\mathbf{p}_3;\ a, b, c, \beta) = E_3 \circ f_k(\mathbf{u};\ a, b, c, \beta). \tag{5.6}$$

Now we prove our main result:

Theorem 5.2. *Let T^2 be a compact 2-manifold with genus 1. Then for sufficiently small $\varepsilon > 0$, there exists a 1-parameter family of immersions $f_k : T^2 \to \mathbb{R}^3(k)$ $(|k| < \varepsilon)$ with constant mean curvature $H \equiv 1/2$.*

Proof. Using Theorem 4.1, we can define smooth functions $\tilde{\theta}_i$ and $\tilde{\tau}_i$ $(i = 2, 3)$ on some neighborhood of $(0, a_0, b_0, 0, 0) \in \mathbb{R}^5$ by

$$\begin{aligned} \tilde{\theta}_i(k, a, b, c, \beta) &= \theta(E_i(k, a, b, c, \beta)) \\ \tilde{\tau}_i(k, a, b, c, \beta) &= \tau(E_i(k, a, b, c, \beta)) \end{aligned} \qquad (i = 2, 3).$$

Thus, to prove the theorem, it suffices to show that the set

$$\{(k, a, b, c, \beta) \in U : \tilde{\theta}_i(k, a, b, c, \beta) \equiv \alpha \in 2\pi\mathbb{Q} \text{ and } \tilde{\tau}_i(k, a, b, c, \beta) = 0\ (i = 2, 3)\}$$

defines a regular curve with respect to k through the point $(0, a_0, b_0, 0, 0)$. To see this, we define a map $\varphi : U \to \mathbb{R}^5$ by

$$\varphi(k, a, b, c, \beta) = (k, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3).$$

In [8] Wente introduced functions θ_1, θ_2, τ_1, and τ_2 of variables a, b, c, and β in such a way that $E_i(a, b, c, \beta)$ $(i = 1, 2)$ are equivalent to $T_0(\theta_i, \tau_i)$, for which he showed that

$$\det\left\{\frac{\partial(\theta_1, \theta_2, \tau_1, \tau_2)}{\partial(a, b, c, \beta)}\right\} \neq 0 \tag{5.7}$$

at $(a_0, b_0, 0, 0)$. It is easily verified that these functions are related to $\tilde{\theta}_2$, $\tilde{\theta}_3$, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ by

$$\begin{aligned} \tilde{\theta}_2(0, a, b, c, \beta) &= \theta_2(a, b, c, \beta), \\ \tilde{\theta}_3(0, a, b, c, \beta) &= \theta_1(a, b, c, \beta) + \theta_2(a, b, c, \beta), \\ \tilde{\tau}_2(0, a, b, c, \beta) &= \tau_2(a, b, c, \beta), \\ \tilde{\tau}_3(0, a, b, c, \beta) &= \tau_1(a, b, c, \beta) + \tau_2(a, b, c, \beta). \end{aligned}$$

Hence we have from (5.7)

$$\begin{aligned} \operatorname{rank}(d\varphi) &= \operatorname{rank}\left\{\frac{\partial(k, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3)}{\partial(k, a, b, c, \beta)}\right\} \\ &= 1 + \operatorname{rank}\left\{\left.\frac{\partial(\tilde{\theta}_2, \tilde{\theta}_3, \tilde{\tau}_2, \tilde{\tau}_3)}{\partial(a, b, c, \beta)}\right|_{k=0}\right\} \\ &= 1 + \operatorname{rank}\left\{\frac{\partial(\theta_1, \theta_2, \tau_1, \tau_2)}{\partial(a, b, c, \beta)}\right\} = 5 \end{aligned}$$

at $(0, a_0, b_0, 0, 0)$. Thus $\varphi^{-1}(k, \alpha, 0, \alpha, 0)$ determines a regular curve on some small neighborhood of $(0, a_0, b_0, 0, 0) \in U$. □

Corollary 5.3. *Any open subset of the 3-sphere or the hyperbolic 3-space contains a torus with constant mean curvature.*

Proof. Let $\{f_k : T^2 \to \mathbb{R}^3(k)\}_{|k|<\varepsilon}$ be as in Theorem 5.2. Then, for sufficiently small ε, the images $\{f_k(T^2)\}_{|k|<\varepsilon}$ are uniformly bounded. Namely, $f_k(T^2)$ is contained in the ball of radius a with respect to g_k for each $k \in (-\varepsilon, \varepsilon)$, where $a > 0$ is a universal constant.

Assume $k < 0$ and define

$$\tilde{f} = \sqrt{|k|} \cdot (\psi_k^{-1} \circ f_k) : T^2 \to M^3(-1),$$

where ψ_k is the stereographic projection (2.1) and $\cdot$ is the scalar multiplication in $M^3(k) \subset \mathbb{L}^4$. Then $\tilde{f}$ gives an immersion of T^2 into the hyperbolic 3-space $M^3(-1)$ with constant mean curvature $1/2\sqrt{|k|}$. Moreover, $\tilde{f}(T^2)$ is contained in the ball of radius $\sqrt{|k|}a$ in $M^3(-1)$, since $f_k(T^2)$ is bounded by the ball with radius a.

Hence, taking a sufficiently small $k < 0$, we can find an immersion of T^2 with constant mean curvature into the hyperbolic 3-space with sufficiently small radius.

Similarly, assuming $k > 0$, we have the conclusion for the 3-sphere. □

By using (5.7), the existence of nonholomorphic harmonic maps of tori generated by any lattice into the unit sphere has been shown in Umehara-Yamada [4], which is based on the fact that Gauss maps of surfaces with constant mean curvature in $\mathbb{R}^3(0)$ are harmonic.

References

1. U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169–192.
2. J. Spruck, *The elliptic sinh-Gordon equation and the construction of toroidal soap bubbles*, Lecture Notes in Math., vol.1340, Springer-Verlag, 1988, pp. 275–381.
3. H. Tasaki, M. Umehara, and K. Yamada, *Deformations of symmetric spaces of compact type to their noncompact duals*, Japan. J. Math. **17** (1991) (to appear).
4. M. Umehara and K. Yamada, *Harmonic non-holomorphic maps of 2-tori into the 2-sphere*, Geometry of Manifolds, Academic Press, 1989, pp. 151–161.
5. R. Walter, *Explicit examples to the H-problem of Heinz Hopf*, Geom. Dedicata **23** (1987), 187–213.
6. ———, *Constant mean curvature tori with spherical curvature lines in noneuclidean geometry*, Manuscripta Math. **63** (1989), 343–363.
7. H. C. Wente, *Counterexamples to a conjecture of H. Hopf*, Pacific J. Math. **121** (1986), 193–243.
8. ———, *Twisted tori in constant mean curvature in* $\mathbf{R}^3$, Seminar on New Results in Nonlinear Partial Differential Equations, Max-Plank-Institut für Mathematik, Bonn, 1987, pp. 1–36.

Institute of Mathematics, University of Tsukuba, Tsukuba Ibaraki 305, Japan

Faculty of General Education, Kumamoto University, Kumamoto 860, Japan